Fourier analysis

Biomedical Signal Processing

Centre for Doctoral Training in Healthcare Innovation

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Fourier analysis

1. Introduction to Fourier analysis, the Fourier series
2. Sampling and Aliasing
3. Discrete Fourier methods, and Applications

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1: Introduction to Fourier analysis

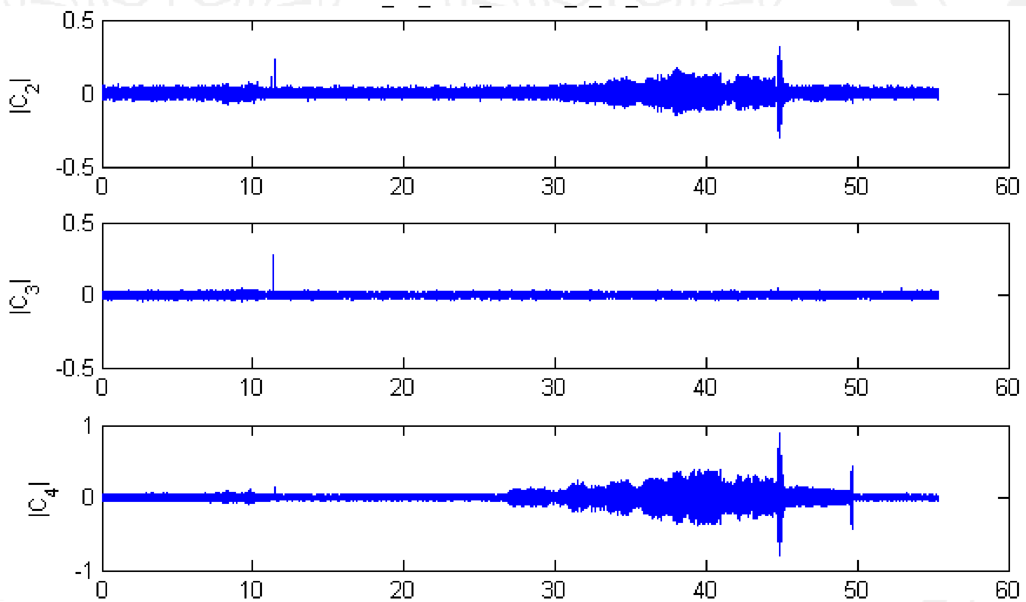
1. Why Fourier?
2. The Fourier series –
   for periodic functions
3. The Fourier transform –
   for non-periodic functions

Why Fourier?

- We are usually very comfortable with the notion of time-series data:
  - the ECG, the EEG
  - movements of a tiny accelerometer mounted in a wristwatch,
    or mounted on the case of a jet engine…

In the time domain

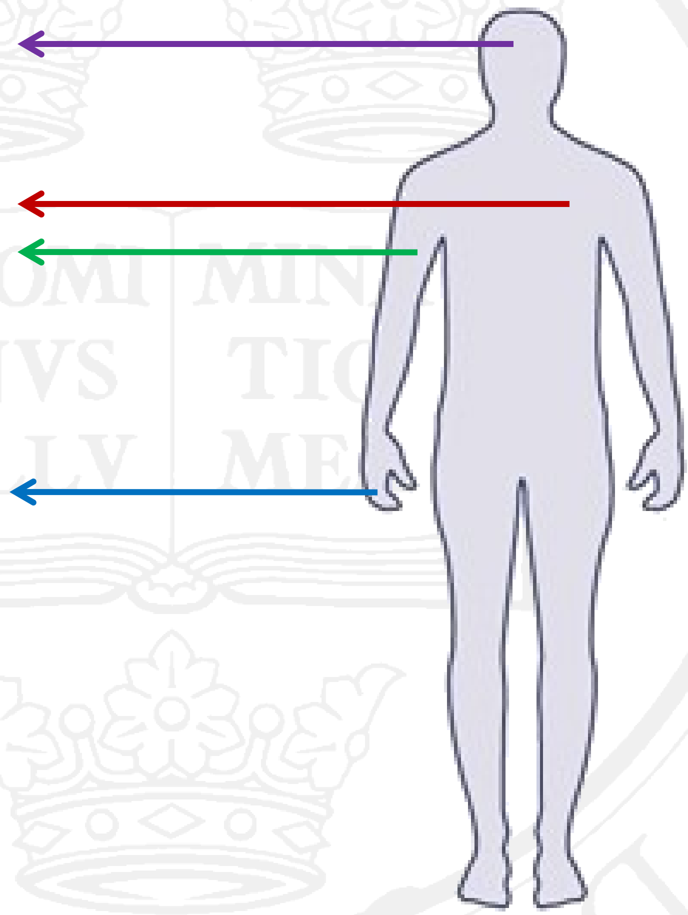
- But how far can analysis in the time domain take us?
- Is there another way of thinking about our data, in a way that may be more useful for our particular application?



- Fourier analysis tells us that we can consider a time-series of data to be a mixture of simple building-blocks ("components")... but what are these components?

Extracting features from data

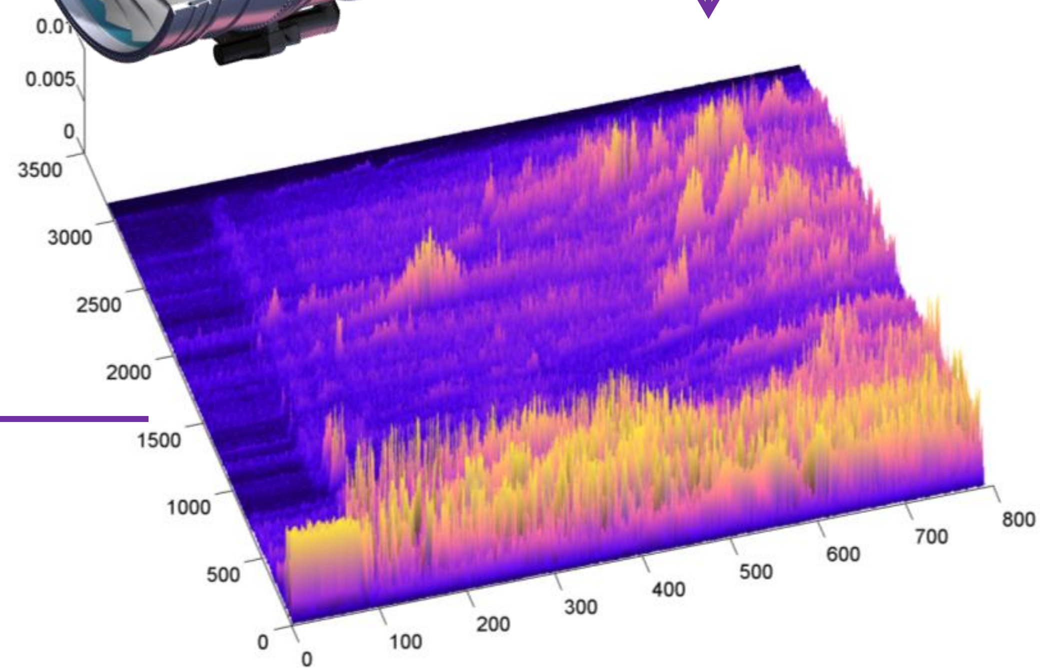
- EEG
- Heart rate
- Breathing rate
- SpO2
- Blood pressure
- Temperature



Extracting features from data

- EEG
- Heart rate
- Breathing rate
- SpO2
- Blood pressure
- Temperature

- Vibration magnitude
- Vibration phase
- Fundamentals
- Harmonics
- Gas pressures
- Gas temperatures



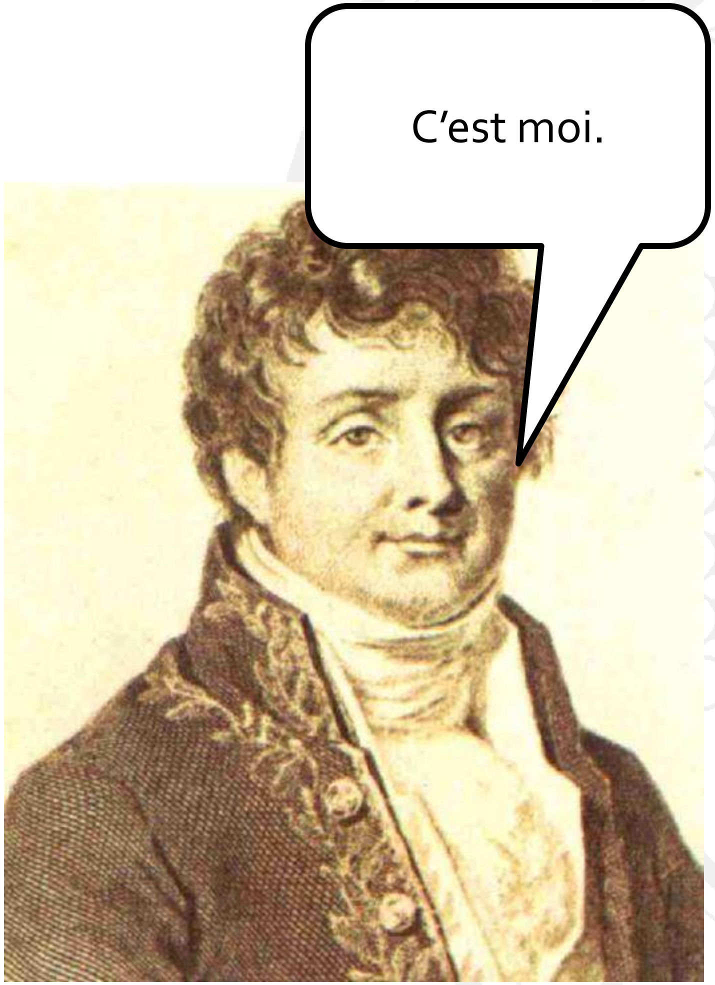
Introducing Fourier...

- A long time ago (1807) in a galaxy far, far away (France), Joseph Fourier solved the heat equation.

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

- This was very exciting, because there was no general solution at the time – only *particular solutions*, such as the case when a sinusoidal heat source was applied to a metal plate.

The Fourier series



- In *Mémoire sur la propagation de la chaleur dans les corps solides* (1807), Fourier postulated what would become known as the **Fourier series**
- He showed that any periodic signal could be described as being a mixture of components – sines and cosines, in particular
- Why is this so terribly exciting?

(Incidentally)

- In what should be one of the most encouraging stories for all scientists, the panel that reviewed Fourier's legendary publication (Lagrange, Laplace, Malus, and Legendre) concluded:

"the manner in which the author arrives at these equations is not exempt of difficulties and … his analysis to integrate them leaves something to be desired on the score of generality and even rigour"

The Fourier series

- If any periodic signal, no matter how complicated, can be represented using a mixture of simple components (sines and cosines)…

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + \sum_{n=1}^{\infty} B_n \sin(n\omega t)$$

- then we can easily analyse the complex signal by analysing its simple components. All we need to do is work out the coefficients
  a_n and b_n
  in the above:

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt \quad n = 0,1,2,\ldots$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt \quad n = 1,2,\ldots$$

The Fourier series

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + \sum_{n=1}^{\infty} B_n \sin(n\omega t)$$

- This is very handy. We have changed from thinking about our signal $f(t)$ in the time-domain to being some mixture of sines and cosines, with fundamental frequency ω and harmonics 2ω, 3ω, etc.

- Note that $\omega = 2\pi / T$, where T is the period of our periodic signal.

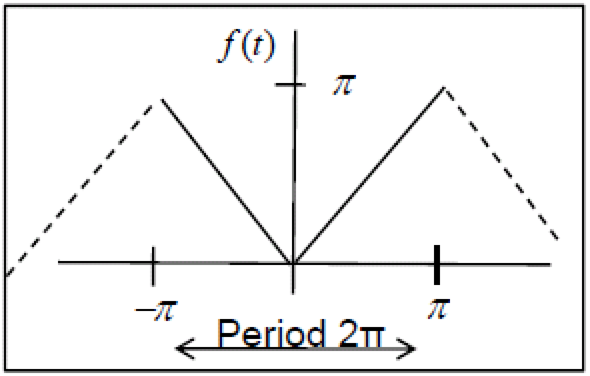
- Also, because $T = 1/f$, where f is the frequency of our periodic signal, we have $\omega = 2\pi f$.

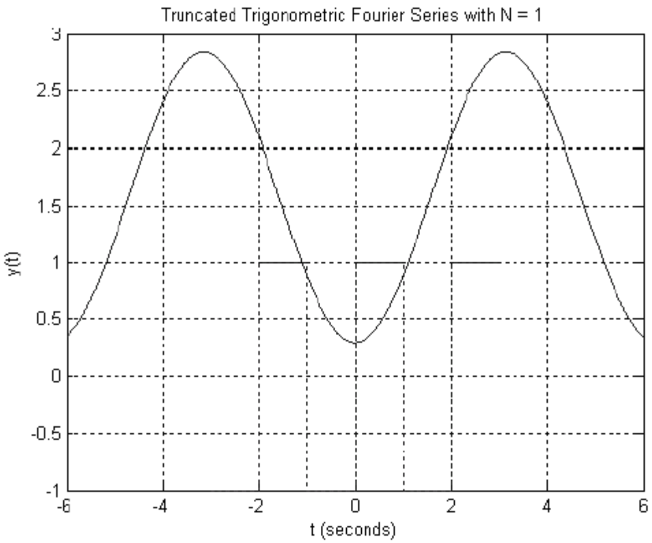
The Fourier series

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + \sum_{n=1}^{\infty} B_n \sin(n\omega t)$$

- Now, this is an infinite sum. That is, we need infinitely many components in our mixture to recreate our original periodic signal.
- In other words, we need to consider sines and cosines of ever-higher frequencies $n\omega$ to recreate our original signal.
- However, we are usually satisfied with a certain degree of accuracy, and so will take only the first N terms in the sum.

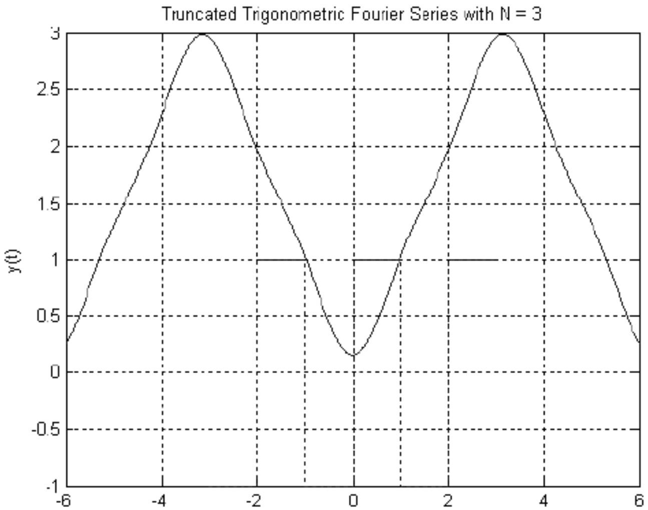
Consider a sawtooth function…

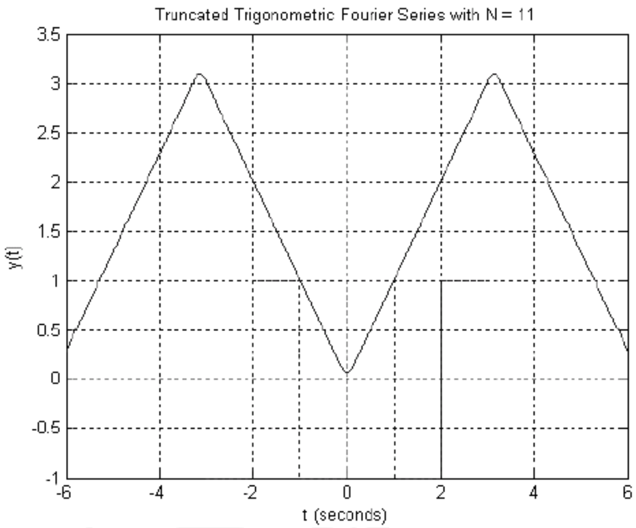




$N = 1$

$N = 3$



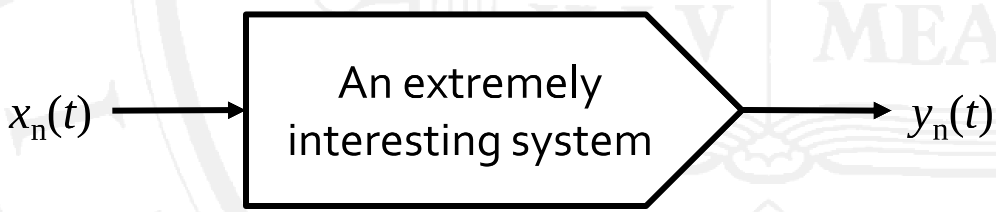


$N = 11$

...and?

- Wouldn't it be nice if we could actually use the Fourier series for something constructive?

- Suppose we have two time-domain signals, $x_1(t)$ and $x_2(t)$. If we pass them through some interesting system, we can find their output in the time domain:



- And if that system is *linear*, we mean that it obeys the following rules:

$$C\,x_n(t) \rightarrow C\,y_n(t) \qquad \text{(Scaling)}$$
$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t) \qquad \text{(Superposition)}$$

- Giving us: $$C\,x_1(t) + D\,x_2(t) \rightarrow C\,y_1(t) + D\,y_2(t)$$

...and?

- Let's say that we know how our system changes the amplitude and frequency of some input sinusoid with frequency ω. This is the system's **frequency response**, given by its **transfer function** $H(\omega)$.
- If we apply an input $X(\omega)$, we can easily find the output:

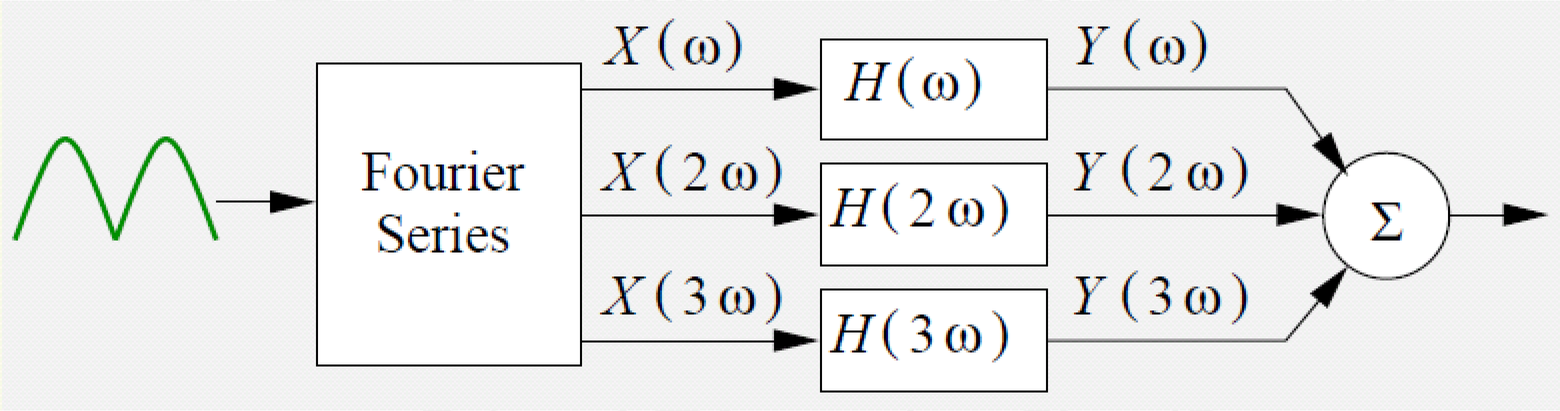
$$Y(\omega) = H(\omega)\, X(\omega)$$

- This just tells us that if we input a sinusoidal waveform with frequency ω, then the system changes its amplitude by some amount, and changes its phase by some amount.
- $|H(\omega)|$ and angle[$H(\omega)$], respectively (**gain** and **phase-shift**)

Using Fourier series

- So if we know how our system changes a single sinusoid $X(\omega)$, and we know from Fourier that any (potentially very complicated) periodic signal can be represented as a mixture of sinusoids…

- …then can't we work out the result of passing our periodic signal through our system? Indeed we can:



Before we move on…

- Happily, Euler’s formula tells us that we can write sums of sines and cosines using exponentials:

$$e^{ix} = \cos x + i \sin x$$

- and so we can write down the **complex Fourier series** using complex numbers:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

Intermission

- Any (potentially very complicated) periodic signal can be represented as a mixture of sinusoidal components

- If we know how a system responds to an input of some frequency ω, then we can:
  - Decompose our signal into sinusoidal components
  - Find out how the system responds to each of those components
  - Add up those responses to give the overall response to our (potentially very complicated) periodic signal

From Series to Transforms

- Recapping: the Fourier series tells us how to represent periodic functions

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

- ...and it turned out that we have a sum of sines and cosines occurring at **discrete frequencies**

- Most useful signals will be non-periodic (EEG, for example). Can we extend Fourier series to allow analysis of such signals? Will the frequency representation still be restricted to discrete frequencies?

The Marquis de Laplace

- Laplace, clearly haven forgiven Fourier his perceived "lack of rigour", considered the extension of Fourier series to non-periodic functions

- His approach was to stop looking for solutions for difficult differential equations. Instead, he transformed them, then considered solutions of the transformed equations – which were much more palatable

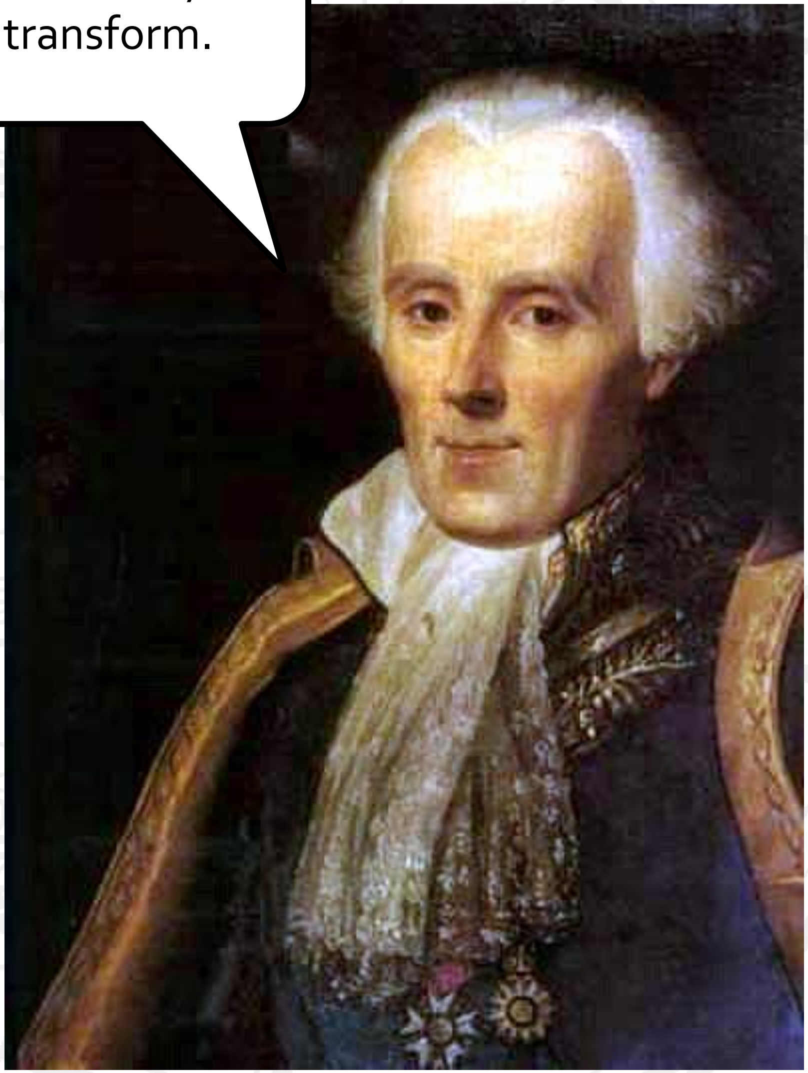
The Laplace transform

- The Laplace transform represents our non-periodic signal $f(t)$ in terms of some interesting continuous variable, $s = \sigma + i\omega$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

- The Fourier transform is a special case of the Laplace, in which $s = i\omega$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$



(Incidentally)

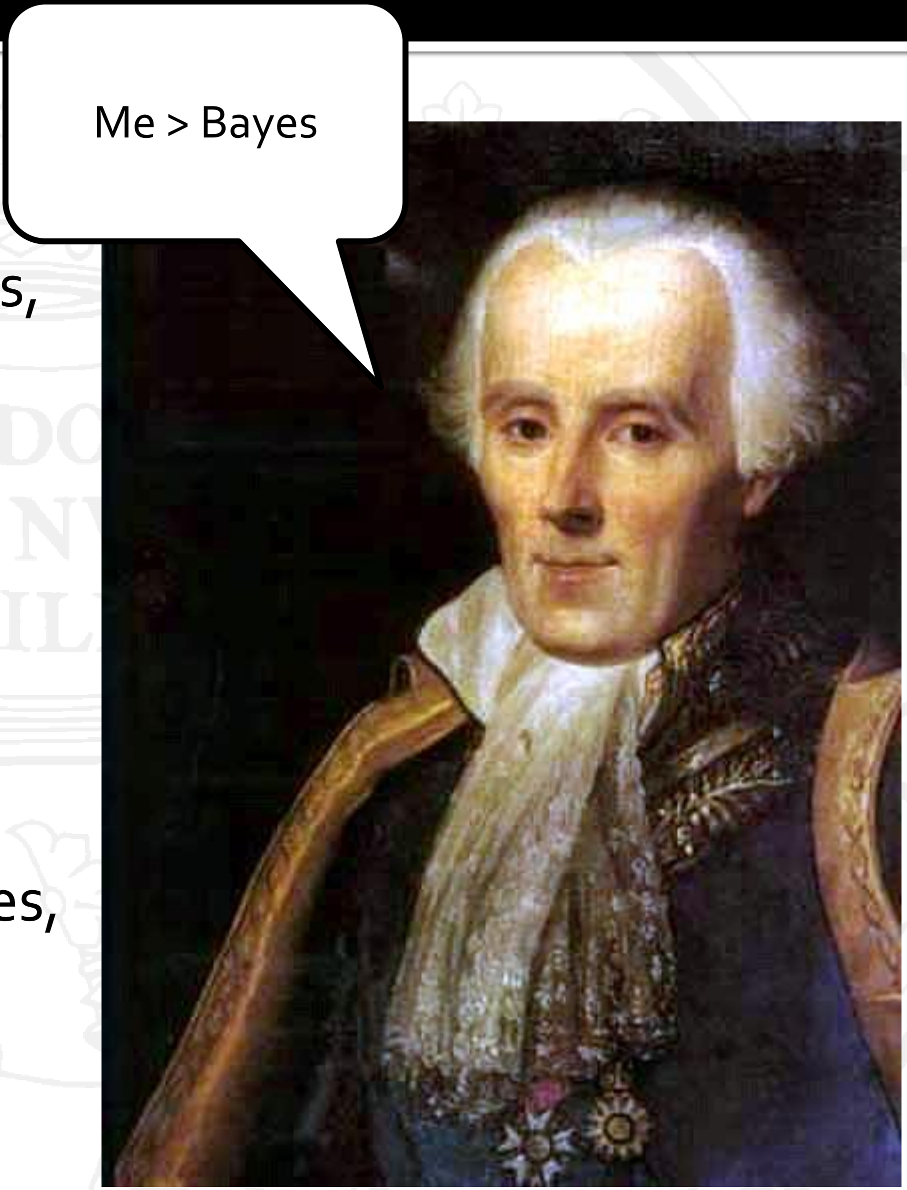
- In his spare time, the "French Newton" almost singlehandedly invented statistics, arguably being the first real Bayesian
(Rev. Bayes was published after his death by a friend)

$$Pr(A_i|B) = \frac{Pr(B|A_i)}{\sum_i Pr(B|A_i)}.$$

- Invented black holes, least squares, scalar potentials, the bikini, etc*

$$\nabla^2 V = 0.$$

*One of these is not true.



The frequency spectrum

- Notice that the Fourier transform has given us a frequency representation of our signal which is **continuous**…

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t$$

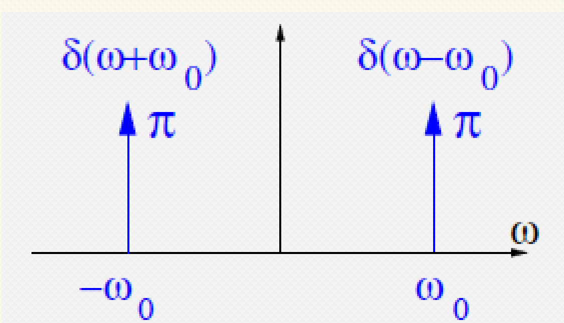
- This $F(\omega)$ is the **frequency spectrum**, corresponding to our original time-series signal. It tells us which frequencies exist in our signal, but is no longer limited to being a sum of a fundamental frequency ω_0 + harmonics $n\omega_0$
  – it is an infinite sum over all frequencies!

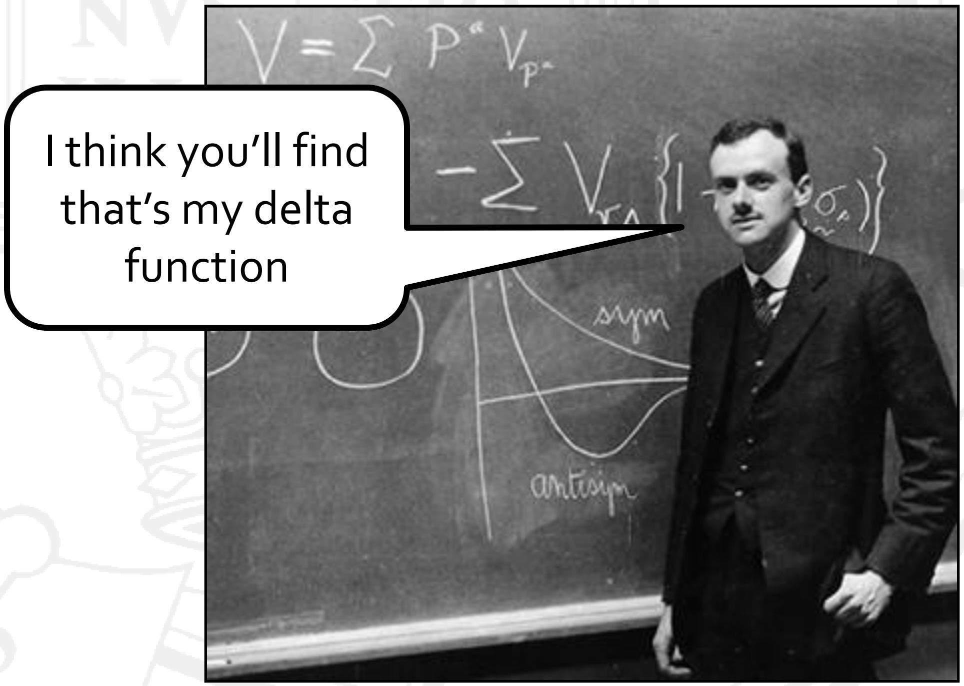
- This allows us to tackle really useful signals, that we typically record from complex systems such as human neural cortices, cardiovascular systems, etc.

The Fourier transform

- The Fourier transform tells us which sinusoids are required in the mixture to construct our original signal
- In the simplest case, it tells us that we just need a single sinusoid if our original signal is a sinusoid!

$$\mathcal{FT}\left[\cos \omega_0 t\right] =$$
$$\pi\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right).$$



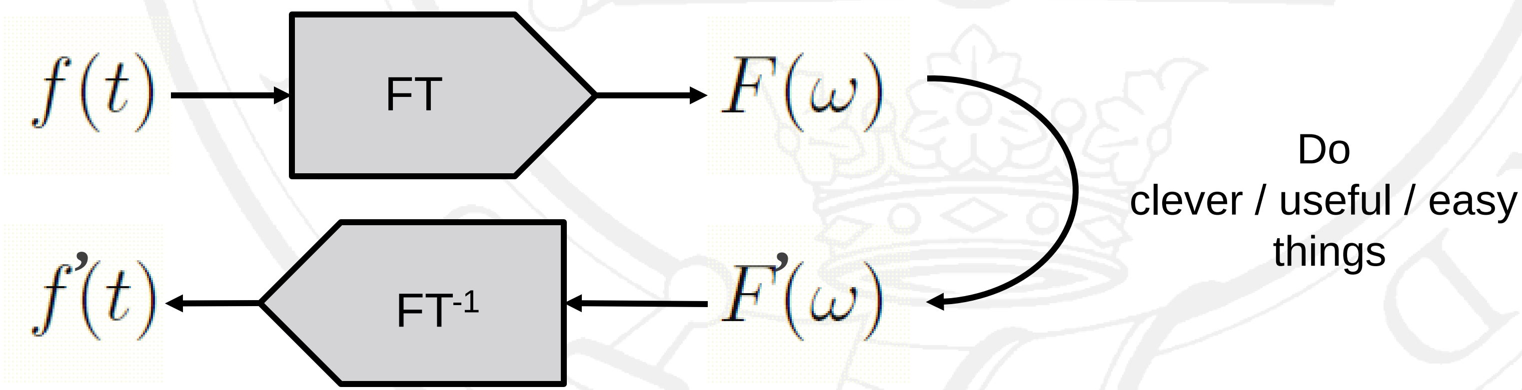


The Fourier transform pair

- The Fourier transform has an associated inverse, such that we can move between the time and frequency representations of our signals

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \quad \Leftrightarrow \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega$$

- Typically (following Laplace's procedure):

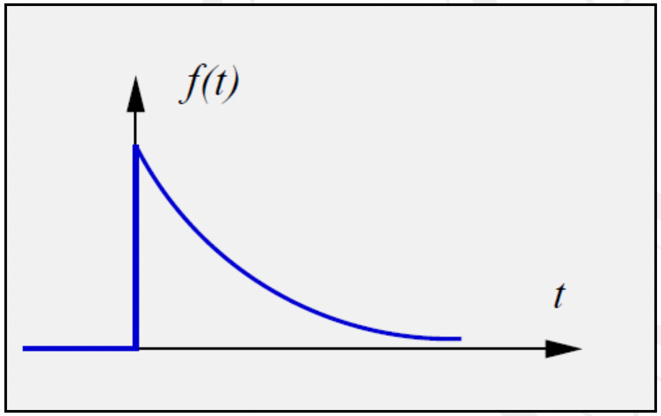


An example Fourier transform

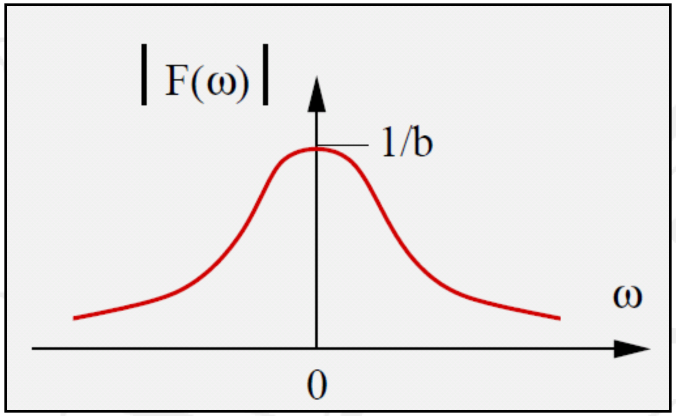
- Let's transform a moderately interesting (non-periodic!) time-series signal, $f(t) = e^{-bt}$ where $t > 0$

$$\mathcal{FT}\left[\mathrm{e}^{-bt}\right] = \int_0^\infty \mathrm{e}^{-bt}\,\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

$$= \frac{-1}{b+\mathrm{i}\omega}\,\mathrm{e}^{-(b+\mathrm{i}\omega)t}\Big|_0^\infty = \frac{1}{b+\mathrm{i}\omega}\,.$$



$\Leftrightarrow$

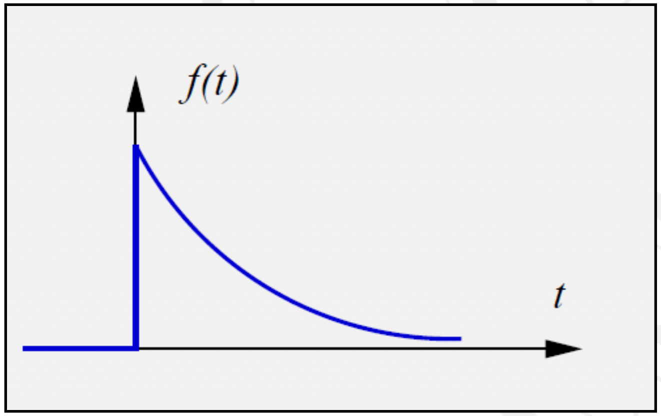


Complex number alert!

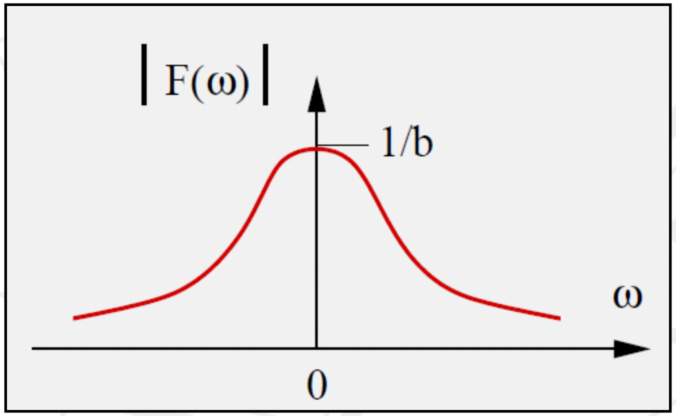
- $F(\omega)$ is a complex spectrum – each frequency ω has both magnitude and phase

$$\mathcal{FT}\left[\mathrm{e}^{-bt}\right] = \frac{1}{b + \mathrm{i}\omega} .$$

- This is **not at all terrifying**. It simply means that every possible ingredient in our continuous mixture of sinusoids is represented, from $\omega = 0$ (d.c.) all the way up to $\omega = \infty$. The magnitude and phase at each ω is given by $F(\omega)$.



$\Leftrightarrow$

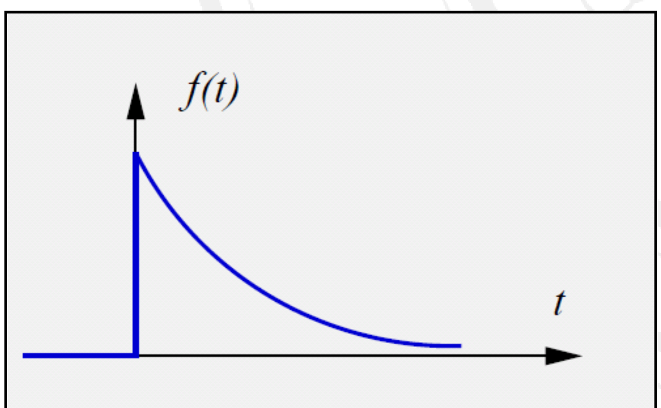


Complex number alert!

- So, we can find the magnitude of the sinusoid at some frequency using $|F(\omega)|$ and the phase using $\angle F(\omega)$.

- Revision: for some complex number $z = x + iy$,
  $|F(\omega)| = \sqrt{(x^2 + y^2)}$, $\angle F(\omega) = \tan^{-1}(y/x)$

- Hence we obtain the aptly named **amplitude spectrum** and **phase spectrum** from the frequency spectrum.



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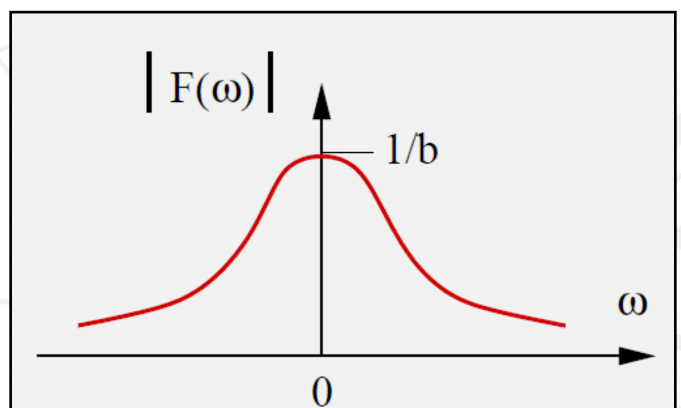
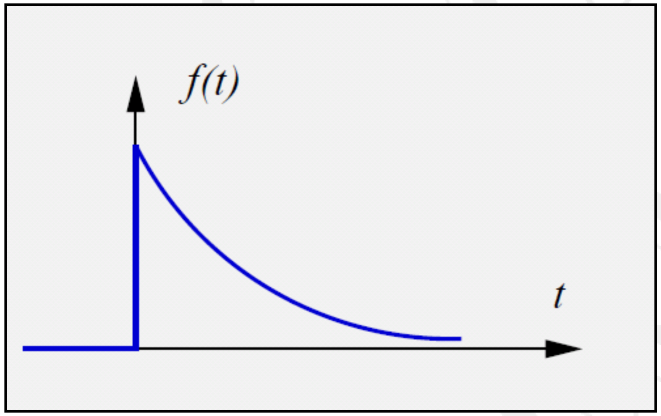


Complex number alert!

- Oh dear: we appear to have sinusoids of negative frequencies $\omega < 0$ in the mixture $F(\omega)$ that we are using to represent our original signal. What does this mean?

- Not much: remember that $\cos(-\omega t) = \cos(\omega t)$ and that $\sin(-\omega t) = -\sin(\omega t)$. In practice, we can safely ignore the $\omega < 0$ half of our frequency/amplitude/phase spectra, as it is always the mirror image of the $\omega > 0$ half.



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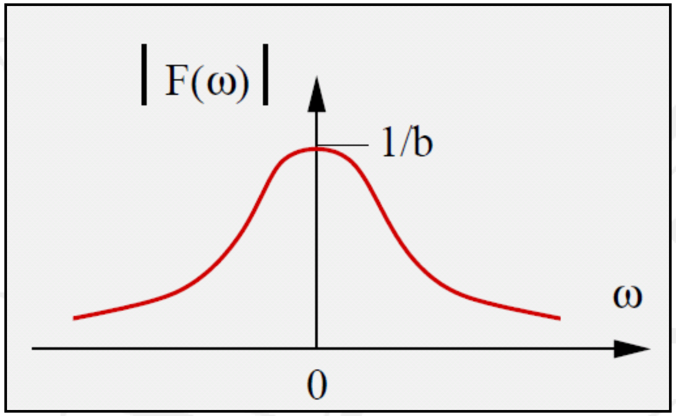
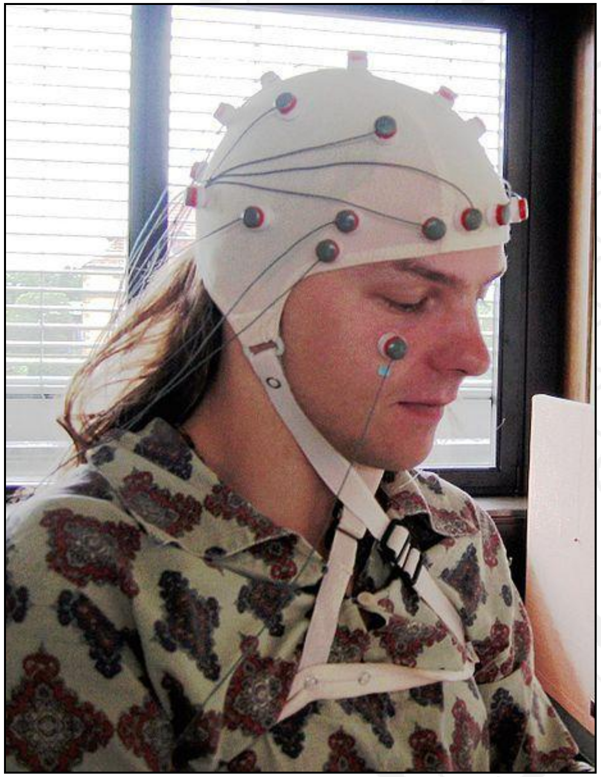
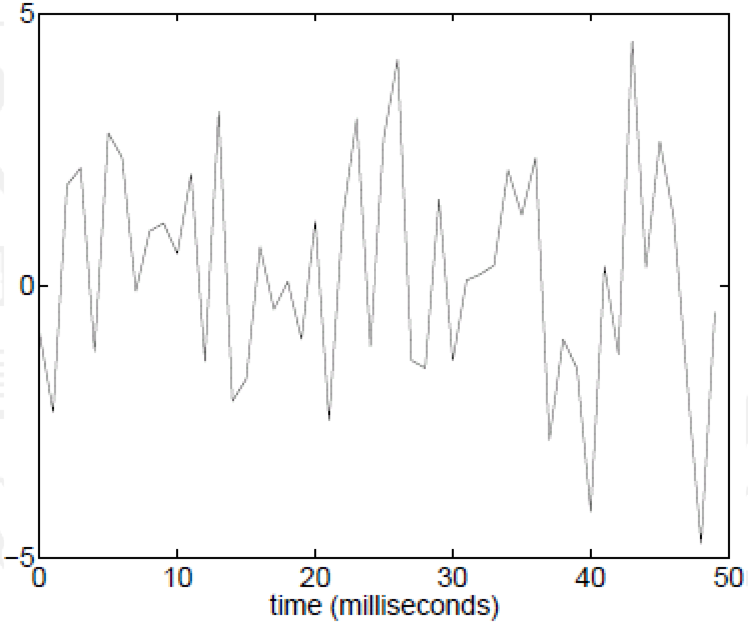


A much better real example

- Suppose that we record the following example after measuring an unfortunate volunteer's EEG for 50 s:

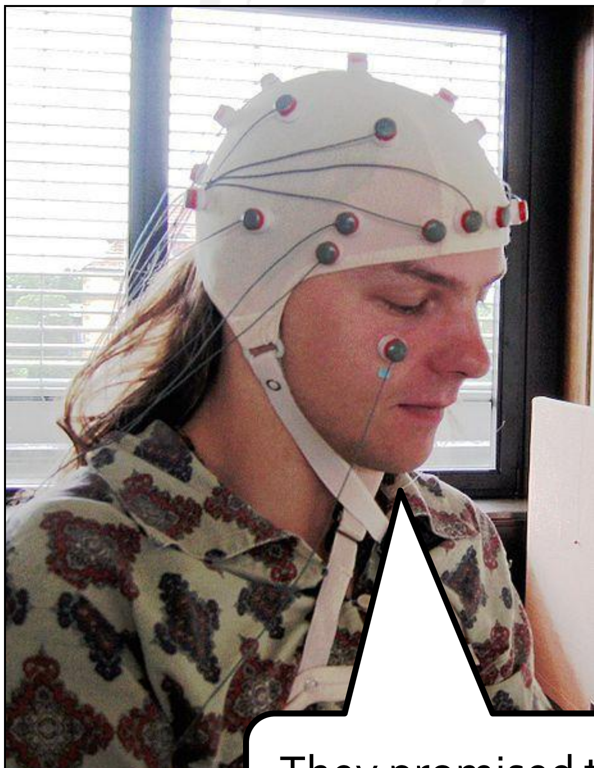




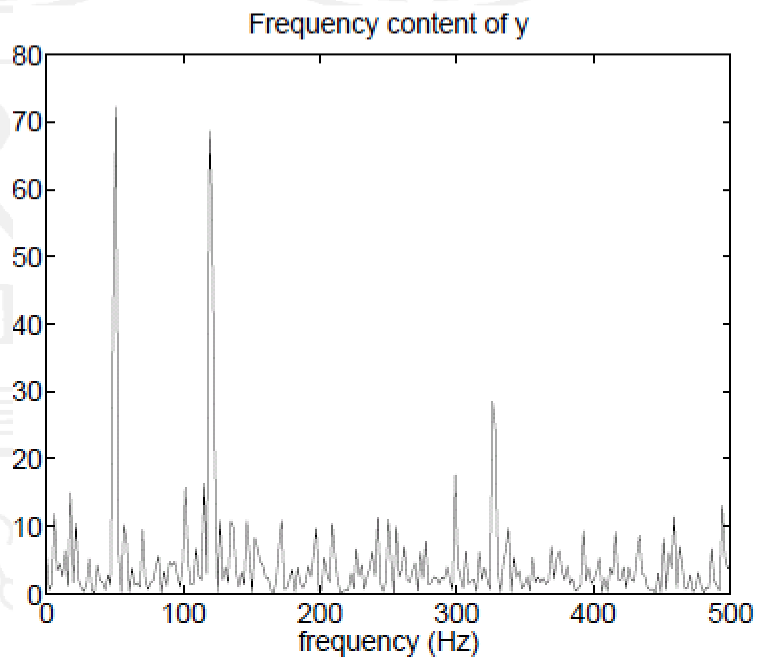
Tip: never wear embarrassing shirts when volunteering for scientific experiments – the photos will be around longer than you imagine

A much better real example

- The amplitude spectrum $|F(\omega)|$ shows us that the signal was mostly generated by three dominant sinusoids, plus noise:



They promised they would delete the photos...

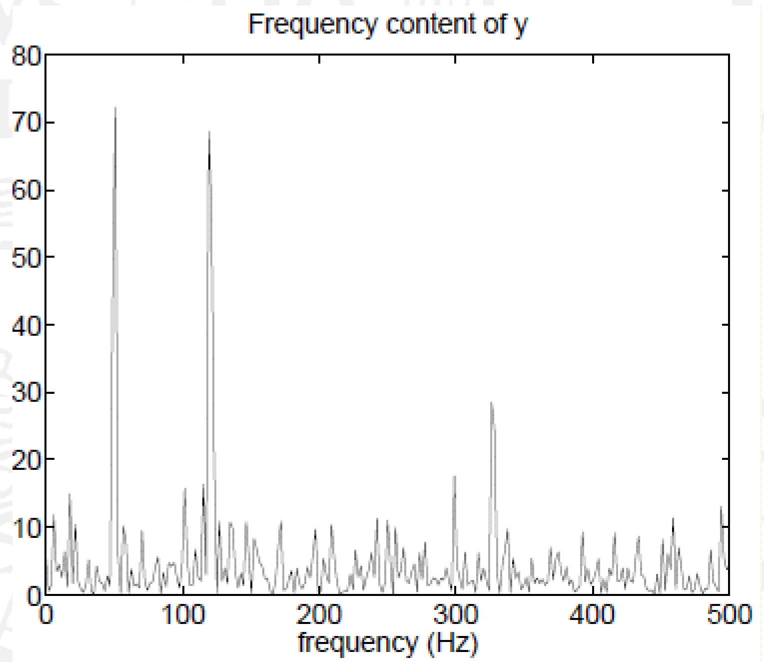


$|F(\omega)|$

Spectral analysis

- Knowing which frequencies were used to make our time-series signal is extremely useful.
  This is, in fact, **spectral analysis**.

- How could we use such knowledge?

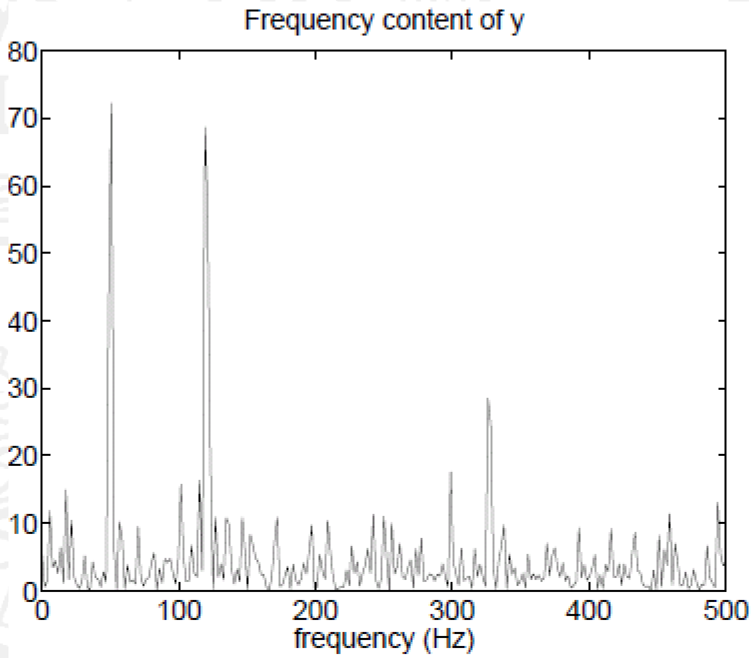


Spectral analysis

- Knowing which frequencies were used to make our time-series signal is extremely useful.
  This is, in fact, **spectral analysis**.

- How could we use such knowledge?

  - Removing "noise"
  - Filtering, to remove high- or low-frequency components
  - Detecting system changes
  - Interacting with other signals
    $F_1(\omega) \times F_2(\omega)$



We now know that...

- Any periodic signal can be represented as a mixture of sinusoidal components (occurring at multiples of some fundamental frequency) using a **Fourier series**

- Any non-periodic signal can be represented as a mixture of sinusoidal components (occurring at *any* frequency) using a **Fourier transform**

- The FT results in frequency spectra which are complex: they contain the magnitude and phase of each component in the mixture required to construct our original signal

- We can consider the amplitude and/or phase spectra

Fourier analysis

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