Last session of TA classes!

We will have a brief review of hard-core Lie groups & algebra!

First, let's discuss the famous problem of non-matrix Lie groups.

$G = \mathbb{R} \times \mathbb{R} \times S^1$ with $(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) =$

$$(x_1 + x_2, y_1 + y_2, e^{i x_1 y_2} u_1 u_2)$$

Let H be the Heisenberg grp $\left\{ \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

Consider $\Phi : H \longrightarrow G$

$$\phi \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} = (a, c, e^{ib})$$

① Homomorphism :

$$\phi \begin{pmatrix} 1 & a_1 & b_1 \\ & 1 & c_1 \\ & & 1 \end{pmatrix} \phi \begin{pmatrix} 1 & a_2 & b_2 \\ & 1 & c_2 \\ & & 1 \end{pmatrix} = (a_1, c_1, e^{ib_1}) \cdot (a_2, c_2, e^{ib_2})$$

$$= (a_1 + a_2, c_1 + c_2, e^{i a_1 c_2} e^{i(b_1 + b_2)})$$

on the other hand: (*)

$$\begin{pmatrix} 1 & a_1 & b_1 \\ & 1 & c_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & b_2 \\ & 1 & c_2 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 + a_2 & b_2 + a_1 c_2 + b_1 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \phi \begin{pmatrix} 1 & a_1 + a_2 & b_1 + b_2 + a_1 c_2 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{pmatrix} = (a_1 + a_2, c_1 + c_2, e^{i b_1 + i b_2 + i a_1 c_2})$$

∎

Let's find $\ker \phi = \{ M \in H \mid \phi(M) = e_{\mathbb{R} \times \mathbb{R} \times S^1} = (0, 0, 1) \}$

since $\phi\begin{pmatrix}1 & a & b\\ & 1 & c\\ & & 1\end{pmatrix} = (a, c, e^{ib}) \equiv (0,0,1)$

so $\longrightarrow$ $\ker\phi = \left\{ \begin{pmatrix}1 & 0 & 2\pi n\\ & 1 & 0\\ & & 1\end{pmatrix} \mid n \in \mathbb{Z} \right\}$

Now let Ψ be a finite-dim representation of G. (i.e. a continuous map of G into some $GL(n, \mathbb{C})$)

so the associated representation (Σ) of H is given by

$$\Sigma = \Psi \circ \Phi$$

The kernel of such representation must include the kernel of Φ.

Let $Z(H)$ be the center of H. By use of $*$ it is

$$Z(H) = \left\{ \begin{pmatrix}1 & 0 & b\\ & 1 & 0\\ & & 1\end{pmatrix} \mid b \in \mathbb{R} \right\}$$

Corollary: For any finite-dimensional rep of H, If

$$\ker\Sigma \supset \ker\phi \longrightarrow \ker\Sigma \supset Z(H).$$

once this is established, we can conclude that there is no faithful finite-dimensional representations of G.

since by above corollary, $\ker\Sigma$ will contain $\ker\phi$ (and $Z(H)$)

Thus:

$$\forall b \in \mathbb{R}: \; I = \Sigma\begin{pmatrix}1 & & b\\ & 1 & \\ & & 1\end{pmatrix} = \psi(0, 0, e^{ib})$$

which means that $\ker\psi$ contains $(0,0,u)$ elements, so ψ is not faithful! $\longrightarrow$ G has no faithful finite-dim. representation.

so elements of G cannot be represented by a matrix, nevertheless

G is a lie group (has manifold structure.)

It just remains to argue Corollary.

Let σ be the associated representation of the lie-Algebra $\mathfrak{h}$ of H.

Basis elements of H are given by :

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

with $[A,C] = B \qquad [A,B] = [C,B] = 0$

so that $[\sigma(A), \sigma(C)] = \sigma(B) \qquad [\sigma(A), \sigma(B)] = [\sigma(B), \sigma(C)] = 0$

we claim that $\sigma(B)$ is nilpotent. In light of SN decomposition, all its eigenvalue are zero.

Let λ be an eigenvalue for $\sigma(B)$ and V_λ its associated eigenspace.

So V_λ is invariant under $\sigma(B)$. since both $\sigma(A), \sigma(C)$ commute with $\sigma(B)$, they also leave V_λ invariant.

The restriction of $\sigma(A), \sigma(B), \sigma(C)$ on V_λ also satisfies the same commutation relations.

By $[\sigma(A)|_{V_\lambda}, \sigma(C)|_{V_\lambda}] = \sigma(B)|_{V_\lambda} \Longrightarrow Tr(\sigma(B)|_{V_\lambda}) = 0$.

on the other hand, V_λ is invariant subspace of $\sigma(B)|_{V_\lambda}$

so $\sigma(B)|_{V_\lambda} = \lambda I$

why? Because $(\sigma(B) - \lambda I)^{k} |v\rangle = 0 \longrightarrow$ on V_λ, $\sigma(B)|_{V_\lambda}$ acts like identity.

$|v\rangle \in V_\lambda$

$$\rightarrow \mathrm{Tr}(\sigma(B)|_{V_\lambda}) = \lambda \underbrace{\dim(V_\lambda)}_{\neq 0} \implies$$

for $\mathrm{Tr}(\sigma(B)|_{V_\lambda}) = 0 \longrightarrow \boxed{\lambda = 0}$

so zero is the only eigenvalue of $\sigma(B) \rightarrow \sigma(B)$ is nilpotent.

Corollary: If X is $\neq 0$ nilpotent matrix, for all $t \neq 0 \rightarrow e^{tX} \neq I$.

since X is nilpotent $\rightarrow$ the expansion terminates.

Let $(e^{tX})_{kl} = P_{kl}(t)$ (polynomial depends on t)

suppose for $t_0 \neq 0 \longrightarrow e^{t_0 X} = I \xrightarrow{(\,)^n} e^{n t_0 X} = I$ $(\forall n)$

In terms of polynomials $\rightarrow$ $P_{kl}(n t_0) = \delta_{kl}$

But a polynomial that has ∞ zeros, must be constant

$\Rightarrow$ As long as there's an $t_0 \in \mathbb{R} - \{0\}$ that $e^{t_0 X} = I \longrightarrow$

$$e^{tX} = I$$

so that $X = \frac{d}{dt}(\exp(tX))\Big|_{t=0} = 0$ ※

so if X is non-zero nilpotent $\rightarrow e^{tX}$ must $\neq I$, for all $t \neq 0$.

Now $\sigma(B)$ is nilpotent (non-zero) matrix. $(B^2 = 0 \rightarrow \sigma(B)^2 = 0)$

and $e^{tB} = \begin{pmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{pmatrix}$

Let Σ be rep of H and $\ker \Sigma \supset \ker \phi$

$$e^{2\pi n \sigma(B)} = \Sigma(e^{2\pi n B}) = I$$

since $\sigma(B)$ is nilpotent (according to the lemma) $\rightarrow \sigma(B)$ has to be zero. $\longrightarrow \forall t \in \mathbb{R}$

$$\Sigma(e^{tB}) = e^{t\sigma(B)} = I$$

$$\Longrightarrow \ker \Sigma \supset Z(H)$$

Let's look at another technical example.

$SL(n,\mathbb{R})$ $(n \geq 2)$ is not simply-connected

while

$SL(n,\mathbb{C})$ $(n \geq 2)$ is simply-connected.

Universal covery of $\tilde{SL}(n,\mathbb{R})$ and

$\Phi : \tilde{SL}(n,\mathbb{R}) \longrightarrow SL(n,\mathbb{R})$ is assumed!

associated $\phi : \tilde{sl}(n,\mathbb{R}) \longrightarrow sl(n,\mathbb{R})$ is isomorphism (lie-alg isomph.)

Theorem: There's no any faithful finite-dim representation of $\tilde{SL}(n,\mathbb{R}) \rightarrow \tilde{SL}(n,\mathbb{R})$ is a non-matrix Lie-grup.

If Π is any finite-dim rep of $\tilde{SL}(n,\mathbb{R})$, thus $\ker \Pi$ contains $\ker \Phi$ $(\Phi : \tilde{SL}(n,\mathbb{R}) \longrightarrow SL(n,\mathbb{R}))$

By above, since $SL(n,\mathbb{R})$ is not simply-connected $\Longrightarrow$ Φ has non-trivial kernel $\Longrightarrow$ Π is not faithful $\rightarrow$ just like above.

The proof is beyond our knowledge and uses homological sequences!

I just wanted to introduce it as another non-matrix lie grp.

The map of Lie groups & algebra:

Matrix Lie groups

گروه اقلیدسی

GL / SL / U / SU / O / SO / Sp / H / E / p(n,1)

گروه پوانکاره

their properties → Connected, Compact, dimension, Homotopy groups, Homology groups, Their relationships, ...

Abstract definition → As a manifold.

Lie Algebra $\xrightarrow{\exp}$ Lie groups.

T_eG is a vector space called Lie-Algebra.

(Namely, all matrixes A such that e^{tA} $(t\in\mathbb{R})$ is in liegrp, is an element of Lie-algebra.)

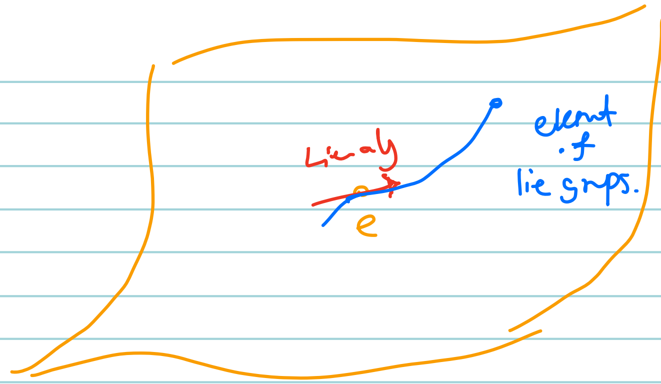
Technically $\mathfrak{g} \cong T_eG$.

It's a vector space of matrices.

Lie group homomorphism Corresponds to lie-alg homomorph.

Exponential mapping:

$\exp : \mathfrak{g} \to G$



Lie alg. properties:

1. $[\cdot,\cdot]$ is linear
2. $[A,B] = -[B,A] \quad \forall A,B \in \mathfrak{g}$
3. $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ (Jacobi)

$$[X_i, X_j] = \underbrace{f_{ij}^{\ k}}_{\text{structure constants.}} X_k$$

There are so many topics about Lie-algebras.

* Complexification
* BCH lemma
* $\oplus$ and $\otimes$ of them.
* Real lie algebra

with some basic knowledge of representation theory:

one can classify lie algebras according to:

I- $\left\{ \begin{array}{l} \text{Simple} \\ \text{semi-simple} \end{array} \right.$

14 52 78 133 248 ...

exceptional lie Algs. (g_2, f_4, e_6, e_7, e_8)

$R \to \begin{cases} \text{simple} \to \text{on invariant - sub algebra.} \\ \text{Semi-simple} \end{cases}$

Then the classical path is to study their reprentations follow ideas, of weyl, verma, Cartan, and find the root systems.

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problems: show that the follow is a Lie-algebra.

vectors with $\times$ product.

sol:

(a) bilinearity : $(\lambda v + \mu w) \times u = \lambda v \times u + \mu w \times u$

$u \times (\lambda v + \mu w) = \lambda\, u \times v + \mu\, u \times w$

(b) $u \times w = -w \times u$ obviously (Componentwise.)

(c) $\underbrace{(u \times v) \times w}_{\cancel{(w \cdot u) v} - \cancel{(w \cdot v) u}} + \overbrace{(v \times w) \times u}^{\cancel{(u \cdot v) w} - \cancel{(u \cdot w) v}} + \underbrace{(w \times u) \times v}_{\cancel{(v \cdot w) u} - \cancel{(v \cdot u) w}} = ?$

$a \times (b \times c) = b (a \cdot c) - c (a \cdot b)$ BAC - CAB

= zero

so it's a lie-algebra!

problem: (1) Determine lie-algebra of H.
(2) prove exp is diffeo from $\mathfrak{h}$ to H.

(1) The bases are simply $\partial_{\epsilon_1}, \partial_{\epsilon_2}, \partial_{\epsilon_3}$ of $\begin{pmatrix} 1 & \epsilon_1 & \epsilon_2 \\ & 1 & \epsilon_3 \\ & & 1 \end{pmatrix}$

$$\to \mathfrak{h} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

$$= \left\{ \begin{pmatrix} 0 & a & b \\ & 0 & c \\ & & 0 \end{pmatrix} ; a, b, c \in \mathbb{R} \right\}$$

(2) $\exp(M) = \sum_n \frac{M^n}{n!}$

Note that $\begin{pmatrix} & a & b \\ & & c \\ & & \end{pmatrix}^3 = 0$

$$\exp\left(\begin{pmatrix} \ddots & a & b \\ & \ddots & c \\ & & \ddots \end{pmatrix}\right) = I + \begin{pmatrix} \ddots & a & b \\ & \ddots & c \\ & & \ddots \end{pmatrix} + \frac{1}{2}\begin{pmatrix} & a & b \\ & & c \\ & & \end{pmatrix}^2$$

$$= \begin{pmatrix} 1 & a & b + \frac{ac}{2} \\ & 1 & c \\ & & 1 \end{pmatrix}$$

problem: prove that $\frac{O(n+1)}{O(n)}$ and $\frac{SO(n+1)}{SO(n)}$ are homogens spaces, which S^n is diffeomphic to them.

By means of the map

$O(n) \longrightarrow O(n+1)$

$$A \longmapsto \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A & \\ 0 & & & \end{array}\right)$$

To prove homogenity :

(i) a C^∞ action of $O(n+1)$ { or $SO(n+1)$ } to S^n.

(ii) It's transitive action

(iii) Isotropy group H_p is isomorphic to $O(n)$ ($SO(n)$.

(i) The action $GL(n+1, \mathbb{R})$ on $\mathbb{R}^{n+1}$ by matrix product,

is C^∞ (why ? It's just linear Combination!)

and since $O(n)$ is closed in $G(n)$, Its restriction

to $O(n+1) \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is C^∞.

And since $O(n+1)$ preserves the length of vectors,

it takes $S^n \cong$ unit vectors in $\mathbb{R}^{n+1}$ to itself.

$\Longrightarrow$ $O(n+1)$ acts on S^n diffeomorphically.

(ii) For any pair $p, q \in S^n$ then indeed exists $A \in O(n+1)$

such that $Ap = q$.

There are actually infinitely many of them.
One can define other elements of A arbitrarly,
so the action is transitive.

$(\widetilde{iii})$ Take (WLOG) $p=(1,0,\dots,0)$

$$H_p = \{A\in O(n+1) \mid Ap=p\}$$

so we have $A=(a_{ij})$ with $a_{11}=1$
$a_{i1}=0$ for $i\geq 2$.

Since $A\in O(n+1) \to A^TA = AA^T = \mathbb{1}$.

So $\underbrace{A^TAp}_{\substack{\text{by def} \\ =p}} = p \Rightarrow A^Tp = p$

$$\Rightarrow H_p = \left\{ \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right) \in O(n+1) \right\}$$

But $A^TA = \mathbb{1} \longrightarrow B^TB = I \longrightarrow B\in O(n)$

$\Rightarrow H_p \simeq O(n)$

Hence the $S^n = \dfrac{O(n+1)}{O(n)}$ is established.

The similar arguments show us that $S^n = \dfrac{SO(n+1)}{SO(n)}$

$\boxtimes$