Riemannian geometry

- Riemannian manifolds
- Connections & Covariant derivative
- Geodesics
- Curvature & Ricci tensors.
- Isometries
- Conformal symmetry & CKVs.

- Homogenous Riemannian manifolds.
- Constant curvature spaces
- metrics on Lie manifolds
- operators ($\vec{\nabla}$, $\nabla\cdot$, $\nabla^2 = \Delta$, $*$-operators.)
- Affine, Killing - conformal - projective, Jacobi-Hamilton vector fields.
- First & second fundamental forms.

Must of the interesting stuff has not been covered in the classroom. So the burden of penetrating through all these topics is on your shoulders.

My Aim is to just some problems, and provide some definitions on new topics ...

Chern - Forms and Chern - classes + RR - GRR - RRH - AS

theorems are a possible direction here!

(problem 1) (on Riemannian Manifolds.)

Let (M,g) be a Riemannian n_manifold. prove that

i) Given $\alpha,\beta \in T_p^*M$, and orthonormal bases $\{e_i\}$, $i=1,\dots,n$ of T_pM,

one has $g^{-1}(\alpha,\beta) = \sum_i \alpha(e_i)\,\beta(e_i)$

(when g^{-1} denotes the Contravariant metric associated g, in physics jargon, its $g^{\mu\nu}$ rather than $g_{\mu\nu}$)

ii) For $X \in T_pM$, one has $g^{-1}(\alpha, X^\flat) = \alpha(X) = g(\alpha^\sharp, X)$

where :

flat, bemol فلت، بمل

$\flat : T_pM \longrightarrow T_p^*M \qquad X^\flat = g(X,\cdot)$

sharp, diese دیز

$\sharp : T_p^*M \longrightarrow T_pM \qquad \alpha^\sharp = g^{-1}(\alpha,\cdot)$

are musical isomorphisms.

i) In general, $g_{ij}(p)$ is a matrix in $\{e_i\}$ bases, then $g^{ij}(p) = (g_{ij}(p))^{-1}$ is a matrix w.r.t dual $\{e^j\}$ bases, base of T_p^*M.

Starting with generic symmetric metric g_{ij}, its bases might not Constitute an orthonormal frame for T_pM, e.g. let $n=3$, then g is given by $g=(g_{ij})$ {Not diagonal!}, then

$$e_1^{\,\mu}\, g_{\mu\nu}\, e_2^{\,\nu} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = g_{12} \neq 0$$

This ailment could be cured by going into bases of eigenvectors of $g_{ij} \rightarrow$ which necessarily fixes $g_{\mu\nu} = \delta_{\mu\nu}$. Hence,

$$g^{-1}(\alpha,\beta) = (g^{ij}\, e_i \otimes e_j)(\alpha_r e^r, \beta_s e^s)$$

$$= g^{ij}\, \alpha_i\, \beta_j = \delta^{ij} \alpha_i\, \beta_j = \sum_i \alpha_i \beta_i$$

$$\overset{\text{by}}{\underset{\text{def}}{=}} \sum_i \alpha(e_i)\, \beta(e_i)$$

ii) $g^{-1}(\alpha, X^{\flat}) = g^{ij}(p)\, \alpha_i\, (X^{\flat})_j$

$$= \underbrace{g^{ij}(p)}\, \alpha_i\, (\underbrace{g_{kj}(p)}_{\text{They contract}})\, X^k$$

$$= \delta^i_k\, \alpha_i\, X^k = \alpha_i X^i = \alpha(X)$$

and $g(\alpha^{\#}, X) = g_{ij}(p)\, (\alpha^{\#})^i\, X^j$

$$= g_{ij}(p)\, g^{ki}(p)\, \alpha_k\, X^j$$

$$= \delta^k_j\, \alpha_k\, X^j = \alpha_j X^j = \alpha(X) \quad \blacksquare$$

(problem 2) (on Connections & Covariant derivative)

$\forall p: \; J_p : T_pM \longrightarrow T_pM \; : J^2 = -1$

Let g be a "Hermitian metric" on an almost Complex manifold (M, J)

i.e $g(JX, JY) = g(X, Y) \quad \forall X, Y \in \mathfrak{X}(M)$

Define $(0,2)$-tensor F:

$$F(X, Y) = g(X, JY) \quad \forall X, Y \in \mathfrak{X}(M)$$

prove the following:

(i) F is skew-symmetric.

(ii) F is J-invariant : $F(JX, JY) = F(X, Y)$

suppose further that ∇ is any metric compatible connection.

(iii) $(\nabla_X F)(Y, Z) = g(Y, (\nabla_X J) Z)$

(iv) $g((\nabla_X J)Y, Z) + g(Y, (\nabla_X J)Z) = 0$

i) Easy! $F(Y,X) = g(Y,JX) = g(JY, J^2X)$
$= -g(JY, X) = -g(X, JY)$
$= -F(X,Y)$

ii) $F(JX,JY) = g(JX, J^2Y) = -g(JX, Y)$
$= -g(J^2X, JY)$
$= g(X, JY)$
$= F(X,Y)$

iii) First notice $\nabla_X \{F(Y,Z)\} = (\nabla_X F)(Y,Z) + F(\nabla_X Y, Z) + F(Y, \nabla_X Z)$

properties of action of ∇ on scalars.

Sometimes called generalize pseudo-Leibnitz rule!

equally

by Compatibility

And $\nabla_X g(Y, JZ) = \cancel{(\nabla_X g)}(Y, JZ) + g(\nabla_X Y, JZ)$
$+ g(Y, \nabla_X JZ)$

$= \underbrace{g(\nabla_X Y, JZ)}_{F(\nabla_X Y, Z)} + g(Y, (\nabla_X J)Z) + \underbrace{g(Y, J\nabla_X Z)}_{F(Y, \nabla_X Z)}$

$+ g(Y, (\nabla_X J)Z)$

Hence $(\nabla_X F)(Y,Z) = g(Y, (\nabla_X J)Z)$

(iv) since $F(X,Y)$ is skew-symmetric by (i), so is $\nabla_X F(Y,Z)$

or a direct computation gives its anti-symmetric property.

by (iii) : $g((\nabla_X J)Y, Z) \overset{\text{sym}}{=} g(Z, (\nabla_X J)Y)$

$\overset{(iii)}{=} \nabla_X F(Z, Y)$

$= -\nabla_X F(Y, Z)$

$= -g(Y, (\nabla_X J)Z)$

Hence, we're done. $g((\nabla_X J)Y, Z) + g(Y, (\nabla_X J)Z) = 0$

(problem 3) (on connection and geodesics)

Consider $\mathbb{R}^3$ with metric $ds^2 = (1+x^2)\,dx^2 + dy^2 + e^2\,dz^2$

(i) Find all Γ-symbols.

(ii) solve geodesic equation.

(iii) Consider $\gamma(t) = (x = t, y = t, z = t)$ curve.

obtain parallel transport of the vector (a, b, c) at origin along γ.

(iv) Is γ, itself, a geodesic?

(v) Find two parallel vector fields $X(t), Y(t)$ on γ, such that $g(X(t), Y(t))$ is constant.

i) $g = \begin{pmatrix} 1+x^2 & & \\ & 1 & \\ & & e^2 \end{pmatrix}$, $g^{-1} = \begin{pmatrix} \frac{1}{1+x^2} & & \\ & 1 & \\ & & e^{-2} \end{pmatrix}$

since the metric is diagonal, according to the prescription in you lecture notes ...

They involve derivative w.r.t other components!

$$\begin{cases} \Gamma^{\lambda}_{\mu\nu} = 0 \\ \Gamma^{\lambda}_{\gamma\gamma} = -\frac{1}{2} g^{\lambda\lambda} \partial_\lambda g_{\gamma\gamma} \\ \Gamma^{\lambda}_{\gamma\lambda} = \partial_\gamma \left(\ln \sqrt{|g_{\lambda\lambda}|} \right) \end{cases}$$

Repeated indices are not summed over!

$$\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda \left(\ln \sqrt{|g_{\lambda\lambda}|} \right)$$

Hence, only nonvanishing Γs are: $\begin{cases} \Gamma^{1}_{11} = \partial_1 \left(\ln (1+x^2)^{\frac{1}{2}} \right) = \dfrac{x}{1+x^2} \\ \Gamma^{3}_{33} = \partial_3 \ln \sqrt{e^z} = \dfrac{1}{2} \end{cases}$

(ii) According to geodesic equation:

$$\frac{d^2x}{dt^2} + \frac{x}{1+x^2} \left(\frac{dx}{dt} \right)^2 = 0$$

$$\frac{d^2y}{dt^2} = 0$$

$$\frac{d^2z}{dt^2} + \frac{1}{2} \left(\frac{dz}{dt} \right)^2 = 0$$

And, solutions are:

$$\frac{x''}{x'} + \frac{x x'}{1+x^2} = 0 \xrightarrow{\int} \ln x' + \frac{1}{2} \ln (1+x^2) = \ln A.$$

$$\Rightarrow x' = \frac{A}{\sqrt{1+x^2}} \longrightarrow \text{integrate} \quad dx \sqrt{1+x^2} = A\, dt$$

$$\rightarrow \frac{1}{2} \left\{ x\sqrt{1+x^2} + \log \left(x + \sqrt{1+x^2} \right) \right\} = At + B$$

and $y(t) = Ct + D$

For $z'' + \frac{1}{2}(z')^2 = 0 \longrightarrow$ let $p = z' \Rightarrow \frac{dp}{dt} + \frac{1}{2}p^2 = 0$

$$\frac{dp}{p^2} + \frac{dt}{2} = 0 \xrightarrow{\int} \frac{1}{p} = \frac{t}{2} + E/2$$

so we find : $\frac{2}{t+E} = z' \longrightarrow z = 2\log(t+E) + 2\log \underbrace{F}_{\text{ثابت}}$

$$z = \log(Ft + \underbrace{G}_{FE})^2$$

(iii) $\nabla_{\gamma'} X = 0$ is equation of parallel transporting X along γ.

$$\frac{dv^i}{dt} + \Gamma^i_{jk} \frac{dx^i(\gamma)}{dt} v^j = 0$$

so we have: $\begin{cases} \frac{dv^1}{dt} + \frac{x}{1+x^2}\frac{dx}{dt}v^1 = 0 \\ \frac{dv^2}{dt} = 0 \\ \frac{dv^3}{dt} + \frac{1}{2}\frac{dz}{dt}v^3 = 0 \end{cases}$ when $x = t$, $y = t$, $z = t$

$$\xrightarrow[\text{to}]{\text{reduces}} \begin{cases} \frac{dv^1}{dt} + \frac{t}{1+t^2}v_1 = 0 \\ \frac{dv^2}{dt} = 0 \\ \frac{dv^3}{dt} + \frac{v^3}{2} = 0 \end{cases}$$

we integrate above equations:

$v^1 \to$ $\log v^1 = \log A - \frac{1}{2}\log(1+t^2)$

with $v^1(0) = a$ initial Condition $\to$ $v^1 = \dfrac{a}{\sqrt{1+t^2}}$

$v^2 \to$ $v^2 = C$ with $v^2(0) = b$ initial condition $\to$ $v^2 = b$

$v^3 \to$ $v^3 = De^{-t/2}$ with $v^3(0) = c$ initial condition $\to$ $v^3 = ce^{-t/2}$

(iv) Does γ satisfy geodesic eq? for example:

$$\underbrace{\cancel{\frac{d^2x}{dt^2}}}_{x=t} + \frac{x}{1+x^2}\left(\cancel{\frac{dx}{dt}}\right)^2 \overset{?}{=} 0$$

zero 1

Does $\dfrac{t}{1+t^2} = 0$ for all t? ✗

so this curve is not geodesic.

(v) In (iii) we found parallel transported VFs.

$$(a,b,c) \xmapsto[\text{transport}]{\text{parallel}} V(t) = \left(\frac{a}{\sqrt{1+t^2}}, b, ce^{-t/2}\right)$$

take $X(0) = (1,0,0)$ $\to$ $X(t) = \left(\dfrac{1}{\sqrt{1+t^2}}, 0, 0\right)$

$Y(0) = (0,1,0)$ $\qquad Y(t) = (0, 1, 0)$

And $g(X(t), Y(t)) = 0$ ☒

(problem 4) (pedagogical on geodesic)

Consider $\mathbb{R}^2$ with usual flat metric, $ds^2 = dx^2 + dy^2$.
Does $\gamma(t) = (x = t^3, y = t^3)$, a geodesic?

Since $\gamma(t) = (t^3, t^3)$ like $y = x$, ... and straight lines are geodesics of the plane, ... it might be true!

$$\frac{d\gamma}{dt} = 3t^2(1,1)$$

→ geodesic equation $\frac{D}{dt}\frac{d\gamma}{dt} = 0$

BUT, $\frac{D}{dt}\frac{d\gamma}{dt} \overset{\text{flat metric}}{=} \frac{d}{dt}(3t^2\partial x + 3t^2\partial y)$

$$= 6t\,\partial x + 6t\,\partial y \neq 0 \;!$$

Hence $\gamma(t)$ is not a geodesic!

observation: geodesics have constant velocity $|\gamma'(t)|$.
but $|\gamma'(t)| = 3\sqrt{2}\,t^2$ ... it's not constant!

The fact that a curve is a geodesic depends both on its shape and its parametrization!

(problem 5) (on isometries)

Let $(\mathbb{H}, g)$ be UHP, define $SL(2,\mathbb{R})$ action on it by

$$z = x + iy, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

$$w = Mz = \frac{az+b}{cz+d}$$

prove that $SL(2,\mathbb{R})$ is group of isometries of $(\mathbb{H}, g)$

Given $z \in UHP$, $z = x+iy \longrightarrow Im(z) = y > 0$

we have $w = \dfrac{az+b}{cz+d} = \dfrac{a(x+iy)+b}{\underbrace{c(x+iy)+d}_{c.c}} \times \dfrac{(cx+d)-icy}{(cx+d)-icy}$

$$= \frac{((ax+b)+iay)(cx+d-icy)}{(cx+d)^2+(cy)^2} =$$

$$\frac{(ax+b)(cx+d)+acy^2+i\{ay(cx+d)-cy(ax+b)\}}{((cx+d)^2+(cy)^2)}$$

$+y(\cancel{ad-bc})^{1}$ since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$

$$Im\, w = \frac{\cancel{aycx} + ayd - \cancel{aycx} - cyb}{((cx+d)^2+(cy)^2)} = \frac{y}{((cx+d)^2+(cy)^2)}$$

$$\longrightarrow Im\, w = \frac{Im\, z}{|cz+d|^2}$$ (takes real line to itself.)

Thus $w \in$ UHP under $SL(2,\mathbb{R})$ linear fractional transformation.

Let us check the metric:

$$g = \frac{1}{y^2}(dx^2 + dy^2) = \frac{dz \wedge d\bar{z}}{\mathrm{Im}^2(z)} = \frac{1}{2}\,\frac{dz \otimes d\bar{z} + d\bar{z} \otimes dz}{\mathrm{Im}^2(z)}$$

On the other hand:

$$dw = \frac{\overset{1}{\cancel{ad-bc}}}{(cz+d)^2}\, dz$$

$$d\bar{w} = \frac{1}{(c\bar{z}+d)^2}\, d\bar{z}$$

So under $z \longrightarrow w = Mz$

$$g \longmapsto \frac{1}{2}\,\frac{\cancel{(cz+d)^2}\, dw \wedge \cancel{(c\bar{z}+d)^2}\, d\bar{w}}{\underbrace{|cz+d|^4}_{\cancel{(cz+d)^2}\ \underbrace{(\overline{cz+d})^2}_{\cancel{(c\bar{z}+d)^2}}}\ \mathrm{Im}^2(w)} = \frac{dw \wedge d\bar{w}}{\mathrm{Im}^2(w)}$$

So it leaves the metric invariant $\Longrightarrow$ $SL(2,\mathbb{R})$ is indeed the isometry group of $\mathbb{H}$ with Poincaré metric.

(problem 6) (Constant curvature spaces)

Related topic: the Riemannian manifold which posseses the maximum number of Killing vectors, is called maximally symmetric space.

By a simple counting argument:

- R is constant.
- $R_{\mu\nu} = \frac{1}{n} R\, g_{\mu\nu}$
- $R_{\mu\nu\lambda\rho} = \frac{R}{n(n-1)} \left(g_{\mu\lambda} g_{\nu\rho} - g_{\nu\lambda} g_{\mu\rho} \right)$

In Lorentzian signature:

$$\text{MSS} \begin{cases} R>0 \longrightarrow dS \\ R=0 \longrightarrow \text{Mink.} \\ R<0 \longrightarrow AdS \end{cases}$$

In Euclidean signature:

$$\text{MSS} \begin{cases} R>0 \longrightarrow \text{sphere} \\ R=0 \longrightarrow \text{Euclidean } \mathbb{R}^n \\ R<0 \longrightarrow \text{saddle or hyperbolic manifolds.} \end{cases}$$

Let (M,g) be a Riemannian manifold of constant curvature.
we define metric $\tilde{g}$ on $M \times M$ by: $(X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M))$

$$\tilde{g}\left((X_1,Y_1), (X_2,Y_2) \right) = g(X_1,X_2) \circ pr_1 + g(Y_1,Y_2) \circ pr_2$$

we say that metric decomposes to $\tilde{g}_{M \times M} = g \oplus g = \begin{pmatrix} g_m & \\ & g_m \end{pmatrix}$

Is $(M \times M)$ a constant curvature space?!

Let $(p,q) \in M \times M$. Then are charts along with coordinates in M

such that $\begin{cases} p \in U \text{ with coordinates } x^1, \dots, x^n \end{cases}$

$\lfloor q \in V$ " " $x^{n+1}, \dots, x^{2n}$

Hence $U \times V$ is a chart on $M \times M$ with $(x^1, \dots, x^{2n})$ Coordinates!

Metric $\tilde{g}$ is : $\tilde{g}_{AB}(x,y) = \begin{pmatrix} g_{ij}(x) & \\ & g_{i+n,j+n}(y) \end{pmatrix}_{n \times n}$

$A, B \in \{1, \dots, 2n\}$

christoffel symbols :

$$\tilde{\Gamma}^{A}_{BC} = \frac{1}{2} \tilde{g}^{AD} \left(\partial_C \tilde{g}_{DB} + \partial_B \tilde{g}_{DC} - \partial_D \tilde{g}_{BC} \right)$$

Since $\tilde{g}$ is block diagonal $\longrightarrow$ only $\tilde{\Gamma}^{i}_{jk}(x,y) = \Gamma^{i}_{jk}(x)$

$\tilde{\Gamma}^{i+n}_{j+n, k+n}(x,y) = \Gamma^{i}_{jk}(y)$

with $i, j, k \in \{1, 2, \dots, n\}$ survives!

Similarly $\tilde{R}^{i}_{jkl}(x,y) = R^{i}_{jkl}(x)$ $\tilde{R}^{i+n}_{j+n, k+n, l+n}(x,y) = R^{i}_{jkl}(y)$

Hence $\tilde{R}$ is not zero.

But if $M \times M$ is going to be a maximally symmetric spaces

then $\tilde{R}^{i}_{jkl} = \tilde{K}(g^{i}{}_{k} \, g_{jl} - g^{i}{}_{l} \, g_{jk})$

$= \tilde{K}(\delta^{i}_{k} g_{jl} - \delta^{i}_{l} g_{jk})$

Note : $g^{\alpha\beta} g_{\beta}{}^{s} = g^{\alpha s} \longrightarrow (\quad)_{n \times n} (\overset{!!}{?}) = (\quad)$ itself by there ruls!

In particular $\rightarrow$ $\tilde{R}^{k+n}_{j, k+n, j} = \tilde{K}(g_{jj}) \equiv 0$

$\Rightarrow \tilde{K} = 0$!

so the product manifold can't be nothing except flat (if possible!)

(problem 7) (on metrics on Lie manifolds!)

Let H be the Heisenberg group $\rightarrow$ $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$, $(x,y,z) \in \mathbb{R}^3$

(i) Find left-invariant metric g on H, with bases dual to left-invariant VFs!

(ii) Find ∇-Connection of g.

(iii) Is (H,g) space of Constant curvature ?!

(i) The basis of left-invariant 1-forms on H, gives us the metric! Just like (dx, dy, dz) in Euclidean space $ds^2 = dx^2 + dy^2 + dz^2$! why it's true? It's had come Lie algebra! and It only works for simple -semi-simple lie algebras! (those without proper ideals!)

we've had left-invariant VFs $\rightarrow$ $\mathcal{B} = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\}$

By a simple linear Combination of $\beta_i = A_i\, dx + B_i\, dy + C_i\, dz$

and applying β_i on Left-inv VFs $\longrightarrow$

$\beta_1 = dx$ $\longrightarrow$ $\beta_1(\partial x) = 1$ and on ∂_y, $x\partial_y + \partial_z = 0$

$\beta_2 = dy - x\,dz$ " " "

$\beta_3 = dz$ " " "

$$\Longrightarrow ds^2 = dx^2 + (dy - x\,dz)^2 + dz^2$$

(ii) Only by hand, the ∇_X could be found or Koszul identity (not metrics needed) ...

$$2g(\nabla_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)$$

we only need to know $\nabla_{X_i} X_j$.

The only non-zero brackets are $[X_1, X_3] = -[X_3, X_1] = X_2$

$$\Longrightarrow g(\nabla_{X_1} X_1, X_i) = 0 \xrightarrow[\text{no degenerate}]{\text{metric's}} \nabla_{X_1} X_1 = 0$$

similarly $\longrightarrow$ $\nabla_{X_1} X_2 = -\frac{X_3}{2}$ and $\nabla_{X_2} X_1 = +\frac{X_3}{2}$

by torsionless connection!

$$\nabla_{X_1} X_3 = -\nabla_{X_3} X_1 = \frac{1}{2} X_2$$

$$\nabla_{X_2} X_3 = -\nabla_{X_3} X_2 = -\frac{1}{2} X_1$$

(iii) Direct computation of curvature shows it's not constant!

It's sectional curvature also isn't constant along any cross-section.

sectional curvature on $K = \langle X_1|_p, X_2|_p \rangle$ plane.

$$K(p) = R(X_1, X_2, X_1, X_2)(p)$$

$$= g\Big(\nabla_{X_1} \cancel{\nabla_{X_2} X_2} - \underbrace{\nabla_{X_2}\underbrace{\nabla_{X_1} X_2}_{-\frac{1}{2}X_3}}_{=\frac{1}{2}(\frac{1}{2})X_1 = \frac{1}{4}X_1} - \nabla_{[X_1, X_2]} X_2, X_1\Big)(p)$$

$$= g\left(\tfrac{1}{4}X', X'\right)(p) = \tfrac{1}{4}$$

take $K' = \langle X_1|_p, X_3|_p \rangle$

$$K(p') = \ldots = -\tfrac{1}{4}$$

(problem 8) (on Operators on Riemannian manifolds)

Remember musical isomorphisms:

$$\flat : T_pM \longrightarrow T_p^*M \qquad X^\flat(Y) = g(X, Y)$$
$$X \longmapsto X^\flat$$

$$\# : T_p^*M \longrightarrow T_pM \qquad \omega^\#(\xi) = g^{-1}(\omega, \xi)$$
$$\omega \longrightarrow \omega^\#$$

Define grad of f as : $\operatorname{grad} f = (df)^\#$

Prove :

(i) $g(\operatorname{grad} f, X) = X(f)$

(ii) $(\partial/\partial x^i)^\flat$

(iii) $(dx^i)^\#$

(iv) write grad f in local Coordinates.

(v) Recover the usual grad in $\mathbb{R}^3$,

$\flat : TM \longrightarrow T^*M$

$X \longmapsto g_{ij} X^i e^j = X_j e^j$

$\# : T^*M \longrightarrow TM$

$\omega \longmapsto g^{ij} \omega_i e_j = \omega^j e_j$

so that $(X^\flat)^\# : X^\flat \xmapsto{\#} (X^\flat)^j e_j$

But $(X^\flat)^j$ by $= X^j$

$\Rightarrow (X^\flat)^\# = X^j e_j = X$!

$\#, \flat$ are inverse of eachother!

(i) $g(\operatorname{grad} f, X) = \underbrace{g((df)^\#, X)}_{\text{by def of } \flat} = ((df)^\#)^\flat (X)$

$= df(X) = X[f]$

☑

(ii) Def: $\left(\frac{\partial}{\partial x^i}\right)^\flat \left(\frac{\partial}{\partial x_j}\right) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$

$= g_{ik}\, dx^k \left(\frac{\partial}{\partial x^j}\right)$

$\longrightarrow \left(\frac{\partial}{\partial x^i}\right)^b = g_{ik}\, dx^k$

(iii) Take # from ii) $\longrightarrow \frac{\partial}{\partial x^i} = g_{ik}\, (dx^k)^\#$

$$\Longrightarrow (dx^j)^\# = g^{ji} \frac{\partial}{\partial x^i}$$

(iv) $\text{grad } f = (df)^\# = \left(\frac{\partial f}{\partial x^i}\, dx^i\right)^\#$

$$= \frac{\partial f}{\partial x^i}\, g^{ji} \frac{\partial}{\partial x^j}$$

$$= g^{ji} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

(v) In $\mathbb{R}^3$ $g^{ij} = 1 \longrightarrow \text{grad } f = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$ ✓

(problem 9) (operator gymnastics)

Consider $\mathbb{R}^n$ with Euclidean metric. and $\omega = dx^1 \wedge \dots \wedge dx^n$

(i) prove that given $\Omega_k \in \Lambda^k(\mathbb{R}^n)$ there's only one $\star\Omega_k$ of degree $n-k$:

$$\star\Omega_k(X_1, \dots, X_{n-k})\,\omega = \Omega_k \wedge X_1^b \wedge \dots \wedge X_{n-k}^b$$

(ii) Hodge star operator is defined by previous formula

$$\star : \Lambda^k \mathbb{R}^n \longmapsto \Lambda^{n-k} \mathbb{R}^n$$

prove: $\star^2 = (-1)^{k(n-k)}$

$\star^{-1} = (-1)^{k(n-k)} \star$

$\Omega_k \wedge (\star \Theta_k) = \Theta_k \wedge (\star \Omega_k)$

(iii) The Codifferential opertor $\delta : \Lambda^k \mathbb{R}^n \longrightarrow \Lambda^{k-1} \mathbb{R}^n$ is defined:

$$\delta = (-1)^{n(k+1)+1} \star d \star$$

prove that $\delta^2 = 0 \longrightarrow$ it establishes a cohomological complex.

(iv) $\Delta : \Lambda^k \mathbb{R}^n \longmapsto \Lambda^k \mathbb{R}^n$ is defined to be:

$$\Delta = (d+\delta)^2 = d\delta + \delta d$$

prove $\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial (x^i)^2}$.

We've only to prove them to a basis of ext. algebra.

Let $\{X^1, \dots, X^n\}$ be an orthonormal bases of VFs on $\mathbb{R}^n$ and $\{\theta^1, \dots, \theta^n\}$ the dual bases.

And also the $i_1 < \dots < i_k$, $j_1 < \dots < j_{n-k}$ orderd indices!

(i) $\star(\theta^{i_1} \wedge \dots \wedge \theta^{i_k})(x_{j_1}, \dots, x_{j_{n-k}})\omega \overset{\text{by def}}{=}$

$(X_{j_1})^\flat = \theta^{j_1}$

$= \theta^{i_1} \wedge \dots \wedge \theta^{i_k} \wedge \theta^{j_1} \wedge \dots \wedge \theta^{j_{n-k}}$

First of all, this vanishes if i, j indicis are not complement.

so the above equation gives:

$$\boxed{*(\theta^{i_1} \wedge \dots \wedge \theta^{i_k}) = \operatorname{sgn}(i_1, \dots, i_k, j_1, \dots, j_{n-k})\, \theta^{j_1} \wedge \dots \wedge \theta^{j_{n-k}}}$$

so we deduce that

$$\{*(\theta^{i_1} \wedge \dots \wedge \theta^{i_k})\}(X_{j_1}, \dots, X_{j_{n-k}})\, \omega =$$

$$\operatorname{sgn}(i^1, \dots, i^k, j_1, \dots, j_{n-k})\, \omega$$

Since action of bases speify a form uniqly, therefore the correspondense is 1-1. $\square$

(ii) From $\boxed{}$ above we take an the $*$.

$$*\{*(\theta^{i_1} \wedge \dots \wedge \theta^{i_k})\} = \operatorname{sgn}(i_1, \dots, i_k, j_1, \dots, j_{n-k}) \times$$

$$*\{\theta^{j_1} \wedge \dots \wedge \theta^{j_{n-k}})$$

$$= \underbrace{\operatorname{sgn}(i_1, \dots, i_k, j_1, \dots, j_{n-k})\, \operatorname{sgn}(j_1, j_2, \dots, j_{n-k}, i_1, \dots, i_k)}_{}\ \theta^{i_1} \wedge \dots \wedge \theta^{i_k}$$

work this out whn both n, k are even/odd / eo / oe

$$(-1)^{k(n-k)}$$

$$\Longrightarrow *^2 = (-1)^{k(n-k)}$$

From $\uparrow$ results $(\times *^{-1}) \to * = (-1)^{k(n-k)} *^{-1}$

$\rightarrow \quad *^{-1} = (-1)^{k(n-k)} * .$

Consider $\Omega_k = \theta^{i_1} \wedge \dots \wedge \theta^{i_k}$

$\Theta = \theta^{j_1} \wedge \dots \wedge \theta^{j_{n-k}}$

$$\rightarrow \Omega_k \wedge (*\Theta)_k = \begin{cases} 0 & \text{if } \{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\} \\ \omega & \text{if } \quad " \quad = \quad " . \end{cases}$$

similarly $$\Theta_k \wedge (*\Omega_k) = \begin{cases} 0 & \text{if } \{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\} \\ \omega & \text{if } \quad " \quad = \quad " . \end{cases}$$

$$\Longrightarrow \Omega_k \wedge (*\Theta)_k = \Theta_k \wedge \Omega_k$$

(iii) $\delta^2 = * d \underbrace{* *}_{*^2} d \, * \qquad \leftarrow \Omega_k \qquad \rightarrow *^2 = (-1)^{(n-k+1)(k-1)}$

$(n-k+1) \leftarrow (n-k)$

$= (-1)^{(n-k+1)(k-1)} * \underbrace{d^2}_{=0} * = 0$ ∎

(iv) Note that $\delta f = 0$ since $*f \propto \omega \rightarrow$

$d * f = d\omega = 0$

$\Longrightarrow \Delta f = (d\delta + \delta d) = \delta d f$

$= - * d * d f$

$$= -\star d \star \frac{\partial f}{\partial x^i} dx^i$$

$$= -\star d \left\{ (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n \right\}$$

$$= -\star (-1)^{i-1} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

$$(-1)^{j-1} \propto (j-1)$$

$$= -\star \left(\sum_{j=1}^{n} \frac{\partial^2 f}{(\partial x^j)^2} \times dx^1 \wedge \dots \wedge dx^n \right)$$

$$= -\sum_{j=1}^{n} \frac{\partial^2 f}{(\partial x^j)^2} \star w = 1$$

$$= -\sum_{j=1}^{n} \frac{\partial^2 f}{\partial (x^j)^2}$$

(problem 10) (Def of Killing KVF)

If $X \in \mathfrak{X}(M)$ is KVF iff $\mathcal{L}_X g = 0$

our starting def is that X is KVF if g is invariant under pull back along its integral curve!

$\rightarrow$ so if ϕ is integral curve of $X \to \phi_t^* g = g$

By def $$\mathcal{L}_X g = \lim_{t \to 0} \frac{\overset{=g}{\phi_t^* g} - g}{t}\Big|_{t=0} = 0$$ ✓

$\leftarrow$ (look at previous notes ...)

since $\mathcal{L}_X g = 0$, For any tensor field K:

$$\phi_s^* (\mathcal{L}_X K) = -\left(\frac{d}{dt}(\phi_t^* K)\right)_{t=s}$$

Hence, by hypothesis $\to$ $0 = \phi_s^* (\underbrace{\mathcal{L}_X g}_{\text{zero}}) = -\left(\frac{d}{dt}(\phi_t^* g)\right)\Big|_{t=s}$

$\to$ $\phi_t^* g$ doesn't depend on t $\to$ $\phi_t^* g$

$= \phi_0^* g = g$

∎

(problem 11) ($\mathbb{R}^3$'s KVFs)

The real Lie algebra generated by

$$\langle \partial_x, \partial_y, \partial_z, x\partial_y - y\partial_x, -y\partial_z + z\partial_y, z\partial_x - x\partial_z \rangle$$

are KVFs of $\mathbb{R}^3$.

let $X = \lambda^i \frac{\partial}{\partial x^i}$ be a KVF $\to$ $L_X g = 0$ $\to$

$$L_X g = \sum_{i,j} \left(\frac{\partial \lambda^j}{\partial x^i} + \frac{\partial \lambda^i}{\partial x^j} \right) dx^i \otimes dx^j$$

we deduce $\to$
$$\begin{cases} \dfrac{\partial \lambda^1}{\partial x^1} = \dfrac{\partial \lambda^2}{\partial x^2} = \dfrac{\partial \lambda^3}{\partial x^3} = 0 \\[2ex] \dfrac{\partial \lambda^1}{\partial x^2} + \dfrac{\partial \lambda^2}{\partial x^1} = \dfrac{\partial \lambda^1}{\partial x^3} + \dfrac{\partial \lambda^3}{\partial x^1} = \dfrac{\partial \lambda^2}{\partial x^3} + \dfrac{\partial \lambda^3}{\partial x^2} = 0 \end{cases}$$

so $\lambda_1(y,z)$, $\lambda_2(x,z)$, $\lambda_3(x,y)$ from first eqs.

The last line ∂_2 (first) & ∂_3 (second)

$$\frac{\partial^2 \lambda^1}{\partial (x^2)^2} = \frac{\partial^2 \lambda^1}{\partial (x^3)^2} = 0$$

$$\Longrightarrow \lambda^1 = A_1 yz + B_1 y + C_1 z + D_1$$

simlry $\longrightarrow$

$$\lambda^2 = A_2 xz + B_2 z + C_2 x + D_2$$

$$\lambda^3 = A_3 xy + B_3 x + C_3 y + D_3$$

plug into last equtions $\longrightarrow$

$$\lambda^1 = -C_2 y + C_1 z + D_1$$

$$\lambda^2 = -C_3 z + C_2 x + D_2$$

$$\lambda^3 = -C_1 x + C_3 y + D_3$$

This gives the genrtors in the statement.

By using $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ one can check that

All these linear combinations actully form

a Lie algebra.

There are a lot of VFs on manifolds.

X is a projective VF if

$$(\mathcal{L}_X \nabla)(Y,Z) = [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X,Z]$$

is of the form $\theta(Y) Z + \theta(Z) Y$

for some $\theta \in \Lambda^1(M)$!

CKVs are easier $\longrightarrow$ X is CKV if $\mathcal{L}_X g = f g$

They preserve the Conformal class of metric on the M.

X is Harmonic VF if $X^\flat$ is ker of Laplace

$\Delta X^\flat = 0$

but how Δ acts on forms?, $\Delta = \cancel{\nabla_r \nabla^r}$

such question are in the realm of Harmonic analysis.

Hamiltonian VFs are defined via symplectic structure on M.

Asymm. non-deg form ω

like Cot. bundle.

$$dH(Y) = \omega(\underbrace{X_H}_{}, Y)$$

Hamiltonian VF of X.

(problem 12) (On Stokes Theorem)

Let M be a Compact Riemannian manifold.

(i) prove that $\delta \propto \star d \star$ is adjoint of the "d"-operator w.r.t. inner product of integration.

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle \qquad \alpha, \beta \in \Lambda^r(M)$$

and vice-versa

(ii) $\Delta = d\delta + \delta d$ is self-adjoint w.r.t inner product of integration

$$\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$$

(i) def of inner product of integration

$$\langle \alpha, \beta \rangle = \int_M (\alpha \wedge \star\beta) \qquad \{\text{on Compact } M\}$$

why on Compact manifold M.

$$0 = \int_{\partial M} (\alpha \wedge \star\beta) = \int_M d(\alpha \wedge \star\beta)$$

$$= \int_M d\alpha \wedge \star\beta + (-1)^r \alpha \wedge d\star\beta$$

$$\delta = (-1)^{n(n-r)+1} \star d \star$$

$$\underbrace{\star^{-1}}\delta = (-1)^{n(n-r+r)+1} d\star$$

$$(-1)^{r(n-r)} \star\delta = (-1)^{\#} d\star$$

$$= \int_M d\alpha \wedge \star\beta - \int_M \alpha \wedge \star\delta\beta$$

$$\Rightarrow \langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$$

(ii) $\langle \Delta\alpha, \beta \rangle = \langle d\delta\alpha + \delta d\alpha, \beta \rangle$

$= \langle d\delta\alpha, \beta \rangle + \langle \delta d\alpha, \beta \rangle$

problem (i) $\downarrow$

$= \langle \delta\alpha, \delta\beta \rangle + \langle d\alpha, d\beta \rangle$

$= \langle \alpha, d\delta\beta \rangle + \langle \alpha, \delta d\beta \rangle$

$= \langle \alpha, (d\delta + \delta d)\beta \rangle = \langle \alpha, \Delta\beta \rangle$ ✓