Identity: $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n}$ $\quad I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $\quad A I_m = A = I_n A$

$A \in \mathbb{R}^{n \times m}$

Diagonal

$$D_{ij} = \begin{cases} d_{ij} & i = j \\ 0 & i \neq j \end{cases} = \begin{bmatrix} d_{11} & & & 0 \\ & d_{22} & & \\ & & \ddots & \\ 0 & & & d_{nn} \end{bmatrix}_{n \times n}$$

$D = \mathrm{diag}(d_{11}, d_{22}, \ldots, d_{nn})$

$I_n = \mathrm{diag}(1, 1, \ldots, 1)$

Transpose: $(A^T)_{ij} = A_{ji}$

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A+B)^T = A^T + B^T$
- $(\alpha A)^T = \alpha A^T$

$(ABC)^T = ?$

$((AB)C)^T = C^T (AB)^T = C^T B^T A^T$

$\Rightarrow$ Symmetric Matrices:

1- square $A \in \mathbb{R}^{n \times n}$

2- $A = A^T$

$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}_{2\times 2}$ $A^T = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

$\Rightarrow$ Anti-Symmetric:

$A = -A^T$

$\Rightarrow A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ ✓

$A = -A^T$

① $A + A^T \Rightarrow$ Symmetric $\boxed{\begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix}} = \begin{bmatrix} 2 & 9 \\ 9 & 8 \end{bmatrix}$ Sym.

$A - A^T \Rightarrow$ anti_symetric $\boxed{\begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix}} = \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$ Anti_sym.

$$\left|\begin{array}{l} A+A^T \\ A-A^T \end{array}\right. \Longrightarrow \frac{1}{2}(\text{sym})+\frac{1}{2}(\text{Anti-sym})=A$$

$$A=\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$$

$\Rightarrow$ The Trace

$A\in\mathbb{R}^{n\times n}$ $\quad tr(A)=\sum_{i=1}^{n}A_{ii}\in\mathbb{R}$ $\quad A=\begin{bmatrix}1 & 2 & 9\\ 4 & -5 & 1\\ 1 & 3 & 2\end{bmatrix}\Rightarrow tr(A)=-2$

1. $tr(A)=tr(A^T)$
2. $tr(A+B)=tr(A)+tr(B)$
3. $tr(\alpha A)=\alpha\, tr(A)$
4. $tr(AB)=tr(BA)\Longrightarrow tr(ABC)=tr(CAB)=tr(BCA)$

Norm vector:

$$\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = x^T x$$

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \qquad p = 1, 2, \infty \qquad l_p\ \text{norm}$$

l_2 norm $\equiv$ اقلیدسی $\qquad \|\vec{x}\|_2 = x^T x$

l_1 norm $\equiv$ منهتن $\qquad \|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$

l_∞ norm $\qquad \|\vec{x}\|_\infty = \max_i |x_i|$

Norm Matrix Frobenius

$$\boxed{\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}^2}}$$

$$\|A\|_F = \sqrt{\operatorname{tr}(A^T A)} = \sqrt{30}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\|A\|_F = \sqrt{1+4+9+16} = \sqrt{30} \cong 5.5$$

$$A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

$$\operatorname{tr}(A^T A) = 30$$

$\Rightarrow$ Linear Independence & $\boxed{\text{Rank}}$

$$\underbrace{\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}}_{\downarrow} \subset \mathbb{R}^m$$

مستقل خطی اند، تنها اگر
هیچ بردار را نتوان از روی ترکیب خطی
بقیه بردارها بنا کرد

$$\nexists i \quad \vec{x}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_j \vec{x}_j$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

مستقل خطی

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{x}_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \quad \vec{x}_3 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\vec{x}_3 = -2\vec{x}_1 + \vec{x}_2$$

$$\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \vec{x}_3$$

وابسته خطی

Rank ⇒ ماتریس

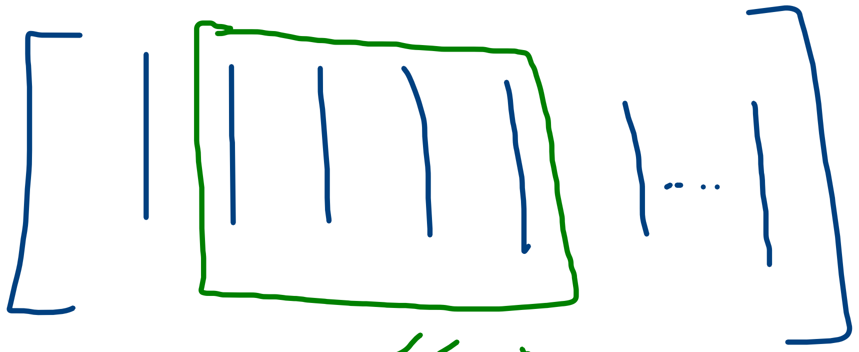
column rank $A \in \mathbb{R}^{n \times m}$

اندازه بزرگترین زیرمجموعه از ستونهای ماتریس A که مستقل خطی باشند.

row rank

اندازه بزرگترین زیرمجموعه از سطرهای ماتریس A که مستقل خطی باشند.

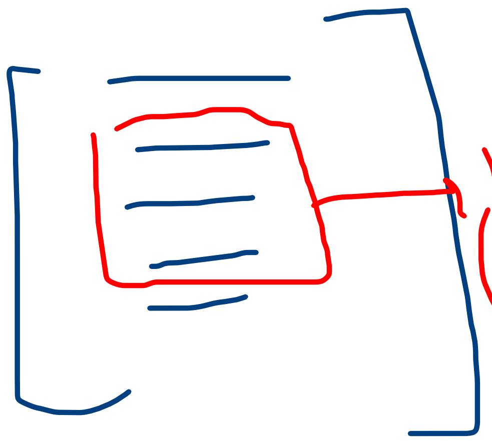
A=



$\Rightarrow \text{rank}_{\text{column}}(A) = 4$

ستونها

A =



سطرها

$\text{rank}_{\text{row}}(A) = 3$

$$\operatorname{rank}(A) = \min\left(\operatorname{rank}_c(A), \operatorname{rank}_r(A)\right)$$

1- $A \in \mathbb{R}^{n\times n} \Rightarrow \operatorname{rank}(A) \leq \boxed{\min(n,m)}$

2- $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

3- $\operatorname{rank}(AB) \leq \min\left(\operatorname{rank}(A), \operatorname{rank}(B)\right)$

4- $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$

if $\operatorname{rank}(A) = \min(m,n)$ *

$\Rightarrow$ full rank ماتریس A

$$A = \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ \cancel{1} & \cancel{7} & \cancel{-3} \\ 2 & 3 & -9 \end{bmatrix} \quad \boxed{3 \times 3}$$

$$\text{rank}_c = 3$$

$$\text{rank}_r = 2$$

$$\text{rank}(A) = 2 < 3 \not\Rightarrow \boxed{\text{full rank}}$$

$$\text{rank}(A) = \min(n, m)$$

$$r_2 = 2r_1 + r_3 \times$$

$\Rightarrow$ Inverse

$$A \in \mathbb{R}^{n \times n} \Rightarrow A^{-1} \Rightarrow$$ ماتریس معکوس

$$\boxed{A^{-1}A = I = AA^{-1}}$$

① A is square لازم ولی ناکافی

② A is fullrank

1- $(A^{-1})^{-1} = A$

2- $(AB)^{-1} = B^{-1}A^{-1}$

3- $(A^{-1})^{T} = (A^{T})^{-1} = A^{-T}$

$$A^{-1} = \frac{1}{|A|} \times adj(A)$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = 1\times(-1) - 2\times(0) + 0\times(-) = -1$$

1) ماتریس علامت $n\times n$ $= S_{ij} = \begin{cases} +1 & (i+j)\,\%\,2 = 0 \\ -1 & (i+j)\,\%\,2 \neq 0 \end{cases} \Rightarrow S_{ij} = (-1)^{i+j}$

$$S_{3\times 3} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$\mathrm{adj}(A)$

---

2) $C_{ij} = |A_{\backslash i, \backslash j}|$

$$C = \begin{array}{c|ccc} & & j & \\ \hline i & \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \end{array}$$

3) $\boxed{\mathrm{adj}(A) = S \odot C^T} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \odot \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

$$\mathrm{adj}(A) = \begin{bmatrix} -1 & -2 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

4) $\boxed{A^{-1} = \frac{1}{-1}\,\mathrm{Adj}(A)} = \begin{bmatrix} +1 & 2 & -2 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$AA^{-1} = A^{-1}A = I = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$[A \,\vdots\, I] \Longleftrightarrow [I \,\vdots\, A^{-1}]$$

$$\xrightarrow{R_3 = R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{I} & \multicolumn{3}{c}{A^{-1}} \\ 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \quad \checkmark$$

Gaussian Elimination

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1 \\ \vdots \qquad\qquad\qquad\qquad \ddots \\ \vdots \\ a_{n1}x_1 + \cdots\cdots + a_{nm}x_m = b_n \end{array}\right.$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nm} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Downarrow$$

$$A\vec{x} = \vec{b} \xRightarrow{\boxed{\text{F.R. } A \sim I}} \boxed{\vec{x} = A^{-1} b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}b$$

$$\boxed{\vec{x} = A^{-1}b}$$