Lecture 7
Probability

Some Basic Concepts Related to Probability

1. Random Experiment

A random experiment is a **process** leading to

- results **in one and only one** of **two or more possible outcomes**,
- possible outcomes are **known in advance**, and
- outcome of a specific trial is **uncertain**.

Examples:

i. A **coin is tossed** and the outcome is either a **head or a tail**.
ii. A **customer enters a cloth store** and either **purchases a shirt or does not**.
iii. The **daily change in an index** of stock market prices is observed and it will **either change or not**.
iv. A **six-sided die** is rolled and possible outcomes are **1, 2, 3, 4, 5, or 6.**

In each of the experiments listed above,

- possible outcomes are at least two,
- we can specify the possible outcomes, and
- one cannot say which outcome will occur in a specific trial

and thus, are examples of random experiments

2. Sample space

- The **set of all possible outcomes** of a random experiment is called a sample space.
- The **symbol S** or ***Ω*** is used to denote the sample space.

Presentation of sample space

Sample space can be presented in three different ways:

i. Set format
ii. Van diagram and
iii. Tree diagram

i. Set Format

Table below lists some examples of experiments, their outcomes, and their sample spaces in set format.

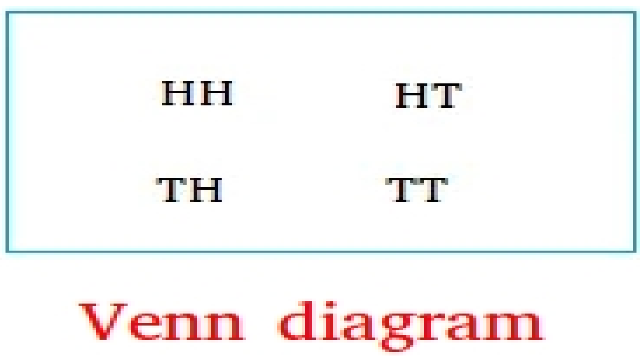
Table: Examples of Experiments, Possible outcomes, and Sample Spaces in set format

Experiment	**Possible outcomes**	**Sample Space**
Toss a coin once	Head, Tail	S = {Head, Tail}
Roll a die once	1, 2, 3, 4, 5, 6	S ={1, 2, 3, 4, 5, 6}
Toss a coin twice	*HH*, *HT*, *TH*, *TT*	S ={*HH*, *HT*, *TH*, *TT*}
Play lottery	Win, Lose	S ={Win, Lose}
Take a test	Pass, Fail	S = {Pass, Fail}
Select a worker	Male, Female	S ={Male, Female}

ii. Venn diagram

A **Venn diagram** is a picture (a closed geometric shape such as a rectangle, a square, or a circle) that depicts all the possible outcomes for an experiment.

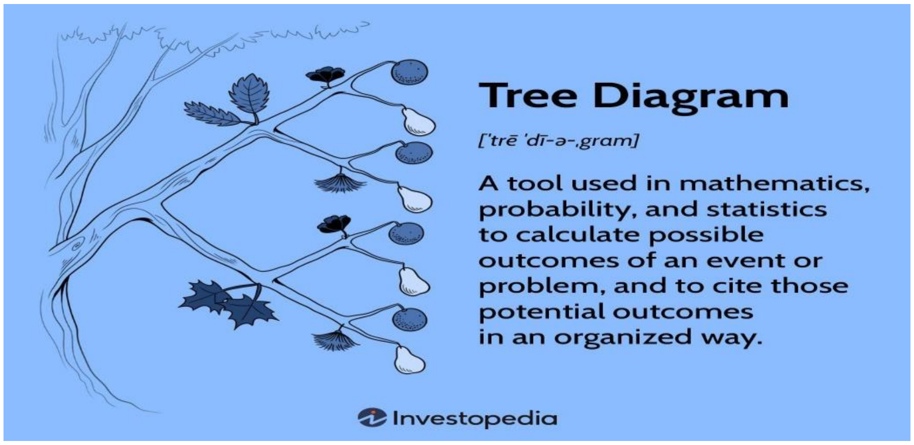
Example: Toss a coin twice



Example: Roll a die once.

iii. Tree Diagram

- In a **tree diagram**, each outcome is represented by a branch of the tree.



- Venn and tree diagrams help us understand probability concepts by presenting them visually.

Example: Draw Venn and tree diagrams for the experiment of tossing a coin twice.

Solution

- This experiment can be split into two parts:
  a. the first toss and
  b. the second toss.
- The Venn diagram (Figure a) and the tree diagram (Figure b) are given below.
- Both of these diagrams show the sample space for this experiment.

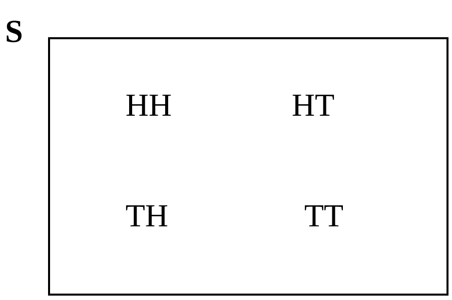


Figure a

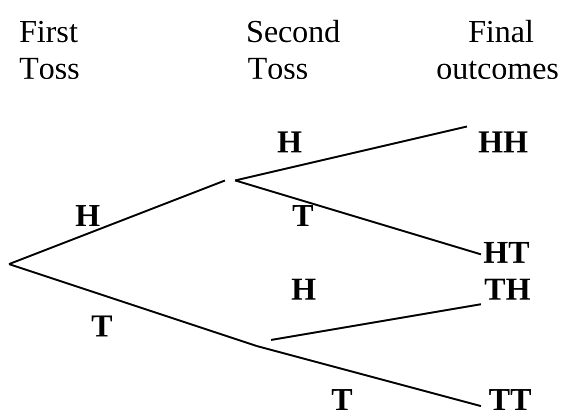
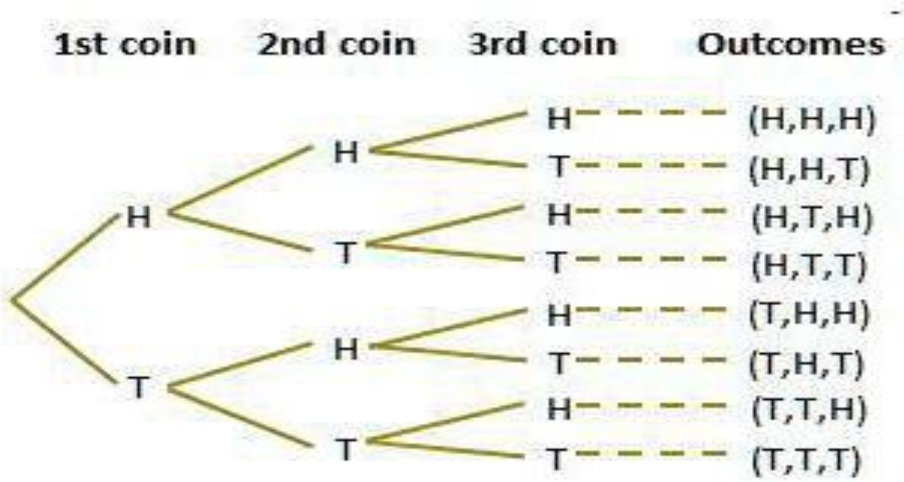


Figure b.

Figure 1 (*a*) Venn diagram and (*b*) tree diagram for two tosses of a coin.

Example: Draw tree diagram for the experiment of tossing a coin three times

Solution: The following figure shows the tree diagram of sample space for the experiment of tossing a coin three times.



Example

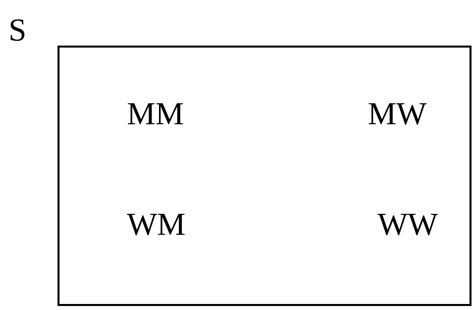
Suppose we randomly select two workers from a company and observe whether the worker selected each time is a man or a woman. Write the sample space of outcomes for this experiment using i. set format ii. the Venn diagram and iii. tree diagram for this experiment.

Solution Let us denote the selection of a man by *M* and that of a woman by *W*. The possible outcomes for this experiment are: *MM*, *MW*, *WM*, *WW*. Hence, the sample space is written in

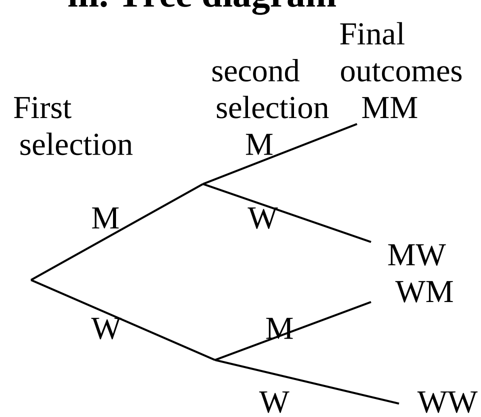
i. set format as

$$S = \{MM, MW, WM, WW\}$$

ii. Venn diagram



iii. Tree diagram



3. Event

- Events in probability can be defined as certain outcomes of a random experiment.
- Events in probability are a subset of the sample space.
- The types of events in probability are
    i. Simple event,
    ii. Compound event,
    iii. Mutually exclusive events,
    iv. Collectively exhaustive events,
    v. Complementary event,
    vi. Sure event,
    vii. Impossible event,
    viii. Equally likely event,
    ix. Independent event, and
    x. Dependent event.

i. Simple Event

An event that includes one and only one of the (final) outcomes for a random experiment is called a *simple event* and is usually denoted by $\boldsymbol{E_i}$.

Example

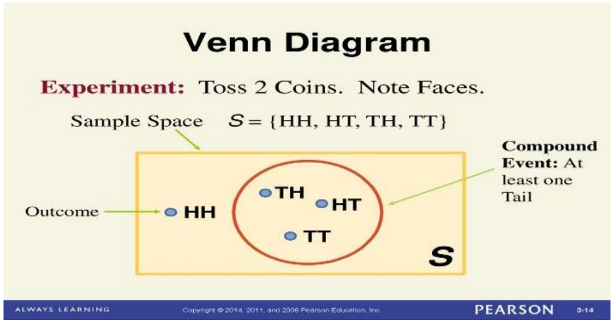
Reconsider the example for selecting two workers from a company and observing whether the worker selected each time is a man or a woman. The possible four outcomes are: *MM*, *MW*, *WM*, and *WW*. Thus, the four simple events for this experiment are as follows:

$$E_1 = \{MM\}, \quad E_2 = \{MW\}, \quad E_3 = \{WM\}, \text{ and } \quad E_4 = \{WW\}$$

ii. Compound Event

- A *compound event* is a collection of more than one outcome for an experiment.
- Compound events are denoted by A, B, C, D,... or by A_1, A_2, A_3,..., and so forth.

Example:



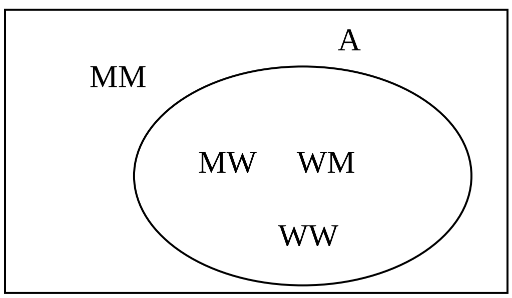
Example: Reconsider Example on selecting two workers from a company and observing whether the worker selected each time is a man or a woman.

- Let *A* be the event that at most one man is selected.
- The set format of the event A is

  A= {MW, WM, WW}

Because event *A* contains more than one outcome, it is a compound event.

- The following Venn diagram gives a graphic presentation of compound event *A*.



Example:

In a group of people, some are in favor of genetic engineering and others are against it. Two persons are selected at random from this group and asked whether they are in favor of or against genetic engineering.

i. How many distinct outcomes are possible?

ii. Draw a Venn diagram and a tree diagram for this experiment.

iii. List all the outcomes included in each of the following events and state whether they are simple or compound events.

   (a) Both persons are in favor of genetic engineering.

   (b) At most one person is against genetic engineering.

   (c) Exactly one person is in favor of genetic engineering.

Solution

i. Let

F = a person is in favor of genetic engineering

A = a person is against genetic engineering

This experiment has the following four outcomes:

FF = both persons are in favor of genetic engineering

FA =the first person is in favor and the second is against

AF =the first person is against and the second is in favor

AA =both persons are against genetic engineering

ii. The Venn and tree diagrams in the following Figure show these four outcomes.

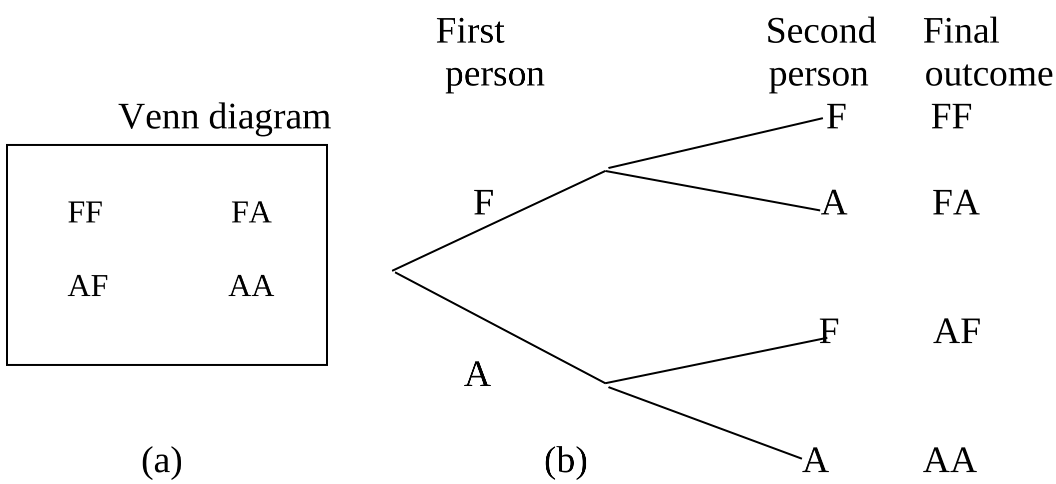


Figure: Venn and tree diagrams

iii. (a) Let A be the event that "both persons are in favor of genetic engineering". Thus,

A = {*FF*}

Because this event includes only one of the final four outcomes, it is a **simple** event.

(b) Let B be the event "at most one person is against genetic engineering". Consequently,

B = {*FF*, *FA*, *AF*}

Because this event includes more than one outcome, it is a **compound** event.

(c) Let C be the event "exactly one person is in favor of genetic engineering". Hence, C includes the following two outcomes:

$$C = \{FA, AF\}$$

Because this event includes more than one outcome, it is a **compound** event.

iii. Mutually Exclusive Events

- If A and B be two events, then they are said to be mutually exclusive if $A \cap B = \Phi$.
- That is, two events are said to be mutually exclusive if they have no common points.

Example: Suppose a coin is tossed twice. Let H and T denote the head and tail of the coin respectively. Then the sample space of the experiment is

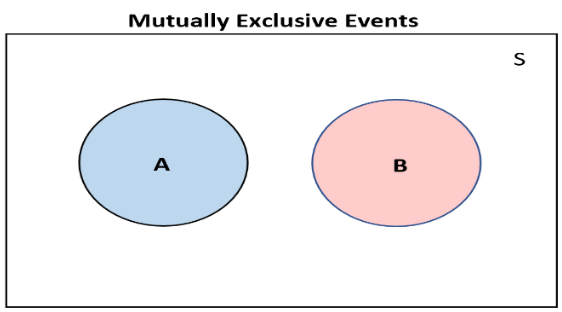
S = {HH, HT, TH, TT}

Let A be the event of head of the first coin and B be the event of tail of the first coin, then A and B will contain the sample points

A = {HH, HT} and

B = {TH, TT}.

Since A and B have no point in common, A and B are said to be mutually exclusive.

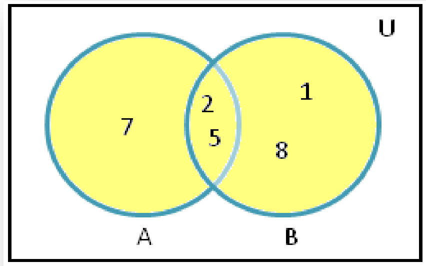


Union

- Let A and B be two events in the sample space, S.
- The union of events A and B, denoted A∪B, is the collection of all outcomes that are elements of one or the other of the sets A and B, or of both of them.
- Hence, the union $A \cup B$ occurs if and only if either A or B or both occur.

Example: Let A be the event of A={2, 5, 7} elements and B be the event of B = {1, 2, 5, 8}, then

$$A \cup B = \{1,2,5,7,8\}$$

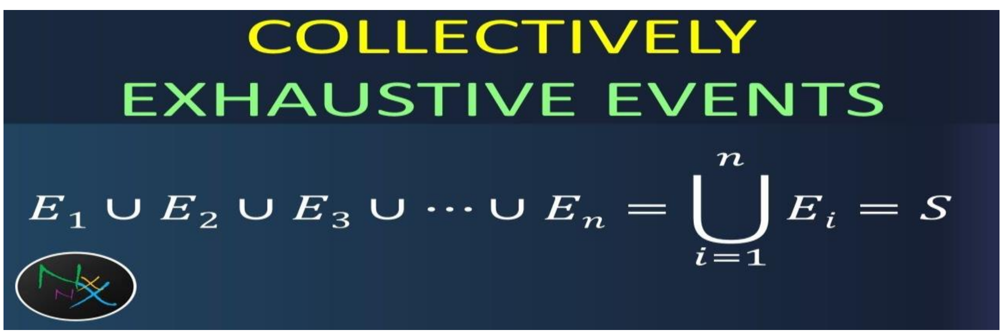


$A \cup B = \{1, 2, 5, 7, 8\}$

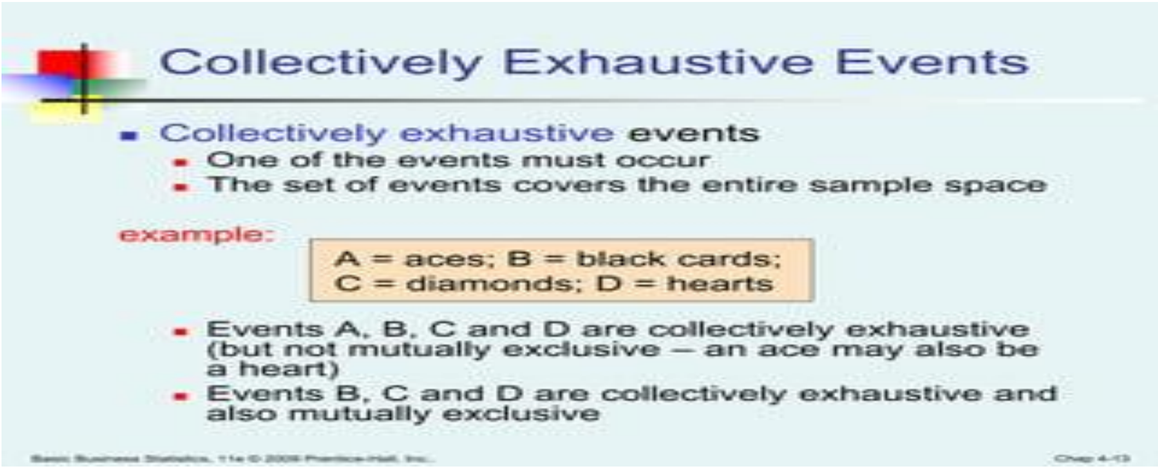
- More generally, given the K events E_1, E_2,……….,E_k, their union, $E_1 \cup E_2 \cup .... \cup E_k$ is the set of all basic outcomes belonging to at least one of these K events.

iv. Collectively Exhaustive Events

- If the union of several events covers the entire sample space, S, we say that these events are mutually collectively exhaustive.
- For example, the events $E_1, E_2, \dots \dots \dots \dots \dots., E_n$ are collectively exhaustive because



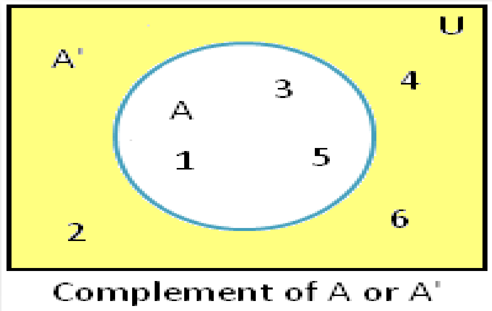
In other words,



v. Complement of an Event

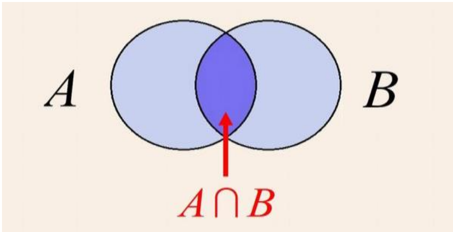
- Let A be an event in the sample space, S. The set of basic outcomes of a random experiment belonging to S but not to A is called the complement of A and is denoted by $\bar{A}$ *or* A' *or* A^c.

Example: A die is rolled. Let A be the event "Number of resulting is even", then what is the complement of A.



Intersection of Two Events

The intersection of events A and B, denoted A∩B, is the collection of all outcomes that are elements of both of the sets A and B.



Example: A die is rolled. Let A be the event "Number resulting is even" and B the even "Number resulting is at least 4." Then

A = {2, 4, 6} and B = {4, 5, 6}.

Find the complement of each event, the intersection and the union of A and B, and the intersection of $\overline{A}$ and B.

Solution: The complements of these events are, respectively

$$\overline{A} = \{1,3,5\} \qquad and \qquad \overline{B} = \{1,2,3\}$$

The intersection of A and B is

$$A \bigcap B = \{4,6\}$$

The union of A and B is

$$A \bigcup B = \{2,4,5,6\}$$

Note also that the events A and $\overline{A}$ are mutually exclusive, since their intersection is the empty set, and collectively exhaustive, because their union is the sample space S; that is,

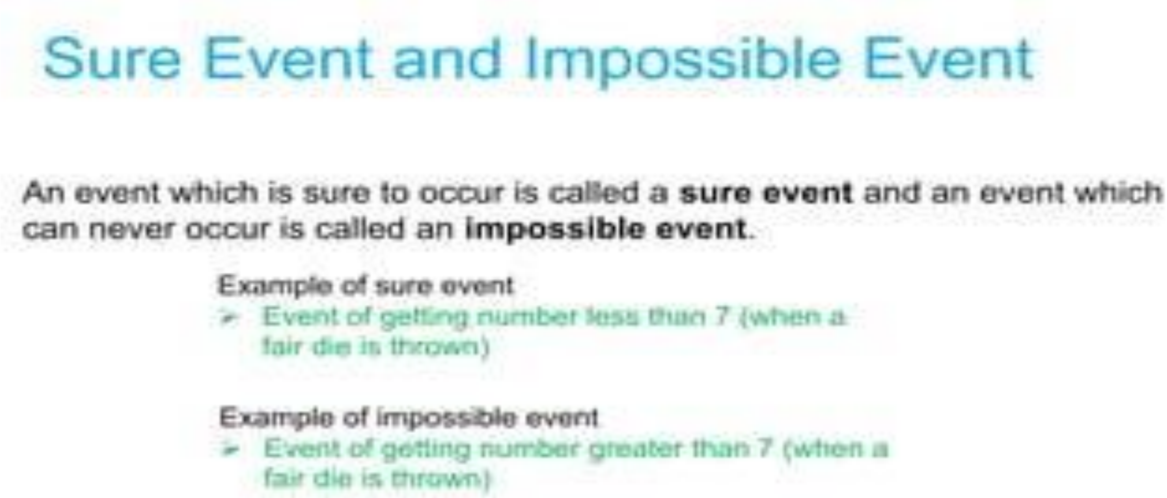
$$A \bigcup \overline{A} = \{1,2,3,4,5,6\} = S$$

vi. Sure Event

- A sure event is defined as the event that always happens.
- Example of a sure event, The probability of getting a number when a die is thrown is a sure event.

vii. Impossible Event

- An event which is impossible to occur, is called an **impossible event**.



Calculating Probability

- **Probability**, which gives the likelihood of occurrence of an event, is denoted by *P*.
- The probability that a simple event *Ei* will occur is denoted by *P*(*Ei*), and the probability that a compound event *A* will occur is denoted by *P*(*A*).

Properties of Probability

1. The probability of an event always lies in the range 0 to 1.

- Whether it is a simple or a compound event, the probability of an event is never less than 0 or greater than 1.
- Using mathematical notation, we can write this property as follows.

$$0 \leq P(E_i) \leq 1$$
$$0 \leq P(A) \leq 1$$

  - An event that cannot occur has zero probability; such an event is called an **impossible event**.
  - An event that is certain to occur has a probability equal to 1 and is called a **sure event**.

- For an impossible event *M*: *P*(*M*) = 0
- For a sure event *C*: *P*(*C*) = 1

2. The sum of the probabilities of all simple events (or final outcomes) for an experiment, denoted by $\sum \boldsymbol{P(Ei)}$, is always 1.

Using mathematical notation,

$$\sum P(E_i) = P(E_1) + P(E_2) + \cdots \ldots \ldots \ldots \ldots \ldots = 1$$

Conceptual Approaches to Define Probability

The three conceptual approaches to probability are

(1) classical probability,
(2) the relative frequency concept of probability, and
(3) the subjective probability concepts.

(1) Classical Probability

- **Equally Likely Outcomes:**
  Two or more outcomes (or events) that have the same probability of occurrence are said to be *equally likely outcomes* (or events).

- The classical probability rule is applied to compute the probabilities of events for an experiment for which all outcomes are equally likely.

Classical Probability Rule to Find Probability

- According to the **classical probability rule**, the probability of a simple event is equal to 1 divided by the total number of outcomes for the experiment.

- In contrast, the probability of a compound event A is equal to the number of outcomes favorable to event A divided by the total number of outcomes for the experiment.

$$P(E_i) = \frac{1}{Total\ number\ of\ outcomes\ for\ the\ experiment}$$

$$P(A) = \frac{\text{Number of outcomes favorable to } A}{\text{Total number of outcomes for the experiment}} = \frac{N_A}{N}$$

Example

In a group of 500 women, 120 have played golf at least once. Suppose one of these 500 women is randomly selected. What is the probability that she has played golf at least once?

Solution

Because the selection is to be made randomly, each of the 500 women has the same probability of being selected. Consequently, this experiment has a total of 500 equally likely outcomes. One hundred twenty of these 500 outcomes are included in the event that the selected woman has played golf at least once. Hence,

$$P(\text{selected woman has played golf at least once}) = \frac{120}{500} = 0.24$$

Example: A small computer store has three Gateway and two Compaq computers in stock on a particular day. Suppose a customer comes into the store to purchase two computers. Any computer on the shelf is equally likely to be selected. What is the probability that the customer will purchase one Gateway and one Compaq computer?

Solution: Let us define the three Gateway computers as, G_1, G_2, and G_3 and two Compaq computers as C_1 and C_2. The sample space, S, contains the following pairs of computers:

$$S = \{G_1C_1, G_1C_2, G_2C_1, G_2,C_2, G_3,C_1, G_3C_2, G_1G_2, G_1G_3, G_2G_3, C_1C_2\}$$

Here N= 10.

If A is the event "One Gateway and one Compaq computers are chosen," then N_A= 6

$$P(A) = \frac{N_A}{N} = \frac{6}{10} = 0.6$$

Formula for Determining the Number of Combinations

The counting process can be generalized by using the following equation to compute the number of combinations of n items taken k at a time:

$$C_k^n = \frac{n!}{k!(n-k)!} \qquad 0! = 1$$

Thus, the number of combinations of the five computers taken two at a time is the number of elements in the sample space:

$$C_2^5 = \frac{5!}{2!(5-2)!} = \frac{5.4.3.2.1}{2.1(3.2.1)} = 10$$

Example: Suppose that Karlyn's store now contains 10 Gateway computers, 5 Compaq Computers, and 5 Acer computers. Susan enters the store and wants to purchase 3 computers. The computers are selected purely by chance from the shelf. Now what is the probability that 2 Gateway computers and 1 Compaq computer are selected?

Solution:

The total number of outcomes in the sample space is

$$N = C_3^{20} = \frac{20!}{3!(20-3)!} = 1{,}140$$

The number of ways that we can select 2 Gateway computers from the 10 available is computed by

$$C_2^{10} = \frac{10!}{2!(10-2)!} = 45$$

Similarly, the number of ways that we can select 1 Compaq computer from the 5 available is computed by

$$C_1^5 = \frac{5!}{1!(5-1)!} = 5$$

Therefore, the number of outcomes that satisfy event A is

$$N_A = C_2^{10} \times C_1^5 = 45 \times 5 = 225$$

Finally, the probability of A = [2 Gateways and 1 Compaq] is

$$P(A) = \frac{N_A}{N} = \frac{225}{1{,}140} = 0.197$$

Relative Frequency Concept of Probability

Suppose we want to calculate the following probabilities:

1. The probability that the next car that comes out of an auto factory is a "lemon"
2. The probability that a randomly selected family owns a home
3. The probability that a randomly selected woman has never smoked
4. The probability that an 80-year-old person will live for at least 1 more year
5. The probability that the tossing of an unbalanced coin will result in a head
6. The probability that a randomly selected person owns a sport-utility vehicle (SUV)

These probabilities cannot be computed using the classical probability rule because the various outcomes for the corresponding experiments are not equally likely. For example, the next car manufactured at an auto factory may or may not be a lemon. The two outcomes, "it is a lemon" and "it is not a lemon," are not equally likely. If they were, then (approximately) half the cars manufactured by this company would be lemons, and this might prove disastrous to the survival of the firm.

- Although the various outcomes for each of these experiments are not equally likely, each of these experiments can be performed again and again to generate data.
- In such cases, to calculate probabilities, we either use past data or generate new data by performing the experiment a large number of times.
- The relative frequency of an event is used as an approximation for the probability of that event.
- This method of assigning a probability to an event is called the **relative frequency concept of probability**.
- Because relative frequencies are determined by performing an experiment, the probabilities calculated using relative frequencies may change almost each time an experiment is repeated.
- For example, every time a new sample of 500 cars is selected from the production line of an auto factory, the number of lemons in those 500 cars is expected to be different.
- However, the variation in the percentage of lemons will be small if the sample size is large.
- Note that if we are considering the population, the relative frequency will give an exact probability.

Using Relative Frequency as an Approximation of Probability

If an experiment is repeated *n* times and an event *A* is observed *f* times, then, according to the relative frequency concept of probability,

$$P(A) = \frac{f}{n}$$

Example

Ten of the 500 randomly selected cars manufactured at a certain auto factory are found to be lemons. Assuming that the lemons are manufactured randomly, what is the probability that the next car manufactured at this auto factory is a lemon?

Solution Let *n* denote the total number of cars in the sample and *f* the number of lemons in *n*. Then,

n = 500 and f = 10

Using the relative frequency concept of probability, we obtain

$$P(\text{next car is a lemon}) = \frac{f}{n} = \frac{10}{500} = .02$$

This probability is actually the relative frequency of lemons in 500 cars. Table below lists the frequency and relative frequency distributions for this example.

Table: Frequency and Relative Frequency Distributions for the Sample of Cars

Car	*f*	*Relative frequency*
Good	*490*	*490/500=.98*
Lemmon	*10*	*10/500=.02*
	n=500	*Sum=1*

The column of relative frequencies in Table is used as the column of approximate probabilities. Thus, from the relative frequency column,

P(next car is a lemon) = 0**.02**
P(next car is a good car) = 0**.98**

- **Note that relative frequencies are not exact probabilities but are approximate probabilities unless they are based on a census**.
- However, if the experiment is repeated again and again, this approximate probability of an outcome obtained from the relative frequency will approach the actual probability of that outcome. This is called the **Law of Large Numbers**.

Subjective Probability
Subjective probability is the probability assigned to an event based on subjective judgment, experience, information, and belief.

Probability Rules
Complement Rule
Let A be an event and $\overline{A}$ its complement. Then the complement rule is

$$P(\overline{A}) = 1 - P(A)$$

The event A and its complement, $\overline{A}$, being mutually exclusive and collectively exclusive

$$P(A \cup \bar{A}) = P(A) + P(\bar{A})$$

and thus

$$P(\overline{A}) = 1 - P(A)$$

Example: Fairselect Inc. is hiring managers to fill four key positions. The candidates are five men and three women. Assuming that every combination of men and women is equally likely to be chosen, what is the probability that at least one woman will be selected?

Solution: We will solve this problem by first computing the probability of the complement of A, "No woman is selected," and then using the complement rule to compute the probability probabilities of one through three women being selected. Using the method of classical probability,

$$P(\overline{A}) = \frac{C_4^5}{C_4^8} = \frac{1}{14}$$

and, therefore, the required probability is

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{14} = \frac{13}{14}$$

The Addition Rule of Probability

Let A and B be two events. Using the addition rule of probabilities, the probability of their union is

$$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$$

Example: A hamburger chain found that 75% of all customers use mustered, 80% use ketchup, and 65% use both. What is the probability that a customer will use at least one of these?

Solution: Let A be the event "Customer uses mustard" and B the event "Customer uses ketchup." Thus, we have

P(A) = 0.75 P(B) = 0.80 and $P(A \bigcap B) = 0.65$

The required probability is

$$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$$
$$= 0.75 + 0.80 - 0.65 = 0.90$$

Conditional Probability

Let A and B be two events. The conditional probability of event A, given that event B has occurred, is denoted by the symbol P(A/B) and is fond to be

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided P(B) > 0}$$

Similarly,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{provided P(A) > 0}$$

Example: A hamburger chain found that 75% of all customers use mustered, 80% use ketchup, and 65% use both. What are the probabilities that ketchup user uses mustard and mustard user uses ketchup?

Solution: Let A be the event "Customer uses mustard" and B the event "Customer uses ketchup." Thus, we have

P(A) = 0.75 P(B) = 0.80 and $P(A \cap B) = 0.65$

The conditional probability that ketchup user uses mustard is the conditional probability of event A, given B.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.65}{0.80} = 0.8125$$

In the same way, the probability that a mustard user uses ketchup is

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.65}{0.75} = 0.8667$$

These calculations can also be developed using the following Table.

Table: Joint Probability for Mustard and Ketchup

	Mustard	No Mustard	Total
Ketchup	0.65	0.45	.080
No Ketchup	0.10	0.10	0.20
Total	0.75	0.25	1.0

Example: Suppose, the residents of a town are classified according to two attributes, the ownership of a house (A) and the ownership of a car (B). The following table summarizes the information:

	A	$\overline{A}$	Total
B	80	100	180
$\overline{B}$	20	800	820
Total	100	900	1000

Suppose we know that a person chosen at random has a house. What is the probability that he has a car too?

Solution: Here

$$P(A) = \frac{100}{1000} = 0.10$$

$$P(B) = \frac{180}{1000} = \frac{18}{100} = 0.18$$

$$P(A \cap B) = \frac{80}{1000} = 0.080$$

Hence the probability that a person has a house has a car too

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.08}{0.10} = \frac{8}{10} = 0.80$$