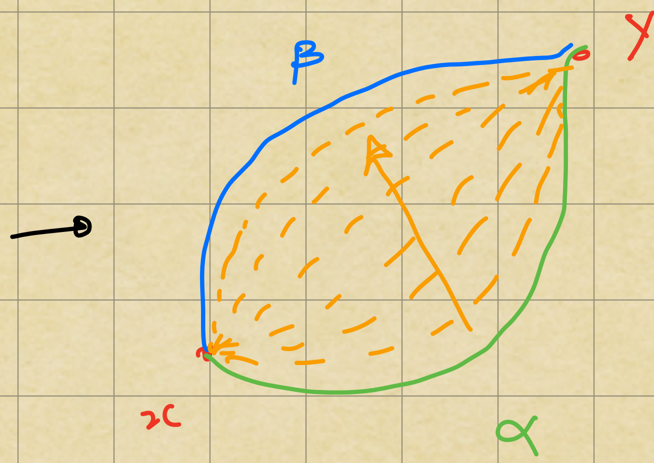
Review of Homotopy :

An equivalence rel. on curve.

$\alpha \sim \beta$



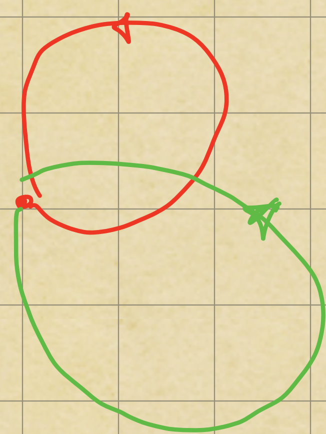
$\exists H : [0,1] \times [0,1] \longrightarrow X$

$H(t,0) = \alpha(t)$

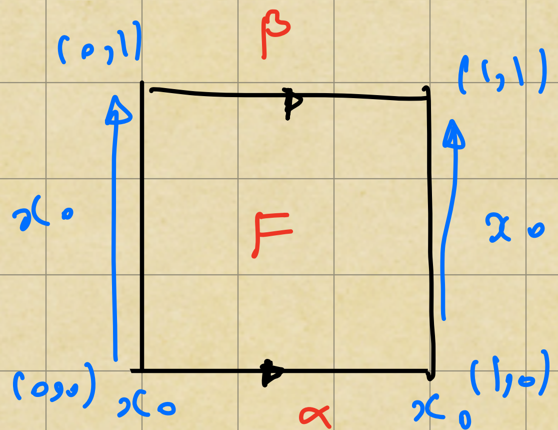
$H(t,1) = \beta(t)$

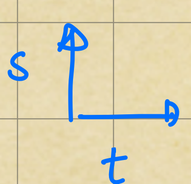
$H(0,s) = x$

$H(1,s) = y$



Look at loops

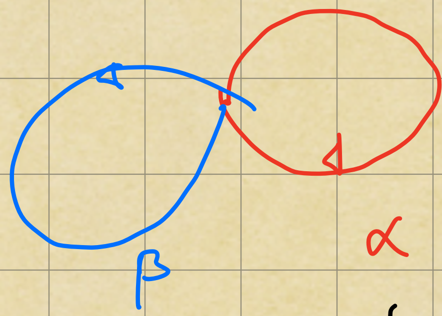


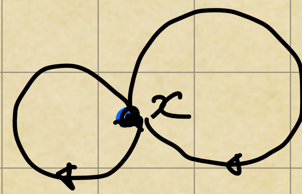


$F(s,t)$

$\pi_1(X, x_0) \rightarrow$

$\pi_1(X, x_0)$





$$(\alpha\beta) = \begin{cases} \beta(2t) & : \\ \alpha(2t-1) & : \end{cases}$$

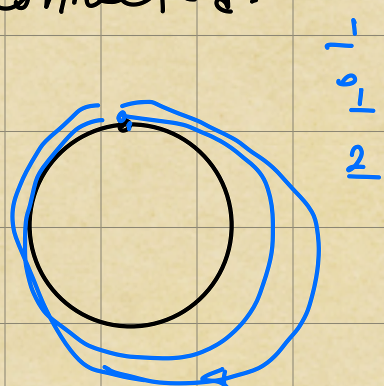
Independent of representative. (well-defined)

i $\cdot\, x_0$ $\alpha(t) = x_0$ $e \in \pi_1(X, x_0)$

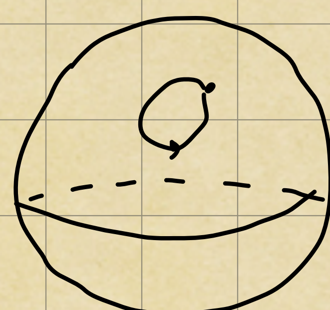
$\pi_1(X, x_0) \simeq \pi_1(X, y_0)$

X is path-connected.

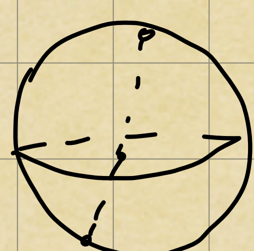
$\pi_1(S^1) = \mathbb{Z}$



$\pi_1(S^2) = \{e\}$



$\pi_1(SO(3)) \simeq \mathbb{Z}_2$



$\pi_1(T^2, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$

$SO(n) \simeq \mathbb{R}P^?$

$SO(\dots)$ J- homomorphism

SO , 6..

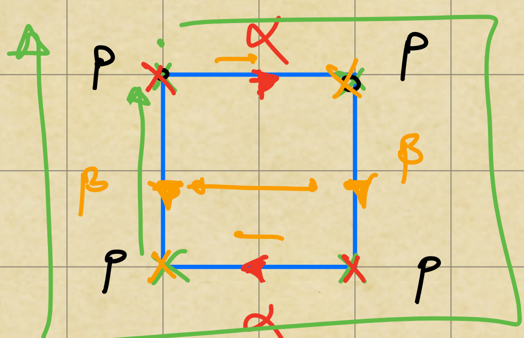
✓ Klein

✓ Mobius strip

✓ $\mathbb{R}P^2$

✓ S^3

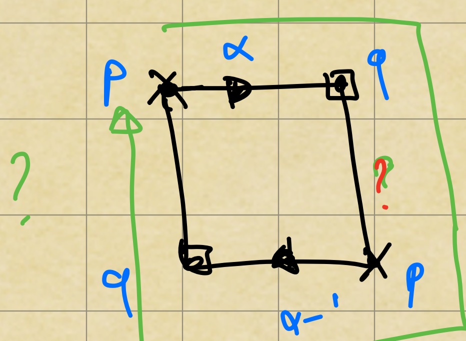
$T_g \oplus$ Klein Bottle



$\alpha\beta\alpha^{-1}\beta^{-1} = e$

$$\pi_1(KB) \cong \frac{\langle \alpha, \beta \rangle}{\{\alpha\beta\alpha\beta^{-1} = e\}}$$ ✓

π_1 (Mobius)

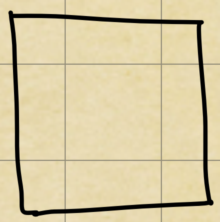


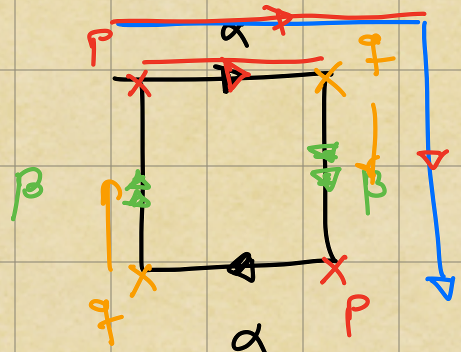
Universal Cover

$$= \frac{\langle \alpha \rangle}{\{\ \}} = \langle \alpha \rangle = \mathbb{Z}$$

$\pi_1(\text{fiw}) \simeq \pi_1(\text{chin-sf Conv})$.

$\pi_1(\mathbb{R}P^2)$





$$\pi_1(\mathbb{R}P^2) = \frac{\langle \alpha\beta, \beta\alpha \rangle}{\underbrace{\alpha\beta\alpha\beta = e}}$$

$\alpha\beta = \beta^{-1}\alpha^{-1}$

$\langle \alpha\beta, \beta\alpha^{-1} \rangle \ \ldots$

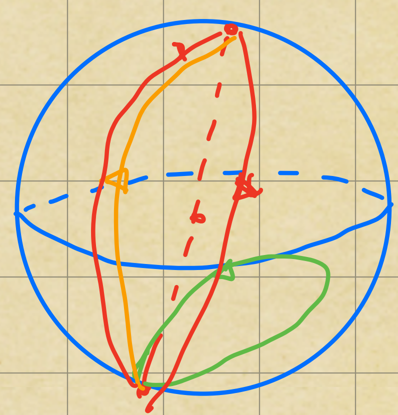
$\mathbb{Z}_2$

$\alpha\beta \qquad \alpha\beta\beta\alpha = \beta^{-1}\alpha^{-1}\beta\alpha$

$\beta\alpha$

$\mathbb{R}P^{2^n} \simeq S^{2^n}$

$\mathbb{R}P^2$



$\langle \alpha\beta,$

✓

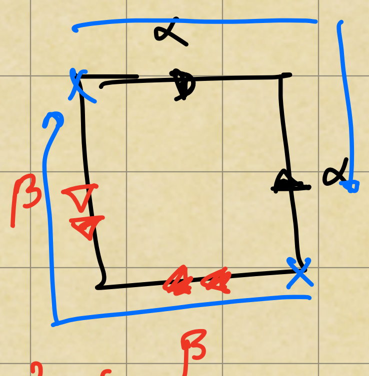
$\mathbb{R}P^n \ncong S^n$

$$\boxed{\pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}_2}$$

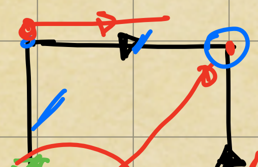
$\mathbb{R}P^n$

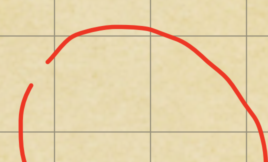
S^n

S^2

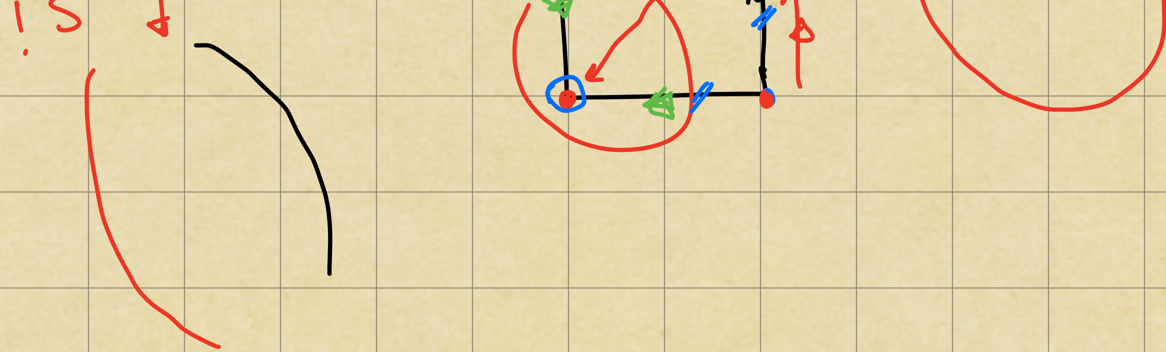


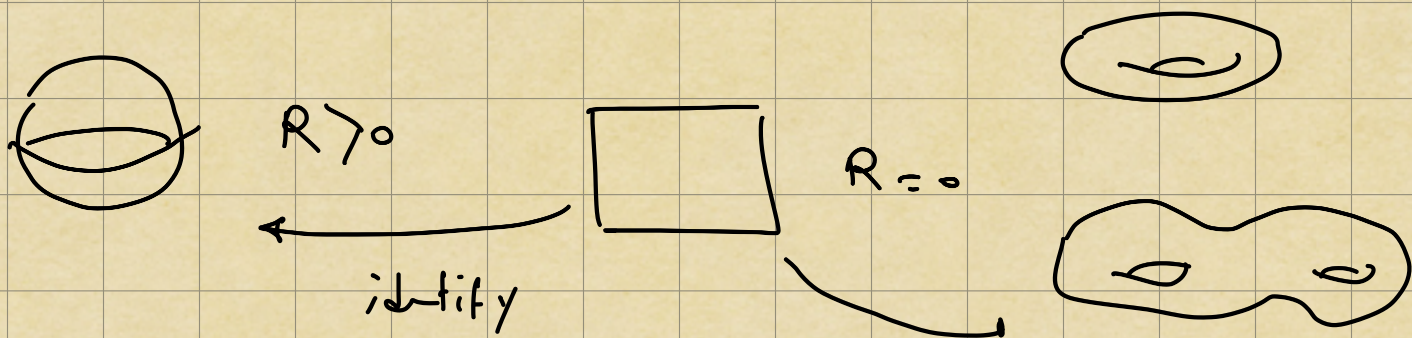
$$\pi_1(S^2) \simeq \frac{\langle \alpha\alpha^{-1}, \beta\beta^{-1} \rangle}{}$$





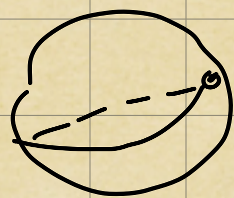
$S^2 \ \{$





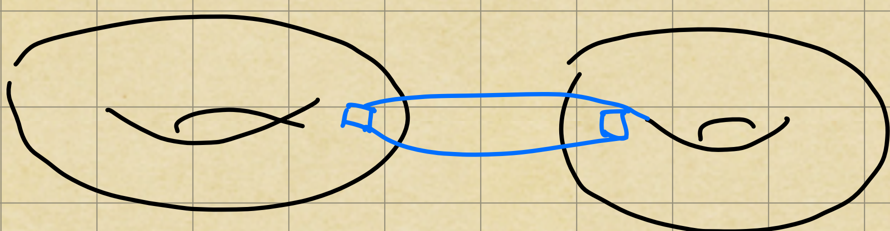
$n \geq 2$

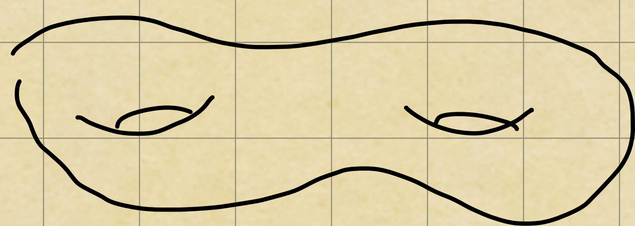
$\pi_1(S^n) \simeq \{e\}$



S^2

$T_g \# \text{Klein}$



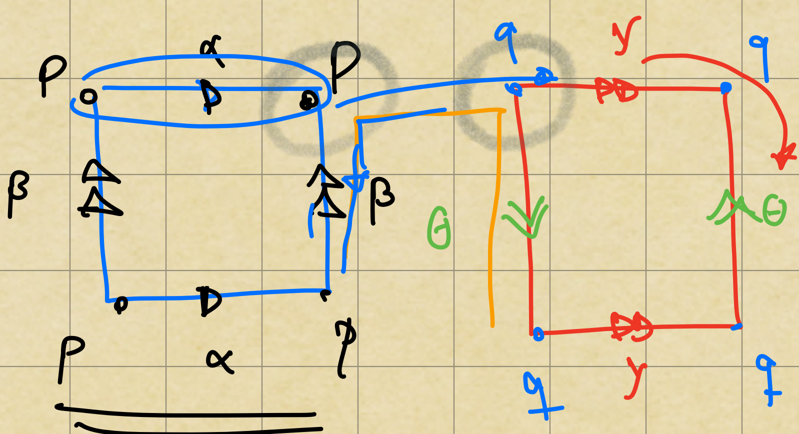


$T_g \# \text{Klein}$ $\qquad$ $T_2 = T_1 \# T_1$

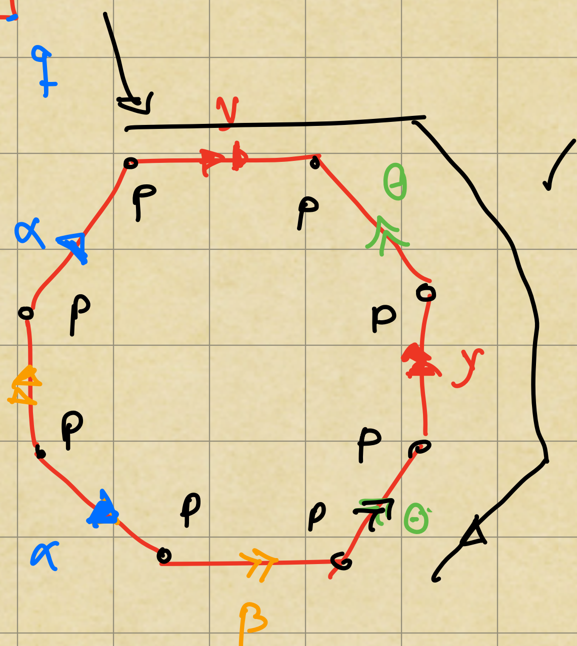
$\pi_1(S_1)\quad \pi_1(S_2) \longrightarrow \pi_1(S_1 \# S_2)$

Seifert van-Kampen

$T_1 \#$ Klein



$T_1 \#$ Klein



$\pi_1(T_1 \#$ Klein$)$

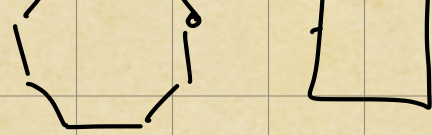
$$= \frac{\langle \alpha, \beta, \gamma, \theta \rangle}{\{\gamma\theta^{-1}\gamma^{-1}\theta^{-1}\beta^{-1}\alpha^{-1}\beta\alpha = e\}}$$

$T_1 \# KB$ $\boxed{?}$

$T_g \#$ Klein

$4g$-gon

$g=2$



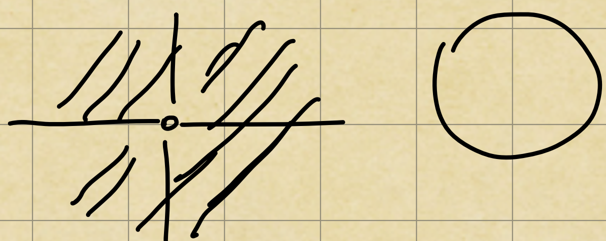
$g = 4$

$g \longrightarrow 4g + 4 = 4(g+1)$

$\mathbb{R}^2 - \{0\} \longrightarrow S^1$

از یک نوع هوتوپی‌اند،

اما Homeomorphic نیستند!

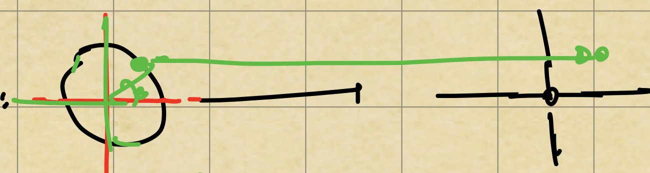


$X = \mathbb{R}^2 - \{0\}$ $\quad$ $Y = S^1$

$f : X \longrightarrow Y$

$g : Y \longrightarrow X$

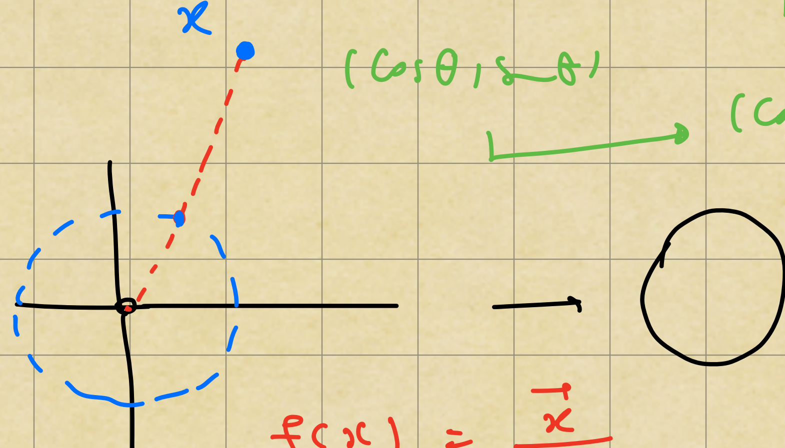
$g :$



$\mathbb{R}^2$ $\quad$ $\mathbb{R}^1 - \{0\}$

$(\cos\theta, \sin\theta) \longmapsto (\cos\theta, \sin\theta)$

$f :$



$f(x) = \dfrac{\vec{x}}{\|x\|}$

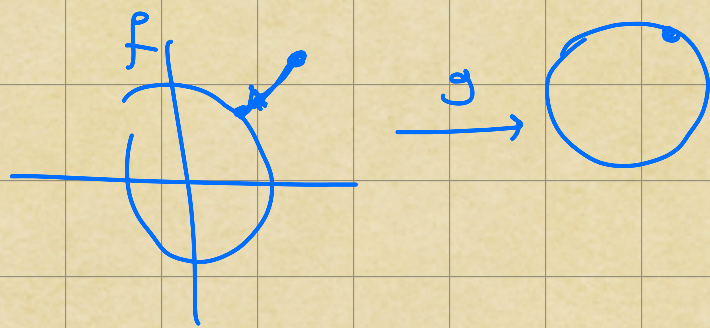
$f \circ g \sim_H id_Y$

$g \circ f \sim_H id_X$

$\curvearrowleft \; g \circ f$

$f \circ g : Y \longrightarrow Y$

$f \circ g \overset{id}{=} id_Y \qquad id_Y \sim id_Y \; ✓$



$\underline{g \circ f} \sim id_X$

$H_1 : X \times [0,1] \longrightarrow X$

$$\boxed{H_1 : (x,t) \longmapsto \frac{t x}{\|x\|} + (1-t)\, x}$$

$x \mapsto \wedge$

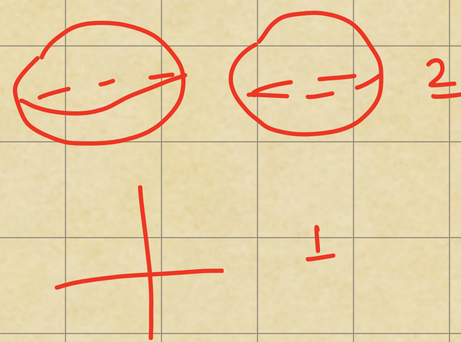
$H_1(x,0) = x =$ ادی ✓ $\qquad H_1$ پیوسته ✓

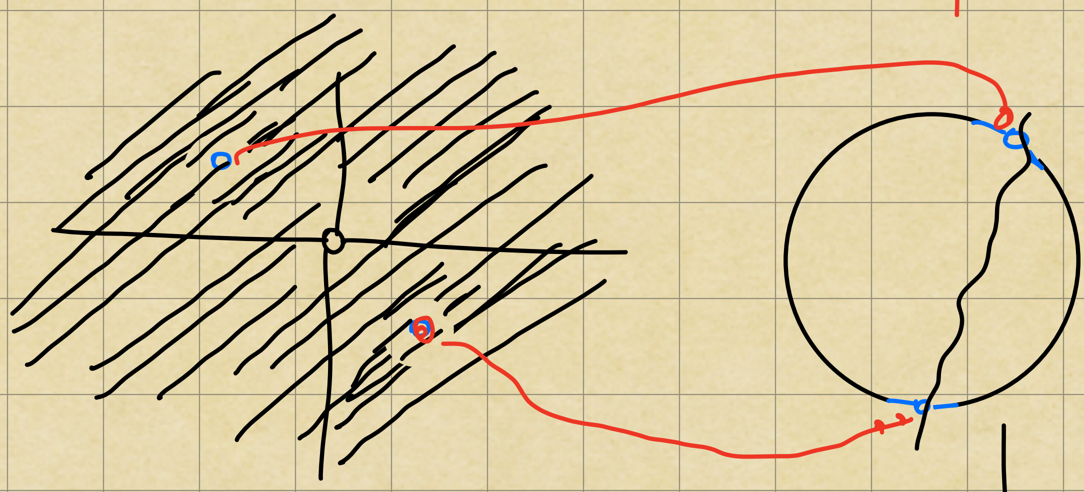
$H_1(x,1) = \dfrac{x}{\|x\|} = g \circ f$

$(\mathbb{R} - \{0\}, S^1)$

Homeomorphism doesn't change connected compact.

$\phi : \underbrace{S_1}_{n} \longrightarrow \underbrace{S_2}_{n}$





$f : S_1 \longrightarrow S_2$

$f^{-1}(U \subset S_2)$

$\downarrow$

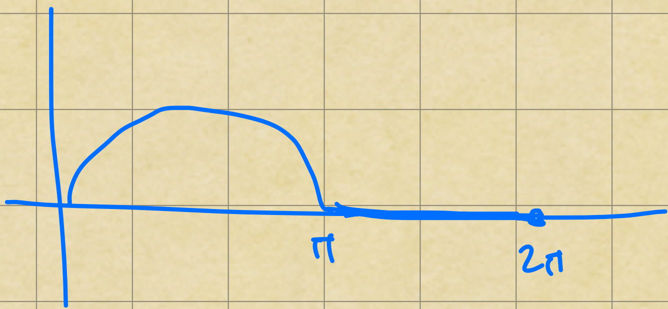
S_1 open ✓

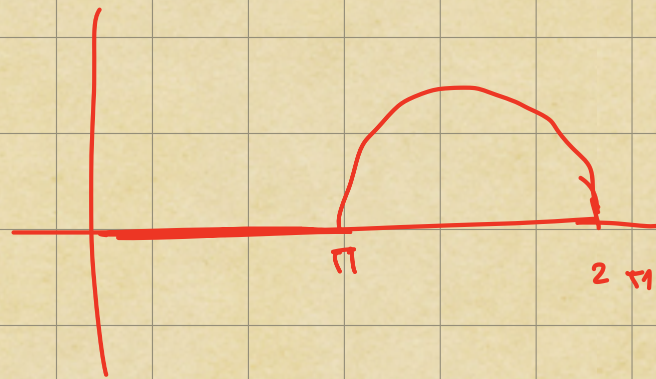
1 $\quad \xrightarrow{\phi} \quad$ 2

$$f(x) = \begin{cases} \sin x & x \in [0, \pi] \\ 0 & x \in (\pi, 2\pi] \end{cases}$$

$$g(x) = \begin{cases} 0 & '' \end{cases}$$

$(\quad -\sin x$ $\qquad$ //





$$H : [0, 2\pi] \times \widetilde{[0,1]} \longrightarrow \mathbb{R}$$

$$H(x,t) = \underline{\underline{t}}\ \underline{\underline{g(x)}} + \underline{\underline{(1-t)}}\ \underline{\underline{f(x)}}$$

$H(x,0) = f(x)$

$H(x,1) = g(x)$

$H(x,t)$ is bicontinuous.

$H(x,t)$ is a homotopy

$$f(x) \sim_H g(x)$$

$\overset{S^1}{\underline{\underline{X}}}$, $\overset{S^1 \times \mathbb{R}}{\underline{\underline{X \times \mathbb{R}}}}$

$f : X \longrightarrow X \times \mathbb{R}$ $\qquad$ $f(x) = (\underset{\in X}{x}, \underset{\in \mathbb{R}}{0})$

$g : X \times \mathbb{R} \longrightarrow X$ $\qquad$ $g(x,r) = x \in X$

$f\circ g : X\times\mathbb{R} \longrightarrow X\times\mathbb{R}$ $\quad$ $f\circ g\,(x,r) = f(x) = (x,0)$

$g\circ f : X \longrightarrow X$; $\quad$ $g\circ f(x) = g(x,0) = x$

$g\circ f = id_X$

$g\circ f \sim_H id_X$

$f\circ g \sim_H id_{X\times\mathbb{R}}$

$$H : (X\times\mathbb{R}) \times [0,1] \longrightarrow X\times\mathbb{R}$$

$$H((x,r),t) = (x, (1-t)r)$$

$H((x,r),0) = (x,r) = id_{X\times\mathbb{R}}$

$H((x,r),1) = (x,0) = f\circ g$ $\quad\rightarrow$ Homotopic

$\Longrightarrow$ X, $X\times\mathbb{R}$ are of the same Homotopy. ✓

$$\pi_1(KB) \cong \frac{\langle a,b\rangle}{(abab^{-1}=e)}$$ $\longrightarrow$ This is $\pi_1(KB)$

$$\frac{\langle ab,\ b\rangle}{\left((ab)ab^{-1} = e\right)}$$

$$(ab)\left(ab\, b^{-1} b^{-1}\right) = e$$

$$(ab)^2\, b^{-2} = e \longrightarrow (ab)^2 = b^2$$

$a \to ab$

$b \to b$

$$\frac{\langle ab,\ b\rangle}{\underline{\underline{\{(ab)^2 = b^2\}}}} = \frac{\langle \alpha, \beta\rangle}{\alpha^2 = \beta^2} \checkmark$$

$$g = \alpha^{i_1} \beta^{\overset{0,1}{j_1}} \alpha^{i_2} \beta^{\overset{0,1}{j_2}} \cdots \alpha^{i_k} \beta^{\overset{0,1}{j_k}}$$

$r_1 = 0, 1$

$$j_1 = 2k_1 + r_1$$

$$\underbrace{\alpha^{i_1} \beta^{2k_1 + r_1}} = \alpha^{i_1} \left(\beta^2\right)^{k_1} \beta^{r_1}$$

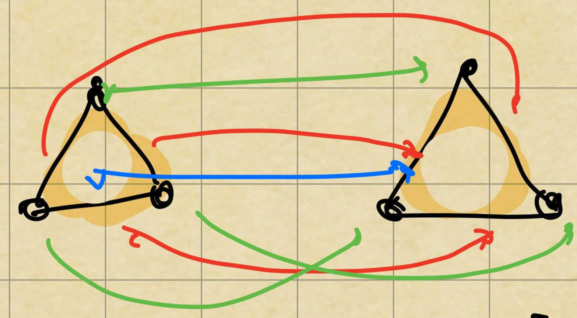
$$= \alpha^{i_1 + 2k_1} \beta^{r_1} \cdots$$

$\chi(S_1 \# S_2) = ?$

$\chi(\text{surface}) = f - e + V$

$(P_0 P_1 P_2)$

$S_1 \# S_2$



S_1
(f_1, e_1, V_1)

$S_2 \; (f_2, e_2, V_2)$

$$S_1 \# S_2 = \begin{cases} V = V_1 + V_2 - 3 \\ e = e_1 + e_2 - 3 \\ f = f_1 + f_2 - 2 \end{cases}$$

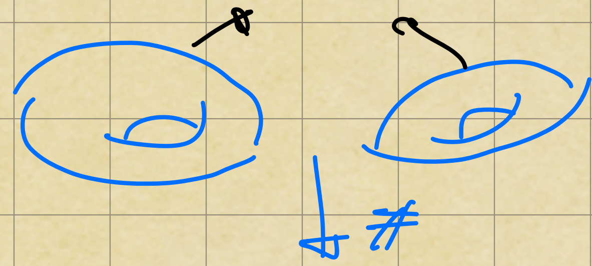
$$\begin{aligned} f - e + V &= f_1 + f_2 - 2 \\ &\quad - (e_1 + e_2 \cancel{- 3}) \\ &\quad + V_1 + V_2 \cancel{- 3} \end{aligned}$$

$$\begin{aligned} &= f_1 - e_1 + V_1 \\ &\quad f_2 - e_2 + V_2 \qquad - 2 \end{aligned}$$

$$= \chi(S_1) + \chi(S_2) - 2$$

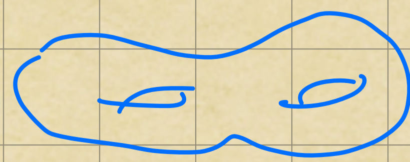
$T^2 \# T^2$

$\chi = ?$



$\chi_{\text{کره}} = 2$

$\chi_{\text{چنبره}} (\ \ ) = 0$



$\chi =$

$$\chi(T^2) \overset{?}{=}$$

$$\chi(T^2 \# T^2) = \chi(\cancel{T^2}) + \chi(\cancel{T^2}) - 2$$

$$\chi(T_{g=2}) = -2$$

$$\chi(\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{g}) =$$

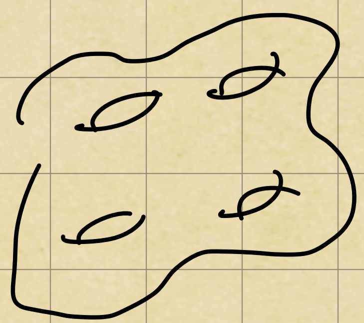
$$= \chi(T^2 \# \cdots \# T^2) - 2$$

$g-1$

$\vdots$

$$= -2(g-1) = \underline{\underline{2-2g}}$$

$\downarrow$



$$= \chi(T_4) = -6$$

$\boxed{g \geqslant 2}$ $\chi_g < 0$

$$\overbrace{\frac{1}{2\pi}\int_{M} \sqrt{g}\, R}^{} = \underbrace{\chi(M)}_{\text{منفی}}$$

R بیشتر جاهاش منفی است

$R=0$ τ

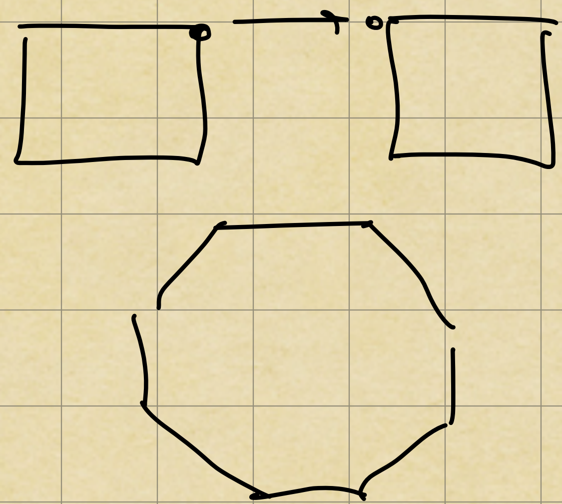
$g \geqslant 2$

$$\mathbb{H} \longrightarrow \frac{d\tau \wedge d\bar{\tau}}{\mathrm{Im}(\tau)^2}$$

Van Kampen

$S_1 \# S_2$

$\pi_1(S_1 \# S_2)$



$\#$

$X = U_1 \boxed{\sqcup} U_2$

π_1

$$\pi_k(Y) = \begin{cases} G & k = n \\ \{e\} & k \neq n \end{cases}$$

Y is called $K(G,k)$ space.

$\pi_0(X) \longleftrightarrow$ connected co

π_1

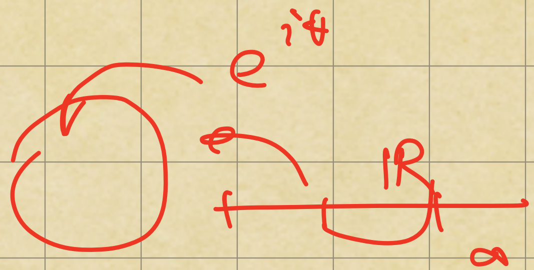
$S = K(\mathbb{Z},1) \qquad \pi_1(S) \cong \mathbb{Z}$

$\underline{\underline{\mathbb{C}P^\infty}} \longrightarrow \lim_{k \to \infty} \mathbb{C}P^k$ $\qquad \pi_2(\mathbb{C}P^\infty) = \mathbb{Z}$

Eilenberg - Maclane

Cell complex

$\underline{\underline{\pi_n(X)}}_{n \geq 2} \simeq \underline{\underline{\pi_n}}_{n\geq 2}$ (universal cover)



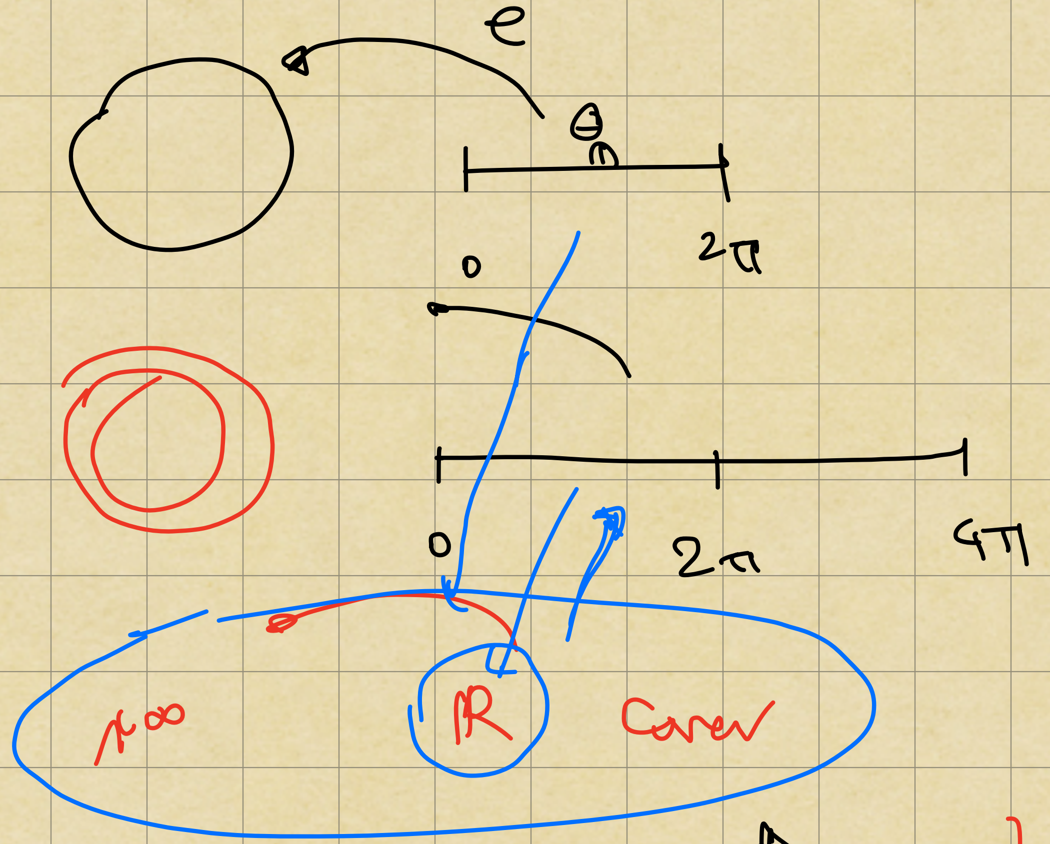
$\cancel{\pi}$

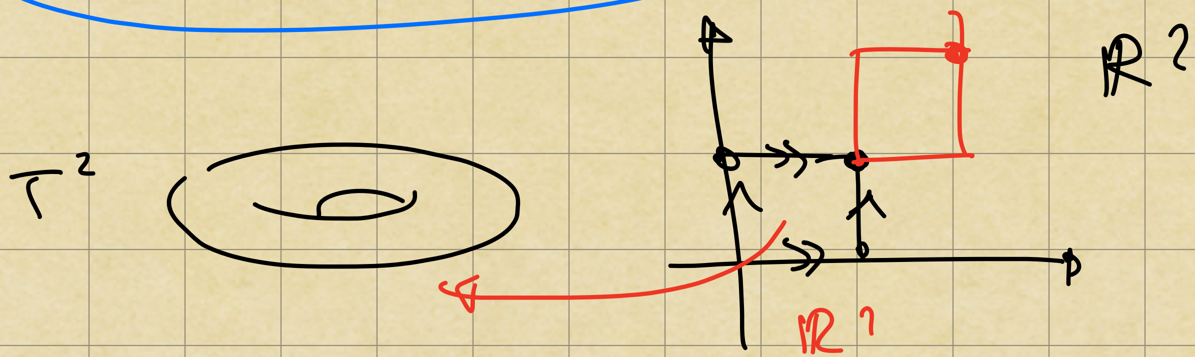
π_1

$\pi_n(\underline{\ \ }x) =$ universal

$\pi_1(X \times Y) = \pi_1(X) \oplus \pi_1(Y)$

$i\theta$





$T_{g \geq 2} \longrightarrow \mathbb{H}$

$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$

$\pi_k(\mathbb{R}P^n) = \{e\} \quad 1 < k < n$

$n = 2$