Generalizations of the Second Mean Value Theorem for Integrals

Chen Hui-Ru[1] and Shang Chan-Juan[2]

[1] College of Mathematics and Information Science,
Huanggang Normal University, Huangzhou 438000, Hubei, China
[2] Institute of Education, Tibet Institute for Nationalities, Xianyang 712000, Shanxi, China

Abstract. The paper systematically summarizes such promotions as conclusion to open interval, to arbitrary point of the interval and to infinite interval. Meanwhile, the paper will further the exploration of new promotion on this basis.

Keywords: mathematical analysis, the second mean value theorem for integrals, open interval, infinite interval.

1 General Generalization

Theorem 1.1 [1]

(1) Suppose $g(x)$ is monotone decreasing and nonnegative on $[a,b]$, $f(x)$ is integrable on $[a,b]$, $n\in R$ and $n\geq 1$, then there exist $\xi\in[a,b]$, cause

$$\int_a^b f(x)g(x)dx = ng(b)\int_a^\xi f(x)dx\,; \tag{1.1}$$

(2) Suppose $g(x)$ is monotone increasing and nonnegative on $[a,b]$, $f(x)$ is integrable on $[a,b]$, $n\in R$ and $n\geq 1$ then there exist $\xi\in[a,b]$, cause

$$\int_a^b f(x)g(x)dx = ng(b)\int_\xi^b f(x)dx\,; \tag{1.2}$$

(3) Suppose $g(x)$ is monotone and nonnegative on $[a,b]$, $f(x)$ is integrable on $[a,b]$, $m,n\in R$ and $m\leq 1\leq n$, then

(i) when $g(x)$ is monotone decreasing , there exist $\eta\in[a,b]$, cause

$$\int_a^b f(x)g(x)dx = ng(a)\int_a^\eta f(x)dx + mg(b)\int_\eta^b f(x)dx\,. \tag{1.3}$$

(ii) when $g(x)$ is monotone increasing , there exist $\xi\in[a,b]$, cause

$$\int_a^b f(x)g(x)dx = mg(a)\int_a^\xi f(x)dx + ng(b)\int_\xi^b f(x)dx\,. \tag{1.4}$$

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The (1)(2) in the second mean value theorem for integrals are respectively special case of the (1)(2) in theorem 1.1 when $n=1$, although in contrast with the (3) in the second mean value theorem for integrals, the (3) in theorem 1.1 add this condition which is $g(x)$ is nonnegative on $[a,b]$, the conclusion is more meticulous than that of the second mean value theorem for integrals. Therefore , theorem 1.1 is the improvement and generalization.

2 Generalizing the Theorem to Open Interval

In the second mean value theorem for integrals,the ξ which meet the conclusion is limited in $[a,b]$, then if the condition remain unchanged, whether could the conclusion of theorem be sharpened to (a,b) ? It turns out to be it can't , that is to say the ξ which meet the second mean value theorem for integrals can't be limited on the open interval (a,b), ξ might not be get in the open interval . The following two examples can explain

Case 2.1

$$f(x)=\sin x,\ g(x)=\begin{cases}0 & x=2\pi\\ -1 & x\neq 2\pi\end{cases}$$

Only when $\xi=0$ or 2π , the second mean value theorem for integrals can be true, while as $\xi \notin (0,2\pi)$.

Case 2.2

In the interval $[a,b]$, $f(x)=1$, $g(x)=\begin{cases}1 & x\in[a,b]\\ 2 & x=b\end{cases}$, if there exist ξ cause

$$\int_a^b f(x)g(x)dx = g(a)\int_a^{\xi} f(x)dx + g(b)\int_{\xi}^{b} f(x)dx\,,$$

that is $b-a=g(a)(\xi-a)+2(b-\xi)=2b-a-\xi$, so there is $\xi=b$, and ξ is on the extreme point of $[a,b]$ from this point , we can think about that under what condition, we can get the ξ in the second mean value theorem for integrals in the open interval (a,b) ? Searching for such condition is undonbtly meaningful.

Theorem 2.1 [2]

(1) Suppose $g(x)$ is incessant on $[a,b]$, $f(x)$ has continual derivative function $f'(x)$ on $[a,b]$, and $f'(x)\geq 0$, $f(a)\geq 0$ there must be $\xi\in[a,b]$, cause

$$\int_a^b f(x)g(x)dx = f(b)\int_\xi^b g(x)dx.$$

(2) $f'(x) \le 0$, $f(b) \ge 0$ then there exist

$$\int_a^b f(x)g(x)dx = f(a)\int_a^\xi g(x)dx \ (a < \xi < b).$$

(3) If $f(a) \ge 0$ is canceled in (1), or $f(b) \ge 0$ is canceled in (2), the conclusion is

$$\int_a^b f(x)g(x)dx = f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx.$$

Although the condition in theorem 2.1 is very strong, the mid value point $\xi \in [a,b]$ in the conclusion can be sharpened to $\xi \in (a,b)$ [3].

Theorem 2.2 [4]

(1) If function $f(x)$ and $g(x)$ are bounded and integrable, function $g(x)$ were monotone in (a,b), $g(a+0) \ne g(b-0)$, then there exist $\xi \in (a,b)$, cause

$$\int_a^b f(x)g(x)dx = g(a+0)\int_a^\xi f(x)dx + g(b-0)\int_\xi^b f(x)dx,$$

(2) Suppose function $f(x)$ and $g(x)$ are bounded and integrable on $[a,b]$, function $g(x)$ is monotone increasing and nonnegative in (a,b), $g(a+0) \ne g(b-0)$, than there exist $\xi \in (a,b)$, cause

$$\int_a^b f(x)g(x)dx = g(b-0)\int_\xi^b f(x)dx.$$

From theorem 2.2 we can know that, under weaker condition of the second mean value theorem for integrals, if we add $g(a) \ne g(b)$, the mid value point $\xi \in [a,b]$ can be sharpened as $\xi \in (a,b)$.

The ξ in theorem 2.2 might not be only , let's look at the example below

Case 2.3

Suppose $f(x) = \sin x$, $g(x) = x$, $x \in [0,2\pi]$. It's easy to know that , when $\xi = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2} \in (0,2\pi)$, all have

$$\int_0^{2\pi} f(x)g(x)dx = g(0+0)\int_0^\xi f(x)dx + g(2\pi-0)\int_\xi^{2\pi} f(x)dx,$$

$$\int_0^{2\pi} f(x)g(x)dx = g(2\pi-0)\int_\xi^{2\pi} f(x)dx.$$

Now the problem is, under what condition the ξ is only? Then let's talk about the sufficient condition for ξ be the only one in the open interval (a,b).

Theorem 2.3 [4]

(1) If function $f(x)$ and $g(x)$ meet the condition of 2.2(1) in $[a,b]$, and $f(x)$ is identically positive (or negative), then there exist only $\xi \in (a,b)$, cause

$$\int_a^b f(x)g(x)dx = g(a+0)\int_a^\xi f(x)dx + g(b-0)\int_\xi^b f(x)dx .$$

(2) If function $f(x)$ and $g(x)$ meet the condition of 2.2(2) in $[a,b]$, and $f(x)$ is identically positive (or negative), then there exist only $\xi \in (a,b)$, cause

$$\int_a^b f(x)g(x)dx = g(b-0)\int_\xi^b f(x)dx .$$

3 Generalizing the Theorem to Any Point on the Interval

The conclusion of the second mean value theorem for integrals just exist one point, obviously it has limit, we hope that the conclusion can be true on any point in the interval , thus we give the conclusion below

Theorem 3.1 [5, 6]
Suppose function $f(x)$ is integrable on the closed interval $[a,b]$.

(i) If function $g(x)$ is (strictly monotore) decreasing on the closed interval $[a,b]$, and $g(x) \ge 0$, $f(x) > 0$ then for arbitrary point $\xi \in [a,b]$, there must be two different points $\alpha, \beta \in [a,b]$, meet the condition $\alpha < \xi < \beta$, cause

$$\int_\alpha^\beta f(x)g(x)dx = g(\alpha)\int_\alpha^\xi f(x)dx .$$

(ii) If function $g(x)$ is (strictly monotone) increasing on the close interval $[a,b]$,and $g(x) \ge 0, f(x) > 0,$ then for arbitrary point $\eta \in [a,b]$, there must be two different points $\alpha, \beta \in [a,b]$, $\alpha < \eta < \beta$, cause

$$\int_\alpha^\beta f(x)g(x)dx = g(\beta)\int_\eta^\beta f(x)dx .$$

(iii) If function $g(x)$ is (strictly monotone) on the closed interval $[a,b]$, $f(x) > 0$ $(f(x) < 0)$, then for arbitrary point $\xi \in [a,b]$,, there must be two different points $\alpha, \beta \in [a,b]$ and $\alpha < \xi < \beta$ cause

$$\int_\alpha^\beta f(x)g(x)dx = g(\alpha)\int_\alpha^\xi f(x)dx + g(\beta)\int_\xi^\beta f(x)dx$$

In theorem 3.1, after adding some condition, we generalize he conclusion to arbitrary point in the interval, but the condition is too sharp, then we get the same result under weaker condition.

Theorem 3.2 [7, 8]

Suppose function $f(x)$, $g(x)$ are integrable and identically positive (or negative) in the closed interval, to arbitrary $\xi \in (a,b)$

(i) If function $g(x)$ is locally strictly monotone decreasing (or increasing) at ξ, then there exist $\alpha, \beta \in [a,b]$,and $\alpha < \xi < \beta$ cause

$$\int_\alpha^\beta f(x)g(x)dx = g(\alpha)\int_\alpha^\xi f(x)dx.$$

(ii) If function $g(x)$ is locally strictly monotone decreasing (or increasing) at ξ , then there exist $\alpha, \beta \in \left[a,b\right]$, and $\alpha < \xi < \beta$ cause

$$\int_\alpha^\beta f(x)g(x)dx = g(\beta)\int_\eta^\beta f(x)dx.$$

We can also use the means above to improve the third form of the second mean value theorem for integrals to:

(iii) Suppose function $f(x)$ is integrable on the closed interval $[a,b]$, $f(x)$ is identically positive (or negative), to arbitrary point $\xi \in (a,b)$, if $g(x)$ is locally strictly monotone at ξ , then there exist $\alpha, \beta \in \left[a,b\right]$, meet the condition $\alpha < \xi < \beta$, cause

$$\int_\alpha^\beta f(x)g(x)dx = g(\alpha)\int_\alpha^\beta f(x)dx + g(\beta)\int_\xi^\beta f(x)dx.$$

4 Generalizing the Theorem o Infinite Interval

In the second mean value theorem for integrals, although finite invertal $[a,b]$ can't be sharpened to (a,b) freely, it can be generalived to $[a,+\infty)$ or $(-\infty,b]$ or $(-\infty,+\infty)$, Next I will give specific explain and proof.

Theorem 4.1[9]

Suppose $g(x)$ is monotone bounded on $[a,+\infty)$, $f(x)$ is integrable on $[a,+\infty)$, and $f(x)$ has no except $+\infty$,then there exist $\xi \in \left[a,+\infty\right)$, cause

$$\int_a^{+\infty} f(x)g(x)dx = g(a)\int_a^\xi f(x)dx + g(+\infty)\int_\xi^{+\infty} f(x)dx \text{ Here}$$

$$g(+\infty) = \lim_{x\to\infty} g(x).$$

Theorem 4.2

Suppose $g(x)$ is monotone bounded on $(-\infty,+\infty)$, $f(x)$ is integrable, and $f(x)$ has except $-\infty$ and $+\infty$,then there exist $\xi \in (-\infty,+\infty)$, cause

$$\int_{-\infty}^{+\infty} f(x)g(x)dx = g(-\infty)\int_{-\infty}^{\xi} f(x)dx + g(+\infty)\int_{\xi}^{+\infty} f(x)dx ,$$

Here $g(+\infty) = g(-\infty) = \lim_{x\to\infty} g(x)$.

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