$A \in \mathbb{R}^{n \times n}$ $\quad x \in \mathbb{R}^n$

Q.f. $\quad \boxed{x^T A x} = \sum_{i=1}^{n} x_i (Ax)_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} A_{ij} x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \in \mathbb{R}$

$\Longrightarrow x^T A x \in \mathbb{R}$

$$x^T A x = (x^T A x)^T = x^T A^T x = \underline{x^T \left(\tfrac{1}{2} A + \tfrac{1}{2} A^T \right) x}$$

$5 = 5^T$ ✓

Positive Definite: $\underline{A}$ P.D. iff $\forall x \in \mathbb{R}^n$ $\vec{x} \neq \vec{0}$ $\quad x^T A x > 0$

Negative Definite: $A \in \mathbb{R}^{n \times n}$ N.P. iff $\forall x \in \mathbb{R}^n, \vec{x} \neq \vec{0}$ $\quad x^T A x < 0$

Positive semi definite: $A \in \mathbb{R}^{n \times n}$ P.S.D iff $\forall x \in \mathbb{R}^n, \vec{x} \neq \vec{0}$ $\quad x^T A x \geq 0$

Negative semi definite: ______ $\quad x^T A x \leq 0$

A P.D. $\Longrightarrow -A$ N.D.

A N.D. $\Longrightarrow -A$ P.D.

A P.S.D $\Longleftrightarrow -A$ N.S.D

نکته: اگر یک ماتریس Positive Def. باشد حتماً "Full rank" است.

? $\boxed{Ax=y}$

$\forall\ A \in \mathbb{R}^{m\times n} \Longrightarrow G = \boxed{A^T} A$ [Gram Matrix] is always positive semidefinite.

if $m \geq n$ then G is Positive definit.

مشتق برداری (Gradient):

$$f: \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{n} \qquad \underline{\underline{A}} \in \mathbb{R}^{m \times n}$$

$$\nabla_A f(A) \in \mathbb{R}^{m \times n} = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \frac{\partial f(A)}{\partial A_{12}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \vdots & & & \\ \frac{\partial f(A)}{\partial A_{m1}} & - & - & \frac{\partial f(A)}{\partial A_{mn}} \end{bmatrix}$$

ابعاد $\nabla_A f(A)$ با ابعاد A

برابر

$$\left(\nabla_A f(A)\right)_{ij} = \frac{\partial f(A)}{\partial A_{ij}}$$

$\vec{x} \in \mathbb{R}^n \Longrightarrow \nabla_{\vec{x}} f(\vec{x}) = \begin{bmatrix} \frac{\partial f(\vec{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\vec{x})}{\partial x_n} \end{bmatrix}$

1. $\nabla_{\vec{x}} \left(f(\vec{x}) + g(\vec{x}) \right) = \nabla_{\vec{x}} f(\vec{x}) + \nabla_{\vec{x}} g(\vec{x})$

2. $\forall t \in \mathbb{R} \quad \nabla_{x} \left(t f(x) \right) = t \nabla_{x} f(x)$

Hessian = (مشتق (۲) [illegible])

$f : \mathbb{R}^n \to \mathbb{R}$

$$H = \nabla_x^2 f(x) \in \mathbb{R}^{n \times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \ddots & & \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

$$H_{ij} = \left(\nabla_x^2 f(x) \right)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$$

$$\boxed{f(\vec{x}) = x^T A x}$$

$$A = \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad n=2$$

$$f(x) = [x_1 \;\; x_2] \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [9x_1 + 6x_2 \quad 6x_1 + 5x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9x_1^2 + \underline{6x_2x_1 + 6x_1x_2} + 5x_2^2$$

$$\boxed{f(\vec{x}) = 9x_1^2 + 12x_1x_2 + 5x_2^2}$$

$$\boxed{G = \begin{bmatrix} 18x_1 + 12x_2 \\ 12x_1 + 10x_2 \end{bmatrix}} = 2Ax$$

$$G = \nabla_{\vec{x}} f(\vec{x}) = \begin{bmatrix} \frac{\partial f(\vec{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\vec{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f(\vec{x})}{\partial x_1} \\ \frac{\partial f(\vec{x})}{\partial x_2} \end{bmatrix}$$

$$\boxed{f(x) = x^T A x} = 9x_1^2 + 12x_1x_2 + 5x_2^2$$

$$H = \left(\nabla_x^2 f(x)\right) = \begin{bmatrix} \dfrac{\partial^2 f(x)}{\partial x_1^2} & \dfrac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f(x)}{\partial x_2^2} \end{bmatrix}$$

① → ②

$$\frac{\partial f(x)}{\partial x_2} = 12x_1 + 10x_2$$

$$\frac{\partial}{\partial x_1}\left(12x_1 + 10x_2\right) = 12$$

$$H = \begin{bmatrix} 18 & 12 \\ 12 & 10 \end{bmatrix}$$

$$\frac{\partial^2 f(x)}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial f(x)}{\partial x_2}\right) =$$

① $\nabla_x b^T x = b$

② $\nabla_x x^T A x = 2Ax$ (if A symmetric)

③ $\nabla_x^2 x^T A x = 2A$ (if A symmetric)

$A = \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $f(x) = x^T A x$

$\nabla_x f(x) = 2Ax$ $\begin{bmatrix} 18 & 12 \\ 12 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18x_1 + 12x_2 \\ 12x_1 + 10x_2 \end{bmatrix}$

$\nabla_x^2 f(x) = 2A$ $\begin{bmatrix} 18 & 12 \\ 12 & 10 \end{bmatrix}$

Least Squares: X

$$\min_x \|Ax-b\|_2^2$$

$$\|Ax-b\|_2^2 = \left(\cancel{\sqrt{(Ax-b)^T(Ax-b)}}\right)^{\cancel{2}}$$

$$= (x^T A^T - b^T)(Ax-b)$$

$$= x^T A^T A x - x^T A^T b - b^T A x + b^T b$$

$$f(x) = x^T \underbrace{A^T A}_{\text{Graph}} x - 2b^T A x + b^T b$$

$$\nabla_x f(x) = 2A^T A x - 2A^T b = 0$$

$$\cancel{2} A^T A x = \cancel{2} A^T b$$

$$x = (A^T A)^{-1} A^T b$$

$$f(x) = 12x^2 - 5x$$

$$g(f(x)) = 7f(x)^2 + 12f(x)$$

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial f} \times \frac{\partial f}{\partial x} = (17\,f(x) + 12) \times (24x - 5)$$

$$= \left(17(12x^2 - 5x) + 12\right)(24x - 5)$$

$$=$$