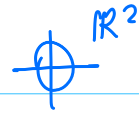
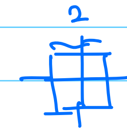
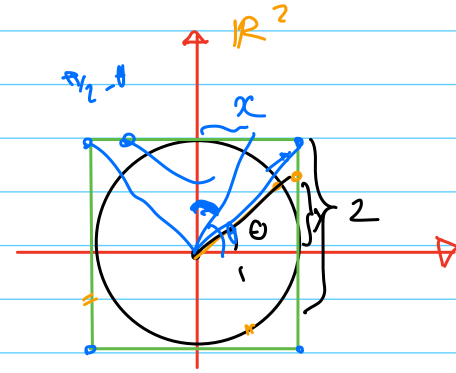
1.

$C \subseteq \mathbb{R}^2$ 

$S \subseteq \mathbb{R}^2$ $\longrightarrow$ $S = \{(x,y) \mid \max(|x|,|y|) = 1\}$ 

Homeomorphic ✓

Diffeomorphic ✗



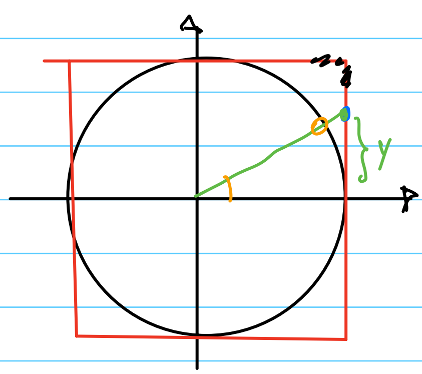
$$f(e^{i\theta}) = \begin{cases} (1, \tan\theta) & -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \\ (\cot\theta, 1) & \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4} \\ (-1, -\tan\theta) & \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4} \\ (-\cot\theta, -1) & \frac{5\pi}{4} \leq \theta \leq \frac{7\pi}{4} \end{cases}$$

$\tan\theta = \frac{y}{1}$

$\tan(\frac{\pi}{2} - \theta) = \frac{x}{1}$

Cot

$f : \text{Cir} \longrightarrow \text{Squ}$ $\uparrow$ f is homeomorphism?



$x = 1, \; y \longrightarrow \theta$

$$f^{-1} = \begin{cases} \tan^{-1}(y) & x = 1, \; |y| \leq 1 \\ \cot^{-1}(x) & y = 1, \; |x| \leq 1 \\ -\tan^{-1}(y) & x = -1, \; |y| \leq 1 \\ -\cot^{-1}(x) & y = -1, \; |x| \leq -1 \end{cases}$$

$\tan\theta = y$

There is no diffeomorphism.

تحریر

F | $\underbrace{\text{subset of } C}$ $\longrightarrow$ Connected neighbourhood of $(1,1)$

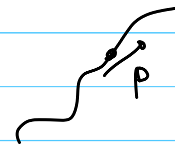
connected

Parametrize it by
$\gamma : (-1,1) \longrightarrow S$

$$F\circ\gamma = \begin{cases} (1,\ y(t)) & -1 < t \leq t_0 \\ (x(t),\ 1) & t_0 < t \leq 1 \end{cases}$$

$\gamma(t_0)\ \nexists \qquad t_0 \in (-1,1)$

$$\frac{d}{dt}(f\circ\gamma)\Big|_{\lambda=0}$$



$\lambda = 0$
$\gamma(\lambda=0) = p$

$\gamma : (-1,1) \xrightarrow{\text{to}} \text{Circle}$

$F\circ\gamma : (-1,1) \longrightarrow \text{square} \qquad$ By def:

$$\frac{d}{dt}(F\circ\gamma) = \begin{cases} (0,\ y'(t)) & -1 < t \leq t_0 \\ (x'(t),\ 0) & t_0 < t \leq 1 \end{cases}$$

@ $t = t_0 \quad x'(t_0) = y'(t_0) = 0$

بردار مماس بر گوشه‌ی مربع $(0,0)$

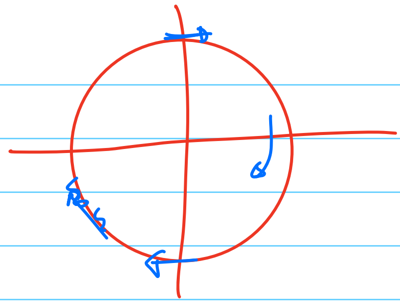
اگر $F: M \to N$ یک دیفئومورفیسم باشد: $F_* : T_pM \longrightarrow T_{F(p)}N$

F^* دوسویی است. (پوشا , $\underline{1-1}$) ker صفر

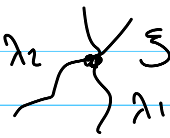
$$\varphi_\alpha \circ \varphi_\beta^{-1} : \underbrace{\varphi_\beta(U_\alpha \cap U_\beta)}_{\text{sub } \mathbb{R}^n} \longrightarrow \underbrace{\varphi_\alpha(U_\alpha \cap U_\beta)}_{\mathbb{R}^n}$$

بردار مماس به ┐ $(0,0)$ $\xleftarrow{\text{تحت}}$ این نمودار با بردار روی دایره، بردار صفر است.

$\gamma'(t)\big|_{t=t_0} = (0,0)$ $\qquad$ $t \to \lambda(t)$



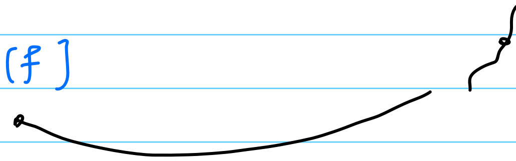
$\gamma(t)$ $\qquad$ $F\circ\lambda(t)$



$$\frac{d}{d\xi}(f\circ\lambda_i)\Big|_{\xi=0}$$

$\partial_i, \partial_j, \dots$

$X[f]$



John. M. Lee

Intro to smooth manifolds

spivack

Calculus on Manifolds

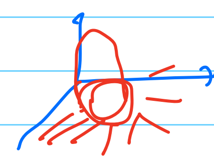
$n\geq 2$

$C^\infty(M)$ infinite-dimensional

Lemma: M a $n\geq 2$-dimensional, $\{f_1,\dots,f_k\} \in C^\infty(M)$

$\mathrm{supp}(f_i) \neq \phi$

$i\neq j \to \mathrm{supp}\, f_i \cap \mathrm{supp}\, f_j = \phi$



($f_1,\dots,f_k$ are linly-indep.)

$a_i\in\mathbb{R}$ $\qquad$ $\sum_{i=1}^{k} a^i f_i = 0$ $\qquad$ $a^i = 0$

$1\leq \alpha\in\mathbb{N}\leq k$ $\qquad$ $x_\alpha \in \mathrm{supp}(f_\alpha)$ $\qquad$ $f_\alpha(x_\alpha)\neq 0$

$f_j(x_\alpha) \overset{\alpha\neq j}{=} 0$

$0 = \left(\sum_{i=1}^{k} a^i f_i\right)(x_\alpha) = \sum_{i=1}^{k} a^i \underbrace{f_i(x_\alpha)}_{\delta_{i\alpha}} = a^\alpha \underbrace{f_\alpha(x_\alpha)}$

$f_k(x_k \neq \alpha)$

$a^\alpha = 0$

α was arbitrary $(x_1, x_2, \dots, x_k) \in (\operatorname{supp} f_1, \dots, \operatorname{supp} f_k)$

$\downarrow \quad a^\alpha = 0 \quad \forall \alpha \in \{1, \dots, k\}$

$\boxed{TK}$ Linear independent.

$n \geq 2$

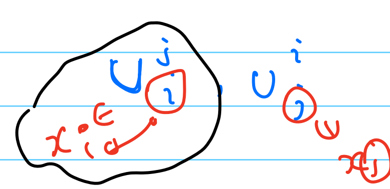
$C^\infty(M)$ is ∞-dimensional. $(M \neq \phi)$

$\underline{C^\infty(M)}$ is $\underline{k\text{-dimensional}}$.

$p \in M \qquad (U_\alpha, \varphi_\alpha)$ on M.

$p \in U$, $\qquad \underline{x_1, \dots, x_{k+1}} \in U$

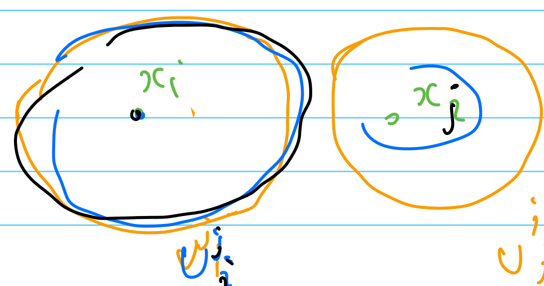
$x_i, x_j \quad i \neq j \quad$ شرط هاسدورف



$U^j_i \cap U^i_j = \phi$

$U_i = \underbrace{\bigcap_{j \neq i} U^j_i}$

اشتراک متناهی باز



U_i is open $\qquad x_i \in U_i^j \quad \forall j \qquad$ نیز U_i

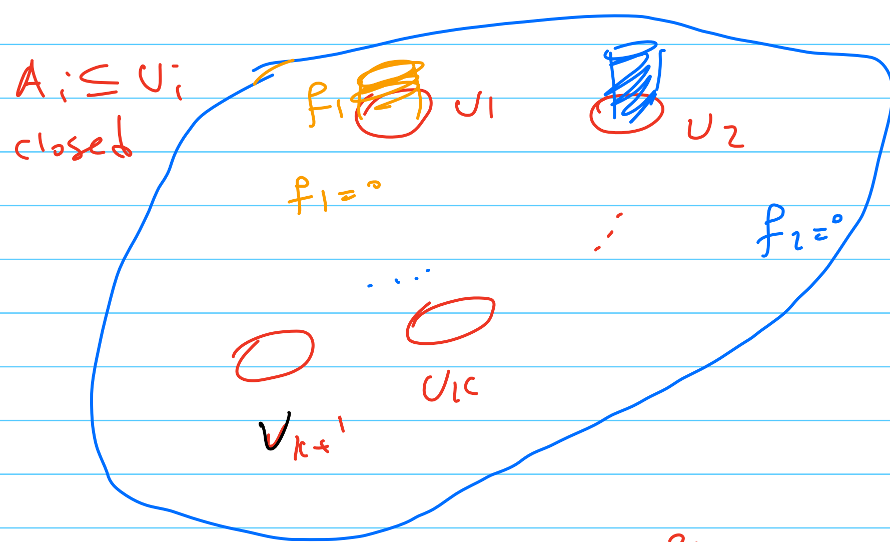
$x_j \notin U_i^j \longrightarrow x_j \notin U_i$

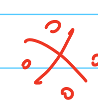
$U_1, U_2, U_3, \dots, U_{k+1}$

همسایگی هایی که دو به دو اشتراک ندارند و همسایگی اند.

تقسیم (partition of unity)

There are smooth functions like $f_1, \dots, f_{k+1}$ which has support on U_i.

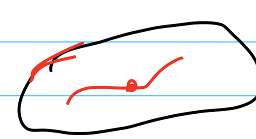




$f_1, \dots, f_{k+1}$ is linearly independent.

For $p \in M$ let G_p is the set of all curves $\gamma : \underset{0}{\overset{\cap}{J}} \longrightarrow M$

$\gamma(0) = p$



A equiv. rel. on G_p

$\gamma_1 \sim \gamma_2 \longleftrightarrow (f\circ\gamma_1)'(0) = (f\circ\gamma_2)'(0)$

f is any function $C^\infty(M)$

V_p denotes equivalence classes.

$\Phi : V_p \longrightarrow T_pM \qquad \Phi[\gamma] = \gamma'(0)$

Φ is well-defined / 1-1 / onto

$[\gamma_1] = [\gamma_2] = [\gamma_3] \qquad \{\gamma_1, \gamma_2, \gamma_3, \dots\}$

$\gamma_1 \neq \gamma_2$

$(f\circ g)' = g' f'\circ g$

$\gamma_1 \sim \gamma_2 \qquad (f\circ\gamma_1)'(0) = (f\circ\gamma_2)'$

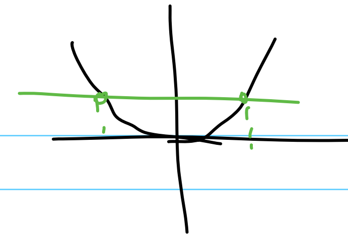
f is arbitrary $\qquad$ chain rule

$f(x) = 1 \qquad \gamma_1'(0) = \gamma_2'(0)$

$\|$

$\phi[\gamma_1] = \phi[\gamma_2]$

1-1 $\quad \underline{\Phi[\gamma_1] = \Phi[\gamma_2]} \xrightarrow{?} [\gamma_1] = [\gamma_2]$



$\gamma_1'(0) = \gamma_2'(0)$

Take f arbitrary

$(f\circ\gamma_1)'(0) = f^*\gamma_1'(0) = f^*\gamma_2'(0) = (f\circ\gamma_2)'(0)$

$M \to \mathbb{R}^n$ $\quad$ Exercise

$(f\circ\gamma_1)'(0) = (f\circ\gamma_2)'(0)$

$\forall f$

$\gamma_1 \sim \gamma_2 \to [\gamma_1] = [\gamma_2]$

$\boxtimes$

$X \overset{?}{=} \Phi([\gamma])$

$X \in T_pM$

There exists $\gamma(0) = p \quad \gamma'(0) = X$

$X = \gamma'(0) = \Phi[\gamma]$

پوشاست ✓

$\boxtimes$

$\forall X \in T_pM$

$\varphi(p) = 0$

$p \in U, \quad (U, \varphi)$

$X = X^i \frac{\partial}{\partial x^i}\Big|_p$

$\gamma : (-\varepsilon, \varepsilon) \to U$

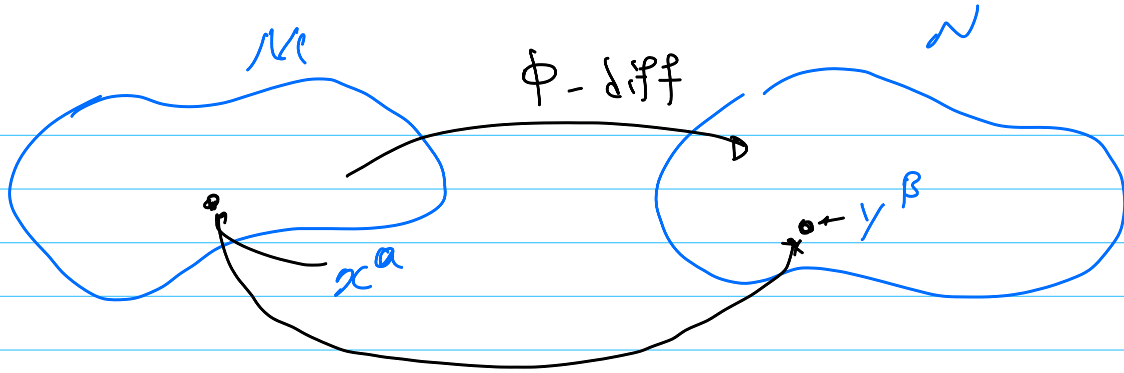
$\gamma(t) = \underbrace{(tX^1, tX^2, \dots, tX^n)}_{\in\mathbb{R}^n}$

$\gamma(t) = \varphi^{-1}(\qquad)$

γ : smooth $\qquad \gamma(0) = \varphi^{-1}(0_n) = p$

$= \gamma'(0) = X$

[illegible]



$$\phi = y^{\beta}(x^{\alpha})$$

$$\frac{\partial y^{\beta}}{\partial x^{\alpha}} \qquad \frac{\partial^{n} y^{\beta}}{\partial x^{\alpha}\, \partial x^{\delta}\, \partial \dots \partial \dots}$$

$$\frac{\partial x^{\alpha}}{\partial y^{\beta}} \dots$$