Methods of Finding Point Estimators

Method of Moments

- The rth population moment about the origin (raw) of a random variable Y is

$$\mu_r' = E(Y^r).$$

- The rth sample moment about the origin (raw) of a random iid (independent & identically distributed) sample is

$$m_r' = \frac{1}{n}\sum_{i=1}^{n} Y_i^r.$$

- If the distribution of Y has K parameters then the method of moment estimators of $\theta_1, \theta_2, \ldots, \theta_K$ are obtained by solving the K-equations:

$$\boxed{\mu_r' = m_r'} \text{ for } r = 1, 2, \ldots, K.$$

- Example: $Y_i \overset{iid}{\sim} \text{Poisson}(\lambda)$; $i=1,2,\dots,n$

- Then $\mu_1' = E(Y^1) = \lambda$

$$m_1' = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}$$

- Setting $\mu_1' = m_1'$, we have

$$\hat{\lambda} = \bar{Y}.$$

- So $\hat{\lambda}_{MOM} = \bar{Y}$.

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• Method of Maximum Likelihood (ML):

⊙ Likelihood function: If the data are independent —

$$\underbrace{L(\theta_1, \ldots, \theta_k; y_1, \ldots, y_n)}_{\downarrow\; L(\underset{\sim}{\theta}; \underset{\sim}{y})} = \prod_{i=1}^{n} \underbrace{f(y_i; \theta_1, \theta_2, \ldots, \theta_k)}_{\text{pdf/pmf}}$$

$$L(\underset{\sim}{\theta}; \underset{\sim}{y})$$

where, $\underset{\sim}{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_k \end{pmatrix}$ $\quad \underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

⊙ Likelihood is Not a probability function.

Illustration with an Example

- Likelihood function is used to find the value of θ for which the observed sample is more likely.

Example: $Y \sim$ Bernoulli (θ) → Prob. of success

- For $n=4$, consider two possible values of θ, say $\theta_1 = 0.3$ and $\theta_2 = 0.8$.

Case I: If $(0,0,0,0)$ is observed which one θ_1 or θ_2 is more likely?	Case II: If $(1,1,1,0)$ is observed which one θ_1 or θ_2 is more likely?
$L(\theta;\underset{\sim}{y}) = \prod_{i=1}^{4} f(y_i;\theta) = \prod_{i=1}^{4} \theta^{y_i}(1-\theta)^{1-y_i} = \theta^{\Sigma y_i}(1-\theta)^{4-\Sigma y_i}$	$L = \theta^{\Sigma y_i}(1-\theta)^{4-\Sigma y_i}$
For $\theta=\theta_1=0.3 \Rightarrow L=(0.3)^0(0.7)^{4-0} = 0.2401$	For $\theta=\theta_1=0.3$: $L=(0.3)^3(0.7)^{4-3}$ $= \cancel{0.0189}$ $= 0.1089$
$\theta=\theta_2=0.8 \Rightarrow L = 0.0016$	$\theta=\theta_2=0.8$: $L=(0.8)^3(0.2)^{4-3}$ $=0.1024$
$\therefore \theta = 0.3$ is more likely.	$\therefore \theta = 0.8$ is more likely

⊙ Description of ML method;
(maximum likelihood)

- MLE's are obtained by maximizing the likelihood $L = L(\underset{\sim}{\theta}; \underset{\sim}{y})$ (or equivalently log likelihood).

- Solving the likelihood equations:

$$\frac{\partial L}{\partial \theta_j} = 0, \quad j = 1, 2, \dots, K$$

Equivalently, $$\frac{\partial \operatorname{Log} L}{\partial \theta_j} = 0, \quad j = 1, 2, \dots, K$$

• Example:

$$Y_i \overset{iid}{\sim} \text{Poisson}(\theta) \ ; \ i=1,2,\cdots,n$$

$$\therefore f(y,\theta) = \frac{e^{-\theta}\theta^{y}}{y!}$$

$$\therefore L(\theta;\underset{\sim}{y}) = \prod_{i=1}^{n} f(y_i;\theta)$$

$$= \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{y_i}}{y_i!}$$

$$= \frac{e^{-n\theta}\,\theta^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}$$

$$\ell = \text{Log}\,L(\theta;\underset{\sim}{y}) = -n\theta + \sum_{i=1}^{n} y_i \log\theta - \log\left(\prod_{i=1}^{n} y_i!\right)$$

$$\therefore \frac{d\ell}{d\theta} = \frac{d\log L(\theta;\underset{\sim}{y})}{d\theta} = 0$$

$$\Rightarrow -n + \left(\sum_{i=1}^{n} y_i\right)\cdot\frac{1}{\theta} = 0$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}$$

$$\therefore \boxed{\hat{\theta}_{MLE} = \bar{Y}}.$$