- Matrix groups.
- Action of grap on sets (Adv Concept.)

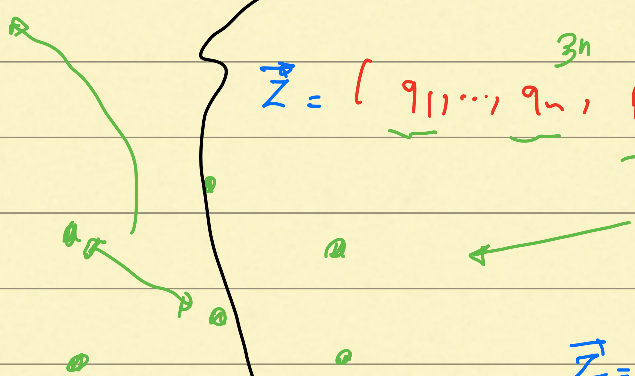
  × symplectic groups. $Sp(2n, F)$

  × Group action from new perspective, $Aut(S_n)$.

Hamiltonian Dynamics :

$H(q^i, p^i)$ $\qquad \dfrac{\partial H}{\partial p_i} = +\dot{q}_i \qquad \dfrac{\partial H}{\partial q_i} = -\dot{p}_i$

$$\vec{z} = (\underbrace{q_1, \dots, q_n}, \underbrace{p_1, \dots, p_n})^T (t) \quad 3n$$



$$\vec{z} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\vec{q}} \\ \dot{\vec{p}} \end{pmatrix} = \dot{\vec{z}} = \underbrace{\left(\begin{array}{c|c} 0 & \mathbb{1}_{n\times n} \\ \hline -\mathbb{1} & 0 \end{array}\right)}_{J} \begin{pmatrix} \frac{\partial H}{\partial \vec{q}} \\ \frac{\partial H}{\partial \vec{p}} \end{pmatrix}$$

$\left(\dfrac{\partial H}{\partial q^1}, \dots, \dfrac{\partial H}{\partial q^n}\right)$ similarly

$$\dot{\vec{z}} = J_{2n\times 2n} \times \left(\frac{\partial H}{\partial \vec{z}}\right)_{2n\times 1}$$

$$\{f, g\}^{H} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

$$\frac{df}{dt} = \dot{f} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t} \qquad \left\{ \begin{array}{l} \dot{x}_i = \{x_i, H\} \\ \dot{p}_i = \{p_i, H\} \end{array} \right.$$

تبدیلاتی که فرم معادله دینامیکی رو عوض نکنند.

$$\tilde{z} = \phi(z)$$

$$(\tilde{z})_{2n\times 1} = \phi_{(2n\times \cancel{2n})} (z)_{\cancel{2n\times 1}}$$

$$\frac{\partial \phi_i}{\partial z_k}$$

$$\dot{\tilde{z}}_i = \left(\frac{\partial \phi}{\partial z}\right)_{ik} (\dot{z})_k$$

$$\left(\frac{\partial \phi}{\partial z}\right)_{ik} \times (J)_{km} \underbrace{\left(\frac{\partial H(z)}{\partial z_m}\right)}_{\text{chain-rule}}$$

$$\tilde{z} = \phi(z)$$

$$\frac{\partial H}{\partial \tilde{z}_n}\left(\frac{\partial \tilde{z}_n}{\partial z_m}\right) = \frac{\partial H}{\partial \tilde{z}_n} \frac{\partial \phi_n}{\partial z_m}$$

$$= \frac{\partial H}{\partial \tilde{z}_n}\left(\frac{\partial \phi}{\partial z}\right)_{nm}$$

$$= \left(\frac{\partial \phi}{\partial z}\right)_{ik} J_{km} \left(\frac{\partial H}{\partial \tilde{z}_n}\right) \underbrace{\left(\frac{\partial \phi}{\partial z}\right)_{nm}}_{\left(\frac{\partial \phi^T}{\partial z}\right)_{mn}}$$

$$\dot{\tilde{z}} = \underbrace{\left(\left(\frac{\partial \phi}{\partial z}\right)_{ik} J_{km} \left(\frac{\partial \phi^T}{\partial z}\right)_{mn} \right)}_{M \quad J \quad M^T} \underbrace{\left(\frac{\partial H}{\partial \tilde{z}_n}\right)}$$

$$MJM^T = J$$

$$J = \left(\begin{array}{c|c} 0 & \mathbb{1}_{n\times m} \\ \hline -\mathbb{1}_{n\times n} & 0 \end{array} \right)_{2n\times 2n}$$

$2n\times 2n$

$\tilde{z} \to Mz \to$ Dynamic remains Hamiltonian.

$$Sp(2n, F) = \{ M \in M_{2n\times 2n}(F) \mid MJM^T = J \}$$

$\mathbb{R}, \mathbb{C}$

$$Z(Sp(2n, \mathbb{C})) = \{\pm I_{2n}\}$$

$S \in Sp(2n, \mathbb{R})$: $S = I + A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$

$S^T J S = J \to$

$$\begin{pmatrix} I + X^T & Z^T \\ Y^T & I + W^T \end{pmatrix} \begin{pmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{pmatrix} \underbrace{\begin{pmatrix} I + X & Y \\ Z & I + W \end{pmatrix}}_{\begin{pmatrix} Z_{n\times n} & I + W \\ -(I+X) & -Y \end{pmatrix}}$$

$$J = \begin{pmatrix} \underline{(I+X^T)Z - Z^T(I+X)} & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

$X \to n^2$

$$\begin{cases} Z^T = Z \\ X^T = X \end{cases}$$

Y, ω

$$\begin{cases} \omega = -X^t \\ Y^t = Y \to \frac{n(n+1)}{2} \\ Z^t = Z \end{cases}$$

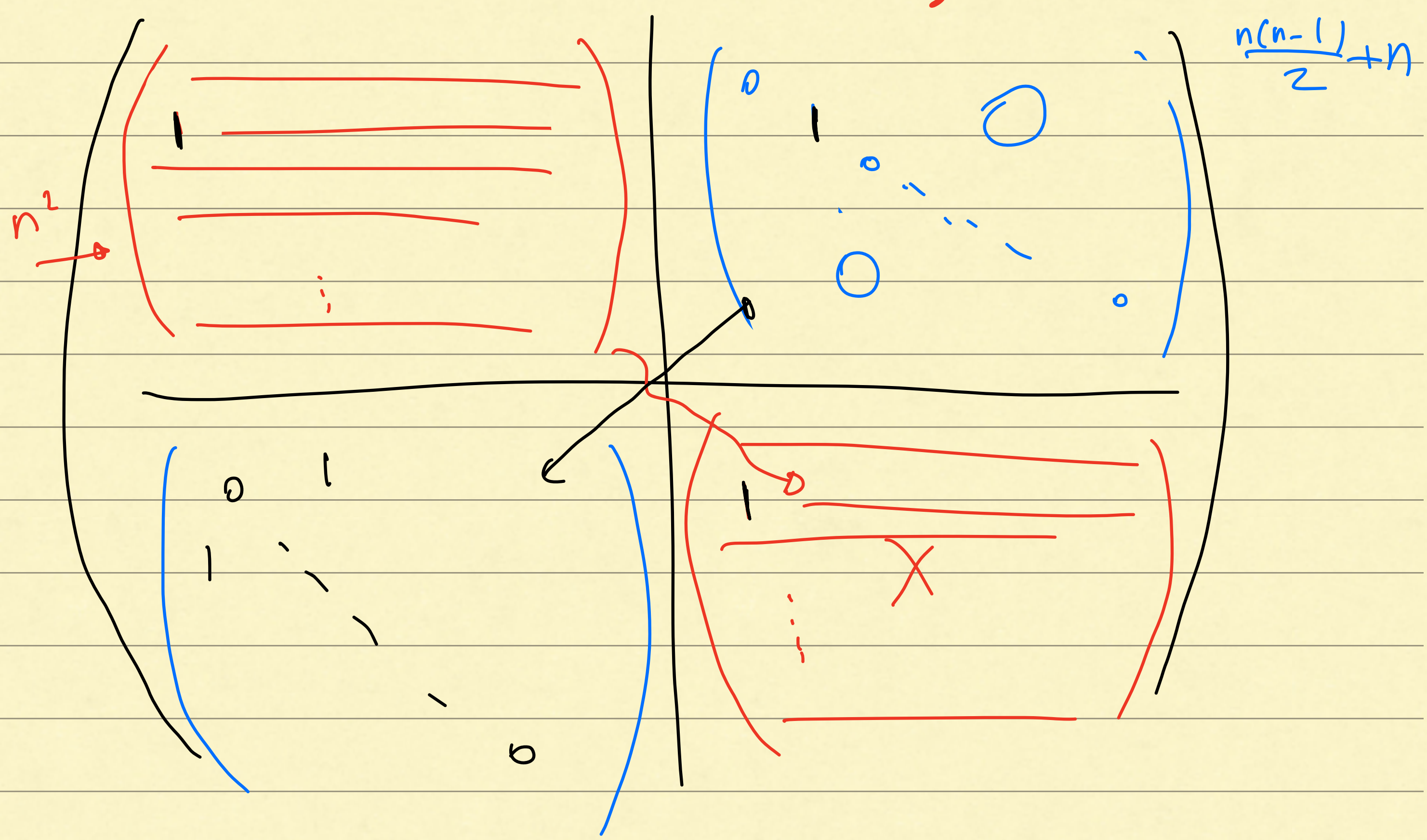
$\to \frac{n(n+1)}{2}$

$n^2 + n(n+1) = 2n^2 + n = n(2n+1)$

$Sp(2n, \mathbb{R})$ هم بعدش با $n(2n+1)$ پارامتر مشخص می‌شود.

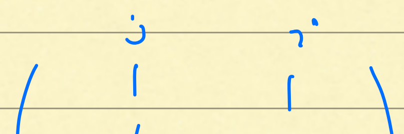
پایه‌ها رو مشخص کنیم:

n^2 matrices $\to$ bases lie algebra.



پایه‌های جبلی:
$sp(2n, \mathbb{R}) \to \boxed{n(2n+1)}$

$$\boxed{E_{ij} = |i\rangle\langle j| - |j\rangle\langle i|}$$



$$E_{ij} = \begin{array}{c} i \\ \\ j \end{array} \begin{pmatrix} & & \vdots & & \\ \cdots & \cdots & \cdots & & 1 \\ & & & & \\ \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

$$\hat{\mathcal{I}} = (\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n)$$

$$\underline{\underline{[\hat{z}_m, \hat{z}^T_n]}} = i\hbar (\mathcal{J})_{mn}$$

properties: $\mathbb{R} \to$ non_Compact $\mathbb{R}$ $(0,1)$ $(1,2)$.

$Sp(2n, \mathbb{R})$: غیرفشرده / همبند / / simple lie Alge

$Sp(2n, \mathbb{C})$: غیرفشرده / همبند / / ...

$$Sp(n) : \underbrace{Sp(2n, \mathbb{C})} \cap \underbrace{U(2n)}_{\text{Comp}} = \checkmark \text{ گروه لی}$$

ارتباط اثر گروه با خودریختی های S_n.

Look at action of G on S as homomorphism to $\underbrace{sym(S)}$

$S = \{ \triangle, \square, \circ, ▱ \}$

$sym(S) = \left\{ \begin{pmatrix} \triangle & \square & \circ & ▱ \\ \circ & \triangle & ▱ & \square \end{pmatrix}, \right.$

$24 = 4! = |Sym(S)|$

$|S| = n$

$\Phi : G \longrightarrow Sym(S) \cong S_n$

$\Phi(g) : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, $\begin{pmatrix} \triangle & \square & \circ & \diamond \\ \circ & \triangle & \diamond & \square \end{pmatrix}$

$g . S$

$\Phi(g)\triangle = \bigcirc$

$\Phi(g)\odot = \diamond$

Definition: Action faithful if ker of homomorphism is trivial

$\forall g,h \in G \quad \Phi(g) = \Phi(h) \Longrightarrow g = h$

سؤال : گروه S_n به چند طریق می‌تواند روی یک مجموعه‌ی $S(|S|=n)$ اثر کند؟

$\sigma . \psi : \underbrace{\sigma \psi \sigma^{-1}}$ $\longleftrightarrow$ اثر مزدوج $\leftarrow$ راه ساده $\underbrace{\text{جایگشت}}$

$S_n \cong Sym(S_n) \to n!$

$S \cong S_n$ $n!$

فرم صریح سؤال : یک اثر $\underline{\underline{\text{faithful}}}$ از S_n بر S ، یک نگاشت ضربی از S_n است (نگاشت 1-1 .)

$\leftarrow$ خودریختی

Faithful action $\longleftrightarrow$ $\underline{\underline{\text{Aut of } S_n}}$

$\underline{\underline{S_n}} \longrightarrow Sym(S) \qquad \sigma . \psi \to \sigma \psi \sigma^{-1}$

خودریختی درونی $g \in G : \quad \underline{\underline{\psi_g(x) : g x g^{-1}}} \qquad \psi_g : G \to G$

$(\mathbb{Z}, +) \to$ $\quad f(n) = -n$ $\quad$ Outer Automorphism.

$\mathbb{Z} \to \mathbb{Z}$

Inner $\cong S_n$

$n \neq 2, 6 \quad Aut(S_n) \cong S_n \to$ Out $\cong \{e\}$

Review of $Aut(S_n)$

$n = 2 \quad Aut(S_2) \cong \{e\} \to$ Inn $\cong$ Out $\cong \{e\}$

$n = 6 \quad Aut(S_6) \cong S_6 \rtimes C_2$

$Aut(G) \cong Inn(G) \rtimes Out(G)$ $\quad$ $Inn(S_6) \cong S_6$

$Out(S_6) \cong C_2$

A_n (Alternating subgroups ...)