Introduction to General Relativity - HW 3 - 401208729

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Q1. Einestein equivalence principle (EEP) states that " gravitational field is equivalent to a non-inertial frame locally, in the sense that we can't distinguish between them by any physical experiment."

It implies that it's impossible to detect the presence of a gravitational field in "small enough regions of spacetime".

The mathematical structure suited to describe such a physical intuition is Riemannian manifold.

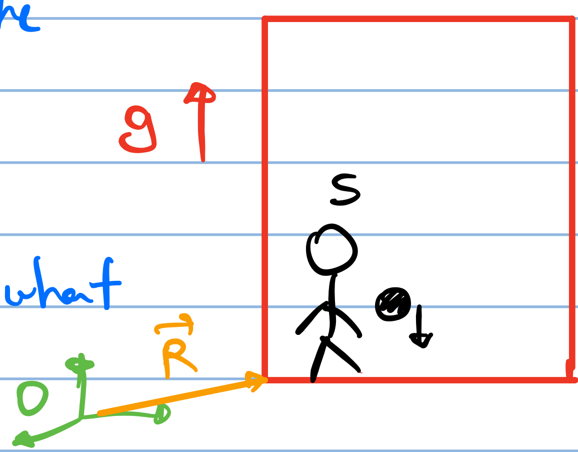
we describe the following hypothetical scenario to elucidate on the meaning of EEP.

1. Consider an observer (S) in a ascending elevator with acceleration, g.

(No gravitational field is present.)

(take the dimensions of the elevator very small!

$l_{elev} \ll \left|\frac{\vec{\nabla} g}{g}\right|$, that's what



we mean small engh regions of space time.)

S, is an non-inertial observer and we know that force in S frame is given by

(from analytical mechanics)

$$F_S = F_O - m\ddot{\vec{R}}$$ (in our case $\ddot{\vec{R}} = \vec{g}$)

If the observer S releases a small ball, according to him, force would be

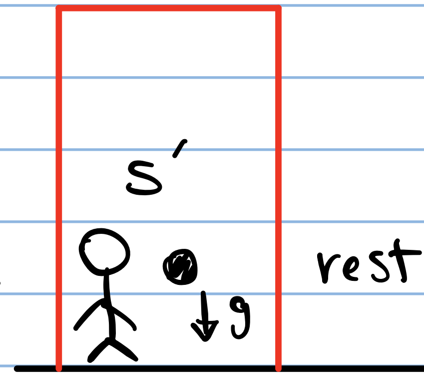
$$\vec{F}_{\substack{\text{on-ball} \\ \text{in } S}} = \cancel{F_{\vec{O}}} - mg = -m\vec{g} \downarrow$$ downward

No force exerts to the ball

So according to S, the ball will fall with $\vec{g}$ acceleration.

2. Consider the same elevator now in rest in the presence of gravitational field.

If the s' observer releases the ball, the ball will fall with acceleration g downward.



(since S' is inertial frame, due to Newton second law $\cancel{m}\vec{a} = -\cancel{m}\vec{g}$ (WEP) $\Rightarrow$

$\vec{a} = -\vec{g}$)

In both cases, the trajectory and the kinematics of the ball is the same, hence observer S, S' are not able to deduce they are in gravitational field (S') or they are in a non-inertial frame.

That's the gist of EEP.

Q2.

a) Easy! according to pset (2). the four-force K^μ is $K^\mu = \gamma\left(\frac{\vec{F}\cdot\vec{u}}{c}, \vec{F}\right)$

when $\vec{F}$ is the usual three-force [or Lorentz force in our case: $\vec{F} = q(\vec{E} + \vec{u}\times\vec{B})$].

Hence:

$$K^\mu = \gamma\left(\frac{1}{c}\left(q(E+\vec{u}\times B)\cdot\vec{u}\right), q(E+u\times B)\right)$$

we know that $(a\times b)\cdot a = 0$ hence

$$K^\mu = \gamma q\left(\frac{\vec{E}\cdot\vec{u}}{c}, \vec{E} + \vec{u}\times\vec{B}\right)$$

Let's check if $q F^{\mu\nu} u_\nu$ equals to this?!

$$F^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & +\frac{\vec{E}}{c} & \\ \hline & 0 & B_z & -B_y \\ -\frac{\vec{E}}{c} & -B_z & 0 & +B_x \\ & +B_y & -B_x & 0 \end{array}\right)$$

due to Carroll.

$$u_\nu = \eta_{\mu\nu} u^\mu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \gamma \begin{pmatrix} c \\ \vec{u} \end{pmatrix} \Longrightarrow$$

$$u_\nu = \gamma(-c, \vec{u})$$

$F^{\mu\nu} u_\nu$ for :

$$\mu = 0 \longrightarrow F^{0\lambda} u_\lambda = \overset{\text{First row}}{\overrightarrow{\left(0, +\frac{\vec{E}}{c}\right)}} \times \gamma \begin{pmatrix} -c \\ \vec{u} \end{pmatrix}$$

$$= \gamma \frac{\vec{E} \cdot \vec{u}}{c}$$

Note that $F^{ij} = +\varepsilon^{ijk} B_k$, so for $\mu = i$

$$F^{i\lambda} u_\lambda = F^{i0} u_0 + F^{ij} u_j$$

velocity components

$$= -\frac{E^i}{c}(-c\gamma) + \gamma \varepsilon^{ijk} B_k (+u_j)$$

$$= \gamma E_i + \gamma \underbrace{\varepsilon^{ijk} u_j B_k}_{\text{cross-product}}$$

$$= \gamma (E_i + (u \times B)_i)$$

$$\text{So} \longrightarrow q F^{\mu\nu} u_\nu = q\gamma \left(\frac{\vec{E} \cdot \vec{u}}{c}, \vec{E} + \vec{u} \times \vec{B} \right)$$

Therefore $K^\mu = q F^{\mu\lambda} u_\lambda$ as desired $\varnothing$.

b) As we know :

$$J^{\mu} = (c\rho, \vec{J})$$

$$A_{\mu} = (\phi/c, \vec{A})$$

$$F^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & +\frac{\vec{E}}{c} & \\ \hline & 0 & B_z & -B_y \\ -\frac{\vec{E}}{c} & -B_z & 0 & B_x \\ & +B_y & -B_x & 0 \end{array}\right)$$

$$\partial_{\mu} = (\tfrac{1}{c}\partial_t, \vec{\nabla})$$

$$\partial_{\mu} F^{\mu 0} = -\mu_0 J^0$$

$$\underbrace{\cancel{\tfrac{1}{c}\partial_0 F^{00}}}_{\text{zero}} + \partial_i F^{i0} = -\mu_0 J^0$$

$$\cancel{+}\ \tfrac{1}{c}\vec{\nabla}\cdot\vec{E} = \cancel{+}\ \mu_0 (c\rho) \Rightarrow$$

$$\vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho(x,t)$$

use : $c = (\varepsilon_0 \mu_0)^{-1/2} \Rightarrow c^2 \mu_0 = \frac{1}{\varepsilon_0}$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho(x,t)}{\varepsilon_0}$$

Gauss's Law.

For i-th component :

$$\partial_\mu F^{\mu i} = -\mu_0 J^i$$

$$\partial_0 F^{0i} + \partial_j F^{ji} = -\mu_0 J^i$$

$$\left(\frac{1}{c}\partial_t\right)\left(+\frac{E_i}{c}\right) + \partial_j\left(+\epsilon^{jik} B_k\right) = -\mu_0 J^i$$

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$$+\frac{1}{c^2}\partial_t E_i - \underbrace{\epsilon^{ijk}\partial_j B_k}_{\text{definition of } (\nabla\times B)_i} = -\mu_0 J^i$$

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$$\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}$$

Amper Law

For the next part, note that

$$F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & -\frac{\vec{E}}{c} & \\ \hline & 0 & B_z & -B_y \\ +\frac{\vec{E}}{c} & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{array}\right)$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0$$

take $\alpha = 0$ and $\beta = i$, $\gamma = j$ $(i \neq j)$ $\Rightarrow$

$$\partial_0 F_{ij} + \partial_j F_{0i} + \partial_i F_{j0} = 0$$

$$\left(\frac{1}{c}\partial_t\right)\left(+\varepsilon^{ijk} B_k\right) + \partial_j\left(-\frac{E_i}{c}\right) + \partial_i\left(+\frac{E_j}{c}\right) = 0$$

$$+\frac{1}{c}\varepsilon^{ijk}\partial_t B_k + \frac{1}{c}\underbrace{\left(\partial_i E_j - \partial_j E_i\right)}_{\text{Rewrite it as } (\nabla\times E)_k\, \varepsilon^{ijk}}$$

$$+\frac{1}{\cancel{c}}\varepsilon^{ijk}\partial_t B_k + \frac{1}{\cancel{c}}\varepsilon^{ijk}(\nabla\times E)_k = 0$$

i,j are free indices, multiply both sides by ε^{ijl} and sum on i,j: $\{\sum_{ij}$ is implicit$\}$,

then use $\varepsilon^{ijk}\varepsilon^{ijl} = \delta_{kl}$ (for $k \neq i$ or j)

(otherwise the equation becomes trivial $0=0$)

$$(\nabla\times E)_l = -\partial_t B_l \quad \text{or}$$

$$(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}\vec{B} \quad \text{Faraday law.}$$

Final set $\alpha = i,\ \beta = j,\ \gamma = k$ to set

(when $i \neq j \neq k$)

$$\partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} = 0$$

$$\partial_i \left(\varepsilon^{jkl} B_l \right) + \partial_k \left(\varepsilon^{ijm} B_m \right) +$$

$$\partial_j \left(\varepsilon^{kis} B_s \right) = 0$$

Multiply both sides by ε^{ijk} and sum on $\sum_{ijk}$ implicitly:

$$\partial_i \Big(\underbrace{\varepsilon^{ijk} \varepsilon^{jkl}}_{*} B_l \Big) + \partial_k \Big(\underbrace{\varepsilon^{ijm} \varepsilon^{ijk}}_{(**)} B_m \Big)$$

$$+ \partial_j \Big(\underbrace{\varepsilon^{kis} \varepsilon^{ijk}}_{(***)} B_s \Big) = 0$$

$(*)$ is zero, unless $l = i$ since i, j, k are permutations of $\{1, 2, 3\}$ so its δ_{kl}.

by the same logic $(**) = \delta_{km}$

$(***) = \delta_{js}$

By plugging these relations, you'll find

$$\partial_i B_i + \partial_j B_j + \partial_k B_k = 0$$

or $\quad 3\,\partial_i B_i = 0 \implies \vec{\nabla}\cdot\vec{B} = 0$

Q3. suppose : $\begin{cases} \nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma^\alpha_{\beta\gamma} A^\gamma \\ \nabla_\mu \Phi = \partial_\mu \Phi \end{cases}$

+ Leibnitz rule $\longrightarrow$

we know that Covariant derivative of any object is just ordinary derivative + connection : $\nabla = \partial + \Gamma$

But this Connection might be diffent for forms.

Let's take : $\nabla_\mu V_\nu = \partial_\mu V_\nu + \gamma^\lambda_{\mu\nu} V_\lambda$

Now let's look at $\nabla_\mu (V_\sigma W^\sigma)$ where W^σ is a ordinary vector-field.

$$\nabla_\mu (V_\sigma W^\sigma) = (\nabla_\mu V_\sigma) W^\sigma + V_\sigma \nabla_\mu W^\sigma =$$

$$\boxed{(\partial_\mu V_\sigma) W^\sigma} + \gamma^\lambda_{\mu\sigma} V_\lambda W^\sigma +$$

$$\boxed{V_\sigma \partial_\mu W^\sigma} + V_\sigma \Gamma^\sigma_{\mu\lambda} W^\lambda$$

on the othe hand since $V_\sigma W^\sigma$ is a scalar, So $\nabla_\mu (V_\sigma w^\sigma) = \partial_\mu (V_\sigma w^\sigma) =$

$$\boxed{(\partial_\mu V_\sigma) w^\sigma} + \boxed{V_\sigma \partial_\mu w^\sigma}$$

Now equating both sides, red and green boxes are equal so $\quad \gamma^{\lambda}_{\mu\sigma} V_\lambda w^\sigma = -V_\sigma \Gamma^{\sigma}_{\mu\lambda} w^\lambda$

$\lambda \leftrightarrow \sigma \quad = \gamma^{\sigma}_{\mu\lambda} V_\sigma w^\lambda$

Now since both V_λ, w^σ were arbitrary, we can take them so that they span a basis at each tangent space $\left\{ T_p M \right\}_{\forall p \in M}$

Hence, $\gamma^{\sigma}_{\mu\lambda} = -\Gamma^{\sigma}_{\mu\lambda}$

So we've shown $\nabla_\beta A_\alpha = \partial_\beta A_\alpha - \Gamma^{\gamma}_{\beta\alpha} A_\gamma$

$\blacksquare$

Q4. suppose we have cylindrical coordinates (r, θ, z), and $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$

so let's compute Γ-symbols.

$$\Gamma^{\lambda}_{\rho\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\rho\nu} \right)$$

Due to metric for just $\partial_r g_{\theta\theta} = 2r \neq 0$ and all the other derivatives vanish.

Let's compute them by hand:

$$\Gamma^{r}_{\theta z} = \frac{1}{2} g^{r\sigma} \left(\cancel{\partial_\theta g_{z\sigma}}^{0} + \cancel{\partial_z g_{\theta\sigma}}^{0} - \cancel{\partial_\sigma g_{\theta z}}^{0} \right)$$

since the metric is diagonal so all $\Gamma^{r}_{\nu\sigma}$ with $\rho \neq \nu \neq \sigma$ will vanish.

Now

sine g is diagonal $\rightarrow \sigma$ must be r.

$$\Gamma^{r}_{\theta\theta} = \frac{1}{2} g^{r\sigma} \left(\cancel{\partial_\theta g_{\sigma\theta}}^{0} + \cancel{\partial_\theta g_{\sigma\theta}}^{0} - \underbrace{\partial_\sigma g_{\theta\theta}}_{\partial_r g_{\theta\theta} = 2r} \right)$$

$= -r$

similarly Γ^r_{zz}, Γ^θ_{zz} since no $\partial_r g_{\theta\theta}$ appears on them, for Γ^θ_{rr} we have:

$$\Gamma^\theta_{rr} = \frac{1}{2}\underbrace{g^{\theta\theta}}_{\text{diagonal metric}}\left(\cancel{\partial_r g_{r\theta}}^{0} + \cancel{\partial_r g_{\theta r}}^{0} - \cancel{\partial_\theta g_{rr}}^{0}\right)$$

so $\Gamma^\theta_{rr} = 0$.

Note that all Γ^r_{rr}, $\Gamma^\theta_{\theta\theta}$, Γ^z_{zz} will vanish since no $\partial_r g_{\theta\theta}$ appears on them.

So, just remains:

$$\Gamma^r_{\theta r} = \frac{1}{2} g^{r\theta}{}^{=r}\overbrace{\left(\partial_\theta g_{r\theta} {=} r + \partial_r g_{\theta\theta} - \partial_\theta g_{r\theta}\right)}^{\text{all vanish}}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{2}\underbrace{g^{\theta\theta}}_{\frac{1}{r^2}}{}^{=\theta}\left(\underbrace{\partial_r g_{\theta\theta}}_{2r} + \cancel{\partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}}^{\text{vanish}}\right)$$

Hence $\Gamma^\theta_{r\theta} = \frac{1}{r}$

you can easily see that Γ^r_{zr}, $\Gamma^\theta_{z\theta}$,

Γ^z_{rz}, $\Gamma^z_{\theta z}$ will vanish.

Non-vanishing Γ-symbols: $\begin{cases} \Gamma^r_{\theta\theta} = -r \\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \end{cases}$

so that we can compute covariant derivt

of $A = (r^2 \cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} + 0\, \hat{z})$

r-th Component:

$$\nabla_r A^r = \partial_r A^r + \Gamma^r_{r\lambda} A^\lambda = \quad (\lambda \text{ is just } \theta)$$

$$\partial_r (r^2 \cos\theta) + \frac{1}{r} A^\theta =$$

$$2r\cos\theta + \frac{\sin\theta}{r}$$

$$\nabla_r A^\theta = \partial_r A^\theta + \Gamma^\theta_{r\lambda} A^\lambda \quad (\text{just } \lambda = \theta)$$

$$\partial_r(\sin\theta) + \frac{1}{r} A^\theta = \frac{\sin\theta}{r}$$

$\nabla_r A^z = 0$ since $A_z = 0$.

Now θ-th Compont:

$$\nabla_\theta A^r = \partial_\theta A^r + \Gamma^r_{\theta\lambda} A^\lambda \qquad (\lambda = \theta)$$

$$= \partial_\theta (r^2 \cos\theta) - r \times A^\theta$$

$$= -r^2 \sin\theta - r\sin\theta$$

$$\nabla_\theta A^\theta = \partial_\theta A^\theta + \Gamma^\theta_{\theta\lambda} A^\lambda \qquad (\lambda = r)$$

$$= \partial_\theta (\sin\theta) + \frac{1}{r}\sin\theta$$

$$= \cos\theta + \frac{\sin\theta}{r}$$

$\nabla_\theta A^z = 0$ since $A_z = 0$

All $\nabla_z A^r$, $\nabla_z A^\theta$, $\nabla_z A^z$ vanish.

since ther no z-dependen in A^r, A^θ, A^z

and all $\Gamma^{z}_{-\,-}$, $\Gamma^{-}_{z,-}$, $\Gamma^{-}_{-,z}$ vanished.

so if we sort $\nabla_\mu A^\nu$ in a matrix,

$$\nabla_\mu A^\nu = \begin{array}{cc} & \nu \longrightarrow \\ \mu \downarrow & \begin{pmatrix} 2r\cos\theta + \frac{\sin\theta}{r} & \frac{\sin\theta}{r} & 0 \\ -r\sin\theta(r+1) & \cos\theta + \frac{\sin\theta}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$