Introduction to General Relativity - HW 1 - 401208729

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Q1. Wave equatio is : $\Box\phi = 0$

Remember : $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu$

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, \underbrace{+1, +1, +1}_{\text{mostly - plus}})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\partial_t, \vec{\nabla}\right)$$

Hence : $\eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{c^2}\partial_t^2 + \nabla^2$

under Galilean tranformations (Along $\hat{x}$ WLOG since we can always rotates our axes.)

$$\begin{cases} x' = x - vt \\ y' = y, \quad z' = z \\ t' = t \end{cases} \rightarrow$$

We do a transformation by $x \rightarrow x'(x, t)$ $y'(y), z'(z)$, $t'(t)$

by chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x}$$

Note that by above transformations

$$x' = x - Vt \xrightarrow{t = t'} x = x' + Vt'$$

Hence $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x'}$

and $\dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial t'}\dfrac{\partial t'}{\partial t} + \dfrac{\partial}{\partial x'}\dfrac{\partial x'}{\partial t}$

$$= \frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'}$$

Substituting into $\Box$,

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\bigcirc\right) + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}\bigcirc\right)$$

$$= -\frac{1}{c^2}\left(\frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'}\right)$$

$$+ \sum_{i=1}^{3}\frac{\partial}{\partial x'_i}\left(\frac{\partial}{\partial x'_i}\right) =$$

$$-\frac{1}{c^2}\left\{\frac{\partial^2}{\partial (t')^2} - 2V\frac{\partial}{\partial t'}\frac{\partial}{\partial x'} + V^2\frac{\partial^2}{\partial (x')^2}\right\}\bigcirc$$

$$+ \underbrace{(\nabla'^2)}_{\text{Laplacian in new Coordinates}}\bigcirc$$

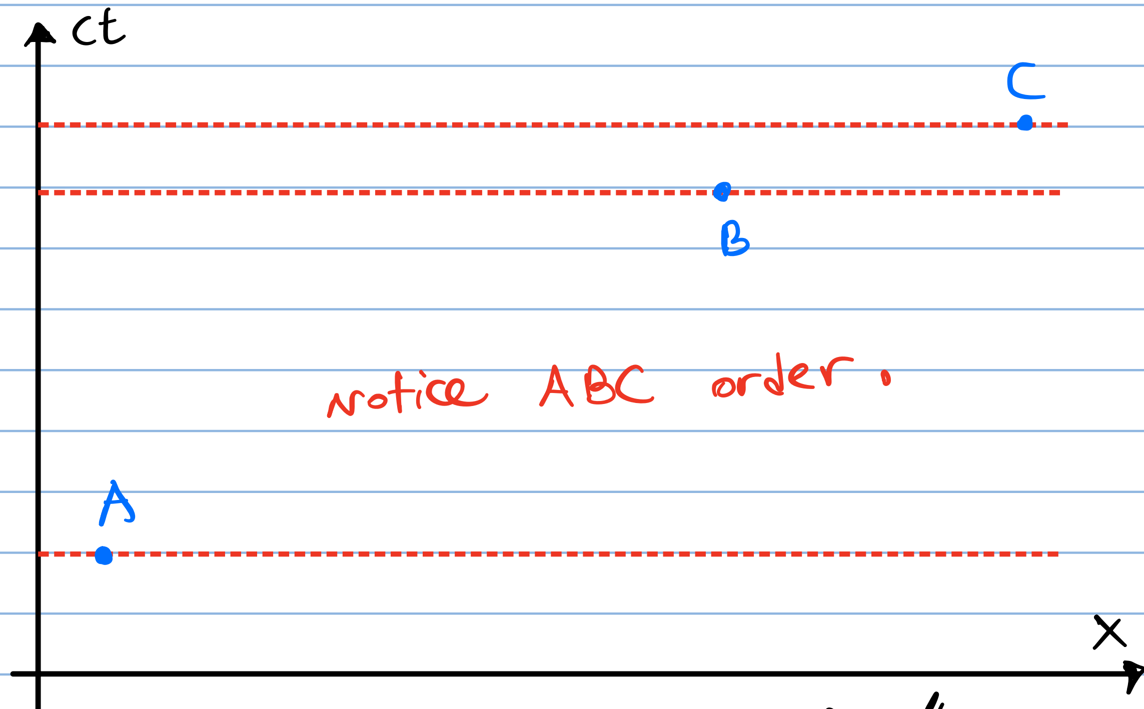
Here, the final form is (for field ψ)

$$\frac{\partial^2}{\partial x'^2}\left(1-\frac{v^2}{c^2}\right)\psi + \frac{\partial^2 \psi}{\partial y'^2} + \frac{\partial^2 \psi}{\partial z'^2}$$

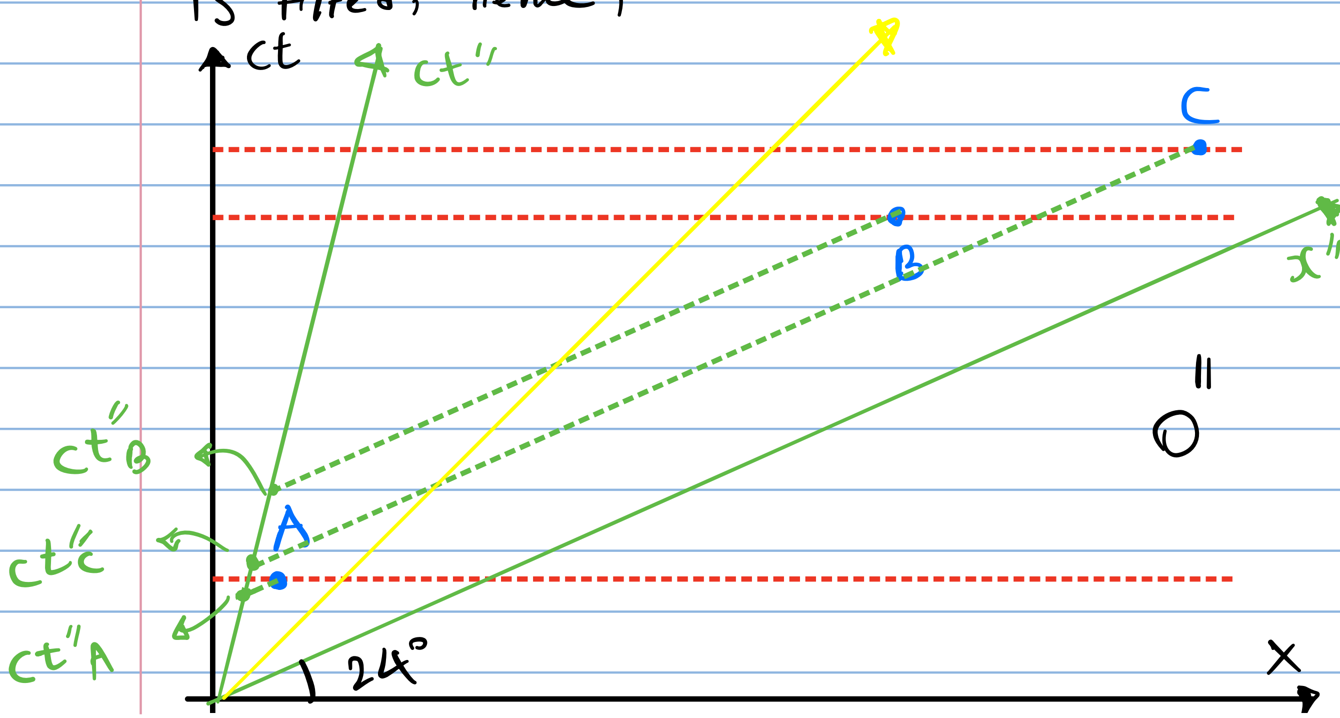
$$-\frac{2v}{c^2}\frac{\partial^2 \psi}{\partial x' \partial t'} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t'^2} = 0$$

The yellow parts are excessive, hence wave eq. isn't invariant under Galilean transformations. $\blacksquare$

Q2. As mentioned, let's draw a diagram,



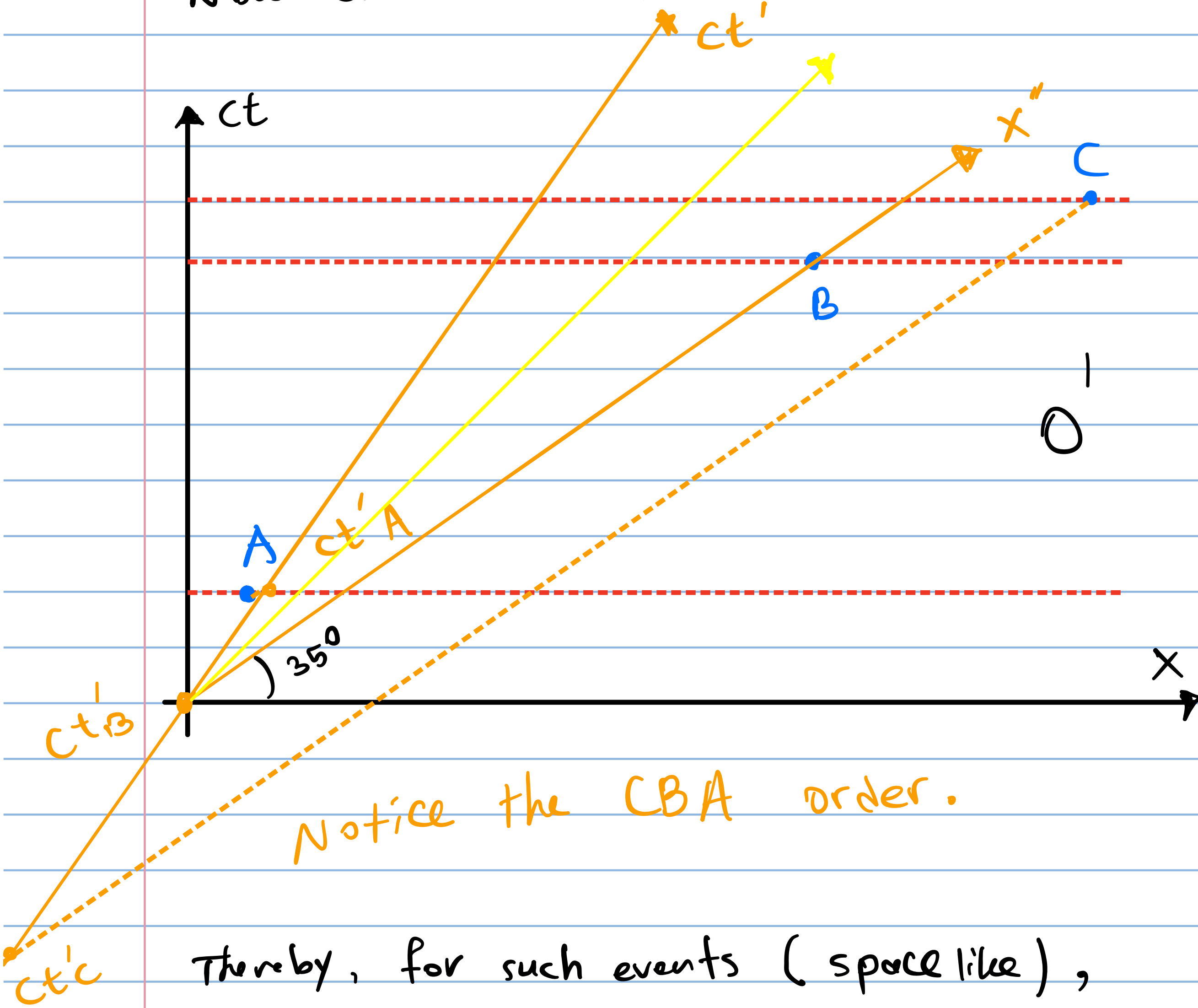
According to Carroll, the axis of O'' observer is tilted, hence,



Notice the ACB order ...

Lines parallel to x' axis have equal time.

Now consider another frame:



Notice the CBA order.

Thereby, for such events (space like), we can find such orderings.

Note: in O' frame, it may look that $ct'_c < 0$ is unacceptable, but with suitable shift in events we can find events that sastisfy such ordering.

I just draw this since my tablets portrait mode allows me.

Q3.

a) $A_\mu A^\mu < 0$

$A_\mu B^\mu = 0$

Since A_μ is time like, by a suitable Lontz Transformation we can rewrite it in the form $A_\mu = (t_A, 0, 0, 0)$.

(This is from rudimentaries of special relativity.)

In this frame inner-product becomes:

$$A_\mu B^\mu = -t_A t_B + \vec{0} \cdot (x_B, y_B, z_B)$$

0

To make $A_\mu B^\mu = 0 \implies t_B = 0$ hence B^μ is space like in this frame, and in all frames due to invarians of inner product $B^\mu B_\mu$.

BUT, we can't conclude any vector, prependicular to space like vector, is time like,

as an example:

$\alpha^{\mu} = (0, 1, 0, 0)$

$\beta^{\mu} = (0, 0, 1, 0)$

Both of which are space like and prependuh.

b) two light like vectors are parallel

$\Updownarrow$

They are prependicular to each other

$\parallel$

parallel $\Rightarrow$ prependicular ($\perp$)

$A_\mu = K B_\mu$, then $A_\mu B^\mu = K B_\mu B^\mu \rightarrow$

since B is light like hence $B_\mu B^\mu = 0 \rightarrow$

A^μ is prependicular to B.

$\parallel \Leftarrow \perp$

$A_\mu B^\mu = 0$ and $A^\mu A_\mu = B_\mu B^\mu = 0$

Go to a frame when, $A_\mu = (1, 1, 0, 0)$

(in $c=1$ units), hence $A_\mu B^\mu = (-t_B + x_B) = 0$

hence $B_\mu = (+t_B, t_B, y_B, z_B)$ in this

frame.

Recall that $B_\mu B^\mu = 0 \longrightarrow -\cancel{t^2_B} + \cancel{t^2_B} + y^2_B + z^2_B = 0$

Hence $y^2_B + z^2_B = 0 \implies y_B = z_B = 0$

Thereby $B_\mu = (+t_B, t_B, 0, 0)$

and since $A_\mu = (1, 1, 0, 0)$

So take $K = \dfrac{1}{t_B} \implies A_\mu = K B_\mu$ $\boxtimes$

c) Notice that by definition, P_μ for an observ

O is $P_\mu = m u_\mu$ where u_μ is $\dfrac{dx^\mu}{d\tau}$, its

4-velocity.

From S.R we know that $u_\mu = \gamma(c, \vec{u})$

hence $u_\mu u^\mu = -\gamma^2(c^2 - u^2) =$

$$-\frac{1}{1-u^2/c^2}(c^2-u^2) = -\frac{c^2}{\cancel{c^2-u^2}}\cancel{(c^2-u^2)}$$

$$= -c^2$$

$$\Rightarrow -u^\mu p_\mu = -m\, u^\mu u_\mu = -m(-c^2)$$

$$= mc^2$$

Actually its the relativistic energy associated with a massive particle ⊠

Q4. Under $x^\mu \to \lambda x^\mu$ (Conformal transf.)

Maxwell eq. are invariant.

$$\nabla \cdot E = 0 \qquad (\rho = 0 \text{ in vacc.})$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \frac{1}{c^2}\frac{\partial}{\partial t}E \qquad (J = 0 \text{ in vacc.})$$

easy! under $x'^\mu = \lambda x^\mu$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\mu} = \lambda \frac{\partial}{\partial x'^\mu}$$

Hence $\begin{cases} \partial_t \to \lambda \partial'_t \\ \partial_{x,y,z} \to \lambda \partial'_{x,y,z} \end{cases}$

substitute into all the Maxwell's equations:

$$0 = \nabla \cdot E = \sum_{i=1}^{3} \partial_i E_i \longrightarrow \lambda \sum_{i=1}^{3} \partial'_i E_i = 0$$

$\Rightarrow$ since $\lambda \neq 0 \rightarrow \nabla' \cdot E = 0$ ✓

$\nabla \cdot B = \nabla' \cdot B = 0$ Similarly.

$$\nabla \times E = -\frac{\partial}{\partial t} B$$

$$\epsilon^{ijk} \partial_i E_j = -\frac{\partial}{\partial t} B_k \longrightarrow \text{in new frame}$$

$$\cancel{\lambda} \epsilon^{ijk} \partial'_i E_j = -\cancel{\lambda} \frac{\partial}{\partial t'} B_k$$

$$\Longrightarrow \nabla' \times \vec{E} = -\partial'_t \vec{B}$$

Finally $\nabla \times B = \frac{1}{c^2} \frac{\partial E}{\partial t}$

$\downarrow$ goes to

$$\cancel{\lambda} \nabla' \times B = \frac{\cancel{\lambda}}{c^2} \frac{\partial E}{\partial t'}$$

So we established the invariance.