Introduction to General Relativity - HW5 - 401208729

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Q1. I show this by two ways, first by the expression of $R^{\rho}{}_{\nu\rho\sigma}$ in terms of connection, and second by using vectorial notations.

$$R_{\rho\alpha\beta\gamma} + R_{\rho\gamma\alpha\beta} + R_{\rho\beta\gamma\alpha} = 0$$

we can multiply by $g^{\sigma\rho}$ and raise the ρ index:

$$\boxed{R^{\sigma}{}_{\alpha\beta\gamma} + R^{\sigma}{}_{\gamma\alpha\beta} + R^{\sigma}{}_{\beta\gamma\alpha} = 0}$$

Notice that since $g_{\mu\nu}$ is nowhere singular, this guarantees that always $R_{\rho\alpha\beta\gamma} + R_{\rho\gamma\alpha\beta} + R_{\rho\beta\gamma\alpha} = 0$, otherwise we are multiplying by zero which ruins our problem.

By $R^{\mu}{}_{\alpha\beta\gamma} = \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\theta}_{\alpha\gamma}\Gamma^{\mu}_{\theta\beta} - \Gamma^{\theta}_{\alpha\beta}\Gamma^{\mu}_{\theta\gamma}$, we can write:

$$R^{\sigma}{}_{\alpha\beta\gamma} + R^{\sigma}{}_{\gamma\alpha\beta} + R^{\sigma}{}_{\beta\gamma\alpha} =$$

$$\cancel{\partial_\beta \Gamma^{\sigma}_{\alpha\gamma}} - \cancel{\partial_\gamma \Gamma^{\sigma}_{\alpha\beta}} + \cancel{\Gamma^{\theta}_{\alpha\gamma}\Gamma^{\sigma}_{\theta\beta}} - \Gamma^{\theta}_{\alpha\beta}\Gamma^{\sigma}_{\theta\gamma}{}_{A}$$

$$\cancel{\partial_\alpha \Gamma^\sigma_{\gamma\beta}} - \cancel{\partial_\beta \Gamma^\sigma_{\gamma\alpha}} + \Gamma^\theta_{\gamma\beta}\Gamma^\sigma_{\theta\alpha} - \cancel{\Gamma^\theta_{\gamma\alpha}\Gamma^\sigma_{\theta\beta}}$$

Cancel

$$\cancel{\partial_\gamma \Gamma^\sigma_{\beta\alpha}} - \cancel{\partial_\alpha \Gamma^\sigma_{\beta\gamma}} + \Gamma^\theta_{\beta\alpha}\Gamma^\sigma_{\theta\gamma} - \Gamma^\theta_{\beta\gamma}\Gamma^\sigma_{\theta\alpha}$$

cancel.

Now look how every thing simplifies due to symmetry of Γ on lower indices. $\Gamma^\#_{ab} = \Gamma^\#_{ba}$.

So it equals zero. ▨

Second argument: Recall from Carroll that,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where X, Y, Z are arbitrary vector fields.

In this notation we have to prove that

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$+ \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X$$

$$+ \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y$$

Factor ...

$$\nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z)$$

$$+ \nabla_Z(\nabla_X Y - \nabla_Y X) - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y =$$

Recall that for torsion-free connection (∇), we have

$$[X,Y]^\mu = X^\alpha \partial_\alpha Y^\mu - Y^\alpha \partial_\alpha X^\mu = X^\alpha \nabla_\alpha Y^\mu - Y^\alpha \nabla_\alpha X^\mu$$

$$= \nabla_X Y^\mu - \nabla_Y X^\mu$$

So rewrite the parenthes as:

$$\nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y]$$

$$- \nabla_{[Y,Z]} X \quad - \nabla_{[Z,X]} Y \quad - \nabla_{[X,Y]} Z$$

Again use the above rule.

$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \qquad \equiv 0$$

But this is nothing but Jacobi identity, which vanishes

So we're done. $\blacksquare$

Finally, $R_{\gamma[\alpha\beta y]} = 0$ since,

$$R_{\gamma[\alpha\beta y]} = \frac{1}{3!}\left\{\begin{array}{l} R_{\gamma\alpha\beta y} + R_{\gamma\beta y\alpha} + R_{\gamma y\alpha\beta} \\ -\underbrace{R_{\gamma\alpha y\beta}} - \underbrace{R_{\gamma\beta\alpha y}} - \underbrace{R_{\gamma y\beta\alpha}} \end{array}\right\}$$

using symmetries of Riemann tensor on ↑ (3-4 symmetry.)

$$= \frac{1}{3!}\left\{\begin{array}{l} R_{\gamma\alpha\beta y} + R_{\gamma\beta y\alpha} + R_{\gamma y\alpha\beta} \\ + R_{\gamma\alpha\beta y} + R_{\gamma\beta y\alpha} + R_{\gamma y\alpha\beta} \end{array}\right\}$$

$$= \frac{1}{3}\left(R_{\gamma\alpha\beta y} + R_{\gamma\beta y\alpha} + R_{\gamma y\alpha\beta}\right) \equiv 0$$

since we've established it on previous part. ☒

Q2. Thanks for the laboring problem!

Christoffel symbols:

$g^{\mu\nu} = (-1, a^{-2}, a^{-2}, a^{-2})$
$g_{\mu\nu} = (-1, a^2, a^2, a^2)$

$$\Gamma^0_{00} = \frac{1}{2} g^{0\sigma}\left(\partial_0 g_{\sigma 0} + \partial_0 g_{0\sigma} - \partial_\sigma g_{00}\right)$$

(σ has to be zero, $\sigma \equiv 0$)

$$= -\tfrac{1}{2}(-1) \times \{\partial_t(-1) + \partial_t(0) - \partial_t(-1)\} = 0$$

$$\Gamma^0_{ij} = \frac{1}{2} g^{0\sigma}\left(\partial_i g_{j\sigma} + \partial_j g_{\sigma i} - \partial_\sigma g_{ij}\right)$$

(σ has to be zero, $\sigma \equiv 0$)

$$= \frac{1}{2}(-1)\{\partial_i(0) + \partial_j(0) - \partial_0(a^2(t)\,\delta_{ij})\}$$

$$= a\dot{a}\,\delta_{ij}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma}\left(\partial_j g_{\sigma k} + \partial_k g_{j\sigma} - \partial_\sigma g_{jk}\right)$$

Note that since $a(t)$ is independent of position, for $\sigma = i$ all partial derivatives vanish

$$\Longrightarrow \Gamma^i_{jk} = 0 \quad (i,j,k \in \{1,2,3\})$$

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{1}{2} g^{i\sigma}\left(\partial_j g_{0\sigma} + \partial_0 g_{j\sigma} - \partial_\sigma g_{j0}\right)$$

($\sigma = i$)

$$= \frac{1}{2} \times \frac{1}{a^2} \times \left(0 + \partial_t(\delta_{ji}\,a^2) - 0\right)$$

$$= \frac{1}{\ } \times 2 \times a\dot{a}\,\delta^i_j = \frac{\dot{a}}{\ }\,\delta^i_j .$$

$2a^2$ ... a ... j

other symbols vanish, also check by mathematica code.

Now Riemann Tensor:

As you know, there are 20 distinct components of Riemann tensor at $d=4$, also symmetries of Riemann tensor help us not to overwork.

Let's see them:

Vanishes for $\theta = 0,1,2,3$!

$$R^0{}_{i0j} = \underbrace{\partial_0 \Gamma^0_{ij}}_{\partial_t (a\dot{a}\,\delta_{ij})} - \partial_j \Gamma^0_{i0} + \underbrace{\Gamma^\theta_{ij} \Gamma^0_{\theta 0}} - \underbrace{\Gamma^\theta_{i0} \Gamma^0_{\theta j}}$$

If $i=j$ then it for $\theta = i = j$ has contribution, it's zero otherwise.

(for $i=j$)

$$= \partial_t (a\dot{a}) - \partial_j(0) + 0 - \frac{\dot{a}}{a} \times a\dot{a} =$$

$$= \cancel{\dot{a}^2} + \ddot{a}a - \cancel{\dot{a}^2} = a\ddot{a}$$

(for $i \neq j$)

$$\partial_t(0) - \partial_j(0) + 0 - 0 = 0$$

This gives $R^0{}_{101}$, $R^0{}_{202}$, $R^0{}_{303}$,

And by Riemann tensor's symmetries,

$$R^{0}{}_{110} = R^{0}{}_{220} = R^{0}{}_{330} = -a\dot{a}$$

Now $R^{i}{}_{00i}$ could be found by tensorial properties of R_{i00i} since:

$$R^{i}{}_{00i} = \underbrace{g^{i\sigma}}_{\text{diagonal}} R_{\sigma 00i} = \frac{1}{a^2} R_{i00i} \xrightarrow[\text{on first pair}]{\text{anti-symmetry}}$$

$$= -\frac{1}{a^2(t)} R_{0i0i} \xrightarrow{\text{upper the index}} -\frac{1}{a^2} \underbrace{g_{r0}}_{\{\text{just } r=0\}} R^{r}{}_{i0i}$$

$$= \frac{1}{a^2} R^{0}{}_{i0i}$$

$$= \frac{1}{a^2} a\ddot{a} = \frac{\ddot{a}}{a}$$

For $R^{1}{}_{212}$ or $R^{1}{}_{313}$ we must ply the relation:

$$R^{1}{}_{212} = \cancelto{0}{\partial_1 \Gamma^{1}_{22}} - \cancelto{0}{\partial_2 \Gamma^{1}_{12}} + \underbrace{\Gamma^{r}_{22}\, \Gamma^{1}_{r1}}_{r=0} - \underbrace{\Gamma^{r}_{21}\, \cancel{\Gamma^{1}_{r2}}}_{\text{no } r \text{ has contribution.}}$$

$$= 0 + 0 + a\dot{a} \times \frac{\dot{a}}{a} + 0 = (\dot{a})^2$$

Similarly $R^{1}{}_{313} = \dot{a}^2$

And by symmetries $\rightarrow \begin{cases} R^{i}{}_{0i0} = -\dfrac{\ddot{a}}{a} \\ R^{1}{}_{221} = R^{1}{}_{331} = -\dot{a}^2 \end{cases}$

For $R^{2}{}_{121}$ we can use

$$R^{2}{}_{121} = g^{2r} R_{r121} = \frac{1}{a^2} R_{2121} = -\frac{1}{a^2} R_{1221} =$$

$$-\frac{1}{a^2} g_{r1} R^{r}{}_{221} = -\frac{\cancel{a^2}}{\cancel{a^2}} \times R^{1}{}_{221} = \dot{a}^2$$

while $R^{2}{}_{112} = -R^{2}{}_{121} = -\dot{a}^2$

By the same logic you can find $\begin{cases} R^{2}{}_{323} = \dot{a}^2 \\ R^{2}{}_{332} = -\dot{a}^2 \end{cases}$

Based on these similar ideas, also

$$\begin{cases} R^{3}{}_{113} = -\dot{a}^2 \\ R^{3}{}_{131} = \dot{a}^2 \end{cases}$$

For any r $=0$

But $R^{3}{}_{232} = \underbrace{\cancel{\partial_3 \Gamma^{3}_{22}}}_{0} - \underbrace{\cancel{\partial_2 \Gamma^{3}_{23}}}_{0} + \underbrace{\Gamma^{r}_{22} \Gamma^{3}_{r3}}_{\text{just } r=0} - \cancel{\Gamma^{r}_{32} \Gamma^{3}_{r2}}$

$$= \frac{\dot{a}}{a} \times a\dot{a} = \dot{a}^2$$

And finally $R^3{}_{223} = -\dot{a}^2$

All other Componets of Riemann tensor vanish, as you can check via mathematica Code.

Now Ricci Tensor $\longrightarrow R_{\mu\nu} = R^\alpha{}_{\mu\alpha\beta}$

Since all $R^\alpha{}_{\rho\beta\sigma}$ vanish for $\rho \neq \sigma \Rightarrow R_{\rho\sigma}$ is diagonal.

$$R_{00} = R^1{}_{010} + R^2{}_{020} + R^3{}_{030} + \overset{\text{zero}}{\cancel{R^0{}_{000}}} =$$

$$= -\frac{3\ddot{a}}{a} + 0$$

$$R_{11} = R^0{}_{101} + \overset{\text{zero}}{\cancel{R^1{}_{111}}} + R^2{}_{121} + R^3{}_{131} =$$

$$a\ddot{a} + 0 + \dot{a}^2 + \dot{a}^2 = 2\dot{a}^2 + a\ddot{a}$$

Similarly $R_{22} = R_{33} = 2\dot{a}^2 + a\ddot{a}$

And Ricci scalar is:

$$g^{\mu\nu} R_{\mu\nu} = -\left(-3\frac{\ddot{a}}{a}\right) + \frac{1}{a^2}\left(6\dot{a}^2 + 3a\ddot{a}\right)$$

$$= \frac{3\ddot{a}}{a} + \frac{6\dot{a}^2 + 3\ddot{a}a}{a^2} = \frac{6\ddot{a}a + 6\dot{a}^2}{a^2}$$

$$= \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2\right)$$

Q3. $\mathcal{L}_\xi T_{\mu\nu} = ?$

we contract $T_{\mu\nu}$ with arbitrary X^μ, Y^ν vector fields, then use the fact that acting $\mathcal{L}_\xi$ on scalar will be like acting ξ-vector on it $\longrightarrow$ By equating both sides we'll do.

$$\mathcal{L}_\xi (\underbrace{T_{\mu\nu} X^\mu X^\nu}_{\text{Scalar}}) = \xi^\rho \nabla_\rho (T_{\mu\nu} X^\mu Y^\nu)$$

Now use a trick! $\longrightarrow$ ∂_μ acts like ∇_μ on scalar

$$= \xi^\rho \nabla_\rho (T_{\mu\nu} X^\mu Y^\nu)$$

use the Leibnitz rule of covariant derivative.

(*)
$$= \xi^\rho T_{\mu\nu;\rho} X^\mu Y^\nu + \xi^\rho \{ T_{\mu\nu} X^\mu{}_{;\rho} Y^\nu + T_{\mu\nu} X^\mu Y^\nu{}_{;\rho} \}$$

In the second scenario, act $\mathcal{L}_\xi$ on each tensor, then use the fact that $\mathcal{L}_\xi X^\mu = [\xi, X]^\mu = \xi^\rho \nabla_\rho X^\mu - X^\rho \nabla_\rho \xi^\mu$

$$\mathcal{L}_\xi (T_{\mu\nu} X^\mu Y^\nu) = (\mathcal{L}_\xi T_{\mu\nu}) X^\mu Y^\nu + T_{\mu\nu} (\mathcal{L}_\xi X^\mu) Y^\nu + T_{\mu\nu} X^\mu \mathcal{L}_\xi Y^\nu$$

$$= (\mathcal{L}_\xi T_{\mu\nu}) X^\mu Y^\nu + T_{\mu\nu} (\xi^\rho \nabla_\rho X^\mu - X^\rho \nabla_\rho \xi^\mu)$$

$$+ T_{\mu\nu} X^\mu (\xi^\rho \nabla_\rho Y^\nu - Y^\rho \nabla_\rho \xi^\nu)$$

$$= (\mathcal{L}_\xi T_{\mu\nu}) X^\mu Y^\nu + T_{\mu\nu} Y^\nu \{ \xi^\rho X^\mu{}_{;\rho} - X^\rho \xi^\mu{}_{;\rho} \}$$

$$T_{\mu\nu} X^\mu \{ \xi^\rho Y^\nu{}_{;\rho} - Y^\rho \xi^\nu{}_{;\rho} \} \quad \cancel{**}$$

Now Comparing both sides of $\cancel{*}$ and $\cancel{**}$, two terms are Common, $T_{\mu\nu} \xi^\rho X^\mu{}_{;\rho} Y^\nu$ and $T_{\mu\nu} X^\mu \xi^\rho Y^\nu{}_{;\rho}$ $\rightarrow$ Hence

$$(\mathcal{L}_\xi T_{\mu\nu}) X^\mu Y^\nu = (\xi^\rho T_{\mu\nu;\rho}) X^\mu Y^\nu$$

$$+ \underbrace{T_{\mu\nu} X^\rho Y^\nu \xi^\mu{}_{;\rho}}_{\mathrm{I}} + \underbrace{T_{\mu\nu} X^\mu Y^\rho \xi^\nu}_{\mathrm{II}}{}_{;\rho}$$

Now, to extract arbitrary $X^\mu Y^\nu$ from both sides manipulate dummy indices $\rightarrow$

term I : $\begin{cases} \mu \longleftrightarrow \alpha \\ \rho \longleftrightarrow \mu \end{cases}$ term II : $\begin{cases} \nu \longleftrightarrow \alpha \\ \rho \longleftrightarrow \nu \end{cases}$

It will be,

$$(\mathcal{L}_\xi T_{\mu\nu}) X^\mu Y^\nu = (\xi^\rho T_{\mu\nu;\rho}) X^\mu Y^\nu$$

$$+ T_{\alpha\nu} X^\mu Y^\nu \xi^\alpha{}_{;\mu} + T_{\mu\alpha} X^\mu Y^\nu \xi^\alpha{}_{;\nu}$$

Now extract arbitrary vector fields X^μ and Y^ν:

$$\mathcal{L}_\xi T_{\mu\nu} = \xi^\rho T_{\mu\nu;\rho} + T_{\alpha\nu}\,\xi^\alpha{}_{;\mu}$$

$$+ T_{\mu\alpha}\,\xi^\alpha{}_{;\nu} \quad \blacksquare$$

Q4. This question is fairly short!

A tangent vector at a curve $x^\mu(\lambda)$ has components

$$V^\beta = \frac{d}{d\lambda} x^\beta(\lambda)$$

Hence, we parallel transport this target vector, along the $x^\mu(\lambda)$ curve itself.

It's a total derivative w.r.t λ.

$$\frac{dx^\mu}{d\lambda} \nabla_\mu \left(\frac{d}{d\lambda} x^\beta \right) = \frac{dx^\mu}{d\lambda} \left\{ \partial_\mu \left(\frac{d}{d\lambda} x^\beta \right) + \Gamma^\beta_{\mu\theta} \frac{dx^\theta}{d\lambda} \right\}$$

$$= \frac{d^2}{d\lambda^2} x^\beta + \Gamma^\beta_{\mu\theta} \frac{dx^\mu}{d\lambda} \frac{dx^\theta}{d\lambda} = 0$$

So geodesic equation is nothing but parallel transport of the target vector to a curve along the curve itself. $\blacksquare$