Posted Pricing for Online Selection: Limited Price Changes and Risk Sensitivity

Hossein Nekouyan* Bo Sun† Raouf Boutaba‡ Xiaoqi Tan§

November 23, 2025

Abstract

Posted-price mechanisms (PPMs) are a foundational tool for online optimization problems, such as online selection and matching. They ensure incentive-compatible decisions in strategic settings and often rely on dynamically increasing prices to balance uncertainty in online settings. While such adaptive pricing achieves strong performance, it also raises practical concerns: dynamically changing prices can create fairness issues due to price discrimination and introduce operational costs from frequent price updates. This paper addresses these challenges by studying posted-price mechanisms that must operate under a limited, pre-specified number of allowed price changes (Δ). We further extend this framework by introducing a second key dimension: risk sensitivity. Instead of evaluating performance in expectation, we use a tail-risk objective—specifically the CVaR_δ (Conditional Value at Risk) of the total social welfare—parameterized by a risk level δ.

We formally introduce the problem class kSelection-(δ, Δ) and propose cPPM-$\boldsymbol{\phi}$, a *correlated* posted-price mechanism that uses a single random seed to correlate posted prices and improve the tail performance of the online algorithm. Our analysis provides performance guarantees under these joint constraints, revealing a clear trade-off between the number of price changes allowed and the risk sensitivity of the algorithm. We also establish optimality results for several important special cases of the problem.

1 Introduction

In this paper, we study posted-price mechanisms (PPMs), a widely used approach in online markets where prices are announced and buyers decide whether to accept or reject them. Such mechanisms are common in real-world systems like airline ticket sales, ride-sharing platforms, and online advertising. In the research literature, PPMs have been extensively studied across various online optimization problems, including online selection [18], online matching [21, 16], and online knapsack [27].

In online settings where both total demand and buyers' valuations are uncertain, PPMs dynamically adjust prices to balance these interacting sources of uncertainty and attain strong performance guarantees relative to a clairvoyant offline benchmark [21, 16, 18]. However, such adaptivity can also induce de facto price discrimination over time, as early arrivals often face lower prices than later ones, raising potential fairness concerns [4, 3, 26, 12]. Practical considerations such as ease of implementation and the operational costs

*University of Alberta. Email: `nekouyan@ualberta.ca`
†University of Ottawa. Email: `bo.sun@uottawa.ca`
‡University of Waterloo. Email: `rboutaba@uwaterloo.ca`
§University of Alberta. Email: `xiaoqi.tan@ualberta.ca`

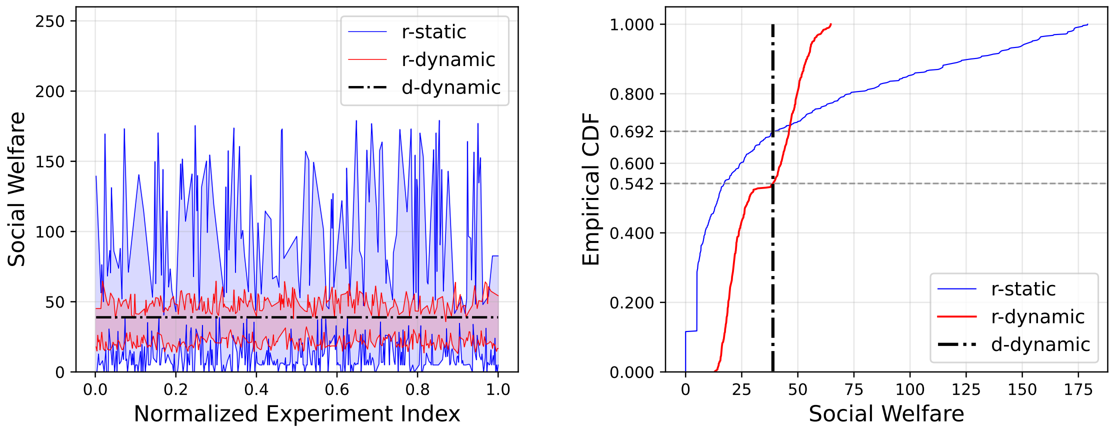

Figure 1: The above plots show the performance of the r-static, r-dynamic, and d-dynamic algorithms on an instance of the kSelection-(δ, Δ) problem. The r-static algorithm, developed in [26], uses a single randomized price. The r-dynamic algorithm, from [18], uses k independent random seeds to generate k randomized, dynamically increasing prices. The d-dynamic algorithm, developed in [29], uses k deterministic dynamically changing prices. The left plot shows the fluctuation in performance of these randomized algorithms over 10^4 independent runs on the same instance, while the right plot shows the CDF of their empirical performance across these runs

associated with frequent price changes [30] further motivate the study of PPMs that allow only limited price adjustments. Similar concerns have recently motivated work on the IID prophet inequality framework [22], which explores PPMs with a restricted number of price updates. These developments prompt the question of whether, within online selection problems, one can design competitive PPMs that remain effective against the clairvoyant optimum while being subject to a *price-change cap*, denoted by Δ, which limits the total number of allowable price updates.

To address price discrimination in online selection problems with limited price changes, recent work by [26] examined several variants of this setting, focusing exclusively on the extreme case of static pricing schemes in which no price changes are permitted (equivalently, the price-change cap is $\Delta = 0$). These algorithms maintain a fixed posted price throughout the horizon and therefore avoid price discrimination entirely. They rely on randomization to balance key uncertainties—such as unknown future demand and valuations—that otherwise hinder decision quality. Although these randomized static pricing algorithms achieve optimal *expected* worst-case performance, they often perform poorly on *tail* outcomes, which occur with non-negligible probability. In Figure 1, the left panel illustrates the fluctuations in performance of the randomized static pricing algorithm developed in [26], referred to as r-static, across multiple independent runs on the same instance of the online k-selection problem. The right panel shows the empirical CDF of its performance. As the figure indicates, r-static exhibits poor tail behavior: in nearly 70% of the runs, its performance falls below that of the deterministic algorithm from [29], referred to as d-dynamic, which employs deterministically increasing prices. Although d-dynamic is not optimal and r-static achieves a higher expected welfare, the deterministic algorithm attains better welfare in almost 70% of the runs. Moreover, r-static yields zero welfare with a non-negligible probability. Figure 1 also reports the performance of the algorithm developed in [18], referred to as r-dynamic, which uses k independently sampled, dynamically increasing prices. While r-dynamic also exhibits weak tail performance, it performs noticeably better in the tail than r-static. These observations collectively point to a trade-off between the number of allowable price changes and tail-risk performance—a trade-off that we examine in detail throughout this work.

The weak tail performance of commonly used randomized algorithms in online settings, with respect

	δ	$\Delta = 0$	$\Delta = k-1$	$1 \le \Delta \le k-2$
κSelection-(δ, Δ)	$\delta = 1$	✓ Optimal [26]	✓ Optimal (**Theorem 1**)	✓ Optimal (**Theorem 1**)
	$\delta \in (0, 1)$	✓ Optimal (**Theorem 2**)	✓ Optimal as $k \to \infty$ (**Theorem 3**)	Best-known (**Theorem 4**)
κSelection-Cost	$\delta = 1$	✓ Optimal [26]	✓ Optimal as $k \to \infty$ [18]	?
κMax-Search	$\delta \in (0, 1]$	Best Known [6]	Optimal for $\delta = 1$, as $k \to \infty$ [19]	?
IID-Prophet	$\delta = 1$	✓ Optimal [17]	✓ Optimal [7]	Best Known [22]

Table 1: Summary of results for various online selection problems under different (δ, Δ) settings and arrival models. All results concern the design of posted-price or threshold-based algorithms. The κSelection-Cost problem, studied in [26, 18], introduces a production cost associated with producing each additional unit of the item. The κMax-Search problem, studied in [6, 19], is a variant of online selection where the decision maker is notified upon the arrival of the last buyer, thereby limiting the uncertainty about total demand. In the IID-Prophet setting studied in [7, 17, 22], buyers' valuations are drawn independently from a common distribution known to the decision maker, allowing the use of distributional information in decision-making.

to tail-risk metrics, is a well-known but still underexplored issue [10]. Recent works [6, 10] have begun addressing this gap by proposing methods to improve the tail performance of randomized online algorithms in settings such as online ski-rental and the 1-max search problem, using risk measures such as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). Motivated by these developments and the poor tail risk performance of r-static algorithm, we also focus on designing *risk-sensitive* posted-price mechanisms, using CVaR as our performance metric. For a risk level (tail probability) $\delta \in (0, 1)$, CVaR_δ evaluates an algorithm's expected performance conditional on the worst δ-fraction of sample paths with respect to its realized objective value. Our goal is to develop posted-price policies that, subject to a price-change cap Δ, achieve strong CVaR_δ-based competitive guarantees relative to the offline clairvoyant optimum.

More formally, we study κSelection-(δ, Δ), a variant of the online k-selection problem defined as follows. A decision maker controls k units of a resource and faces a sequence of buyers arriving online, each with a *private* valuation. Upon each arrival, the mechanism posts a price, and the buyer accepts if and only if their valuation exceeds the posted price. The objective is to maximize CVaR_δ of the total social welfare, defined as the sum of the valuations of buyers who accept the posted prices. A central feature of κSelection-(δ, Δ) is the *price-change cap* Δ, which limits the total number of price updates the mechanism may perform over the horizon. This constraint induces a fundamental trade-off between price flexibility and risk-sensitive performance. As Δ increases, the mechanism can adapt prices more finely and thereby achieve tighter CVaR_δ-competitive ratios. However, this flexibility comes at the cost of allowing prices to vary more substantially over time, which may introduce undesirable forms of price discrimination between early and late buyers.

1.1 Our Contribution and Techniques

In the following, we elaborate on the contributions of this work to the existing literature on online optimization and posted pricing mechanisms. We emphasize that the κSelection-(δ, Δ) problem is newly introduced in this paper. However, several of its special cases have been studied in prior work. To build toward our main results, we first analyze these special cases and then generalize our findings to the most comprehensive form of the problem.

We begin with the risk-neutral setting (i.e., $\delta = 1$), where the CVaR_δ metric reduces to *expected* efficiency.

In this case, our goal is to focus solely on the price-change cap and design PPMs with arbitrary caps Δ that achieve optimal competitiveness. Prior work [26] provides an optimal PPM for the special case $\Delta = 0$, but their techniques do not extend to settings with more permitted price changes. In contrast, we develop a *family* of PPMs that achieve the *optimal* competitive ratio for every price-change cap $\Delta \in \{0, 1, \ldots, k-1\}$. The techniques introduced in this section form the basis for extending the price-change cap constraint to the risk-sensitive regime.

We then turn to the risk-sensitive setting (i.e., $\delta \in (0,1)$), beginning with the fully-static case $\Delta = 0$. This simple setting helps build toward the most general case. Recent work [6] studies the one-max search problem—a close variant of kSelection-(δ, Δ) when $k = 1$—and introduces a single-threshold randomized algorithm. Although their model differs slightly from ours and does not allow a direct comparison, their algorithm is not optimal. In contrast, for $\Delta = 0$, we derive the *optimal risk-sensitive static price*, achieving the tightest CVaR_δ-competitive ratio among all online algorithms that use a single static price for every $\delta \in (0,1)$. Next, we examine the special case $\Delta = k-1$, which permits a fully-dynamic pricing scheme. This setting removes the price-change cap and isolates the challenge of designing risk-sensitive algorithms under the CVaR_δ metric. Here, we design a PPM that achieves *exact optimality in the large-inventory limit* $(k \to \infty)$ for every $\delta \in (0,1)$. Finally, synthesizing insights from these two extreme cases, we present a *general* Δ*-level pricing design* that accommodates any capped number of price changes for arbitrary $\delta \in (0,1)$. This framework yields monotonic improvements in CVaR_δ-competitiveness as the allowed number of price changes Δ increases or as the tail probability increases. Although we cannot establish optimality in this general setting, the design closely mirrors the optimal constructions from the previous two cases. We thus conjecture that, with a more refined analysis, optimality can be proven here as well.

The technical contributions of this work can be summarized from two main perspectives:

Correlated Posted Pricing. We design a PPM in which all pricing levels are coupled through a single random seed drawn from a uniform distribution over $[0,1]$. This seed jointly determines the posted prices at every level. From a risk perspective, such correlation induces a family of pricing profiles that remain more balanced across different realizations of randomness, thereby reducing the variability of the algorithm's total welfare (or revenue) across sample paths. When evaluating the CVaR_δ objective—namely, the expected performance over the worst δ-fraction of outcomes—the algorithm loses little performance because the remaining $(1-\delta)$-fraction of sample paths does not diverge significantly from the worst δ-fraction. Moreover, correlating the pricing levels enforces smoother price changes across realizations, leading to more consistent allocation decisions. In the special case $\Delta = k-1$, we further show that this correlation scheme yields randomized integral decisions that closely track the fractional decisions of an algorithm that uses the same pricing functions in the fractional version of the kSelection-(δ, Δ) problem. In this sense, the correlation mechanism can be interpreted as a rounding scheme that employs correlation to convert fractional allocations into integral ones without loss. Viewed this way, our design contributes to a growing line of work on online rounding schemes that leverage correlation, including [15, 20, 13]. We also demonstrate that both the correlation mechanism and the proof technique developed for the $\Delta = k-1$ case extend to the open question posed in [18]: constructing a PPM that achieves optimal performance in both small and large inventory regimes for the online selection variant studied in that work.

Risk-Sensitive Randomized Online Primal-Dual. From a methodological standpoint, to establish the competitiveness of our algorithm, we introduce a dual linear program specifically tailored to the Δ-capped setting. This dual program includes one variable per price *level* (a total of $\Delta+1$ variables), where each variable intuitively corresponds to the price associated with that level of price change. To analyze the algorithm's performance, we develop a randomized online primal–dual (R-OPD) framework adapted to the risk-sensitive setting. The R-OPD approach constructs dual variables as functions of the algorithm's randomized decisions. In standard online settings [11, 8], update rules for dual variables depend on random decisions along each sample path, and the final dual values are obtained by taking expectations over all realizations of randomness. In the risk-sensitive setting, however, this procedure must be modified so that dual updates are performed *only over the worst δ-fraction of sample paths*. These updates are designed to ensure two key properties: (i) the expected dual objective matches the CVaR_δ performance of the algorithm, and (ii) the dual feasibility constraints hold in expectation. This tail-focused adaptation of the R-OPD method, combined with the Δ-level dual structure, allows us to establish CVaR_δ competitiveness guarantees for all results derived in this work. To balance the uncertainty inherent in the kSelection-(δ, Δ) problem, our algorithms are randomized and become controllably, yet progressively, more aggressive along certain sample paths. Because the worst $(1-\delta)$-fraction of paths is disregarded under the CVaR_δ metric, the design of the pricing functions naturally leads to a system of delay differential equations [24] parameterized by the risk level δ. These equations induce a *memory effect* in the algorithm's behavior, determining how rapidly the pricing functions adapt based on previously posted and accepted prices, while effectively ignoring the $(1-\delta)$-fraction of paths where performance is minimized.

1.2 More Related Work

Pricing with Limited Price Changes.

Within the operations research literature, [4] study dynamic pricing with unknown demand under a limited number of allowable price levels, designing policies that strategically schedule a few exploratory prices before exploitation, achieving near-optimal performance while respecting a cap on the number of price changes. They motivate this setting by noting that in many markets, firms cannot change prices freely—due to operational frictions, adverse customer reactions, or additional costs associated with price adjustments discussed in [30]—which effectively constrains the number of feasible price updates. Motivated by this perspective, several subsequent works have explored similar price-change–capped settings in both theoretical and applied contexts (e.g., [3, 2]).

Risk-Aware Randomized Algorithms. Recent work by [10] studied the online ski rental problem under a tail-risk constraint specified by the pair (γ, δ), designing online algorithms such that the probability of the worst-case competitive ratio exceeding γ is at most δ. Another recent work, [6], investigates a range of online problems—such as ski rental and one-max search—under the CVaR risk metric. Moreover, risk-aware algorithm design has gained considerable attention in the learning theory community, where CVaR has been explored in various settings, including Markov Decision Processes (MDPs) [9, 5], online submodular optimization [25], and multi-armed bandit problems [28, 1].

2 Problem Setting and Main Results

Let us introduce the problem of kSelection-(δ, Δ) as follows. A seller has k identical units and faces T buyers arriving one by one. When buyer t arrives, the seller posts a price p_t. Buyer t has a private value v_t and accepted the price if $v_t \geq p_t$; otherwise the buyer leaves. The price may change at most Δ times over the horizon, i.e., $\sum_{t=1}^{T-1} \mathbf{1}_{\{p_t \neq p_{t+1}\}} \leq \Delta$.

Let $x_t \in \{0, 1\}$ indicate whether buyer t receives an item. Buyer t's utility is given by $u_t = (v_t - p_t)x_t$, the seller's revenue is $r = \sum_{t=1}^{T} p_t x_t$, and the total social welfare—defined as the sum of buyer utilities and seller revenue—is $r + \sum_{t=1}^{T} u_t = \sum_{t=1}^{T} v_t x_t$. Let $P = \{p_t\}_{t=1}^{T}$ denote the vector of prices posted by an online algorithm $\mathtt{ALG}$. In the online setting, the seller must determine each price p_t without knowledge of future valuations $\{v_{t'}\}_{t'>t}$ or even the total number of arrivals T. Since the algorithm may randomize its pricing decisions to manage uncertainty in buyer valuations and total demand, P is treated as a random vector.

For an instance $I = \{v_t\}_{t=1}^{T}$ of kSelection-(δ, Δ), let $\mathtt{ALG}(I, P)$ denote the random variable representing the total social welfare achieved by algorithm $\mathtt{ALG}$ on instance I under the random price vector P. Let $F_{\mathtt{ALG}(I,P)}$ be the cumulative distribution function (CDF) of this random variable

We use Conditional Value at Risk (CVaR) as our risk metric as it is tail-sensitive, coherent, and convex, which makes it tractable for optimization [14, 23]. Following the standard definition in [14, 23], we define CVaR_δ as follows:

Definition 1. *Given $\delta \in (0,1]$ and for a reward-type random variable X, we define* CVaR_δ *as*

$$\mathsf{CVaR}_\delta[X] = \sup_{\tau \in \mathbb{R}} \left\{ \tau - \tfrac{1}{\delta}\,\mathbb{E}\big[(\tau - X)_+\big] \right\},$$

where δ specifies the risk level (tail probability) and $(x)_+ = \max\{x, 0\}$.

We adopt the reward-based formulation of CVaR_δ since the objective in this work represents a reward (social welfare) rather than a loss. Moreover, if the cumulative distribution function $F_{\mathtt{ALG}(I,P)}$ corresponding to the random objective value of an online algorithm ALG is strictly increasing and continuous, the CVaR_δ performance of that algorithm is given by $\mathsf{CVaR}_\delta[\mathtt{ALG}(I,P)] = \frac{1}{\delta}\int_0^\delta F^{-1}_{\mathtt{ALG}(I,P)}(\eta)\,d\eta$, where F^{-1} denotes the inverse cumulative distribution (quantile) function. Intuitively, CVaR_δ measures the algorithm's expected performance over the worst δ-fraction of its sample paths.

The objective is to design an online algorithm that minimizes its CVaR_δ-competitive ratio, denoted by CVaR_δ-CR, defined as

$$\mathsf{CVaR}_\delta\text{-}\mathsf{CR}(\mathtt{ALG}) = \sup_{I \in \mathcal{I}} \frac{\mathtt{OPT}(I)}{\mathsf{CVaR}_\delta[\mathtt{ALG}(I)]},$$

where $\mathtt{OPT}(I)$ denotes the offline clairvoyant optimum computed as the summation of the k highest values, i.e., $\mathtt{OPT}(I) = \max_{\{x_t\}} \sum_{t=1}^T v_t x_t$, subject to $\sum_{t=1}^T x_t \le k,\ x_t \in \{0,1\},\ \forall t$.

Without any additional information regarding the buyers valuations, no online algorithm attains a bounded competitive ratio for kSelection-(δ, Δ). We therefore assume that buyer values are bounded within a range whose bounds are known the decision maker.

Assumption 1. *In* kSelection-(δ,Δ), *buyer valuations satisfy $v_t \in [L, U]$ for all $t \in [T]$.*

In the following section, we present a posted pricing mechanism that specifies how the posted prices at different price levels are generated and correlated across these levels.

2.1 The Algorithm: Correlated PPMs with Limited Price Changes (cPPM-$\boldsymbol{\phi}$)

We introduce a correlated posted pricing mechanism, denoted by cPPM-$\boldsymbol{\phi}$, which is formalized in Algorithm 1. The mechanism is defined by a set of $\Delta + 1$ pricing functions $\boldsymbol{\phi} = \{\phi_i\}_{i=1}^{\Delta+1}$ and a corresponding reservation vector $\{q_i\}_{i=1}^{\Delta+1}$, where each q_i represents the number of units reserved to be sold at price level i using the pricing function ϕ_i. Each pricing function ϕ_i is nondecreasing on the interval $[0,1]$, and higher price levels dominate the lower ones throughout the entire range. Formally, $\phi_i(0) \le \phi_i(1) \le \phi_{i+1}(0) \le \phi_{i+1}(1),\ \forall i \in [\Delta]$.

The algorithm begins by sampling a single random seed R uniformly from the interval $[0,1]$. Using this random seed, it correlates the prices across different price levels by setting the posted prices as $\phi_j(R)$ for each $j \in [\Delta+1]$. Depending on the number of units sold and the corresponding price level from which the reserved units are drawn to be allocated, the algorithm posts the appropriate price to each arriving buyer t and updates the resource utilization based on the buyer's decision to accept or reject the offer.

Algorithm 1: Correlated PPM with Pricing Functions $\boldsymbol{\phi}$ (cPPM-$\boldsymbol{\phi}$)

Input: A set of pricing functions $\boldsymbol{\phi} = \{\phi_i\}_{i=1}^{\Delta+1}$, reservation vector $\{q_i\}_{i=1}^{\Delta+1}$

Initialize: Sample the random seed $R \sim \mathcal{U}(0, 1)$ Initialize $y_1 \leftarrow 0$

for *each buyer* $t = 1, 2, \ldots$ **do**

  Let $j_t \leftarrow \max \left\{ j \in [\Delta + 1] \,\middle|\, y_t \geq \sum_{l=1}^{j-1} q_l \right\}$

  Post price $p_t = \phi_{j_t}(R)$ to buyer t

  if *buyer t accepts the price p_t* **then**

    Set $x_t \leftarrow 1$

  else

    Set $x_t \leftarrow 0$

  end

  Update $y_{t+1} \leftarrow y_t + x_t$

end

In contrast to the algorithm in [18], which samples an independent random seed for each unit of the item—resulting in independently sampled prices across different items—Algorithm 1 employs a single random seed to correlate the prices across all levels. This correlation ensures that, across different sample paths of the randomized algorithm, the prices vary more smoothly. In other words, Algorithm 1 generates pricing profiles $\{\phi_j(R)\}_{j\in[\Delta+1]}$ that become gradually more aggressive as the value of R increases from 0 to 1. This design helps balance the trade-off between serving lower-valued and higher-valued buyers. The smooth evolution of prices across sample paths also induces a desirable monotonic behavior in the algorithm's resource utilization, making it easier to analyze and determine the probability that each buyer's request is accepted or rejected. On the other hands, the correlation helps to synchronize posted prices such that total welfare of the algorithm varies only slightly across sample paths. As a result, when evaluating the CVaR_δ objective—i.e. by computing the expected performance over the worst δ-fraction of sample paths—the algorithm loses little, because the remaining $(1-\delta)$-fraction of outcomes is not much better than the worst δ-fraction.

In the following sections, we study different variants of the kSelection-(δ, Δ) problem, starting with a warm-up case where $\delta = 1$ and the price-change cap can take any arbitrary value. The intuition and design developed in this section will serve as a foundation for the analyses that follow. In addition, we refer to cPPM-$\boldsymbol{\phi}$ with a single pricing function (i.e., $\Delta = 0$) as **fully-static pricing**, and with k pricing functions (i.e., $\Delta = k - 1$) as **fully-dynamic pricing**. Otherwise, for any $\Delta = 1, \cdots, k-2$, we refer to cPPM-$\boldsymbol{\phi}$ as the **Δ-dynamic pricing** scheme.

2.2 Main Results

In this section, we present the main theoretical results on the CVaR_δ-CR performance of cPPM-$\boldsymbol{\phi}$ for the kSelection-(δ, Δ) problem. We progressively develop the pricing function designs used by cPPM-$\boldsymbol{\phi}$, starting from the risk-neutral case and moving toward the general risk-aware formulation. Each subsection introduces the motivation, key result, and (when applicable) the analytical structure underlying the design.

Risk-Neutral Posted Pricing with Limited Price Changes. We begin with the risk-neutral case, where $\delta = 1$. In this setting, the objective reduces to the standard worst-case competitive ratio analysis, where the algorithm's performance is evaluated in terms of its expected value. The work of [26] establishes an optimal static pricing algorithm for the case $\Delta = 0$. However, the problem of designing posted pricing mechanisms that achieve the optimal competitive ratio remains open for cases with a higher number of price-change caps. The following theorem provides the optimal pricing function design for cPPM-$\boldsymbol{\phi}$ such that algorithm achieves the tightest possible CVaR_δ-CR among all online algorithms for the kSelection-(δ, Δ) problem.

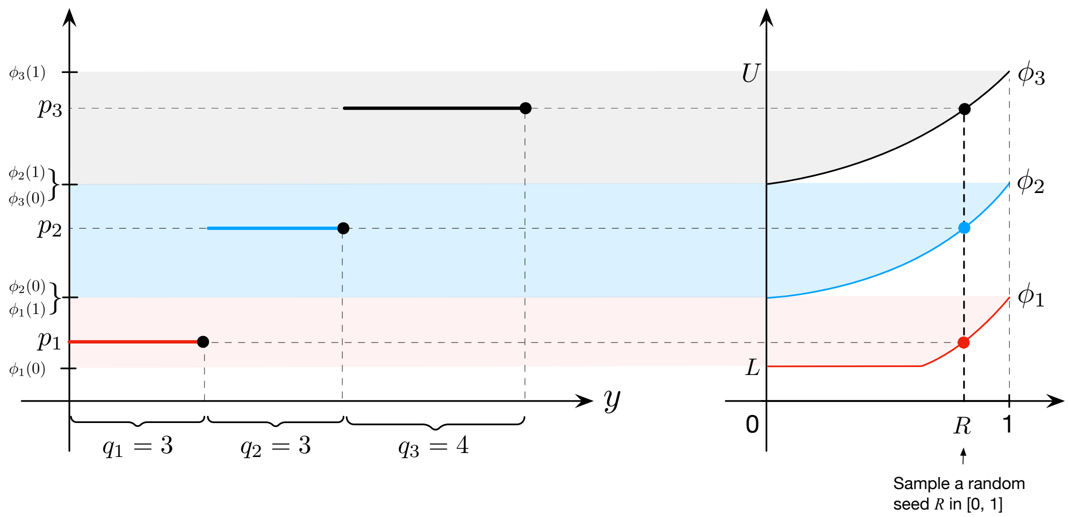

Figure 2: Illustration of cPPM-$\boldsymbol{\phi}$ with $\Delta = 2$ (i.e., dynamic pricing with three total price levels), total units $k = 10$, and reservation vector $\{q_1 = q_2 = 3,\ q_3 = 4\}$. When a random seed $R \sim \mathcal{U}(0, 1)$ is sampled, the three prices p_1, p_2, and p_3 are generated according to the pricing functions ϕ_1, ϕ_2, and ϕ_3, respectively. By construction, the pricing functions satisfy $L = \phi_1(0) \le \phi_1(1) = \phi_2(0) \le \phi_2(1) = \phi_3(0) \le \phi_3(1) = U$, which ensures that $p_1 \le p_2 \le p_3$ always holds.

Theorem 1. *Consider* κSELECTION-(δ, Δ) *with* $\delta = 1$ *and any given price-change cap* $\Delta \in \{0, 1, \ldots, k-1\}$. *Let* $\{q_j\}_{j\in[\Delta+1]}$ *be any reservation vector satisfying* $q_1 \le q_2 \le \cdots \le q_{\Delta+1}$ *and* $\sum_{j=1}^{\Delta+1} q_j = k$, *cPPM-$\boldsymbol{\phi}$ is* $\alpha^\star$*-competitive, where* $\alpha^\star = 1 + \ln(\theta)$, *if for all* $j \in [\Delta+1]$, *the pricing function* ϕ_j *is given by*

$$\phi_j(x) = \begin{cases} L & \text{if } \dfrac{\sum_{l=1}^{j-1} q_l + q_j x}{k} \in \left[0, \frac{1}{\alpha^\star}\right), \\ L \cdot \exp\left(\alpha^\star \cdot \dfrac{\sum_{l=1}^{j-1} q_l + q_j x}{k} - 1\right) & \text{if } \dfrac{\sum_{l=1}^{j-1} q_l + q_j x}{k} \in \left[\frac{1}{\alpha^\star}, 1\right]. \end{cases} \tag{1}$$

By the lower bound established by prior studies [26], cPPM-$\boldsymbol{\phi}$ is optimal with the pricing functions given by Theorem 1. The intuition behind the above pricing function design and the proof of the theorem are provided in Section 3. The results obtained in this section serve as the foundation for extending the analysis to risk-aware settings, where the objective incorporates risk sensitivity.

Risk-Sensitive Fully-Static and Fully-Dynamic Pricing. Trying to build-up our results for most general case of κSELECTION-(δ, Δ) problem, we consider the simplest non-trivial case, where no price changes are allowed, i.e., $\Delta = 0$. Here, we derive the optimal static pricing function that obtains the tightest CVaR$_\delta$-CR performance across all static pricing algorithms.

Theorem 2 (RISK-SENSITIVE FULLY-STATIC PRICING)**.** *Consider* κSELECTION-(δ, Δ) *with* $\delta \in [0, 1]$ *and* $\Delta = 0$. *cPPM-$\boldsymbol{\phi}$ achieves the optimal CVaR$_\delta$-CR, denoted by* α_δ^{SP}, *among all static pricing schemes if* $q_1 = k$ *and the single price function* ϕ *is given by*

$$\phi(x) = \begin{cases} L & x \in \left[0, 1 - \delta + \delta \cdot \lfloor \frac{1}{\alpha_\delta^{SP}} \rfloor\right], \\ L\left[1 + \displaystyle\sum_{j=1}^{\left\lfloor \frac{x - \delta(1 - 1/\alpha_\delta^{SP})}{1-\delta} \right\rfloor} \frac{\left(\frac{\alpha_\delta^{SP}}{\delta}\right)^j}{j!} \left(x - 1 + \delta\left(1 - \frac{1}{\alpha_\delta^{SP}}\right) - j(1-\delta)\right)^j\right] & x \in \left[1 - \delta + \delta \cdot \lfloor \frac{1}{\alpha_\delta^{SP}} \rfloor, 1\right], \end{cases}$$

where $\alpha_\delta^{\mathsf{SP}}$ is the solution to the following equation:

$$\sum_{j=1}^{\lfloor \frac{\delta(1-1/\alpha_\delta^{\mathsf{SP}})}{1-\delta} \rfloor} \frac{\left(\frac{\alpha_\delta^{\mathsf{SP}}}{\delta}\right)^j}{j!} \left(\delta\left(1-\frac{1}{\alpha_\delta^{\mathsf{SP}}}\right) - j(1-\delta)\right)^j = \theta - 1. \tag{2}$$

Since the right-hand side of Eq. (2) is monotonically increasing with respect to α, the equation admits a unique solution. As $\delta \to 1$, the value of α^* converges to $1 + \ln\left(\frac{U}{L}\right)$. Consequently, the fully-static pricing scheme designed according to Theorem 3 coincides with the optimal pricing design of the online algorithm for the kSelection-(δ, Δ) problem when $\delta = 1$ and $\Delta = 0$, as established in Theorem 1. The detailed intuition behind the design of above pricing design and the proof leading to the above result are provided in Section 4.1.

Next, we study another extreme case where $\Delta = k - 1$, meaning that the price-change constraint in kSelection-(δ, Δ) is relaxed and the algorithm becomes fully dynamic—allowed to adjust prices up to $k-1$ times. In the following theorem, we provide a design for the pricing functions in cPPM-$\boldsymbol{\phi}$ to obtain certain level of CVaR$_\delta$-CR.

Theorem 3 (Risk-Sensitive Fully-Dynamic Pricing)**.** *Consider kSelection-(δ, Δ) with $\delta = [0, 1]$ and $\Delta = k - 1$. The CVaR$_\delta$-CR of cPPM-$\boldsymbol{\phi}$ is $\alpha_\delta^{\mathsf{DP}}$ if (i) $\alpha_\delta^{\mathsf{DP}}$ is given by*

$$\alpha_\delta^{\mathsf{DP}} = \frac{kU\delta}{\sum_{i=1}^k \int_0^\delta \phi_i(\eta)\, d\eta}, \tag{3}$$

and (ii) the pricing functions $\boldsymbol{\phi} = \{\phi_i\}_{i \in [k]}$ are recursively designed as follows:

- *For all $i \in \left\{1, 2, \ldots, \left\lfloor \frac{k}{\alpha_\delta^{\mathsf{DP}}} \right\rfloor\right\}$, the pricing function is a constant: $\phi_i(x) = L$.*
- *For $i = \left\lfloor \frac{k}{\alpha_\delta^{\mathsf{DP}}} \right\rfloor + 1$, the pricing function $\phi_i(x)$ is defined as*

$$\phi_i(x) = \begin{cases} L & x \in \left[0,\ 1 - \delta + A\delta\right], \\ \frac{\alpha_\delta^{\mathsf{DP}}}{k\delta}\left(\lfloor k/\alpha_\delta^{\mathsf{DP}} \rfloor \cdot L \cdot \delta + \int_0^{\delta - 1 + x} \phi_i(\eta)\, d\eta\right) & x \in \left[1 - \delta + A\delta,\ 1\right], \end{cases}$$

  where $A = \frac{k}{\alpha_\delta^{\mathsf{DP}}} - \lfloor \frac{k}{\alpha_\delta^{\mathsf{DP}}} \rfloor$ and $\alpha_\delta^{\mathsf{DP}}$ is given by Eq. (3)*.*

- *For all $i \in \left\{\left\lfloor \frac{k}{\alpha_\delta^{\mathsf{DP}}} \right\rfloor + 2, \ldots, k\right\}$, the pricing function $\phi_i(x)$ is given by*

$$\phi_i(x) = \frac{\alpha_\delta^{\mathsf{DP}}}{k\delta}\left(\sum_{j=1}^{i-1} \int_x^{\min\{1,\, x+\delta\}} \phi_j(\eta)\, d\eta + \sum_{j=1}^{i} \int_0^{\max\{0,\, \delta-1+x\}} \phi_j(\eta)\, d\eta\right).$$

We note that there exists a unique competitive ratio $\alpha \geq 1$ and a corresponding set of pricing functions $\{\phi_i\}_i$ that satisfy the system of recursive equations given in the theorem above. However, obtaining closed-form expressions for these functions from the associated system of differential equations requires substantial computation, which is beyond our scope. For a given set of parameter values defining the problem setting, the worst-case CVaR$_\delta$-CR of cPPM-$\boldsymbol{\phi}$ can instead be computed numerically using a binary search procedure.

For example, Figure 3 illustrates the performance of cPPM-$\boldsymbol{\phi}$ for the case $L = 1$ and $U = 100$ under three different values of the tail probability δ. As shown, as $\delta \to 1$, the competitiveness of cPPM-$\boldsymbol{\phi}$ improves and approaches the lower bound $1 + \ln(U/L)$ (established by [26]). It can be verified that as $\delta \to 1$, the

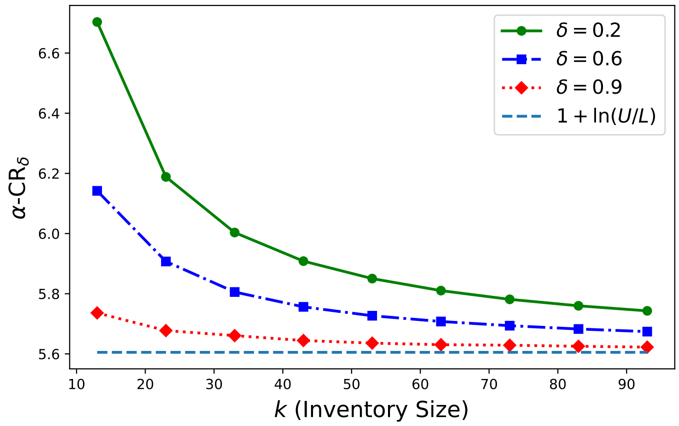

Figure 3: Worst-case CVaR$_\delta$-CR of cPPM-$\boldsymbol{\phi}$ for $\delta \in \{0.2, 0.6, 0.9\}$ with $L = 1$, $U = 100$, and k ranging from 3 to 100. The pricing functions are designed according to Theorem 3.

above pricing-function design smoothly reduces to the design given in Theorem 1 for the case $\Delta = k - 1$. Thus, under this design and as $\delta \to 1$, cPPM-$\boldsymbol{\phi}$ achieves $1 + \ln(U/L)$-CVaR$_\delta$-CR. Moreover, as illustrated in Figure 3, the performance of cPPM-$\boldsymbol{\phi}$ improves as the value of k increases. In fact, we can establish that cPPM-$\boldsymbol{\phi}$, under the pricing design in Theorem 3, becomes optimal in the large-inventory regime as $k \to \infty$.

Proposition 1. *Following the fully-dynamic pricing scheme in Theorem 3, the CVaR$_\delta$-CR of cPPM-$\boldsymbol{\phi}$ is asymptotically optimal as $k \to \infty$, namely, cPPM-$\boldsymbol{\phi}$ attains the smallest possible CVaR$_\delta$-CR among all online algorithms for any confidence level $\delta \in [0, 1]$ when $k \to \infty$.*

Risk-Sensitive Δ-Dynamic Pricing: A General Framework. Building on the insights from the two special cases of risk-sensitive setting, we extend the framework to the general case with $\Delta \geq 1$. The pricing functions are now recursively determined through a system of delayed differential equations that capture the effect of the tail probability δ and correlation among pricing levels.

Theorem 4 (Risk-Sensitive Δ-Dynamic Pricing)**.** *Consider* kSelection-(δ, Δ) *with $\delta \in [0, 1]$ and any number of price changes $\Delta \in \{1, \dots, k-1\}$. Let $\{q_j\}_{j \in [\Delta+1]}$ be a reservation vector satisfying $q_1 = \lceil \frac{k}{\alpha} \rceil$, $q_2 \leq \cdots \leq q_{\Delta+1}$, and $\sum_{j=1}^{\Delta+1} q_j = k$. The CVaR$_\delta$-CR of cPPM-$\boldsymbol{\phi}$ is $\alpha_\delta^{\Delta\text{-}DP}$ if (i) $\alpha_\delta^{\Delta\text{-}DP} \geq 1$ is the unique solution satisfying the following equation*

$$\alpha_\delta^{\Delta\text{-}DP} = \frac{2Uk\delta}{\left\lceil \frac{k}{\alpha_\delta^{\Delta\text{-}DP}} \right\rceil \cdot L \cdot \delta + \sum_{j=2}^{\Delta+1} \int_0^\delta q_j\, \phi_j(\eta)\, d\eta} \tag{4}$$

and (ii) the pricing functions $\boldsymbol{\phi} = \{\phi_i\}_{i \in [\Delta+1]}$ are recursively designed as follows: set $\phi_1(x) = L$, and for all $i \in \{2, \dots, \Delta+1\}$, define

$$\phi_i(x) = \frac{\alpha_\delta^{\Delta\text{-}DP}}{2k\delta} \left(\left\lceil \frac{k}{\alpha_\delta^{\Delta\text{-}DP}} \right\rceil \cdot L \cdot \delta + \sum_{j=2}^{i-1} \int_x^{\min\{1,\, x+\delta\}} q_j\, \phi_j(\eta)\, d\eta \;+\; \sum_{j=2}^{i} \int_0^{\max\{0,\, \delta-1+x\}} q_j\, \phi_j(\eta)\, d\eta \right).$$

Following the method of steps, we emphasize that the design of each pricing function ϕ_i is recursively determined by the values of the preceding pricing functions $\phi_{i'}$ for all indices $i' < i$. Furthermore, it can be

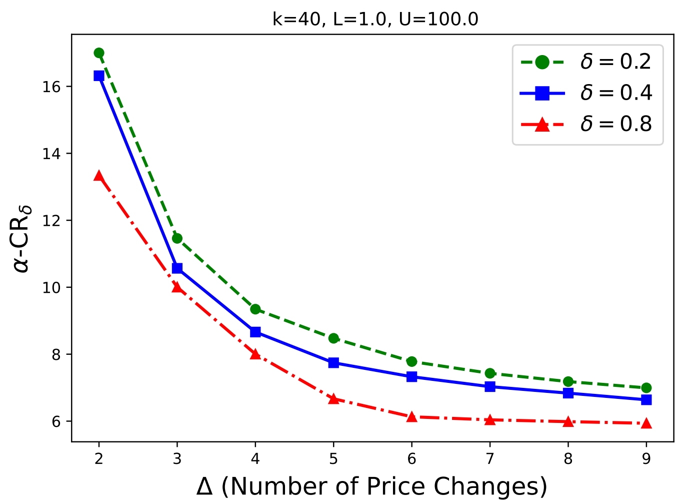

Figure 4: Worst-case CVaR_δ-CR of cPPM-$\boldsymbol{\phi}$ for $\delta \in \{0.2, 0.4, 0.8\}$ with $L = 1$, $U = 100$, and $k = 40$, where the design of the pricing functions are according to Theorem 4, and we have $\max_{i,j} |q_i - q_j| \leq 1$

verified that all pricing functions are monotonically increasing with respect to the parameter α. Hence, there exists a unique value of α for which the boundary condition $\phi_{\Delta+1}(k) = 1$ is satisfied.

For a given vector of reserved quantities $\{q_j\}_{j\in[\Delta+1]}$, obtaining a closed-form expression for the set of pricing functions $\{\phi_i\}_{i\in[\Delta+1]}$ according to the above design, for arbitrary inventory size k, number of price changes Δ, and a given set of relational constraints among the reserved quantities, is computationally intensive; hence, we omit it here. However, the smallest feasible value of α for which there exists a feasible design of the pricing functions following the theorem above can be computed numerically using a binary search procedure. Figure 4 illustrates the worst-case CVaR_δ-CR of cPPM-$\boldsymbol{\phi}$, where the pricing functions are designed according to the theorem above, for three representative cases with $\delta \in \{0.2, 0.4, 0.8\}$. The curves correspond to the setting with $L = 1$, $U = 100$, and $k = 40$. As observed, increasing the number of allowed price changes leads to tighter performance guarantees. Moreover, for lower risk levels (smaller values of δ), cPPM-$\boldsymbol{\phi}$ achieves a better worst-case competitive ratio, as the algorithm becomes less sensitive to risk and can be more aggressive in design of the posted prices. The intuition behind the above pricing function design and the proof of Theorem 4 are provided in Section 4.3.

3 kSelection-(δ, Δ) with $\delta = 1$: Optimal Risk-Neutral PPMs with Limited Price Changes

In this section, we revisit the risk-neutral setting where $\delta = 1$. We provide a detailed discussion of the pricing design introduced in Theorem 1, explain the intuition behind this construction, and present a complete proof of the theorem.

3.1 Intuition Behind the Design

To elaborate on the intuition behind the pricing design in Theorem 1, consider the fractional relaxation of kSelection-(δ, Δ), where the integrality constraint $x_t \in \{0, 1\}$ is relaxed to $x_t \in [0, 1]$. Let $\hat{y}_t := \sum_{s<t} \hat{x}_s$ denote the algorithm's cumulative (fractional) allocation just before buyer t arrives. In this relaxation, the optimal online algorithm proposed by [31] determines the fractional allocation $\hat{x}t$ for buyer t, given their

valuation v_t, according to the following utility maximization rule:

$$\hat{x}_t = \arg\max_{x\in[0,1]} \left\{ v_t x - k \int_{\hat{y}_t/k}^{(\hat{y}_t+x)/k} \phi(\eta)\, d\eta \right\}, \quad \text{where } \phi(x) = \begin{cases} L, & \text{if } x \in \left[0, \frac{1}{\alpha}\right), \\ L \cdot \exp(\alpha x - 1), & \text{if } x \in \left[\frac{1}{\alpha}, 1\right]. \end{cases}$$

Intuitively, $\phi(\eta)$ represents the marginal price at the normalized utilization level $\eta \in [0,1]$, and the integral term captures the total cost of allocating an additional x units when the current utilization is $\hat{y}_t/k$.

To see how pricing design arise naturally from this perspective, partition the inventory into two quotas, q_1 and q_2, such that $q_1 + q_2 = k$. There exists an equivalent form of the above maximization expression that yields the same fractional decision. By an appropriate change of variables, one can verify that $\hat{x}_t$ can also be expressed as follows:

$$\begin{aligned} \hat{x}_t = \arg\max_{x\in[0,1]} \Big\{ v_t x - q_1 \cdot \int_{\min\{\hat{y}_t/q_1,\,1\}}^{\min\{1,(\hat{y}_t+x)/q_1\}} \phi\big((q_1/k)\,\eta\big)\, d\eta - \\ \mathbf{1}\{\hat{y}_t + x > q_1\} \cdot q_2 \cdot \int_{\max\{0,\frac{\hat{y}_t-q_1}{q_2}\}}^{\min\{1,(\hat{y}_t+x-q_1)_+/q_2\}} \phi\big((q_1/k) + (q_2/k)\,\eta\big)\, d\eta \Big\}. \end{aligned}$$

The first integral prices the portion of the allocation that lies within the first q_1 units, using the curve $\eta \mapsto \phi\big((q_1/k)\eta\big)$. Once the cumulative allocation reaches q_1 (i.e., when $\hat{y}_t \geq q_1$), this term becomes inactive. The second integral prices the spillover into the next q_2 units using a shifted curve $\eta \mapsto \phi\big((q_1/k) + (q_2/k)\eta\big)$, which only contributes when the decision interval $(\hat{y}_t, \hat{y}_t + x)$ crosses the boundary at q_1. Moreover, the pricing of each segment of the items is normalized so that the fraction of that segment lies within the unit interval $[0,1]$. For example, for the first q_1 units, the integration over the curve $\eta \mapsto \phi\big((q_1/k)\eta\big)$ is normalized to the range $[0,1]$. Consequently, the design of the pricing functions in Theorem 1 is inspired by this structure—where the pricing for the first q_1 units and the subsequent q_2 units follows directly from the utility-maximization formulation described above.

3.2 Proof of Theorem 1

Before, getting into the details of the proof, we first introduce a set of notations that will be used throughout the remainder of the paper.

Notation. Let $y_T^{(r)}$ denote the number of units sold when the random seed R realizes to $r \in [0,1]$. This quantity is deterministic given some instance I of the kSelection-(δ, Δ) problem as input. For simplicity we drop the subscript T and write $y^{(r)}$. Let $y^* := \max_{r\in[0,1]} y^{(r)}$ be the maximum over realizations of the random seed R, and let $r^* \in [0,1]$ denote the largest value of the random seed R under which the algorithm fully utilizes the reserved units from the first up to the i^*-th price level, i.e., $r^* = \max\{\, r \in [0,1] : y^{(r)} \geq \sum_{i=1}^{i^*} q_i \,\}$. Furthermore, let

$$i^* := \begin{cases} 0, & \text{if } y^* < q_1, \\ \max\left\{ i \in \{1, 2, \ldots, \Delta+1\} \;\middle|\; y^* \geq \sum_{l=1}^{i} q_l \right\}, & \text{otherwise}, \end{cases}$$

which denotes the highest price level such that, under the realization $R = r^*$, the algorithm fully allocates all reserved units from price levels 1 through i^*. Furthermore, let us define the function $\phi_i^* : [L, U] \to [0,1]$ where $\phi_i^*(v) = \sup\{x \in [L,U] | \phi(x) \leq v\}$ is defined as the general inverse of the ϕ_i function.

Proof Road-Map. We first establish two key structural properties of Algorithm 1. The first is a monotonicity property concerning the number of items sold, and the second provides a lower bound on the utilization level of the algorithm across all realizations of the random seed R. We then formulate a dual linear program whose optimal objective value serves as an upper bound on the offline optimal welfare. Next, following a randomized online primal–dual (R-OPD) approach, we define dual variable updates as functions of the realized random seed and construct a candidate dual solution. We then show that the dual constraints are $\alpha^\star$-feasible in expectation under these updates and that each buyer contributes at most v_t in expectation to the dual objective. By weak duality, this implies that the algorithm achieves at least a $1/\alpha^\star$ fraction of the offline optimal value, thereby establishing its $\alpha^\star$-competitiveness.

3.2.1 Detailed Proof of Theorem 1

Let us fix an input instance I of the kSelection-(δ, Δ) problem and prove Algorithm 1 is $\alpha^\star = 1 + \ln(U/L)$-competitive on this instance of the problem. Although Algorithm 1 is randomized, its correlated pricing scheme ensures that the number of units sold does not vary significantly across different sample paths. In other words, the total number of sold units remains close to the maximum $\sum_{i=1}^{i^*} q_i$ achieved across all realizations of the random seed. This observation is made precise through the following two properties.

Monotonic Utilization. We establish a monotonicity property for the number of units sold by Algorithm 1 as a function of the realized random seed R. Specifically, we show that the utilization of the algorithm by the arrival of the t-th buyer, denoted $y_t^{(r)}$, is *nondecreasing* in r. This monotonicity arises from the fact that the pricing profiles posted by the algorithm become progressively more aggressive as the random seed R increases from 0 to 1.

Lemma 1. *For any $r_1, r_2 \in [0, 1]$ with $r_1 \le r_2$, and any buyer $t \in [T]$, we have $y_t^{(r_1)} \ge y_t^{(r_2)}$.*

The proof of above lemma can be found in Appendix A. Thus, following the above lemma, we can see that $y^{(r)} \ge \sum_{i=1}^{i^*} q_i$ for all values of $r \in [0, r^*]$.

Lower bound on the utilization $y^{(r)}$. Given the constraint on the reservation vector $q_1 \le q_2 \le \cdots \le q_{\Delta+1}$, and noting that the level-wise pricing functions are nondecreasing across levels—i.e., $\phi_j(\cdot) \le \phi_{j+1}(\cdot)$ by design—we obtain the following lower bound on the total utilization $y^{(r)}$.

Lemma 2. *If $i^* \ge 2$, then for all $r \in [0, 1]$, we have $y^{(r)} \ge \sum_{i=1}^{i^*-1} q_i$, under the reservation-vector constraint $q_1 \le q_2 \le \cdots \le q_{\Delta+1}$.*

The proof of above lemma can be found in Appendix B. In the following, we utilize the above two properties to prove the optimality of the competitive ratio for the pricing design given in Theorem 1.

Dual LP designed to Δ-cap price-change setting. Consider the following dual linear program (LP), which upper bounds the offline optimum:

$$\min_{u_t, \lambda_j} \quad \sum_{t \in [T]} u_t + \sum_{j=1}^{\Delta+1} \lambda_j \cdot q_j \qquad \text{s.t.} \quad v_t \le u_t + \frac{1}{k} \sum_{j=1}^{\Delta+1} \lambda_j \cdot q_j, \quad \forall t \in [T]. \tag{5}$$

It can be verified that the optimal objective value of this LP provides an upper bound on the performance of the offline clairvoyant algorithm. Following the economic interpretation of the randomized primal–dual framework presented in [11], we can interpret the variable λ_j as the *price* associated with the j-th set of reserved units at the j-th price level, and the variable u_t as the *utility* of buyer t resulting from participating in the pricing scheme implied by the primal–dual construction.

Dual variables construction. Following the randomized primal–dual framework, we construct, for each realization of the random seed $R = r$, a corresponding set of dual variables $\lambda_j^{(r)}, u_t^{(r)}$. The final dual variables are then defined as their expectations over the random seed such that $\lambda_j = \mathbb{E}_R[\lambda_j^{(R)}]$ and $u_t = \mathbb{E}_R[u_t^{(R)}]$, where the expectation is taken with respect to the random seed R.

Initialize all dual variables to zero. Then for a realization of the random seed $R = r$, let us update the dual variables $\lambda_j^{(r)}$ as follows:

$$\lambda_i^{(r)} = \begin{cases} \phi_i(r), & i \in \{1, 2, \ldots, i^* - 1\}, \\ \phi_i(r), & i = i^*,\ r \in [0,\ r^*], \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Furthermore, if buyer t receives one unit from the i-th price level, set

$$u_t^{(r)} = \begin{cases} v_t - \phi_i(r), & \text{if } i < i^* \text{ or } (i = i^* \text{ and } r \le r^*), \\ v_t, & \text{otherwise.} \end{cases} \tag{7}$$

The update in Eq. (6) mirrors the posted price at level i whenever, under realization r, the reserved units at that level are fully utilized. By the lower bound established in Lemma 2, which guarantees that $y^{(r)} \ge \sum_{l=1}^{i^*-1} q_l$ for all $r \in [0, 1]$, and based on the structural monotonicity proved in Lemma 1, the reserved units for the i^*-th level are fully utilized for all $r \in [0, r^*]$. The update in Eq. (7) sets $u_t^{(r)}$ to the buyer's utility when buyer t is allocated a unit from a fully utilized price level (that is, a level $i < i^*$, or level i^* when $r \le r^*$), according to the price posted at that level and the buyer's valuation. Taking expectations of these per-realization dual variables over R produces the final dual solution $(\{u_t\}, \{\lambda_j\})$ used in the OPD analysis.

Showing that $\sum_{t\in[T]} u_t + \sum_{j=1}^{\Delta+1} \lambda_j\, q_j = \mathbb{E}_R\left[\mathtt{ALG}^{(R)}(I)\right]$. We next show that the dual objective value of the solution obtained from the above updates equals the expected performance of Algorithm 1 on instance I. It suffices to prove that, under any realization $R = r$, $\sum_{t\in[T]} u_t^{(r)} + \sum_{j=1}^{\Delta+1} \lambda_j^{(r)}\, q_j = \mathtt{ALG}^{(r)}(I)$.

Fix a realization $R = r$. Let $B^{(r)}$ denote the set of buyers who are allocated a unit under this realization. For each price level j, let $B_j^{(r)} \subseteq B^{(r)}$ denote the subset of buyers served from the j-th level's reserved units, so that $|B_j^{(r)}|$ represents the number of j-level units actually sold under r.

From Lemma 2, we know that the algorithm fully utilizes the first $(i^* - 1)$ levels of reserved units for all realizations of R. Furthermore, following the definition of r^* and the monotonicity established in Lemma 1, all reserved units up to the i^*-th level are also fully utilized for realized values of the random seed in the range $[0, r^*]$. Thus, we have $|B_j^{(r)}| = q_j,\ \forall j \in \{1, \ldots, i^* - 1\}$, and $|B_{i^*}^{(r)}| = q_{i^*},$ if $r \le r^*$.

Since the dual variable $u_t^{(r)}$ is updated only for buyers who receive an item under realization r, as specified by Eq. (7), we can express the total contribution from the $u_t^{(r)}$-variables as

$$\begin{aligned} \sum_{t\in[T]} u_t^{(r)} &= \sum_{j=1}^{\Delta+1} \sum_{t\in B_j^{(r)}} v_t \\ &= \sum_{j=1}^{i^*-1} \sum_{t\in B_j^{(r)}} (v_t - \phi_j(r)) + \mathbf{1}\{r \le r^*\} \sum_{t\in B_{i^*}^{(r)}} (v_t - \phi_{i^*}(r)) + \mathbf{1}\{r > r^*\} \sum_{t\in B_{i^*}^{(r)}} v_t + \sum_{j=i^*+1}^{\Delta+1} \sum_{t\in B_j^{(r)}} v_t, \\ &= \sum_{j=1}^{\Delta+1} \sum_{t\in B_j^{(r)}} v_t - \sum_{j=1}^{i^*-1} q_j \phi_j(r) - \mathbf{1}\{r \le r^*\}\, q_{i^*} \phi_{i^*}(r). \end{aligned}$$

Here, the second equality follows from the dual update rule in Eq. (7), by decomposing the summation over all price levels and separating the case of the i^*-th price level based on the realized value of the random seed. The third equality follows from the sizes of the sets $B_j^{(r)}$ established above.

Using the dual update rule for the $\lambda_j^{(r)}$ variables from Eq. (6), the total dual contribution associated with these variables is given by

$$\sum_{j=1}^{\Delta+1} \lambda_j^{(r)} q_j = \sum_{j=1}^{i^*-1} q_j \phi_j(r) + \mathbf{1}\{r \le r^*\}\, q_{i^*} \phi_{i^*}(r).$$

Combining the above two expressions for $\sum_t u_t^{(r)}$ and $\sum_j \lambda_j^{(r)} q_j$, we obtain

$$\sum_{t \in [T]} u_t^{(r)} + \sum_{j=1}^{\Delta+1} \lambda_j^{(r)} q_j = \sum_{i=1}^{\Delta+1} \sum_{t \in B_i^{(r)}} v_t = \mathtt{ALG}^{(r)}(I),$$

which shows that the dual objective under realization r exactly equals the realized welfare of the algorithm. Taking expectations with respect to R on both sides of the equality above yields the desired identity.

Dual constraints feasibility in expectation ($\alpha^\star$-feasibility). For every buyer $t \in [T]$, we show that the dual constraint in Eq. (5) corresponding to this buyer is $\alpha^\star$-feasible in expectation; that is, $\mathbb{E}_R \left[u_t^{(R)} + \frac{1}{k} \sum_{j=1}^{\Delta+1} \lambda_j^{(R)} q_j \right] \ge \frac{v_t}{\alpha^\star}$. Establishing this inequality completes the randomized online primal–dual analysis and proves the $\alpha^\star$-competitiveness of Algorithm 1 based on the framework used in [8]. Consider a buyer t with valuation v_t such that, for some $i \in [\Delta + 1]$, we have $\phi_i(0) \le v_t \le \phi_i(1)$. Depending on the value of i and v_t, we prove the dual constraint feasibility under several different scenarios. We analyze one scenario below and defer the remaining cases to appendix.

Case I: Either $i \le i^* - 1$, or $i = i^*$ and $\phi_i^*(v_t) \le r^*$.

From the dual update rule defined in Eq. (6), and noting that for all realizations of $R \in [0, 1]$, the reserved units corresponding to the first $i^* - 1$ price levels are fully sold, and for $R \in [0, r^*]$, the reserved units at the i^*-th price level are also fully sold, we have

$$\begin{aligned}
\mathbb{E}\left[u_t + \frac{\sum_{l=1}^{\Delta+1} \lambda_l q_l}{k}\right] &\ge \sum_{l=1}^{i^*-1} \frac{q_l}{k} \int_0^1 \phi_l(\eta)\, d\eta + \frac{q_{i^*}}{k} \int_0^{r^*} \phi_{i^*}(\eta)\, d\eta \\
&= \sum_{l=1}^{i^*-1} \int_{\sum_{m=1}^{l-1} \frac{q_m}{k}}^{\sum_{m=1}^{l} \frac{q_m}{k}} \phi(\eta)\, d\eta + \int_{\sum_{l=1}^{i^*-1} \frac{q_l}{k}}^{\sum_{l=1}^{i^*-1} \frac{q_l}{k} + \frac{q_{i^*}}{k} r^*} \phi(\eta)\, d\eta \\
&= \int_0^{\sum_{l=1}^{i^*-1} \frac{q_l}{k} + \frac{q_{i^*}}{k} r^*} \phi(\eta)\, d\eta \\
&= \frac{\phi\left(\sum_{l=1}^{i^*-1} \frac{q_l}{k} + \frac{q_{i^*}}{k} r^*\right)}{\alpha^\star} \\
&\ge \frac{v_t}{\alpha^\star}.
\end{aligned}$$

The first inequality follows from the dual update rule in Eq. (6). The first equality holds by the construction of the pricing functions ϕ_j, as described in Theorem 1 (see Eq. (1)).

To save space, we defer the proof of the feasibility of the dual constraints for the remaining cases $\left((i = i^*, \phi_i^{-1}(v_t) > r^*) \text{ or } i > i^* \right)$ to Appendix C. The proof structure closely follows that of the cases described above. Considering all cases together, the $\alpha^\star$-feasibility of the dual constraint corresponding to each buyer t is thus verified. Consequently, the $\alpha^\star$-competitiveness of Algorithm 1 follows. Based on the lower bound of $1 + \ln(U/L)$ established in prior work [26] for the attainable competitive ratio of any online algorithm for the k-selection problem, which is a special case of kSelection-(δ, Δ), cPPM-$\boldsymbol{\phi}$ attains the optimal worst-case competitive ratio when $\delta = 1$ and value of Δ varies in the range $\{0, 1, \ldots, k - 1\}$.

4 kSelection-(δ, Δ) with $\delta \in (0, 1]$: Risk-Sensitive Results with CVaR$_\delta$-CR Guarantees in Fully-Static, Fully-Dynamic, and Δ-Dynamic Pricing Setting (Theorems 2–4)

In this section, we first revisit Theorems 2–3, which present an optimal *static pricing algorithm* and a fully dynamic pricing scheme for cPPM-$\boldsymbol{\phi}$. In the first case, cPPM-$\boldsymbol{\phi}$ achieves the tightest CVaR$_\delta$-CR among all posted pricing mechanisms (PPMs) that maintain a single price throughout the horizon (i.e., $\Delta = 0$). In the second case, it achieves exact optimality in large-inventory settings where $k \to \infty$. Building on the results from these two cases, we then extend our analysis to prove the result for the most general form of the kSelection-(δ, Δ) problem, Theorem 4, where we design Δ-dynamic PPMs. In the following, we discuss the intuition behind these pricing designs, explain how they are derived, and outline the proofs behind the two main theorems.

4.1 Proof of Theorem 2

In this section, we first give an intuition behind the pricing design in Theorem 2. To do so, we first establish a lower bound on the performance of all static pricing mechanisms. Subsequently, we present a proposition that characterizes the pricing design for an α-CVaR$_\delta$-CR static pricing mechanism, motivated by this lower-bound analysis, and demonstrate how the optimality result established in Theorem 2 follows from these findings.

4.1.1 Intuition behind Pricing Design in Theorem 2

The design of the pricing function in Theorem 2 is inspired by the approach in [26, 18], which derives lower bounds on the competitive ratio of all online algorithms by identifying the optimal online algorithm over a class of hard instances. The behavior of this optimal algorithm on such instances, in turn, motivates the construction of the pricing function for our optimal static algorithm. Thus, following this approach, for some value of $\epsilon \geq 0$, consider the following hard instance $\mathcal{I}^{(\epsilon)}$ defined as:

$$\mathcal{I}^{(\epsilon)} = \Big\{ \underbrace{L, \ldots, L}_{k \text{ buyers}}, \underbrace{L+\epsilon, \ldots, L+\epsilon}_{k \text{ buyers}}, \ldots, \underbrace{L+j\cdot\epsilon, \ldots, L+j\cdot\epsilon}_{k \text{ buyers in stage } L+j\cdot\epsilon}, \ldots,$$
$$\underbrace{L+\lfloor (U-L)/\epsilon \rfloor \cdot \epsilon, \ldots, L+\lfloor (U-L)/\epsilon \rfloor \cdot \epsilon}_{k \text{ buyers}} \Big\}.$$

Let $V^{(\epsilon)} = \{L, L+\epsilon, \ldots, L+\lfloor (U-L)/\epsilon \rfloor \cdot \epsilon\}$ denote the set of all possible buyer valuations appearing in $\mathcal{I}^{(\epsilon)}$. For any $v \in V^{(\epsilon)}$, let $\mathcal{I}_v^{(\epsilon)}$ denote the subset of buyers in $\mathcal{I}^{(\epsilon)}$ consisting of all arrivals up to and including the k buyers with valuation v. We derive the optimal static pricing algorithm on the class of instances $\{\mathcal{I}_v^{(\epsilon)}\}_{v \in V^{(\epsilon)}}$. Since the instance $\mathcal{I}_v^{(\epsilon)}$ is identical to $\mathcal{I}_{v'}^{(\epsilon)}$ up to the arrival of the k buyers with valuation v, for some values of $v \leq v'$ in $V^{(\epsilon)}$, the online algorithm cannot distinguish between these instances. Therefore, to analyze the performance of an online algorithm on the class $\{\mathcal{I}_v^{(\epsilon)}\}_{v \in V^{(\epsilon)}}$, we can equivalently assume that the entire instance $\mathcal{I}^{(\epsilon)}$ is revealed to the algorithm, although the sequence may stop at any stage. In other words, to achieve α-CVaR$_\delta$-CR competitiveness, the algorithm must guarantee an expected welfare of at least $k \cdot v/\alpha$, $\frac{1}{\alpha}$ fraction of optimal clairvoyant, by the end of the stage in $\mathcal{I}^{(\epsilon)}$ where k buyers with valuation v arrive. This requirement follows because the input sequence may end after any stage corresponding to the arrival of k buyers with valuation $v \in V^{(\epsilon)}$, thereby yielding an instance from the class $\{\mathcal{I}_v^{(\epsilon)}\}_{v \in V^{(\epsilon)}}$.

Let the random variable P represent the price posted by a static pricing algorithm on the given instance $\mathcal{I}^{(\epsilon)}$. Since the algorithm is static, we have $p_t = P$ for all $t \in \{1, 2, \dots, T\}$. Define the function $\psi : [L, U] \to [0, 1]$ such that $\psi(x)$ denotes the probability that the algorithm posts a price less than or equal to x, i.e., $\Pr[P \le x] = \psi(x)$. The following condition must hold for a static pricing algorithm with pricing distribution characterized by ψ to be α-CVaR_δ-CR on the class of hard instances $\{\mathcal{I}_v^{(\epsilon)}\}_{v\in V^{(\epsilon)}}$:

Proposition 2. *For the class of hard instances $\{\mathcal{I}_v^{(\epsilon)}\}_{v\in V^{(\epsilon)}}$, a static pricing algorithm with price distribution function ψ must satisfy the following constraints to be α-CVaR_δ-CR competitive:*

$$\psi(L) \ge 1 - \delta + \delta \cdot \frac{1}{\alpha},$$

$$L \cdot \frac{1}{\delta} \cdot \min\{\delta - 1 + \psi(v), \psi(L)\} + \int_{\eta=\psi(L)}^{\max\{\psi(L),\delta-1+\psi(v)\}} \frac{1}{\delta} \cdot \psi^*(\eta) d\eta \ge \frac{v}{\alpha}, \quad \forall v \in V^{(\epsilon)}$$

where in above the function $\psi^*(x) = \sup\{v \in [L, U] | \psi(v) \le x\}$

Proof of above proposition can be found in Appendix D. In the above, the first inequality follows from the fact that, with probability at least $1 - \delta + \delta \cdot \frac{1}{\alpha}$, any online algorithm must post the price L. Otherwise, the CVaR_δ performance of the online algorithm on instance $I_L^{(\epsilon)}$ would fall below $\frac{k\cdot L}{\alpha}$, which is the benchmark value required for the algorithm to be α-CVaR_δ-CR. Furthermore, the left-hand side of the second inequality represents the CVaR_δ objective value of an online algorithm up to the end of the stage in which k buyers with valuation v arrive, where the algorithm's randomness in the posted price is captured by the function ψ.

Without loss of generality, as $\epsilon \to 0$, we can assume that the function ψ, corresponding to the optimal online algorithm on the class of hard instances $\{\mathcal{I}_v^{(\epsilon)}\}_{v\in V^{(\epsilon)}}$, is a continuous strictly increasing function. Obtaining the function ψ that satisfies the above inequality for all $v \in [L, U]$ with the smallest possible value of α naturally motivates a corresponding design for the pricing function ϕ. By setting $\phi = \psi^{-1}$ and enforcing the inequality to hold with equality, we obtain the following design of ϕ that achieves α-CVaR_δ-CR.

Proposition 3 (Risk-Sensitive Static Pricing)**.** *Consider the* kSelection-(δ, Δ) *problem with $\delta \in [0, 1]$ and $\Delta = 0$. The CVaR_δ-CR of* **cPPM-$\boldsymbol{\phi}$** *is $\alpha_\delta^{\mathsf{SP}}$ (with $\alpha_\delta^{\mathsf{SP}} \ge 1$) (i) where $\alpha_\delta^{\mathsf{SP}}$ is a solution to following equation*

$$\alpha_\delta^{\mathsf{SP}} = \frac{U\delta}{\displaystyle\int_0^\delta \phi(\eta)\, d\eta} \tag{8}$$

and (ii) the pricing function ϕ is designed as

$$\phi(x) = \begin{cases} L, & \text{for } x \in \left[0,\ 1 - \delta + \delta \cdot 1/\alpha_\delta^{\mathsf{SP}}\right], \\ \dfrac{\alpha_\delta^{\mathsf{SP}}}{\delta} \displaystyle\int_0^{\delta-1+x} \phi(\eta)\, d\eta, & \text{for } x \in \left[1 - \delta + \delta \cdot 1/\alpha_\delta^{\mathsf{SP}},\ 1\right]. \end{cases}$$

It is worth noting that differentiating both sides of the equality given in above proposition for the design of ϕ function yields the delayed differential equation $\phi'(x) = \frac{\alpha_\delta^{\mathsf{SP}}}{\delta} \cdot \phi(\delta - 1 + x)$, with the initial condition $\phi(x) = L$ for all $x \in [0,\ 1 - \delta + \delta \cdot \frac{1}{\alpha_\delta^{\mathsf{SP}}}]$.

Proof of Theorem 2. Putting everything together, by employing the *method of steps* [24] and Taylor expansion, we can derive the closed-form design of the pricing function ϕ presented in Theorem 2, based on the system of delayed differential equations described in the proposition. Furthermore, we can show that the design established in Theorem 2 achieves the *optimal* CVaR_δ-CR by identifying the smallest value of α for which a feasible auxiliary function ψ exists, subject to the constraints in Proposition 2. Since the proof relies on straightforward mathematical arguments, we defer the full proof of Theorem 2 to Appendix G. In what follows, we proceed to prove Proposition 3 using the risk-sensitive R-OPD framework.

4.1.2 Proof of Proposition 3

The proof uses a modified version of R-OPD method. The central idea of R-OPD for problems that aim to maximize the expected reward is to specify, for each sample path of the randomized algorithm, an update rule for the dual variables of the LP as a function of the algorithm's decisions along that path. These path-wise dual updates are chosen so that the dual objective accumulated on each realized path is equal to the social welfare achieved by the online algorithm on that same path.

In the kSelection-(δ, Δ) problem, where the objective is to maximize the CVaR_δ of the algorithm's social welfare, the standard randomized primal–dual update scheme is modified. Specifically, dual updates are performed only along the worst δ-fraction of sample paths—those realizations of the algorithm's randomization for which the objective value is minimized and smaller than that of all other sample paths. Each sample path of Algorithm 1 corresponds to a particular realization of the random seed R. Once R is realized as $R = r$, the algorithm's performance $\mathtt{ALG}^{(r)}$ becomes deterministic. Accordingly, to adapt the R-OPD approach for establishing the CVaR_δ-CR-competitiveness of Algorithm 1, we perform dual updates only for those realizations $R = r$ that belong to the set $\mathcal{S}$. When the function $F_{\mathtt{ALG}(I)}$, corresponding to the CDF of the algorithm's social welfare on an input instance I of kSelection-(δ, Δ) problem, is continuous, this set is defined as:

$$\mathcal{S} = \left\{ r \in [0,1] \;\middle|\; F_{\mathtt{ALG}(I)}(\mathtt{ALG}^{(r)}) \le \delta \right\},$$

where $\delta \in (0,1)$ is the risk level associated with the CVaR_δ metric. However, if this function is not continuous, we must carefully handle potential discontinuities of the CDF around the tail probability δ. In that case, we define

$$\mathcal{S} = \left\{ r \in [0,\tilde{r}] \;\middle|\; F_{\mathtt{ALG}(I)}\left((\mathtt{ALG}^{(r)})^-\right) \le \delta \right\},$$

where $\tilde{r}$ is given by

$$\tilde{r} = \inf_{r\in[0,1]} \left\{ \int_{x=0}^{r} \mathbf{1}_{\left\{ F_{\mathtt{ALG}(I)}\left((\mathtt{ALG}^{(x)})^-\right) \le \delta \right\}} \, dx \ge \delta \right\}.$$

The quantity $\tilde{r}$ is the smallest cutoff in the random-seed interval $[0,1]$ such that the total measure of seed values in $[0,\tilde{r}]$ that produce the worst δ-fraction of welfare outcomes is exactly δ.

Proof. Let us define y^* and r^* according to the notation given for the proof of Theorem 1. Furthermore, the monotonicity condition in Lemma 1 holds for Algorithm 1 following the design given in Proposition 3. Furthermore, let us define the function $\phi^* : [L,U] \to [0,1]$ where $\phi^*(v) = \sup\{x \in [L,U] | \phi(x) \le v\}$ is defined as the general inverse of the ϕ function. Based on the value of y^*, let us consider the following cases:

Case I: Consider the case that $y^* < k$. In this case, the total number of buyers in instance I must be exactly equal to k^*. Otherwise, when the random seed R is realized as $R = 1-\delta+\delta/\alpha_\delta^{\mathsf{SP}}$, the posted price becomes L. By Assumption 1, all buyers in instance I have valuations at least equal to L and therefore will accept the posted price. Consequently, under the realization $R = 1-\delta+\delta/\alpha_\delta^{\mathsf{SP}}$, more than k^* units of the resource would be allocated, which contradicts the definition of y^*. Hence, there must exist exactly k^* buyers in instance I.

Furthermore, for any buyer t with valuation v_t, a unit of the item is allocated whenever $R \in [0, \phi^*(v_t)]$, since in this range the posted price remains below v_t and the total number of units sold never reaches k for any realized value of R as there are less than k buyers in instance I. Therefore, the worst δ-fraction of realizations of R—those that minimize the objective value of Algorithm 1,correspond to the range $[1-\delta, 1]$. Thus, we have:

$$\mathsf{CVaR}_\delta[\mathtt{ALG}(I,P)] = \sum_{i=1}^{k^*} v_i \cdot (\phi^*(v_i) - (1-\delta)) \ge \sum_{i=1}^{k^*} \int_{\eta=0}^{\phi^*(v_i)-1+\delta} \phi(\eta) = \sum_{i=1}^{k^*} \frac{v_i}{\alpha_\delta^{\mathsf{SP}}},$$

where in above the second equality follows from the design of ϕ function given in Proposition 3. On the other hand, since the total number of buyer in instance I is equal to $k^* < k$, $\texttt{OPT}(I) = \sum_{i=1}^{k^*} v_i$, the $\alpha_\delta^{\mathsf{SP}}$-$\mathsf{CVaR}_\delta$-$\mathsf{CR}$ of Algorithm 1 is established.

Case II: Consider the case that $y^* = k$. The following LP provides an upper bound on the offline optimal value for the kSelection-(δ, Δ) problem with $\Delta = 0$:

$$\min_{u_t, \lambda} \quad \sum_{t \in [T]} u_t + k \cdot \lambda \qquad \text{s.t.} \quad v_t \le u_t + \lambda, \quad \forall t \in [T].$$

To establish the $\alpha_\delta^{\mathsf{SP}}$-competitiveness of Algorithm 1, we employ the modified R-OPD framework where the dual update rules for the dual variables are performed for the values of the random seed belonging to the set $\mathcal{S}$, ensuring that the dual constraints hold in expectation.

Following this framework, we construct, for each realization of the random seed $R = r$, a corresponding set of dual variables $\{\lambda^{(r)}, u_t^{(r)}\}$. The final dual variables are then defined as their expectations over the random seed, where $\lambda = \mathbb{E}_R[\lambda^{(R)}]$, $u_t = \mathbb{E}_R[u_t^{(R)}]$, where the expectation is taken with respect to the random seed R. We begin by initializing all dual variables to zero. For each realized value of the random seed R, we perform the following dual updates.

Dual update rules. Suppose the random seed is realized as $R = r$. If $r \in \mathcal{S}$, we proceed with the dual update; otherwise, we skip it. The variable $\lambda^{(r)}$ is updated as follows:

$$\lambda^{(r)} = \begin{cases} \frac{\phi(r)}{\delta}, & \text{if } r \le r^*, r \in \mathcal{S} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Next, for each buyer t, we update the dual variable $u_t^{(r)}$ as follows. If buyer t receives an allocation of one unit then, we have

$$u_t^{(r)} = \begin{cases} \frac{1}{\delta} \cdot (v_t - \phi(r)), & \text{if } r \le r^*, \ r \in \mathcal{S} \\ \frac{v_t}{\delta}, & \text{otherwise.} \end{cases} \tag{10}$$

Under these updates, and noting that the total number of sold units equals k for all $r \in [0, r^*]$, the total dual objective for any $r \in \mathcal{S}$, equals the algorithm's objective under that realization: $\left(\sum_{t \in [T]} u_t^{(r)} + k \cdot \lambda^{(r)}\right) = \frac{1}{\delta} \cdot \texttt{ALG}^{(r)}(I)$. Also, since dual updates occur only for realizations of R within intervals in $\mathcal{S}$, we have $\mathbb{E}_R[\sum_{t \in [T]} u_t^{(R)} + k \cdot \lambda^{(R)}] = \mathsf{CVaR}_\delta[\texttt{ALG}^{(R)}(I)]$.

Dual constraint expected $\alpha_\delta^{\mathsf{SP}}$-feasibility. We prove that the dual constraint in the dual LP corresponding to each buyer $t \in [T]$ is $\alpha_\delta^{\mathsf{SP}}$-feasible in expectation; that is, $\mathbb{E}_R[u_t^{(R)} + \lambda^{(R)}] \ge \frac{v_t}{\alpha_\delta^{\mathsf{SP}}}$. Combining this $\alpha_\delta^{\mathsf{SP}}$-feasibility with the fact that the expected dual objective equals the CVaR_δ performance of Algorithm 1, $\mathbb{E}_R[\sum_{t \in [T]} u_t^{(R)} + k \cdot \lambda^{(R)}] = \mathsf{CVaR}_\delta[\texttt{ALG}^{(R)}(I)]$, it follows by weak duality that the algorithm achieves the $\alpha_\delta^{\mathsf{SP}}$-$\mathsf{CVaR}_\delta$-$\mathsf{CR}$ guarantee.

Let us consider the following two subcases to prove the $\alpha_\delta^{\mathsf{SP}}$-feasibility of the dual constraints. Let $w = \inf\{ r \in [0,1] \mid \phi(r) \ge v_t \}$. Following from Eq. (8), which enforces $\phi(1) = U$, such a value of w always exists. Depending on the value of w, we analyze two cases. First, consider the case where $w \le r^*$:

$$\mathbb{E}_R\left[\sum_{t \in [T]} u_t^{(R)} + k \cdot \lambda^{(R)}\right] \ge \frac{1}{\delta} \cdot \int_{\eta=0}^{r^*-(1-\delta)} \phi(\eta) d\eta = \frac{\phi(r^*)}{\alpha_\delta^{\mathsf{SP}}} \ge \frac{v_t}{\alpha_\delta^{\mathsf{SP}}}.$$

The first inequality above holds since there must exist a subrange of size $r^* - (1-\delta)$ within the range $[0, r^*]$ inside the set $\mathcal{S}$, as the tail probability is set to be equal to δ. Furthermore, based on the dual updates above and the fact that the ϕ function is increasing, in the worst case this subrange corresponds to the interval $[0, r^* - (1-\delta)]$. Then, in this case, since by the design of the ϕ function we have $w \geq 1 - \delta + \delta \cdot (\frac{1}{\alpha_\delta^{\mathsf{SP}}})$, and $r^* \geq w$, the inequality follows. Thus, based on the dual updates defined in Eq. (9), the first inequality holds. The last equality follows from the design of the ϕ function.

For the second case, consider $w > r^*$. A unit is allocated to buyer t for all realizations $R \in [r^*, w]$, because the posted price is below v_t on this range and the utilization satisfies $y^{(R)} < y^{(r^*)} = k$ based on Lemma 1 and the definition of r^*. Therefore, by the dual updates in Eqs. (9)-(10), we have:

$$\sum_{t \in [T]} u_t^{(r)} + k \cdot \lambda^{(r)} \geq \begin{cases} \frac{\phi(r)}{\delta}, & \text{for all } r \in [0, r^*],\ r \in \mathcal{S}, \\ \frac{v_t}{\delta} \geq \frac{\phi(r)}{\delta}, & \text{for all } r \in [r^*, w],\ r \in \mathcal{S}. \end{cases}$$

Thus, in the worst case, we have:

$$\mathbb{E}_R\left[\sum_{t \in [T]} u_t^{(R)} + k \cdot \lambda^{(R)}\right] \geq \frac{1}{\delta}\left(\int_{\eta=0}^{w-1+\delta} \phi(\eta) d\eta\right) = \frac{\phi(w)}{\alpha_\delta^{\mathsf{SP}}},$$

where the final equality follows from the design of the function ϕ in Proposition 3. This concludes the proof of $\alpha_\delta^{\mathsf{SP}}$-feasibility of the dual constraints for each buyer t. □

4.2 Proof of Theorem 3 and Proposition 1

In this section, we revisit Theorem 3, which provides a design for an online mechanism that uses k pricing functions and achieves exact optimal performance under the CVaR_δ metric in the large-inventory regime where $k \longrightarrow \infty$, as established by Proposition 1. Theorem 3 focuses on another special case of the KSELECTION-(δ, Δ) problem in which the online algorithm is allowed up to $k-1$ price changes. In this setting, the price-change cap constraint is effectively relaxed, allowing the decision maker to employ a fully dynamic pricing scheme. The proof of Proposition 1 follows from Theorem 3 using simple mathematical arguments, and we defer the full proof to Appendix H. Below, we first outline the proof road map for Theorem 3, and then, in the subsequent section, we provide the detailed proof.

Proof Road map. We next provide a proof of Theorem 3 that departs from the randomized primal–dual framework used earlier and instead focuses on interpreting the correlated pricing scheme in Algorithm 1 as a rounding method for fractional allocations. This proof approach illustrates how the correlated pricing scheme employed by Algorithm 1 naturally induces randomized integral decisions that losslessly round those of a fractional algorithm using the same set of pricing functions. Hence, the correlated posted-pricing scheme in Algorithm 1 can be viewed not only as a mechanism for ensuring incentive compatibility but also as a rounding scheme that converts fractional decisions into randomized integral ones without any loss in expected performance.

4.2.1 Detailed Proof

Fractional Algorithm Consider the following algorithm, denoted by `ALG-FRAC`, which uses the set of pricing functions $\{\phi_i\}_{i=1}^k$ to generate the fractional allocation $\hat{x}_t$ for each arriving buyer t as follows:

$$\hat{x}_t = \arg\max_{\{x \in [0,1]\}} \left[v_t \cdot x \int_{\hat{y}_t - \lfloor \hat{y}_t \rfloor}^{\min\{1, \hat{y}_t - \lfloor \hat{y}_t \rfloor + x\}} \phi_{\kappa_t}(\eta) d\eta + \int_0^{[\hat{y}_t - \lfloor \hat{y}_t \rfloor + x - 1]^+} \phi_{\kappa_t + 1}(\eta) d\eta\right], \tag{11}$$

where $\hat{y}_t$ denotes the cumulative fractional allocation upon the arrival of buyer t, that is, $\hat{y}_t = \sum_{\tau=1}^{t} \hat{x}_\tau$, and $\kappa_t = \lceil \hat{y}_t \rceil$ is the index of next unit of item a fraction of which is allocated to buyer t in case $\hat{x}_t \neq 0$. Also, $\hat{y}_t - \lfloor \hat{y}_t \rfloor$ corresponds to the portion of κ_t-th unit that is already allocated. This utility–maximization rule is standard for producing fractional allocations in online selection and matching problems and is similar to the equation introduced in Section 3.1 for generating fractional allocation. At each arrival, the fractional quantity allocated to buyer t comes from portions of the κ_t-th and κ_t-st units of the resource. Thus, the pricing functions associated with these units determine the fractional allocation.

Rounding Fractional Decisions $\hat{x}_t$ Losslessly We now show that the correlated pricing scheme used by cPPM-$\boldsymbol{\phi}$ performs a randomized rounding of the fractional decisions produced by `ALG-FRAC` such that a unit of item is allocated to each buyer t with probability at least $\hat{x}_t$ equal to the fractional allocation generated by `ALG-FRAC`.

Lemma 3. *cPPM-$\boldsymbol{\phi}$ allocates a unit of the item to buyer t with probability at least $\hat{x}_t$. More specifically, upon the arrival of buyer t (assuming the random seed $R = r$), the following holds:*

- *If $\hat{x}_t + \hat{y}_t \leq \kappa_t$ and $r \in \big[\hat{y}_t - \lfloor \hat{y}_t \rfloor,\ \hat{y}_t + \hat{x}_t - \lfloor \hat{y}_t \rfloor\big)$, then a unit of the item is allocated to buyer t.*
- *If $\hat{x}_t + \hat{y}_t > \kappa_t$ and $\big(r \in \big[0,\ \hat{y}_t + \hat{x}_t - \lfloor \hat{y}_t \rfloor - 1\big]$ or $r \in \big[\hat{y}_t - \lfloor \hat{y}_t \rfloor,\ 1\big]\big)$, then a unit of the item is allocated to buyer t.*

Proof. Consider a buyer t in instance I whose fractional allocation in nonzero, in other words we have $\hat{x}_t \neq 0$.

(Case I) Suppose $\hat{y}_t + \hat{x}_t \leq \kappa_t$. We first show that if the random seed R lies in $\big[\hat{y}_t - \lfloor \hat{y}_t \rfloor,\ \hat{y}_t + \hat{x}_t - \lfloor \hat{y}_t \rfloor\big]$, then the κ_t-st unit of the item will be available at the arrival of buyer t. Furthermore, by Eq. (11), we have $v_t \geq \phi_{\kappa_t}(x)$ for all $x \in \big[\hat{y}_t - \lfloor \hat{y}_t \rfloor,\ \hat{y}_t + \hat{x}_t - \lfloor \hat{y}_t \rfloor\big]$. Thus, the valuation of buyer t exceeds the posted price for the κ_t-st unit for every realization of the random seed in this interval and since at least one unit among the first κ_t units will be available at the arrival of buyer t, a unit will be allocated to buyer t.

We now show, by contradiction, that the κ_t-st unit is always available at the arrival of buyer t for every value of the random seed in the specified range. Assume, to the contrary, that for some r in this interval, cPPM-$\boldsymbol{\phi}$ has already allocated the first κ_t units to buyers who arrived before buyer t. Then there must exist a sequence of κ_t such buyers, where the j-th buyer in the sequence has valuation at least $\phi_j(r)$. Since these buyers are part of instance I, feeding them to the fractional algorithm `ALG-FRAC` would, by Eq. (11), cause the total fractional utilization to exceed $\hat{y}_t$ and reach at least $\lfloor \hat{y}_t \rfloor + r > \hat{y}_t$ prior to the arrival of buyer t, contradicting the definition of $\hat{y}_t$. Therefore, for every $r \in \big[\hat{y}_t - \lfloor \hat{y}_t \rfloor, \hat{y}_t + \hat{x}_t - \lfloor \hat{y}_t \rfloor\big]$, the κ_t-st unit must still be available at the arrival of buyer t. Thus, with probability at least $\hat{x}_t$, cPPM-$\boldsymbol{\phi}$ allocates a unit to buyer t.

(Case II) Suppose now that $\hat{y}_t + \hat{x}_t > \kappa_t$. Then one of the first κ_t units is allocated to buyer t whenever $R \in \big[\hat{y}_t - \lfloor \hat{y}_t \rfloor,\ 1\big]$, and one of the $(\kappa_t + 1)$ units is allocated when $R \in \big[0,\ \hat{y}_t + \hat{x}_t - \lfloor \hat{y}_t \rfloor - 1\big]$. The argument mirrors the reasoning in Case I. Hence, with probability at least $\hat{x}_t$, buyer t receives one unit. □

Upper-bounding $\texttt{OPT}(I)$ Consider the following two cases. **Case 1:** $\hat{y}_T = k$**.** From Eq. (3), we have $\phi_k(1) = U$, and therefore, we can simply upper-bound $\texttt{OPT}(I) \leq k \cdot U = k \cdot \phi_{\kappa_T}(\hat{y}_T)$. **Case 2:** $\hat{y}_T < k$**.** Since the total utilization of `ALG-FRAC` never exceeds $\hat{y}_T$, there cannot be k buyers in instance I with valuation greater than $\phi_{\kappa_T}(\hat{y}_T - \lfloor \hat{y}_T \rfloor) < U$; otherwise, by Eq. (11), the utilization of `ALG-FRAC` would exceed $\hat{y}_T$, a contradiction. On the other hand, there may be fewer than k such buyers. Thus, we can upper-bound the offline optimum as $\texttt{OPT}(I) \leq k \cdot \phi_{\kappa_T}(\hat{y}_T - \lfloor \hat{y}_T \rfloor) + \sum_{t \in [T]} v_t - \phi_{\kappa_T}(\hat{y}_T - \lfloor \hat{y}_T \rfloor)$. Thus, in both cases, the same general upper bound $\phi_{\kappa_T}(\hat{y}_T - \lfloor \hat{y}_T \rfloor) + \sum_{t \in [T]} v_t - \phi_{\kappa_T}(\hat{y}_T - \lfloor \hat{y}_T \rfloor)$ on $\texttt{OPT}(I)$ holds.

Computing CVaR_δ of cPPM-$\boldsymbol{\phi}$ on instance I In order to obtain a lower-bound for the CVaR_δ social welfare of cPPM-$\boldsymbol{\phi}$ on instance I, we need to establish the following two facts.

Fact 1. For each buyer t with valuation v_t greater than $\phi_{\kappa_T}(\hat{y}_T - \lfloor \hat{y}_T \rfloor)$, the fractional allocation satisfies $\hat{x}_t = 1$. This is because the total utilization level never exceeds $\hat{y}_T$, and the marginal price of the resource remains below v_t, thus based on Eq. (11), $\hat{x}_t = 1$. Hence, ALG-FRAC allocates a full unit of the resource to every such buyer. Moreover, for each buyer with $\hat{x}_t = 1$, Lemma 3 implies that, for every realized value of the random seed, cPPM-$\boldsymbol{\phi}$ also allocates a unit of the item to that buyer. Thus, across all sample paths of the randomized algorithm, the social welfare of cPPM-$\boldsymbol{\phi}$ is always incremented by v_t.

Fact 2. For any value $r \in [0, 1]$, define the subset of buyers $\hat{B}_T^{(r)}$ as

$$\hat{B}_T^{(r)} = \left\{ t \in [T] \;\middle|\; \hat{x}_t \neq 0,\ \sum_{t'<t} \hat{x}_{t'} \leq i + r < \sum_{t' \leq t} \hat{x}_{t'} \text{ for some } i \in \{0, 1, \ldots, k\} \right\}.$$

The set $\hat{B}_T^{(r)}$ consists of those buyers whose fractional allocation is nonzero and for whom r falls inside the fractional portion of a unit allocated to them. By Lemma 3, for every realization $R = r$, a unit of item is allocated to all buyers in the set $\hat{B}_T^{(r)}$ by cPPM-$\boldsymbol{\phi}$. Furthermore, since the total fractional allocation is $\hat{y}_T$ and each $\hat{x}_t \in [0, 1]$, the size of $\hat{B}_T^{(r)}$ satisfies

$$|\hat{B}_T^{(r)}| = \lfloor \hat{y}_T \rfloor + 1 \quad \text{for } r \in [0, \hat{y}_T - \lfloor \hat{y}_T \rfloor], \qquad |\hat{B}_T^{(r)}| = \lfloor \hat{y}_T \rfloor \quad \text{for } r \in [\hat{y}_T - \lfloor \hat{y}_T \rfloor, 1].$$

Thus, when the random seed lies in $[0, \hat{y}_T - \lfloor \hat{y}_T \rfloor]$, the first $\lfloor \hat{y}_T \rfloor + 1$ units are sold at the price levels determined by the pricing functions corresponding to levels 1 through $\lfloor \hat{y}_T \rfloor + 1$, evaluated at the realized value of r. Similarly, when $r \in [\hat{y}_T - \lfloor \hat{y}_T \rfloor, 1]$, the first $\lfloor \hat{y}_T \rfloor$ units are sold at the price levels determined by the pricing functions corresponding to levels 1 through $\lfloor \hat{y}_T \rfloor$, again evaluated at the realized value of r.

Putting together Fact 1 and 2, we can lower-bound CVaR_δ performance of cPPM-$\boldsymbol{\phi}$ over different realization of random seed R as follows:

$$\mathtt{ALG}^{(r)}(I) \geq \begin{cases} \displaystyle\int_0^{\hat{y}_T - \lfloor \hat{y}_T \rfloor} \left(\sum_{i=1}^{\lfloor \hat{y}_T \rfloor + 1} \phi_i(r) \right) dr \ + \ \int_{\hat{y}_T - \lfloor \hat{y}_T \rfloor}^{1} \left(\sum_{i=1}^{\lfloor \hat{y}_T \rfloor} \phi_i(r) \right) dr \ + \\ \qquad\qquad\qquad\qquad \displaystyle\sum_{t \in [T]} v_t - \phi_{\kappa_T+1}(\hat{y}_T - \lfloor \hat{y}_T \rfloor), \quad r \in [0,\ \hat{y}_T - \lfloor \hat{y}_T \rfloor], \\ \displaystyle\int_{\hat{y}_T - \lfloor \hat{y}_T \rfloor}^{1} \left(\sum_{i=1}^{\lfloor \hat{y}_T \rfloor} \phi_i(r) \right) dr \ + \ \sum_{t \in [T]} v_t - \phi_{\kappa_T+1}(\hat{y}_T - \lfloor \hat{y}_T \rfloor), \qquad r \in [\hat{y}_T - \lfloor \hat{y}_T \rfloor,\ 1]. \end{cases}$$

Thus, even in the worst case, we can lower-bound CVaR_δ as follows:

$$\mathsf{CVaR}_\delta[\mathtt{ALG}] \geq \frac{1}{\delta} \cdot \left(\int_{r=0}^{\min\{0, \delta - (1 - \hat{y}_T + \lfloor \hat{y}_T \rfloor)\}} \sum_{i=1}^{\lfloor \hat{y}_T \rfloor + 1} \phi_i(r)\, dr + \right.$$
$$\left. \int_{r = \hat{y}_T - \lfloor \hat{y}_T \rfloor}^{\min\{\delta + \hat{y}_T - \lfloor \hat{y}_T \rfloor, 1\}} \sum_{i=1}^{\lfloor \hat{y}_T \rfloor} \phi_i(r)\, dr + \delta \cdot \sum_{t \in [T]} v_t - \phi_{\kappa_T+1}(\hat{y}_T - \lfloor \hat{y}_T \rfloor) \right).$$

It can be verified that, based on the design of the ϕ functions given in Theorem 3, the right-hand side of the above inequality is exactly $\frac{k}{\alpha_\delta^{\mathsf{DP}}} \cdot \phi_{\kappa_T+1}(\hat{y}_T - \lfloor \hat{y}_T \rfloor) + \sum_{t \in [T]} v_t - \phi_{\kappa_T+1}(\hat{y}_T - \lfloor \hat{y}_T \rfloor)$. Therefore, $\mathsf{CVaR}_\delta[\mathtt{ALG}] \geq \frac{k \cdot \phi_{\kappa_T+1}(\hat{y}_T - \lfloor \hat{y}_T \rfloor) + \sum_{t \in [T]} v_t - \phi_{\kappa_T+1}(\hat{y}_T - \lfloor \hat{y}_T \rfloor)}{\alpha_\delta^{\mathsf{DP}}} \geq \frac{1}{\alpha_\delta^{\mathsf{DP}}} \cdot \mathtt{OPT}(I)$, and the $\alpha_\delta^{\mathsf{DP}}$ of cPPM-$\boldsymbol{\phi}$ over all instances of kSelection-(δ, Δ) problem is established.

4.3 Revisiting Theorem 4

In this section, we revisit the pricing design for cPPM-$\boldsymbol{\phi}$ given in Theorem 4, where the price-change cap Δ can take any value within the range $\{1, 2, \dots, k-1\}$. So far, we have analyzed the case of $\delta = 1$ and derived the pricing function design for cPPM-$\boldsymbol{\phi}$ that achieves the optimal CVaR$_\delta$-CR for any number of price changes in the risk-neutral setting. We then examined two extreme cases: in the first, where $\Delta = 0$, we derived the optimal risk-sensitive static pricing function design for all $\delta \in (0, 1)$; and in the second, where $\Delta = k-1$, we studied the fully dynamic setting and obtained an algorithm that achieves exact optimality in large-inventory regimes. Building on the insights from these special cases, Theorem 4 extends our framework to design the set of pricing functions $\{\phi_i\}_{i\in[\Delta+1]}$ for the general case where $\Delta \geq 1$. The construction of pricing functions at each level follows a system of delayed differential equations, where each function includes a delayed term that depends on the tail probability δ and the pricing behavior of the preceding price levels. In what follows, we present the intuition behind the design of these pricing functions.

Intuition behind design. Let r^*, i^*, and y^* denote the instance-dependent parameters that characterize the performance of cPPM-$\boldsymbol{\phi}$ on a given instance I, as defined in the proof of Theorem 1. Following the pricing design in Theorem 4, we can establish both the monotonicity property from Lemma 1 and a lower bound on the number of sold units similar to Lemma 2. Under these properties, the algorithm fully allocates all units up to the i^*-th price level for all realizations of the random seed within the interval $[0, r^*]$. Additionally, using the lower bound from Lemma 2, we know that for any realization of the random seed R, the algorithm must sell all reserved units up to the (i^*-1)-th price level. These units are allocated at the corresponding prices $\{\phi_i(R)\}_{i\in[i^*]}$. These two properties together allow us to construct a nontrivial lower bound on the algorithm's revenue across all sample paths. Because the performance of cPPM-$\boldsymbol{\phi}$ is evaluated using the CVaR$_\delta$ metric, we only focus on the worst δ-fraction of sample paths. For realizations of the random seed greater than r^*, the algorithm sells no more than the total number of reserved units up to the i^*-th price level. Consequently, at most $\sum_{i=1}^{i^*} q_i$ buyers can have valuation at least $\phi_{i^*}(r)$. This observation allows us to upper-bound the revenue of the offline optimal benchmark on instance I. Combining these insights yields a system of delayed differential equations, parameterized by δ, which govern the structure of the pricing functions. The complete proof of Theorem 4 appears in Appendix I. The proof applies a modified version of the randomized online primal–dual (R-OPD) framework, using the dual linear program in Eq. (5). Crucially, the dual updates are applied only to the worst δ-fraction of sample paths—those realizations in which the algorithm's objective value is minimized relative to all others.

5 Conclusion

This paper addresses critical practical limitations of traditional posted-price mechanisms (PPMs) by introducing and analyzing a new framework for online selection. Motivated by the fairness and operational challenges associated with highly dynamic pricing, we study the design of PPMs under the dual constraints of a limited number of price changes (Δ) and a risk-sensitive CVaR$_\delta$ objective. Our primary contribution is the development of cPPM-$\boldsymbol{\phi}$, a novel correlated posted-price mechanism that manages risk by using a single random seed to correlate posted prices. Our theoretical analysis provides performance guarantees for this mechanism and formally establishes the inherent trade-off between the number of allowable price updates, the degree of risk sensitivity, and worst-case competitive performance.

There are several open problems to be addressed in future research. One direction is to perform a tighter analysis of the CVaR$_\delta$-CR competitiveness of cPPM-$\boldsymbol{\phi}$ in the most general case of the problem; we conjecture that the CVaR$_\delta$-CR induced by the pricing design in Theorem 4 can be improved to establish optimality. Another promising direction is extending this risk-sensitive, price-change constrained framework to other fundamental online problems, such as online matching or knapsack. In addition, exploring alternative correlation structures or other classes of risk-sensitive objectives could yield further insights. Finally, an important practical extension is to study these problems in a data-driven setting, where buyer value distributions are unknown and must be learned over time.

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A Proof of Lemma 1

Let $B_t^{(r_2)}$ be the set of buyers to whom a unit is allocated by the arrival of buyer t under realization r_2. Consider the first buyer t' in $B_t^{(r_2)}$. If $y_{t'}^{(r_1)} = 0$ at the arrival of t', then t' accepts the posted price because $\phi_1(r_1) \leq \phi_1(r_2) \leq v_{t'}$, and thus $y_{t'}^{(r_1)}$ increases to 1. Proceeding inductively, suppose that upon the arrival of the ℓ-th buyer in $B_t^{(r_2)}$ we have $y_{t'}^{(r_1)} = \ell - 1 < y_{t'}^{(r_2)} = \ell$. Then $v_{t'}$ is at least the posted price for the ℓ-th unit under realization $R = r_2$. Since the posted prices are lower under r_1 than under r_2, following from the nondecreasing property of the pricing functions, buyer t' also accepts under r_1, implying $y_{t'}^{(r_1)} = \ell$. Hence, $y_t^{(r_1)} \geq y_t^{(r_2)}$.

B Proof of Lemma 2

For each i, let B_i^* be the set of buyers who, under realization r^*, were allocated one of the q_i reserved units at the i-th price level. By definition, for each $i \in [i^*]$, $|B_i^*| \geq q_i$. From the pricing design in Theorem 1, for all $t \in B_i^*$ we have $v_t \geq \phi_{i-1}(1)$. Let t' be the first buyer in B_2^*. Fix any $r \in [0, 1]$ and suppose $y_{t'}^{(r)} < q_1$ at the arrival of t'. Since $|B_2^*| = q_2 \geq q_1$, and each buyer in B_2^* has valuation at least $\phi_1(1)$ while the posted price for the first q_1 units is $\phi_1(R) \leq \phi_1(1)$ (as ϕ_1 is increasing), every buyer in B_2^* accepts the price for those first q_1 units. Thus these q_1 units are fully sold by the arrival of the last buyer in B_2^*. Repeating the same argument inductively for all $i \leq i^*$ yields, for any $r \in [0, 1]$, $y^{(r)} \geq \sum_{i=1}^{i^*-1} q_i$, as claimed.

C Continuation of the Proof of Theorem 1

Let us now continue the proof of the feasibility of the dual constraints for each buyer t by considering the remaining cases.

Case II: $i = i^*$ and $\phi_i^{-1}(v_t) > r^*$.

In this case, we consider the following two subcases.

Subcase I: For some realized value of $R \in [r^*, \phi_i^{-1}(v_t))$, buyer t is allocated a unit from among the first $\sum_{l=1}^{i-1} q_l$ reserved units. We show that, with probability one, buyer t is always allocated a unit.

Consider such an $r' \in [r^*, \phi_i^{-1}(v_t))$ where buyer t receives a unit from the first $\sum_{l=1}^{i-1} q_l$ reserved units when $R = r'$. By Lemma 2, for all $r \in [r', 1]$, the utilization level at the arrival of buyer t satisfies $y_t^{(r)} \leq y_t^{(r')}$. Therefore, for any realization $R \in [r', 1]$, at least one unit from the first $\sum_{l=1}^{i-1} q_l$ units is always available for allocation to buyer t. Moreover, for any realization of R within $[0, r')$, the utilization level $y_{t-1}^{(R)}$ is strictly less than $\sum_{l=1}^{i} q_l$. Otherwise, following an argument similar to that in Lemma 1, it would imply that $y_{t-1}^{(r')} \geq \sum_{l=1}^{i-1} q_l$, contradicting the assumption that under r', buyer t is allocated a unit from the first $\sum_{l=1}^{i-1} q_l$ units. Hence, from the above analysis, buyer t is allocated a unit with probability one.

Using the dual update rules in Eq. (7) and Eq. (6), we obtain

$$\begin{aligned}
\mathbb{E}\left[u_t + \frac{1}{k}\sum_{l=1}^{\Delta+1}\lambda_l q_l\right] \geq{} & \sum_{l=1}^{i^*-1}\frac{q_l}{k}\int_0^1 \phi_l(\eta)\,d\eta + \frac{q_{i^*}}{k}\int_0^{r^*}\phi_{i^*}(\eta)\,d\eta \\
& + v_t - \int_0^{r^*}\phi_{i^*}(\eta)\,d\eta - \int_{r^*}^1 \phi_{i^*-1}(\eta)\,d\eta \\
={} & \sum_{l=1}^{i^*-1}\int_{\sum_{m=1}^{l-1}\frac{q_m}{k}}^{\sum_{m=1}^{l}\frac{q_m}{k}}\phi(\eta)\,d\eta + \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}}^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}\phi(\eta)\,d\eta \\
& + v_t - \int_0^{r^*}\phi_{i^*}(\eta)\,d\eta - \int_{r^*}^1 \phi_{i^*-1}(\eta)\,d\eta \\
\geq{} & \int_0^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}\phi(\eta)\,d\eta + \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}\phi_{i^*}^{-1}(v_t)}\phi(\eta)\,d\eta \\
={} & \frac{1}{1+\ln\left(\frac{U}{L}\right)}\,\phi\left(\sum_{l=1}^{i^*-1}\frac{q_l}{k} + \frac{q_{i^*}}{k}r^*\right) \geq \frac{v_t}{1+\ln\left(\frac{U}{L}\right)}.
\end{aligned}$$

In the above derivation: The first two terms follow from the dual update rule in Eq. (6), together with the facts that $y_T^{(r)} \geq \sum_{l=1}^{i^*-1} q_l$ for all $r \in [0,1]$, and $y_T^{(r)} \geq \sum_{l=1}^{i^*} q_l$ for all $r \in [0, r^*]$. The last three terms on the left-hand side of the first inequality follow from the dual update for buyer t in Eq. (7). The second inequality follows from the construction of the functions ϕ_j in Theorem 1. The final inequality is implied by the following lemma.

Lemma 4. *For any buyer t with valuation $v_t \in [\phi_{i^*}(0),\, \phi_{i^*}(1)]$, such that $\phi_{i^*}^{-1}(v_t) \geq r^*$, the following inequality holds:*

$$v_t - \int_{r^*}^1 \phi_{i^*-1}(\eta)\,d\eta - \int_0^{r^*}\phi_{i^*}(\eta)\,d\eta \geq \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}\phi_{i^*}^{-1}(v_t)}\phi(\eta)\,d\eta.$$

The proof of the above lemma is provided in Appendix E. The argument follows from the construction of the pricing functions described in Theorem 1. This concludes the proof of dual constraint feasibility for buyer t, whose valuation satisfies the conditions of Case 2, Subcase 1.

Subcase 2: For no value of $r \in [r^*,\, \phi_i^{-1}(v_t))$ is a unit from the first $\sum_{l=1}^{i-1} q_l$ reserved units allocated to buyer t.

In this case, whenever $r \in [r^*,\, w]$, buyer t is allocated a unit from the reserved units corresponding to the i^*-th price level. From the dual updates, we have

$$\begin{aligned}
\mathbb{E}\left[u_t + \frac{1}{k}\sum_{l=1}^{\Delta+1}\lambda_l q_l\right] &\geq \sum_{l=1}^{i^*-1}\frac{q_l}{k}\int_0^1 \phi_l(\eta)\,d\eta + \frac{q_{i^*}}{k}\int_0^{r^*}\phi_{i^*}(\eta)\,d\eta + v_t(w-r^*) \\
&= \sum_{l=1}^{i^*-1}\int_{\sum_{m=1}^{l-1}\frac{q_m}{k}}^{\sum_{m=1}^{l}\frac{q_m}{k}}\phi(\eta)\,d\eta + \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}}^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}\phi(\eta)\,d\eta + v_t(w-r^*) \\
&\geq \int_0^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}\phi(\eta)\,d\eta + \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}w}\phi(\eta)\,d\eta \\
&= \frac{1}{1+\ln\left(\frac{U}{L}\right)}\,\phi\left(\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}w\right) = \frac{v_t}{1+\ln\left(\frac{U}{L}\right)}.
\end{aligned}$$

Case III: $i = i^*+1$ and $\phi_i^{-1}(v_t) < r^*$. Let $w = \phi_i^{-1}(v_t)$.

Subcase 1: For some $r \in [0, w]$, one of the first $\sum_{l=1}^{i^*} q_l$ reserved units is allocated to buyer t.

Then, following the same reasoning as in Subcase 1 of Case 2, a unit of the item is allocated to buyer t with probability one. Hence, we have

$$\begin{aligned}
\mathbb{E}\left[u_t + \frac{1}{k}\sum_{l=1}^{\Delta+1}\lambda_l q_l\right] &\geq \sum_{l=1}^{i^*-1}\frac{q_l}{k}\int_0^1 \phi_l(\eta)\,d\eta + \frac{q_{i^*}}{k}\int_0^{r^*}\phi_{i^*}(\eta)\,d\eta \\
&\quad + v_t - \int_0^{r^*}\phi_{i^*}(\eta)\,d\eta - \int_{r^*}^1\phi_{i^*-1}(\eta)\,d\eta \\
&= \sum_{l=1}^{i^*-1}\int_{\sum_{m=1}^{l-1}\frac{q_m}{k}}^{\sum_{m=1}^{l}\frac{q_m}{k}}\phi(\eta)\,d\eta + \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}}^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}\phi(\eta)\,d\eta \\
&\quad + v_t - \int_0^{r^*}\phi_{i^*}(\eta)\,d\eta - \int_{r^*}^1\phi_{i^*-1}(\eta)\,d\eta \\
&\geq \int_0^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}\phi(\eta)\,d\eta + \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}\phi_{i^*}^{-1}(v_t)}\phi(\eta)\,d\eta \\
&= \frac{1}{1+\ln\left(\frac{U}{L}\right)}\,\phi\left(\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*\right) \geq \frac{v_t}{1+\ln\left(\frac{U}{L}\right)}.
\end{aligned}$$

The first inequality and the second equality follow by the same reasoning as in the preceding cases. Furthermore, the second inequality follows from the lemma stated below.

Lemma 5. *For any buyer t with valuation $v_t \in [\phi_{i^*+1}(0),\ \phi_{i^*+1}(1)]$, such that $\phi_{i^*+1}^{-1}(v_t) \leq r^*$, the following inequality holds:*

$$v_t - \int_{\eta=r^*}^1 \phi_{i^*-1}(\eta)\,d\eta - \int_{\eta=0}^{r^*}\phi_{i^*}(\eta)\,d\eta \geq \int_{\eta=\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}\cdot r^*}^{\sum_{l=1}^{i^*}\frac{q_l}{k}+\frac{q_{i^*+1}}{k}\cdot w}\phi(\eta)\,d\eta.$$

The proof of above lemma can be found in Appendix F which follows from the pricing function design in Theorem 1.

Subcase 2: For no value of $r \in [0, w]$ is any of the first $\sum_{l=1}^{i^*} q_l$ reserved units allocated to buyer t. Then, it must be that for $r \in [r^*, 1]$, a unit from the reserved units at the i^*-th price level is allocated to buyer t, and for $r \in [0, w]$, a unit from the $(i^* + 1)$-th price level is allocated to buyer t. Following the dual updates described above, we obtain

$$\begin{aligned}
\mathbb{E}\left[u_t + \frac{1}{k}\sum_{l=1}^{\Delta+1}\lambda_l q_l\right] &\geq \sum_{l=1}^{i^*-1}\frac{q_l}{k}\int_0^1 \phi_l(\eta)\,d\eta + \frac{q_{i^*}}{k}\int_0^{r^*}\phi_{i^*}(\eta)\,d\eta + v_t\cdot(1 - r^* + w) \\
&\geq \sum_{l=1}^{i^*-1}\int_{\sum_{m=1}^{l-1}\frac{q_m}{k}}^{\sum_{m=1}^{l}\frac{q_m}{k}}\phi(\eta)\,d\eta + \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}}^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}\phi(\eta)\,d\eta + v_t\cdot(1 - r^* + w) \\
&\geq \int_0^{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}\phi(\eta)\,d\eta + \int_{\sum_{l=1}^{i^*-1}\frac{q_l}{k}+\frac{q_{i^*}}{k}r^*}^{\sum_{l=1}^{i^*}\frac{q_l}{k}+\frac{q_{i^*+1}}{k}w}\phi(\eta)\,d\eta \\
&= \frac{1}{1+\ln\left(\frac{U}{L}\right)}\,\phi\left(\sum_{l=1}^{i^*}\frac{q_l}{k} + \frac{q_{i^*+1}}{k}w\right) \geq \frac{v_t}{1+\ln\left(\frac{U}{L}\right)}.
\end{aligned}$$

Case IV: Either $i = i^* + 1$ and $\phi_i^{-1}(v_t) > r^*$, or $i > i^* + 1$.

The proof for this case follows by reasoning analogous to the previous cases. Therefore, considering all four cases, the $1 + \ln\left(\frac{U}{L}\right)$-feasibility of the dual constraint corresponding to each buyer t is verified. Consequently, the $1 + \ln\left(\frac{U}{L}\right)$-competitiveness of Algorithm 1 follows.

D Proof of Proposition 2

Let ALG be an online algorithm whose distribution over static posted prices on the class of hard instances is characterized by the function ψ. Consider an input instance $I_v^{(\epsilon)}$ for some $v \in V^{(\epsilon)}$. By the definition of ψ, the CVaR_δ performance of the algorithm on this instance is

$$\begin{aligned}
\mathsf{CVaR}_\delta[\mathtt{ALG}(I_v^{(\epsilon)})] &= L\cdot k\cdot\frac{1}{\delta}\cdot\min\{\,\delta - 1 + \psi(v),\ \psi(L)\,\} \\
&\quad + \int_{\eta=\psi(L)}^{\max\{\,\psi(L),\ \delta-1+\psi(v)\,\}} k\cdot\frac{1}{\delta}\cdot\psi^*(\eta)\,d\eta,
\end{aligned}$$

where $\psi^*(x) = \sup\{\,u \in [L, U] \mid \psi(u) \leq x\,\}$.

On the other hand, for the algorithm to be α-competitive on the instance $I_v^{(\epsilon)}$, it must satisfy $\mathsf{CVaR}_\delta[\mathtt{ALG}(I_v^{(\epsilon)})] \geq \frac{k\cdot v}{\alpha}$. Therefore, enforcing this inequality for all $v \in V^{(\epsilon)}$ yields precisely the set of constraints stated in Proposition 2.

E Proof of Lemma 4

Set $A = \sum_{l=1}^{i^*-1}\frac{q_l}{k}$, $w = \phi_{i^*}^{-1}(v_t)$. Then

$$\begin{aligned}
&v_t - \int_{r^*}^1 \phi_{i^*-1}(\eta)\,d\eta - \int_0^{r^*}\phi_{i^*}(\eta)\,d\eta \\
&\quad = \phi\left(A + w\,\frac{q_{i^*}}{k}\right) - \frac{k}{\alpha\,q_{i^*-1}}\Big(\phi(A) - \phi\big(A - (1-r^*)\frac{q_{i^*-1}}{k}\big)\Big) - \frac{k}{\alpha\,q_{i^*}}\Big(\phi\big(A + r^*\frac{q_{i^*}}{k}\big) - \phi(A)\Big),
\end{aligned}$$

where the first equality follows from the definition of ϕ_j in Eq. (1).

To prove the lemma it suffices to show

$$\phi\left(A + w\,\frac{q_{i^*}}{k}\right) - \frac{k}{\alpha\, q_{i^*-1}}\Big(\phi(A) - \phi\big(A - (1-r^*)\frac{q_{i^*-1}}{k}\big)\Big)$$
$$- \frac{k}{\alpha\, q_{i^*}}\Big(\phi\big(A + r^*\frac{q_{i^*}}{k}\big) - \phi(A)\Big) \ge \frac{1}{\alpha}\left(\phi\left(A + w\,\frac{q_{i^*}}{k}\right) - \phi\left(A + r^*\,\frac{q_{i^*}}{k}\right)\right).$$

where in above the right-hand-side follows from the definition of the ϕ function.

Define

$$F = \phi\left(A + w\,\frac{q_{i^*}}{k}\right) - \frac{k}{\alpha\, q_{i^*-1}}\Big(\phi(A) - \phi\big(A - (1-r^*)\frac{q_{i^*-1}}{k}\big)\Big) - \frac{k}{\alpha\, q_{i^*}}\Big(\phi\big(A + r^*\frac{q_{i^*}}{k}\big) - \phi(A)\Big)$$
$$- \frac{1}{\alpha}\left(\phi\left(A + w\,\frac{q_{i^*}}{k}\right) - \phi\left(A + r^*\,\frac{q_{i^*}}{k}\right)\right).$$

We show $F \ge 0$ for all admissible instance-dependent parameters.

$$\begin{aligned}
F &= \left(1 - \frac{1}{\alpha}\right)\phi\left(A + w\,\frac{q_{i^*}}{k}\right) + \frac{1}{\alpha}\,\phi\left(A + r^*\,\frac{q_{i^*}}{k}\right) \\
&\quad - \frac{k}{\alpha\, q_{i^*}}\Big(\phi\big(A + r^*\frac{q_{i^*}}{k}\big) - \phi(A)\Big) - \frac{k}{\alpha\, q_{i^*-1}}\Big(\phi(A) - \phi\big(A - (1-r^*)\frac{q_{i^*-1}}{k}\big)\Big) \\
&= \phi\left(A + r^*\,\frac{q_{i^*}}{k}\right)\left[\left(1 - \frac{1}{\alpha}\right)e^{\alpha(w-r^*)} + \frac{1}{\alpha}\right. \\
&\quad \left. - \frac{k}{\alpha\, q_{i^*}}\Big(1 - e^{-\alpha r^*\frac{q_{i^*}}{k}}\Big) - \frac{k}{\alpha\, q_{i^*-1}}\Big(e^{-\alpha r^*\frac{q_{i^*}}{k}} - e^{-\alpha\left(r^*\frac{q_{i^*}}{k} + (1-r^*)\frac{q_{i^*-1}}{k}\right)}\Big)\right] \\
&\ge \phi\left(A + r^*\,\frac{q_{i^*}}{k}\right)\left[1 - \frac{k}{\alpha\, q_{i^*}}\Big(1 - e^{-\alpha r^*\frac{q_{i^*}}{k}}\Big) - \frac{k}{\alpha\, q_{i^*-1}}\Big(e^{-\alpha r^*\frac{q_{i^*}}{k}} - e^{-\alpha\left(r^*\frac{q_{i^*}}{k} + (1-r^*)\frac{q_{i^*-1}}{k}\right)}\Big)\right] \\
&\ge \phi\left(A + r^*\,\frac{q_{i^*}}{k}\right)\left[1 - \frac{k}{\alpha\, q_{i^*}}\cdot\alpha r^*\frac{q_{i^*}}{k} - \frac{k}{\alpha\, q_{i^*-1}}\cdot\alpha(1-r^*)\frac{q_{i^*-1}}{k}\right] \\
&= \phi\left(A + r^*\,\frac{q_{i^*}}{k}\right)\,\big(1 - r^* - (1-r^*)\big) = 0.
\end{aligned}$$

where the first inequality follows from $w \ge r^*$ (so $e^{\alpha(w-r^*)} \ge 1$), and the second inequality uses the bound $1 - e^{-x} \le x$ for $x \ge 0$. Moreover, we can verify that $e^{-\alpha r^*\frac{q_{i^*}}{k}} - e^{-\alpha\left(r^*\frac{q_{i^*}}{k} + (1-r^*)\frac{q_{i^*-1}}{k}\right)} \le \alpha(1-r^*)\frac{q_{i^*-1}}{k}$, since

$$e^{-\alpha r^*\frac{q_{i^*}}{k}} - e^{-\alpha\left(r^*\frac{q_{i^*}}{k} + (1-r^*)\frac{q_{i^*-1}}{k}\right)} = \int_{\eta = -\alpha\left(r^*\frac{q_{i^*}}{k} + (1-r^*)\frac{q_{i^*-1}}{k}\right)}^{-\alpha r^*\frac{q_{i^*}}{k}} e^{\eta}\, d\eta \le \alpha(1-r^*)\frac{q_{i^*-1}}{k},$$

where the above inequality holds because the integration interval lies within $\eta \in (-\infty, 0]$, and $e^{\eta} \le 1$. Thus, $F \ge 0$, which completes the proof of the lemma.

F Proof of Lemma 5

Let us set $A = \sum_{l=1}^{i^*-1} \frac{q_l}{k},\ w = \phi_{i^*}^{-1}(v_t)$.. Then

$$
\begin{aligned}
&v_t - \int_{r^*}^{1} \phi_{i^*-1}(\eta)\, d\eta - \int_{0}^{r^*} \phi_{i^*}(\eta)\, d\eta \\
&= \phi\left(A + \frac{q_{i^*}}{k} + w\, \frac{q_{i^*+1}}{k}\right) - \frac{k}{\alpha\, q_{i^*-1}} \left(\phi(A) - \phi\left(A - (1-r^*)\frac{q_{i^*-1}}{k}\right)\right) - \frac{k}{\alpha\, q_{i^*}} \left(\phi\left(A + r^* \frac{q_{i^*}}{k}\right) - \phi(A)\right).
\end{aligned}
$$

Subtract the right-hand side of the lemma's inequality from the left-hand side and denote the result by F. Equivalently,

$$
\begin{aligned}
F =& \phi\left(A + \frac{q_{i^*}}{k} + w\, \frac{q_{i^*+1}}{k}\right) - \frac{k}{\alpha\, q_{i^*-1}} \left(\phi(A) - \phi\left(A - (1-r^*)\frac{q_{i^*-1}}{k}\right)\right) - \frac{k}{\alpha\, q_{i^*}} \left(\phi\left(A + r^* \frac{q_{i^*}}{k}\right) - \phi(A)\right) \\
&- \frac{1}{\alpha} \left(\phi\left(A + \frac{q_{i^*}}{k} + w\, \frac{q_{i^*+1}}{k}\right) - \phi\left(A + r^* \frac{q_{i^*}}{k}\right)\right).
\end{aligned}
$$

We show that $F > 0$ for all admissible values of the instance–dependent parameters; hence the lemma's inequality follows. Indeed,

$$
\begin{aligned}
F =& \phi\left(A + \frac{q_{i^*}}{k} + w\, \frac{q_{i^*+1}}{k}\right) - \frac{k}{\alpha\, q_{i^*-1}} \left(\phi(A) - \phi\left(A - (1-r^*)\frac{q_{i^*-1}}{k}\right)\right) - \frac{k}{\alpha\, q_{i^*}} \left(\phi\left(A + r^* \frac{q_{i^*}}{k}\right) - \phi(A)\right) \\
&- \frac{1}{\alpha} \left(\phi\left(A + \frac{q_{i^*}}{k} + w\, \frac{q_{i^*+1}}{k}\right) - \phi\left(A + r^* \frac{q_{i^*}}{k}\right)\right) \\
=& \phi\left(A + \frac{q_{i^*}}{k} + r^* \frac{q_{i^*+1}}{k}\right) \Bigg[\left(1 - \frac{1}{\alpha}\right) e^{\frac{q_{i^*+1}}{k}(w-r^*)} + \frac{1}{\alpha}\, e^{-(1-r^*)\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} \\
&\qquad - \frac{k}{\alpha\, q_{i^*-1}} \left(e^{-\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} - e^{-\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k} - (1-r^*)\frac{q_{i^*-1}}{k}}\right) \\
&\qquad - \frac{k}{\alpha\, q_{i^*}} \left(e^{-(1-r^*)\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} - e^{-\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}}\right)\Bigg] \\
\geq& \phi\left(A + \frac{q_{i^*}}{k} + r^* \frac{q_{i^*+1}}{k}\right) \Bigg[\left(1 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\, e^{-(1-r^*)\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} \\
&\qquad - \frac{k}{\alpha\, q_{i^*-1}}\, e^{-\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} \left(1 - e^{-(1-r^*)\frac{q_{i^*-1}}{k}}\right) - \frac{k}{\alpha\, q_{i^*}}\, e^{-(1-r^*)\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} \left(1 - e^{-r^* \frac{q_{i^*}}{k}}\right)\Bigg] \\
\geq& \phi\left(A + \frac{q_{i^*}}{k} + r^* \frac{q_{i^*+1}}{k}\right) \Bigg[\left(1 - \frac{1}{\alpha}\right) \\
&+ \frac{1}{\alpha}\, e^{-(1-r^*)\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} - \frac{1-r^*}{\alpha}\, e^{-\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} - \frac{r^*}{\alpha}\, e^{-(1-r^*)\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}}\Bigg] \\
\geq& \phi\left(A + \frac{q_{i^*}}{k} + r^* \frac{q_{i^*+1}}{k}\right) \left[\left(1 - \frac{1}{\alpha}\right) + \frac{1}{\alpha}\, e^{-(1-r^*)\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}} - \frac{1}{\alpha}\, e^{-(1-r^*)\frac{q_{i^*}}{k} - r^* \frac{q_{i^*+1}}{k}}\right] \\
=& \phi\left(A + \frac{q_{i^*}}{k} + r^* \frac{q_{i^*+1}}{k}\right) \left(1 - \frac{1}{\alpha}\right) \geq 0
\end{aligned}
$$

where in the first inequality we used $w \geq r^*$ (so $e^{\frac{q_{i^*+1}}{k}(w-r^*)} \geq 1$) and the identity $e^{-x} - e^{-(x+y)} = e^{-x}\,(1 - e^{-y})$, and in the second we used $1 - e^{-x} \leq x$ for $x \geq 0$. Since $\alpha > 1$ (hence $1 - \frac{1}{\alpha} > 0$) and $\phi(\cdot) \geq 0$, the claim follows. This completes the proof.

G Proof of Theorem 2

We first derive the closed-form expression of the pricing function ϕ using the design specified in Proposition 3 for an arbitrary value of α, and in particular for $\alpha = \alpha^*$. We then apply Proposition 2 to establish that α^* serves as a lower bound on the performance of any online algorithm.

Closed-form Design of ϕ for the Upper-Bound α^*

We begin by defining the parameters $\tau = 1 - \delta,\ c = \frac{\alpha}{\delta}$. Consider the function $\phi : [0, 1] \to \mathbb{R}_{\geq 0}$, defined implicitly based on Proposition 3 as follows:

$$\phi(x) = \begin{cases} L, & x \in [0, b], \\ \dfrac{\alpha}{\delta} \displaystyle\int_0^{x-\tau} \phi(\eta)\, d\eta, & x \in [b, 1], \end{cases} \tag{$\star$}$$

where the breakpoint $b = 1 - \delta + \frac{\delta}{\alpha}$.

For $x \in [b, 1]$, the function ϕ satisfies the following delayed differential equation (DDE):

$$\phi'(x) = \frac{\alpha}{\delta}\, \phi(x - \tau) = c\, \phi(x - \tau). \tag{DDE}$$

This is a linear DDE with constant delay τ and constant history $\phi(x) = L$ for all $x \leq b$. We use the method of steps to derive an explicit closed-form expression for $\phi(x)$. Let $z = x - b \in [0,\ 1 - b]$, and define the *delay exponential function* as

$$E_c(t) = 1 + \sum_{j=1}^{\lfloor t/\tau \rfloor} \frac{c^j}{j!} (t - j\tau)^j, \qquad (t \geq 0).$$

Given the constant history $\phi = L$ on $(-\infty, b]$, the unique solution of (DDE) on $[b, 1]$ is

$$\phi(x) = \phi(b)\, E_c(x - b) = L\, E_c(x - b).$$

Expanding E_c yields a finite, piecewise-polynomial form:

$$\boxed{\phi(x) = \begin{cases} L, & 0 \leq x \leq b, \\ L\left[\, 1 + c(x - b) + \dfrac{c^2}{2!}\big(x - b - \tau\big)_+^2 + \cdots + \dfrac{c^n}{n!}\big(x - b - (n-1)\tau\big)_+^n \right], & b \leq x \leq 1, \end{cases}}$$

where $(u)_+ = \max\{u, 0\}$ and $n = \left\lfloor \frac{x-b}{\tau} \right\rfloor + 1$, ensuring the summation is finite for every x.

To determine the optimal value of α^* for which a feasible design of the pricing function ϕ exists according to the construction in Theorem 2, we impose the boundary condition $\phi(1) \geq U$. Hence, α^* is defined as the smallest value of α satisfying this condition. By substituting $x = 1$ into the expression for $\phi(x)$, and noting that the right-hand side of the equation is monotonically increasing in α, we obtain that α^* is the solution to the following equation:

$$\phi(1) = L\left[1 + \sum_{j=1}^{\left\lfloor \frac{\delta(1-1/\alpha)}{1-\delta} \right\rfloor} \frac{\left(\frac{\alpha}{\delta}\right)^j}{j!} \left(\delta(1 - \frac{1}{\alpha}) - j(1 - \delta)\right)^j \right] = U.$$

Thus, α^* is the smallest value of α that satisfies the above equation.

Lower Bound on α^* Based on Proposition 2 and the definition of the function ψ, we can, without loss of generality, assume that ψ is strictly increasing. Using a change of variables, we obtain the following constraint on the design of ψ^{-1}:

$$\int_{\nu=0}^{\delta-1+\psi(v)} \Big[\psi^{-1}(\nu)\cdot \mathbf{1}\{\nu \geq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\} + L\cdot \mathbf{1}\{\nu \leq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\}\Big] d\nu \geq \frac{v}{\alpha}, \qquad \forall v \in [L,U].$$

For a given α, if there exists a feasible design of ψ that satisfies the above inequality, then there must exist a corresponding inverse function ψ^{-1}. Without loss of generality, we can also assume that $\psi^{-1}(1) = U$. Rewriting the above constraints yields:

$$\int_{\nu=0}^{\delta-1+x} \Big[\psi^{-1}(\nu)\cdot \mathbf{1}\{\nu \geq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\} + L\cdot \mathbf{1}\{\nu \leq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\}\Big] d\nu \geq \frac{\psi(x)}{\alpha}, \qquad \forall x \in [\psi(L),1]. \tag{12}$$

From this, we can upper-bound $\psi(1)$ as:

$$U = \psi(1) \leq \alpha\cdot \int_{\nu=0}^{\delta} \Big[\psi^{-1}(\nu)\cdot \mathbf{1}\{\nu \geq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\} + L\cdot \mathbf{1}\{\nu \leq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\}\Big] d\nu,$$

which leads to the following lower bound on α:

$$\frac{U}{\int_{\nu=0}^{\delta} \Big[\psi^{-1}(\nu)\cdot \mathbf{1}\{\nu \geq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\} + L\cdot \mathbf{1}\{\nu \leq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\}\Big] d\nu} \leq \alpha. \tag{13}$$

Computing the above lower bound requires knowledge of ψ^{-1} within the range $[0,\delta]$. By substituting the upper bound implied by the constraint in Eq. (12), for the values of ψ^{-1}, we can derive the lower bound. Thus, let us consider the ψ^{*-1} function for which the inequalities in Eq. (12) are tight. Thus, the design for ψ^{*-1} is as follows:

$$\psi^{*-1}(x) = \begin{cases} L, & x \in [0, 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor], \\ \displaystyle\int_{\nu=0}^{\delta} \Big[\psi^{*-1}(\nu)\cdot \mathbf{1}\{\nu \geq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\} + L\cdot \mathbf{1}\{\nu \leq 1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor\}\Big] d\nu, & x \in [1-\delta+\delta\cdot\lfloor\tfrac{1}{\alpha}\rfloor, 1]. \end{cases} \tag{14}$$

Constructing ψ^{*-1} according to the above and substituting into Eq. (13) yields a lower bound on α achievable by any online algorithm. It can be verified that this design of ψ^{*-1} is equivalent to the pricing function ϕ characterized in Proposition 3, and the resulting expression in Eq. (13) produces the lower-bound value equal to α^* following the same procedure as before.

H Proof of Proposition 1

In the following, we will prove that for values of $\alpha \geq 1+\ln(\frac{U}{L})$, according to the pricing design in Theorem 4, we will have $\phi_k(1) \geq U$, and thus there exists a design that obtains $1+\ln(\frac{U}{L})$-CVaR_δ competitive. Following the well-established lower-bound $1+\ln(\frac{U}{L})$, the proof of above theorem follows.

Let us set $m = \lfloor \frac{k}{\alpha} \rfloor$ and for $i \geq m+2$ define

$$S_i := \sum_{r=1}^{i} \int_0^{\delta} \phi_r(\eta)\, d\eta, \qquad b_i = \phi_i(0), \qquad c_i = \phi_i(1).$$

Evaluating ϕ_i at $x = 0$ and $x = 1$ gives

$$b_i = \frac{\alpha}{k\,\delta}\, S_{i-1}, \qquad c_i = \frac{\alpha}{k\,\delta}\, S_i \qquad (i \geq m+2), \tag{15}$$

and since $\phi_r = L$ for $1 \leq r \leq m$, we have the base mass

$$S_{m+1} \;\geq\; \sum_{r=1}^{m} \int_0^{\delta} L\, d\eta \;=\; L \cdot m\,\delta. \tag{16}$$

Monotonicity. We claim each ϕ_i is nondecreasing on $[0, 1]$. This is clear for $i \leq m$; for $i \geq m+1$ it follows by induction: if all ϕ_r with $r < i$ are nondecreasing, then in case $x + \delta \leq 1$, then we will have:

$$\phi_i'(x) = \frac{\alpha}{k \cdot \delta} \cdot \left(\sum_{r=1}^{i-1} \phi_r(x+\delta) - \phi_r(x) \right) > 0$$

where in above the inequality follows from the induction hypothesis and thus $\phi_i(x)$ for $x \in [0, 1-\delta]$ is nondecreasing. Furthermore, for $x + \delta > 1$, we will have:

$$\phi_i'(x) = \frac{\alpha}{k \cdot \delta} \cdot \left(\sum_{r=1}^{i} \phi_r(x+\delta-1) - \sum_{r=1}^{i-1} \phi_r(x) \right) > 0$$

where in above the inequality follows from the fact that $\phi_r(1) = \phi_{r+1}(0)$ and the induction hypothesis. thus $\phi_i(x)$ for $x \in [1-\delta, 1]$ is nondecreasing.

A covering inequality on $[0, \delta]$. If $g : [0, 1] \to \mathbb{R}_{\geq 0}$ is nondecreasing, then for every $x \in [0, \delta]$,

$$\int_x^{\min\{1,\, x+\delta\}} g(\eta)\, d\eta \;+\; \int_0^{\max\{0,\, \delta-1+x\}} g(\eta)\, d\eta \;\;\geq\;\; \int_0^{\delta} g(\eta)\, d\eta. \tag{17}$$

Indeed, if $x \leq 1-\delta$ the second integral vanishes and translating a length-δ window to the right can only increase the integral of nondecreasing g. If $x > 1-\delta$, the left side equals $\int_0^1 g - \int_{x+\delta-1}^{x} g$, and among all intervals of length $1-\delta$ the integral of g is minimized on $[0, 1-\delta]$, giving $\int_{x+\delta-1}^{x} g \leq \int_{\delta}^{1} g$ and hence (17).

Multiplicative growth of S_i. Applying (17) to each nondecreasing ϕ_r $(r < i)$ design given in Theorem 4, we obtain for any $x \in [0, \delta]$:

$$\phi_i(x) \;\geq\; \frac{\alpha}{k\,\delta} \sum_{r=1}^{i-1} \int_0^{\delta} \phi_r(\eta)\, d\eta \;=\; \frac{\alpha}{k\,\delta}\, S_{i-1} \;=\; b_i.$$

Integrating over $[0, \delta]$ yields

$$S_i = S_{i-1} + \int_0^{\delta} \phi_i(\eta) d\eta \;\geq\; S_{i-1} + \delta\, b_i = S_{i-1} + \delta \cdot \frac{\alpha}{k\,\delta}\, S_{i-1} = \left(1 + \frac{\alpha}{k}\right) S_{i-1} \qquad (i \geq m+2). \tag{18}$$

Wrapping Up. Iterating (18) from $i = m+2$ to $i = k$ and using (16) gives

$$S_k \;\geq\; S_{m+1} \left(1 + \frac{\alpha}{k}\right)^{k-(m+1)} \;\geq L \cdot m\,\delta\, \left(1 + \frac{\alpha}{k}\right)^{k-m-1}.$$

Thus, by (15),

$$\phi_k(1) = c_k = \frac{\alpha}{k\,\delta} S_k \ \geq \ L \cdot \frac{\alpha m}{k} \left(1 + \frac{\alpha}{k}\right)^{k-m-1}.$$

Since $m = \lfloor k/\alpha \rfloor$, once $k \to \infty$, we have $\frac{\alpha m}{k} \to 1$ and

$$\lim_{k\to\infty} \phi_k(1) = \lim_{k\to\infty} L \cdot \left(1 + \frac{\alpha}{k}\right)^{k-m-1} = \lim_{k\to\infty} L \cdot \exp\Big((k-m-1)\log(1+\alpha/k)\Big) = L \cdot \ e^{\,\alpha-1}.$$

Based on above, for values of $\alpha = 1 + \ln(\frac{U}{L})$, we will have $\phi_k(1) \to U$, satisfying Eq. (3), and thus there exists a feasible solution for the design of $\{\phi_i\}_{i\in[\Delta+1]}$ for value of $\alpha = 1 + \ln(\frac{U}{L})$ satisfying the bound condition in Eq. (3). On the other hand, since the $1 + \ln(\frac{U}{L})$ is an established lower-bound for the performance of any online algorithm for kSelection-(δ, Δ) problem, the optimality of cPPM-$\boldsymbol{\phi}$ based on the design given in Theorem 3 follows.

I Proof of Theorem 4

Let us define y^*, r^*, and i^* as in the proof of Theorem 1, where y^* denotes the highest number of units sold across all sample paths of Algorithm 1 under the pricing scheme specified above. The value r^* represents the realization of the random seed R for which the number of sold units equals y^*, that is, $y_T^{(r^*)} = y^*$. Similarly, i^* denotes the highest price level such that, under the realization $R = r^*$, the algorithm fully allocates all reserved units from price levels 1 through i^*.

Furthermore, the same monotonicity and lower-bound results as those stated in Lemma 1 and Lemma 2 can be established here by following analogous proof arguments, given the constraints on the reservation vector $\{q_i\}_{i\in[\Delta+1]}$ and the pricing design in the theorem above. The only distinction between the constraint set of reserved quantities $\{q_i\}_{i\in[\Delta+1]}$ and that in Theorem 1 is that here $q_1 = \lceil \frac{k}{\alpha} \rceil$, and the monotonicity property applies to the remaining reserved quantities $\{q_i\}_{i\in\{2,\ldots,\Delta+1\}}$. However, since the pricing function ϕ_1 is fixed to the constant value L, it follows that the same monotonicity and lower-bound results (Lemma 1 and Lemma 2) continue to hold.

Given an instance I of the problem, $\texttt{OPT}(I)$ can be upper-bounded using the dual LP given in Eq. (5). We will restate the LP in the following:

$$\min_{u_t,\lambda_j} \quad \sum_{t\in[T]} u_t + \sum_{j=1}^{\Delta+1} \lambda_j \cdot q_j \quad \text{s.t.} \qquad v_t \leq u_t + \frac{1}{k}\sum_{j=1}^{\Delta+1} \lambda_j \cdot q_j, \quad \forall t \in [T].$$

We begin by initializing all dual variables to zero. Let us define the set $\mathcal{S}$ as defined for the proof of Theorem 2 such that

$$\mathcal{S} = \left\{ r \in [0, \tilde{r}] \;\middle|\; F_{\texttt{ALG}(I)}((\texttt{ALG}^{(r)})^-) \leq \delta \right\},$$

where in above $\tilde{r} = \inf_{r\in[0,1]} \left\{ \int_{x=0}^{r} \mathbf{1}\{ F_{\texttt{ALG}(I)}((\texttt{ALG}^{(x)})^-) \leq \delta \}\, dx \geq \delta \right\}$.

Suppose the random seed is realized as $R = r$. If $r \in \mathcal{S}$, we proceed with updating the dual variables; otherwise, no update is performed. The dual variables $\{\lambda_i^{(r)}\}_{i=1}^{\Delta+1}$ are updated as follows:

$$\lambda_i^{(r)} = \begin{cases} \frac{\phi_i(r)}{2}, & r \in \mathcal{S},\ i \in \{1, 2, \ldots, i^* - 1\}, \\ \frac{\phi_i(r)}{2}, & r \in \mathcal{S},\ i = i^*,\ r \in [0,\, r^*], \\ 0, & \text{otherwise}. \end{cases} \tag{19}$$

Next, consider a buyer t who receives an allocation of one unit when the random seed is realized as $R = r$. Suppose this unit is allocated from the reserved quantity associated with the i-th price level. Then, under the realization $R = r$, where $r \in \mathcal{S}$, we update the dual variable $u_t^{(r)}$ as follows:

$$u_t^{(r)} = \begin{cases} v_t - \frac{\phi_i(r)}{2}, & \text{if } r \in \mathcal{S} \text{ and } \big(i < i^* \text{ or } (i = i^* \text{ and } r \le r^*)\big), \\ v_t, & \text{otherwise.} \end{cases} \tag{20}$$

It can be verified that, under these updates—together with the lower bound established in Lemma 2 and the monotonicity property in Lemma 1—the total objective value of the dual solution equals the algorithm's objective when the random seed is realized as $r \in \mathcal{S}$. In other words, we have $\sum_{t\in[T]} u_t^{(r)} + \sum_{j=1}^{\Delta+1} \lambda_j^{(r)} \cdot q_j = \texttt{ALG}^{(r)}(I)$. Since the dual updates are performed only for values of $r \in \mathcal{S}$, it follows that $\mathsf{CVaR}_\delta[\texttt{ALG}] = \mathbb{E}_R\left[\sum_{t\in[T]} u_t^{(R)} + \sum_{j=1}^{\Delta+1} \lambda_j^{(R)} \cdot q_j\right]$. Next, we show that for all buyers $t \in [T]$, the dual constraint in the above dual LP is α-feasible in expectation; that is, $\mathbb{E}_R\left[u_t^{(R)} + \frac{1}{k}\sum_{j=1}^{\Delta+1} \lambda_j^{(R)} \cdot q_j\right] \ge \frac{v_t}{\alpha}$, thereby completing the primal–dual analysis and establishing the α-competitiveness of the algorithm.

Consider a buyer t with valuation v_t such that, for some $i \in [\Delta + 1]$, we have $\phi_i(0) \le v_t \le \phi_i(1)$. To prove the α-feasibility of the dual constraints, we analyze the following cases.

Case I: Either $i \le i^* - 1$, or $i = i^*$ and $\phi_i^{-1}(v_t) \le r^*$. Based on the dual updates defined in Eq. (19), and given that, for all realizations of $R \in [0, 1]$, the reserved units corresponding to the first $i^* - 1$ price levels are fully sold, while for $R \in [0, r^*]$, the reserved units at the i^*-th price level are also exhausted, we have:

$$\frac{\sum_{l=1}^{\Delta+1} \lambda_l^{(r)} \cdot q_l}{k} = \begin{cases} \dfrac{\sum_{l=1}^{i^*} q_l \cdot \phi_l(r)}{2\cdot k}, & \text{if } r \in [0, r^*] \text{ and } r \in \mathcal{S}, \\ \dfrac{\sum_{l=1}^{i^*-1} q_l \cdot \phi_l(r)}{2\cdot k} w, & \text{if } r \in [r^*, 1] \text{ and } r \in \mathcal{S}. \end{cases}$$

Thus, in the worst-case, we will have:

$$\begin{aligned} \mathbb{E}_R\left[u_t^{(R)} + \frac{\sum_{l=1}^{\Delta+1} \lambda_l^{(R)} \cdot q_l}{k}\right] &\ge \frac{1}{2\cdot k\cdot\delta}\left(\sum_{r=1}^{i^*-1}\int_{r^*}^{\min\{1,\, r^*+\delta\}} \phi_r(\eta)\, d\eta + \sum_{r=1}^{i^*}\int_0^{\max\{0,\, \delta-1+r^*\}} \phi_r(\eta)\, d\eta\right) \\ &= \frac{\phi_{i^*}(r^*)}{\alpha} \ge \frac{v_t}{\alpha}. \end{aligned}$$

The first inequality above follows directly from the design of the pricing functions specified in Theorem 4, while the second inequality follows from the condition defined for Case I. Therefore, the dual constraint is α-feasible in expectation in this case.

Case II: $i = i^*$ and $\phi_i^{-1}(v_t) > r^*$. Let $w = \phi_i^{-1}(v_t)$. Given the update rules defined in Eq. (20)-Eq. (19), we have:

$$u_t^{(r)} + \frac{\sum_{l=1}^{\Delta+1} \lambda_l^{(r)} \cdot q_l}{k} \ge \begin{cases} \dfrac{\sum_{l=1}^{i^*} q_l \cdot \phi_l(r)}{2\cdot k}, & \text{if } r \in [0, r^*] \text{ and } r \in \mathcal{S}, \\ v_t - \dfrac{1}{2}\cdot\phi_{i^*-1}(r) + \dfrac{\sum_{l=1}^{i^*-1} q_l \cdot \phi_l(r)}{2\cdot k} w, & \text{if } r \in [r^*, w] \text{ and } r \in \mathcal{S}, \\ \dfrac{\sum_{l=1}^{i^*-1} q_l \cdot \phi_l(r)}{2\cdot k}, & \text{if } r \in [w, 1] \text{ and } r \in \mathcal{S}. \end{cases}$$

Following the fact that $v_t - \frac{1}{2}\cdot\phi_{i^*-1}(r) \ge \frac{q_{i^*}\cdot\phi_{i^*}(r)}{2\cdot k}$ for values of $r \in [r^*, w]$, in the worst case, we will have:

$$
\begin{aligned}
\mathbb{E}_R\left[u_t^{(R)} + \frac{\sum_{l=1}^{\Delta+1} \lambda_l^{(R)} \cdot q_l}{k}\right] \geq & \frac{1}{2 \cdot k\delta} \sum_{j=1}^{i^*-1} \int_w^{\min\{1,\, w+\delta\}} \phi_j(\eta)\, d\eta + \sum_{j=1}^{i^*} \int_0^{\max\{0,\, \delta-1+w\}} \phi_j(\eta)\, d\eta \\
= & \frac{\phi_{i^*}(w)}{\alpha} = \frac{v_t}{\alpha}.
\end{aligned}
$$

where in above the first inequality follows from the design of the pricing functions in above theorem and the second inequality follows from the condition set be case one. Thus, the dual constraint holds in this case. The proof of α-feasibility for the dual constraints corresponding to the remaining cases, where $i > i^*$, follows analogously from the above analysis.