Introduction to General Relativity - HW 2 - 401208729

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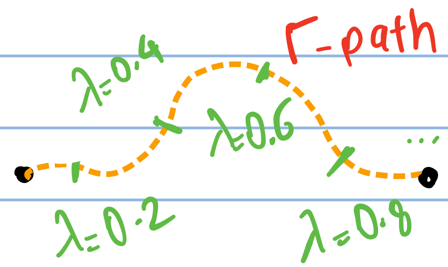
Q1.



$$S = \alpha \int_\Gamma d\tau$$

a) Let's parametrize path Γ by λ parameter.

Hence, the taken path is $x^\mu(\lambda)$.

(By definition of proper time, $ds^2 = c^2 d\tau^2$, so $ds = c\, d\tau$)

$$\underbrace{ds^2}_{\text{line-element squared.}} = c^2 d\tau^2 = \eta_{\mu\nu} \frac{dx^\mu(\lambda)}{d\lambda} \frac{dx^\nu(\lambda)}{d\lambda} \times d\lambda^2$$

$$\Rightarrow d\tau = \frac{1}{c} \times \sqrt{+\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

$$\rightarrow \quad S = \frac{\alpha}{c} \int_\Gamma \left(+\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda$$

Notice that, If we're going to end up with Lagrangian at "non-relativistic" limit, we should choose "time" as our parameter.

or $\lambda = t$

Thereby,

$$S = \frac{\alpha}{c} \int_{\Gamma} \left(+\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right)^{1/2} dt$$

Now use $dt = \gamma d\tau$

$$S = \frac{\alpha}{c} \int_{\Gamma} \frac{1}{\gamma} \left(+\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)^{1/2} d\tau$$

Remember that $\frac{dx^\mu}{d\tau} = u^\mu \longrightarrow$ so $+\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} =$

$\eta_{\mu\nu}$ has mostly minus or $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$$+\eta_{\mu\nu} u^\mu u^\nu = \gamma^2 (c^2 - u^2) = \frac{1}{\cancel{1 - u^2/c^2}} \times c^2 \times \cancel{\left(1 - \frac{u^2}{c^2}\right)}$$
$$= c^2$$

Hence,

$$S = \alpha \int_{\Gamma} \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt$$

in non-rel. limit, $v/c \ll 1$, hence

$$S = \alpha \int_{\Gamma} \left(1 - \frac{v^2}{2c^2}\right) dt$$

$$= \int_{\Gamma} \left(\alpha - \frac{\alpha v^2}{2c^2}\right) dt$$

The first term is Constant and accounts for rest - mass Contribution to Lagrangian.
(Notice that we can add any Constant term to Lagrangian regardless of its value, hence its not a significant contribution to the classical action).

To obtain $\frac{1}{2}mv^2$ as "Non-rel." Lagrangian we have to:

$$-\frac{\alpha \cancel{v^2}}{\cancel{2}c^2} = \frac{1}{\cancel{2}} m \cancel{v^2} \Longrightarrow \alpha = -mc^2 \quad \blacksquare$$

b) start with τ - parametrization:

$$S_0 = -mc^2 \int_\Gamma \overbrace{\sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}}^{L} \, d\tau$$

$$x^\mu \to x^\mu + \delta x^\mu$$

Notice that since $\delta \sqrt{\#} = \frac{\delta \#}{2\sqrt{\#}}$,

Hence the roots of $\delta\sqrt{\#}$ is equal to

the roots of $\delta \# = 0$, we can safely neglect the square root.

$$S_0 \to S' = -mc^2 \int_\Gamma \Big(+ \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

$$+ 2\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d(\delta x^\nu)}{d\tau}$$

$$+ O((\delta x^\mu)^2) \Big) \, d\tau$$

$$= S_0 - 2mc^2 \int_\Gamma \Big(\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \underbrace{\frac{d(\delta x^\nu)}{d\tau}}_{\text{By part Integral}} \Big) d\tau$$

$\cancel{+ O(\delta^2)}$

neglect

$$\Rightarrow \delta S = S' - S_0 = -2mc \int_\Gamma \frac{d}{d\tau}\Big(\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \Big) d\tau$$

$$+ 2mc \int_\Gamma \eta_{\mu\nu} \frac{d}{d\tau}\Big(\frac{dx^\mu}{d\tau} \Big) d\tau$$

This is $\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \Big|_{\tau=\tau_i}^{\tau=\tau_f}$

since δx^μ (variations) vanish at endpoints,

$$\left(\delta x^\mu \Big|_{\tau=\tau_i} = \delta x^\mu \Big|_{\tau=\tau_f} = 0 \right)$$

Here it vanishes.

Therefore for obeying Hamilton's principle ($\delta S = 0$) for any δx^μ, $\frac{d}{d\tau}\left(\underbrace{\frac{dx^\mu}{d\tau}}_{\text{this is } u^\mu} \right) = 0$

$$\Rightarrow \frac{d}{d\tau} u^\mu = 0 \qquad \blacksquare$$

Q2. $K^\mu = \frac{d}{d\tau} P^\mu = \frac{d}{d\tau}(m_0 u^\mu) \xrightarrow{dt = \gamma\, d\tau \text{ from S.R}}$

$\gamma \frac{d}{dt}(m_0 u^\mu) \xrightarrow[\text{where}]{} \begin{cases} u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v}) \\ \gamma = (1-(v/c)^2)^{-1/2} \end{cases}$

Note that $\vec{F}$ is not the usual force, which is $m\vec{a}$, but is $\vec{F} = \frac{d}{dt}(m_0 \gamma \vec{v})$ $= \frac{d}{dt}(\vec{p})$ by definition of S.R. books.

Hence:

$$K^\mu = \gamma \frac{d}{dt}\{m_0 \gamma (c, \vec{v})\}$$

$$= \gamma \left(\frac{d}{dt}(m_0 \gamma c), \underbrace{\frac{d}{dt}(m_0 \gamma \vec{V})}_{\text{This is } \vec{F} \text{ (new def.)}} \right)$$

Let's check $\frac{d}{dt}(m_0 \gamma c)$, this should be $\frac{\vec{F} \cdot \vec{u}}{c}$.

$$\frac{d}{dt}(\gamma) = \frac{d}{dt}\left(1-(v/c)^2\right)^{-1/2} = \frac{dv}{dt} \times v/c^2 \times \left(1-(v/c)^2\right)^{-3/2}$$

$$= a \times \beta/c \times \gamma^3 \qquad \boxed{\beta = v/c}$$

where $a = \frac{d|v|}{dt}$ is magnitude of accelration vector.

$$\Rightarrow \frac{d}{dt}(m_0 \gamma c) = m_0 c \times a \times \beta/c\, \gamma^3$$
$$= m_0 a \beta \gamma^3$$

On the other side,

$$\vec{F} = m_0 \left(\left(\frac{d}{dt}\gamma \right) \vec{v} + \gamma \vec{a} \right)$$
$$= m_0 \left(a \times \beta/c \times \gamma^3 \vec{v} + \gamma \vec{a} \right)$$

Hence, $\vec{F}.\vec{v}$ is equal to :

$$\vec{F}\cdot\vec{u} = m_0 \left(a \beta/c\, \gamma^3 v^2 + \gamma \vec{a}\cdot\vec{v} \right)$$

To be able to proceed further we should think how to alter $\vec{a}.\vec{v}$ term?

$$\vec{v}\cdot\vec{v} = v^2 \longrightarrow d/dt (\vec{v}\cdot\vec{v}) = 2v \frac{dv}{dt}$$

$$\rightarrow \cancel{2}\, \vec{a}\cdot\vec{v} = \cancel{2} v\, dv/dt \rightarrow \vec{a}\cdot\vec{v} = |a|\,|v|$$

Replace it in below :

$= m_0 \left(a \frac{\beta}{c} \gamma^3 v^2 + \gamma a v \right) \longrightarrow$ Factor

$= m_0 \gamma v a \left(1 + \gamma^2 \frac{\beta v}{c} \right) =$

$= m_0 \gamma a v \left(1 + \frac{\beta^2}{1 - \beta^2} \right) =$

$= m_0 \gamma a v \left(\frac{1 - \cancel{\beta^2} + \cancel{\beta^2}}{1 - \beta^2} \right)$

$= m_0 a \times v \times \gamma^3$

Therfore $\frac{\vec{F} \cdot \vec{u}}{c}$ is $m_0 a \beta \gamma^3$, or

equal to $\frac{d}{dt} (m_0 \gamma c)$.

Indeed, $K^\mu = \gamma \left(\frac{\vec{F} \cdot \vec{u}}{c}, \vec{F} \right)$

Q3. $u^\mu . u_\mu = -c^2 \rightarrow$ derivative w.r.t τ,

$$2\frac{du^\mu}{d\tau} . u_\mu = 0 \rightarrow a^\mu . u_\mu = 0$$

Hence a^μ is $\perp$ to u_μ.

For proving that a^μ is space likes

since u^μ is time likes go to a frame where it is of form $u^\mu = (c, 0, 0, 0)$.

(This is always possible.)

Now if $a^\mu = (a^0, a^1, a^2, a^3)$, then

for $a^\mu . u_\mu = 0 \rightarrow c . a^0 + \cancel{0 . a} = 0$

hence $a_0 = 0 \rightarrow a^\mu = (0, a^1, a^2, a^3)$

$$\Rightarrow a^\mu a_\mu = +(a^1)^2 + (a^2)^2 + (a^3)^2 > 0$$ spacelik

(Notice that a^i are not usual acceleration, we don't need them too.)

b) $a^\gamma = \frac{d}{d\tau} u^\gamma$

For a single particle, we can follow its trajectory and $\frac{d}{d\tau} u^\gamma$ is meaningful.

But for a fluid we can no longer keep track of particles, hence it's better to re-interprete

$\frac{d}{d\tau}$ → chain rule $\frac{d}{d\tau} = \underbrace{\frac{dx^\alpha}{d\tau}}_{u^\alpha} \frac{\partial}{\partial x^\alpha}$

This means that to find acceleration in fluid, just you should have velocity field and its gradient.

$$\Rightarrow a^\gamma = \frac{d}{d\tau} u^\gamma = u^\alpha \partial_\alpha u^\gamma$$

Q4. $T^{\mu\nu} = (\rho_0 + P) U^\mu U^\nu + P\eta^{\mu\nu}$ and $P = P(\rho)$

For dust, $P = 0$: $T^{\mu\nu} = \rho_0 U^\mu U^\nu$

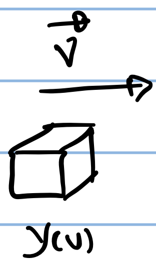
Consider $u^\alpha = \gamma(c, \vec{u})$ in any frame,

$$\rightarrow T^{\mu\nu} = \underbrace{\rho_0 \gamma^2}_{\rho(x)} \begin{pmatrix} c^2 & cu_x & cu_y & cu_z \\ cu_x & u_x^2 & u_x u_y & u_x u_z \\ cu_y & u_x u_y & u_y^2 & u_y u_z \\ cu_z & u_z u_x & u_z u_y & u_z^2 \end{pmatrix}$$

Note that we call $\rho_0 \gamma^2 = \rho$, which is density in the observer's frame.

One can justify its dependence on γ^2 by this spectacular argument:

$\rho_0 = \dfrac{m_0}{V_0}$ → In any frame, you should multiply rest-mass (m_0) by γ to accout for the relativistic transformation of mass.



Besides, length along the motion will be Contracted by $\frac{1}{\gamma}$ factor.

$$\rho_{new-frame} = \frac{\gamma m_0}{\frac{1}{\gamma} V_0} = \gamma^2 \rho_0 \qquad \blacksquare$$

Hence, $T^{\mu\nu}$ is energy-momentum Tensor in any frame!

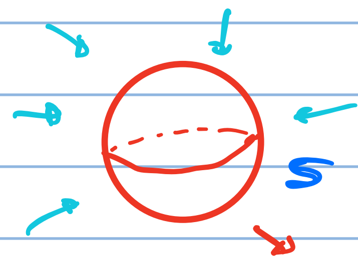
Now let's calculate $\partial_\mu T^{\mu\nu} = 0$ for $\nu = 0$ →

$$\partial_\mu T^{\mu 0} = \frac{1}{\cancel{c}} \frac{\partial}{\partial t}(\rho \cancel{c^2}) + \cancel{c}\,\vec{\nabla}\cdot(\rho \vec{u}) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\rho \vec{u}) = 0$$

This is continuity equation for ideal fluid!

Describing the flow of fluid within an infinitesimal region.



The flux of fluid in the surface S ~ density's rate of change!

or integral form: $\frac{\partial}{\partial t} \int_V \rho(\vec{x})\, d^3x = -\oint_{S=\partial V} \rho \vec{u}\cdot d\vec{S}$

Now let's workout the i-components.

$$\partial_\mu T^{\mu i} = \partial_0 T^{0i} + \partial_j T^{ji} = \frac{1}{\cancel{c}} \frac{\partial}{\partial t}(\rho \cancel{c} u^i) + \partial_j(\rho u^i u^j) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho u^i) + \partial_j(\rho u^i u^j) = 0$$

These equations govern the momentum of the fluid, also they are the famous Nävier – stokes equation that are derived from Newton's second law.

We can also utilize definition of energy – momentum tensor's components to interpret equations:

$$T^{00} = \rho \times c^2 \sim \frac{m}{V} c^2 \sim \frac{E}{V}$$ so this the Relativistic energy density of the fluid.

$$T^{0i} = c \rho u^i \sim c \times \frac{m u^i}{V} \sim \frac{\text{Momentum}}{\text{vol.}}$$

so this is momentum density of the fluid.

$$T^{ij} = \rho u^i u^j = \frac{m u^i u^j}{V}$$, this indicates how much momentum – flux passes through the j-th direction!

Thereby, $\partial_\mu T^{\mu 0}$ relates momentum density to energy density $\rightarrow$ hence its conservation equation.

$\partial_\mu T^{\mu i}$ Relates momentum density to momentum flux which is known as momentum-current conservation.