$AB \neq BA \Rightarrow$ [illegible] $\Big\}$ $AB = BA$

$x^T y \in \mathbb{R} = \sum_i x_i y_i$ inner product dot product

$$x \cdot y = x^T y = \sum_i x_i y_i = y^T x$$

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outer product $x \in \mathbb{R}^n$ $y \in \mathbb{R}^m$

$$x y^T \in \mathbb{R}^{n \times m}$$

Matrix-Vector Products:

$A \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^{n}$

$\boxed{y \in \mathbb{R}^{m}}$

$y = Ax$

$$y = Ax = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix} = y$$

$\mathbb{R}$

$$\boxed{y_i = a_i^T x}$$

$$y = Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \overbrace{\underbrace{\begin{bmatrix} | \\ a_1 \\ | \end{bmatrix}}_{\text{بردار}} \overset{\mathbb{R}}{x_1} + \underbrace{\begin{bmatrix} | \\ a_2 \\ | \end{bmatrix}}_{\text{بردار}} x_2 + \cdots + \underbrace{\begin{bmatrix} | \\ a_n \\ | \end{bmatrix}}_{\text{بردار}} x_n} = y$$

y ⇐ ترکیب خطی از ستون‌های A که ضرایب آن y

مقادیر بردار x است.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad x = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

$$y_i = a_i^T x \Rightarrow y = \begin{bmatrix} a_1^T x \\ a_2^T x \end{bmatrix} = \begin{bmatrix} 18 \\ 36 \end{bmatrix}$$

$$a_1^T \quad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \overset{x}{\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}} = 18$$

$$a_2^T \quad \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \underset{x}{\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}} = 36$$

$$y = \begin{bmatrix} a_1 \end{bmatrix} x_1 + \begin{bmatrix} a_2 \end{bmatrix} x_2 + \begin{bmatrix} a_3 \end{bmatrix} x_3$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} \times (-1) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \times (2) + \begin{bmatrix} 3 \\ 6 \end{bmatrix} \times (5)$$

$$= \begin{bmatrix} -1 \\ -4 \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \end{bmatrix} + \begin{bmatrix} 15 \\ 30 \end{bmatrix} = \begin{bmatrix} 18 \\ 36 \end{bmatrix}$$

$$y = Ax \in \mathbb{R}^m \qquad A \in \mathbb{R}^{m\times n} \qquad x \in \mathbb{R}^n$$

$$y^T = x^T A \qquad A \in \mathbb{R}^{m\times n} \qquad x \in \mathbb{R}^m$$

$$y^T = x^T A = x^T \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \qquad (1\times m)\times(m\times n) = (1\times n)$$

$$y^T = x^T A = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} = x_1 \begin{bmatrix} - & a_1^T & - \end{bmatrix} + x_2 \begin{bmatrix} - & a_2^T & - \end{bmatrix} + \cdots + x_m \begin{bmatrix} - & a_m^T & - \end{bmatrix}$$

Matrix-Matrix product

$A \in \mathbb{R}^{m \times n}$ $\quad B \in \mathbb{R}^{n \times p}$

dot prod

$$C = A \times B = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & & \ddots & \\ a_m^T b_1 & \cdots & & a_m^T b_p \end{bmatrix}$$

$$\boxed{C_{ij} = a_i^T b_j} \checkmark$$

ماتریس

$$C = A \times B = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \sum_{i=1}^{n} a_i b_i^T$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & -1 & 3 & 2 \\ -1 & 3 & 4 & -5 \end{bmatrix}$$

$$C = A \times B =$$

$$C = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} \begin{bmatrix} -1 & 3 & 4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 & 5 \\ 8 & 12 & 16 & 20 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 6 & 4 \\ 5 & -5 & 15 & 10 \end{bmatrix} + \begin{bmatrix} -3 & 9 & 12 & -15 \\ -6 & 18 & 24 & -30 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 16 & 22 & -6 \\ 7 & 25 & 55 & 0 \end{bmatrix}$$

$$C = AB = A\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{bmatrix} \checkmark \Rightarrow c_i = Ab_i$$

$$C = AB = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} B = \begin{bmatrix} — & a_1^T B & — \\ — & a_2^T B & — \\ & \vdots & \\ — & a_m^T B & — \end{bmatrix} \checkmark \Rightarrow c_i^T = a_i^T B$$

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① $AB \neq BA$

② $(AB)C = A(BC)$

③ $A(B+C) = AB + AC$

A Identity & Diagonal Matrix

1. $I \in \mathbb{R}^{n\times n}$ / I_n

$$I_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. $AI = IA = A$

$$A + AB = A(I+B)$$

$$AI + AB = A(I+B)$$

$$A = \mathrm{diag}(d_1, d_2, \ldots, d_n)$$

$$A_{ij} = \begin{cases} d_i & i=j \\ 0 & i\neq j \end{cases}$$

$$\Rightarrow \quad I = \mathrm{diag}(1, 1, \ldots, 1)$$

$$\mathrm{diag}(5, -1, 2, 3) = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(13) The Transpose

$$A \in \mathbb{R}^{m \times n} \Rightarrow A^T \in \mathbb{R}^{n \times m}$$

$$(A^T)_{ij} = A_{ji}$$

(1) $(A^T)^T = A$

(2) $(AB)^T = B^T A^T$

(3) $(A+B)^T = A^T + B^T$

(C) Symmetric Matrix, Anti-symmetric

(1) $A \in \mathbb{R}^{n\times n}$, (2) $A = A^T$ (3) $A = -A^T$

$$\Longrightarrow A \in \mathbb{R}^{n\times n} \Longrightarrow \begin{cases} A+A^T \Rightarrow \text{Symmetric} \\ A-A^T \Rightarrow \text{Anti-Symetric} \end{cases}$$

$$\Rightarrow \quad A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

(P) The Trace

(1) $A \in \mathbb{R}^{n \times n}$ $\quad tr(A) = \sum_{i=1}^{n} A_{ii}$

$\rightarrow tr(A) = tr(A^T)$

$\rightarrow tr(A+B) = tr(A) + tr(B)$

$\rightarrow tr(\alpha A) = \alpha \, tr(A)$

$\rightarrow AB \in \mathbb{R}^{m \times m} \Rightarrow tr(AB) = tr(BA)$

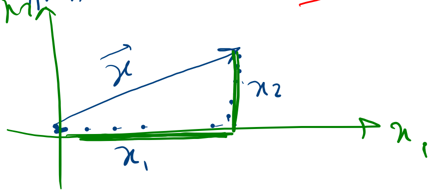
$\Rightarrow AB C \ldots X \in \mathbb{R}^{m \times m} \Rightarrow tr(ABC) = tr(CAB) = tr(BCA)$

(E) Norm $\longrightarrow$ $f(x): \mathbb{R}^n \to \mathbb{R}$ طول length

$\rightarrow$ Euclidean distance

$\boxed{\ell_2\text{-norm}}$

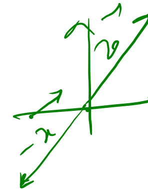
$\|x\|$



$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2}$$

$x \in \mathbb{R}^n$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$



$$\|x\|_2^2 = \sum_{i=1}^{n} x_i^2 = x^T x$$

$$\|x\|_2 = \sqrt{x^T x}$$

1. $\forall x \in \mathbb{R}^n, \; f(x) \geq 0$
2. $f(x) = 0$ — iff $x = 0$
3. $x \in \mathbb{R}^n, \alpha \in \mathbb{R} \Rightarrow f(\alpha x) = |\alpha| f(x) \Rightarrow f(x) = f(-x)$
4. $x, y \in \mathbb{R}^n, \; f(x+y) \leq f(x) + f(y)$

$$l_2 = \|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_i x_i^2}$$ فاصله اقلیدسی

$$l_1 = \|x\|_1 = \sum_{i=1}^{n} |x_i|$$ فاصله منهتن , city block dis.

$$\boxed{l_\infty = \|x\|_\infty = \max_i |x_i|}$$

$$l_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \Rightarrow p \geq 1$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2} = \sqrt{tr(A^T A)}$$ Frobenius norm

$$A \in \mathbb{R}^{m \times n}$$