- ☐ Mobius strip ✓ (Comment Φ-structure ...) $\longrightarrow$ orientable
- ☐ An exercise (d, $\wedge$, i_X, ...)
- ☐ Lie derivative identities
- ☐ $[X, Y] = 0 \longleftrightarrow \Phi_{X,t} \circ \Phi_{Y,s} = \Phi_{Y,s} \circ \Phi_{X,t}$

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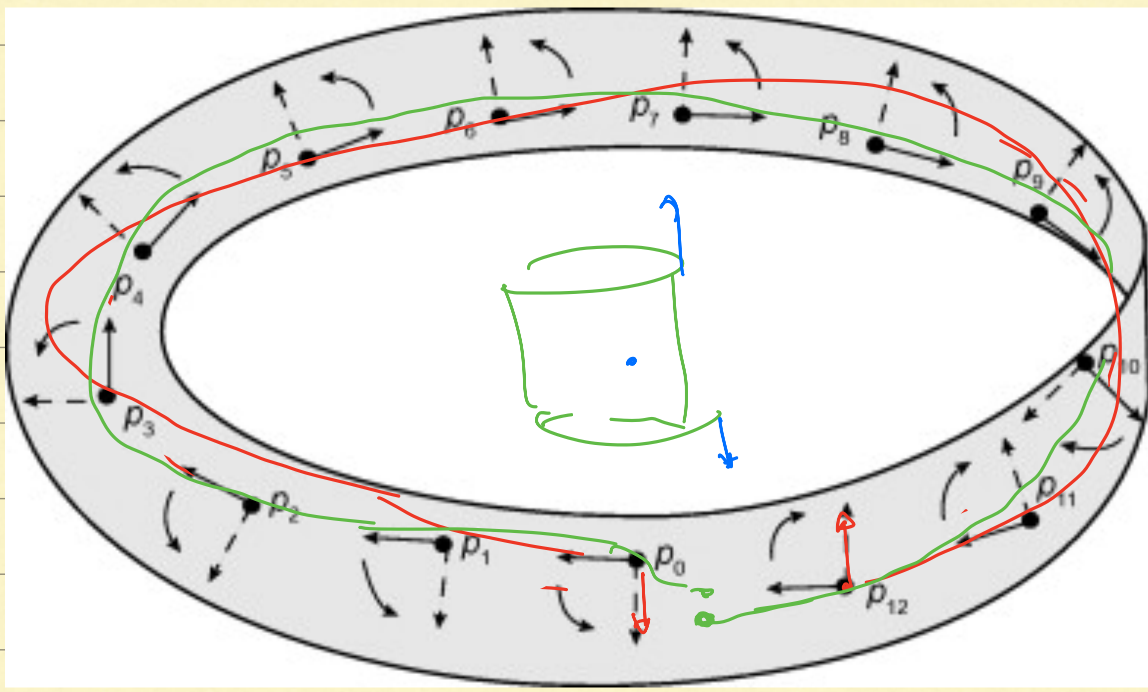
https://www.youtube.com/watch?v=2nUXOiYxT98



$\forall$ all chart !

$\phi_\alpha \circ \phi_\beta^{-1}$

$\phi_\beta(U_\alpha \cap U_\beta)$

$\downarrow$

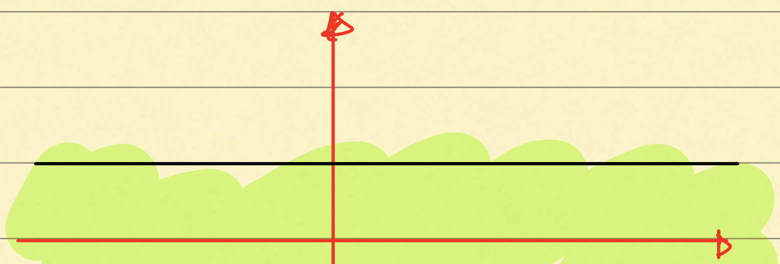
$\phi_\alpha(U_\alpha \cap U_\beta)$

$J(\phi_\alpha \circ \phi_\beta^{-1}) > 0$

$M = \{ (x, y) \in \mathbb{R}^2 \mid -\infty < x < +\infty , -1 < y < 1 \}$

$L > 0 : \quad (x + L, -y) \sim (x, y)$

$\pi : M \longrightarrow \hat{M}$: [illegible]



utilized $(U_\alpha, \phi_\alpha)_{\alpha \in I}$.

$x \in \mathbb{R}$ $\quad \underbrace{\pi : (x, 0)} \longrightarrow \hat{M}$

$\exists \alpha : \alpha \in I \quad \pi(x, \cdot) \in \underline{U_\alpha} \, , \, \underline{\phi_\alpha}$

$f : \phi_\alpha^{-1} \circ \pi$

This is diffeomorphism between $M, \hat{M}$

$J(x) := \operatorname{sgn}\left(\underbrace{\det(f(x, \cdot))}\right)$

Exercise: $J(x)$ is locally constant.

$f(x + L, -y) = -f(x, y)$

$a \qquad -a$

$J(x + L) = -J(x)$

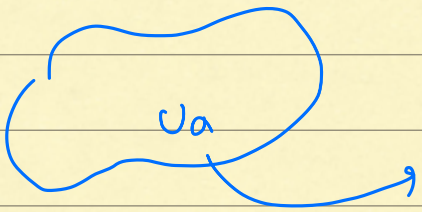
$J(x)$ is locally Cnst.

$J(x) \equiv 0$

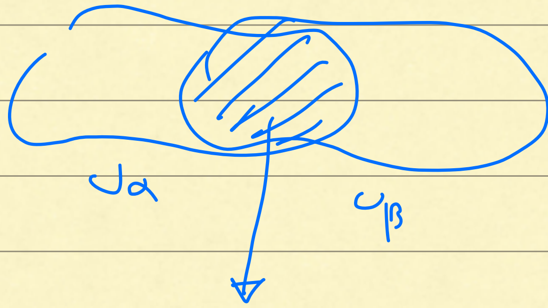
ϕ diffeo: $\quad d\Phi \quad \left(\frac{\partial y^\alpha}{\partial x^\beta} \right) \longrightarrow$ 1-1 onto ...

Corollary: Complex-manifolds are orientable.

Role model $\mathbb{R}^n \longrightarrow \mathbb{C}$



$\mathbb{C}^n$



$\varphi_\beta \circ \varphi_\alpha^{-1}$: is holomorphic (C-R-conditions)

$\mathbb{C}^1 \quad \mathbb{C}^1 \qquad \mathbb{C}^n$; $p \in M$: $z^1, \dots, z^n$

Cordinate $\to$ charcter

$\underset{U_\alpha}{(z_\alpha)}, \underset{U_\beta}{(z_\beta)} \mapsto$ on $U_\alpha \cap U_\beta \Rightarrow \quad z_\alpha \circ z_\beta^{-1}$: $h_{\alpha\beta}$

$h_{\alpha\beta} = u_{\alpha\beta}^{(x,y)} + i\, v_{\alpha\beta}^{(x,y)}$

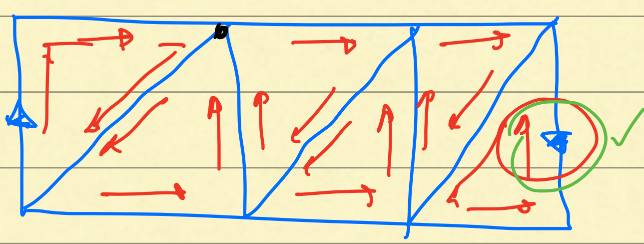
$$\text{Jacobian}\,(h_{\alpha\beta}) = \det \begin{pmatrix} \dfrac{\partial u_{\alpha\beta}}{\partial x} & \dfrac{\partial v_{\alpha\beta}}{\partial x} \\[2ex] \dfrac{\partial u_{\alpha\beta}}{\partial y} & \dfrac{\partial v_{\alpha\beta}}{\partial y} \end{pmatrix}$$

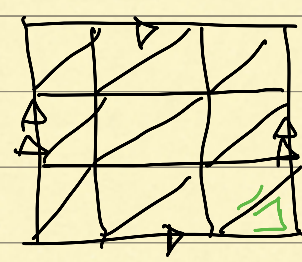
$z = x + iy$

$$g(z) := u(x,y) + i\, v(x,y)$$

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

$$\partial_x u \underset{\partial_x u}{(\partial_y v)} - \partial_x v \underset{-\partial_x v}{(\partial_y u)} = \underbrace{(\partial_x u)^2 + (\partial_x v)^2}_{\neq 0} > 0$$

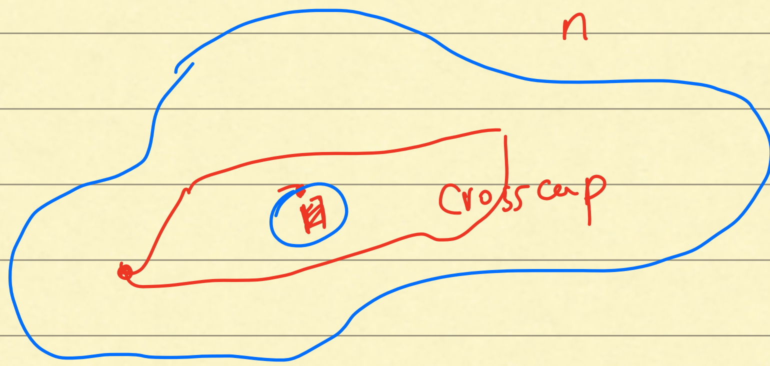




Mobius strip ...

$\mathbb{RP}^2$:

Mobius $\longrightarrow$ Fundamental non-orient. mfld.



$i_X,\ d,\ \mathcal{L},\ [\ \ ], \dots$

e.g. $\mathcal{L} d$?

$$\mathcal{L}_X \omega = \underbrace{(d\, i_X + i_X\, d)}\,\omega$$

$$\omega = f \; dx^i \wedge \dots \wedge \dots$$

$r \in \Omega^k(X)$ is exact. $\qquad r = d\xi \qquad \xi \in \Omega^{k-1}(X)$

$\omega \in \Omega^\ell(X)$ is closed. $\qquad d\omega = 0$

$r \wedge \omega$ is exact. $\qquad r \wedge \omega \in \Omega^{k+\ell}(X)$

$$— \in \Omega^{k+l-1}(X)$$

$$\underset{\sim (r,)}{L_{-}} \qquad d(—) = r \wedge \omega$$

$$— = \xi \wedge \omega \to \underline{\underline{d(\xi \wedge \omega)}} = \underset{r}{\underbrace{d\xi}} \wedge \omega + \cancel{(-1)^{k-1} \xi \wedge d\omega}$$

$$d(\xi \wedge \omega) = \underbrace{r \wedge \omega} \quad \text{exact}$$

$$\underline{\text{Exact} \wedge \text{closed} \equiv \text{exact}}$$

De Rham $\equiv$ Dual singular Homology

$$\frac{\text{Closed}}{\text{Exact}} \qquad H^0 \wedge H^0$$

~~Homology!~~

$$\bigoplus_{k} H^0(k)$$

$$\mathcal{L}_{fX} Y = \underbrace{[fX, Y]} = f\mathcal{L}_X Y - \underline{\underline{df(Y)}} X$$

$$[fX, Y]^{\beta} = (fX)^{\alpha} \partial_{\alpha} Y^{\beta} - Y^{\alpha} \partial_{\alpha} (fX)^{\beta}$$

$$f X^{\alpha} \partial_{\alpha} Y^{\beta} - f Y_{\alpha} \partial_{\alpha} X^{\beta}$$

$$- Y^{\alpha} X^{\beta} \partial_{\alpha} f \qquad Y[f]$$

$$f([X,Y]^\beta) - \underbrace{Y(f)}_{\text{??}} X^\beta$$

$$\Rightarrow \quad [fX, Y] = \underbrace{f[X,Y]}_{\mathcal{L}_X Y} - \underbrace{Y(f)}\, X$$

p

$$Y(f) = df(Y)$$

df

i_X

$$\underbrace{(\mathcal{L}_{fX}\,\omega)(Y)} = (f\,\mathcal{L}_X\,\omega + \omega(X)\, df)(Y) \quad (8.50)$$

$$\mathcal{L}_{fX}\underbrace{(\omega(Y))} - \omega(\underbrace{\mathcal{L}_{fX} Y}_{[fX,Y]})$$

$$(fX)(\omega(Y)) - \omega([fX,Y]) =$$

$$(fX)(\omega(Y)) - \omega(f[X,Y] - \underbrace{Y(f)}\, X)$$

$$f\,\underbrace{X}(\underbrace{\omega(Y)}) - f\,\omega([X,Y]) + \underbrace{Y(f)}_{df(Y)}\,\omega(X)$$

$$f(\underbrace{\mathcal{L}_X\,\omega(Y) - \omega(\mathcal{L}_X Y)}) + df(Y)\,\omega(X)$$

8.50

$$\big(f(\mathcal{L}_X\omega)Y\big) + df(Y)\,\omega(X)$$

$$= \big(f\,\mathcal{L}_X\omega + \omega(X)\,df\big)\,Y$$

$\varphi : M \to N$ M, N C^∞, manifold,

$$X \in \mathcal{X}(M) \qquad (\varphi_* X)_x \equiv \varphi^*\big(\cancel{X}_{\underbrace{\varphi^{-1}(x)}}\big), \quad x \in N$$

1) $(\varphi_* X)_x \in T_x N$

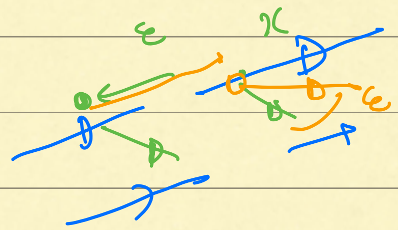
2) $\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y]$

$X, Y \in \mathcal{X}(M)$ φ_t is flow X.

$$\varphi_s^*\Big(\underbrace{(\mathcal{L}_X Y)}\,\varphi^{-1}_s(p)\Big) = \lim_{t\to 0} \frac{1}{t}\Big(\varphi_s^* Y_{\varphi^{-1}_s(p)} - \varphi^*_{s+t} Y_{\varphi^{-1}_{s+t}(p)}\Big)$$

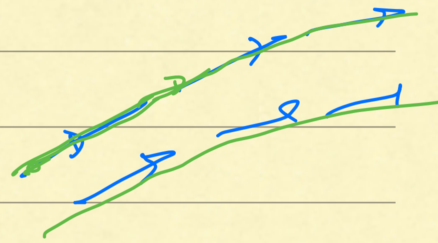
ادعا : $\varphi_t^* X\big| = X$

$$\mathcal{L}_X Y = \lim_{\varepsilon \to 0} \frac{Y|_x - (\varphi_\varepsilon)^* Y|_{\varphi_{-\varepsilon}(x)}}{\varepsilon}$$



$\mathcal{L}_X X = 0$

$[X, X] = 0$



$$Y|_x = (\sigma_\varepsilon)^* Y|_{\sigma_{-\varepsilon}(x)}$$

$$\varphi_s \cdot (\mathcal{L}_X Y) = \varphi_s \cdot [X, Y] = [\varphi_s \cdot X, \varphi_s \cdot Y]$$

$$[X, \varphi_s \cdot Y]$$

$$= \mathcal{L}_X (\varphi_s \cdot Y)$$

$$\mathcal{L}_X (\varphi_s^* Y_{\varphi_s^{-1}(p)})$$

$$= \lim_{t \to 0} \frac{1}{t} \left\{ \varphi_s^* Y_{\varphi_s^{-1}(p)} - \varphi_t^* \left(\varphi_s^* Y_{\varphi_s^{-1}(p)} \right)_{\varphi_t^{-1}(p)} \right\}$$

$$(f \circ g)^* = g^* \circ f^*$$

$$\sigma_t \circ \sigma_s = \sigma_{t+s}$$

$$\lim_{t \to 0} \frac{1}{t} \left\{ \varphi_s^* Y_{\varphi_s^{-1}(p)} - \varphi_{t+s}^* Y_{\varphi_{s+t}^{-1}(p)} \right\}$$

$\varphi_t \quad \psi_s$

$$[X, Y] = 0 \longleftrightarrow \underbrace{\varphi_t \circ \psi_s = \psi_s \circ \varphi_t}$$

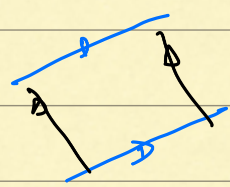
$\leftarrow$ Exercise : $\begin{cases} \varphi_t \cdot Y = Y \\ \psi_s \cdot Y = Y \end{cases}$

$$\psi_s \circ \varphi_t^* \quad (\varphi_t \circ \psi_s)(Y)$$

$$\psi_s^*(\varphi_t^*(Y))$$

$\Downarrow$

$\vdots$



$$\varphi_t \cdot Y = Y|_p$$

$$[X, Y] = \mathcal{L}_X Y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\frac{\cancel{Y|_p} - \varphi_t^* Y_{\varphi_t^{-1}(p)}}{\varepsilon} \right)$$

$= 0$

$$\overset{?}{\Longrightarrow} \quad \varphi_s^* \left((\mathcal{L}_X Y)_{\varphi_s^{-1}(p)} \right) = \lim_{t \to 0} \frac{1}{t} \left(\varphi_s^* Y_{\varphi_s^{-1}(p)} \right.$$

$\varphi_s \cdot ([X, \cancel{Y}])$ $\qquad \varphi_{s+t} \, Y \, \varphi^{-1}_{s+t}(p)$

je

$$= \frac{d}{dt}(\varphi_t \cdot Y)\Big|_{t=s}$$

$\varphi_t \cdot Y = \text{const at}$

$\varphi_0 \cdot Y = Y \longrightarrow \underbrace{\varphi_t \cdot Y = Y}$

$$\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$$

$$\varphi \circ \psi = \psi \circ \varphi$$

$$\varphi = \psi^{-1} \circ \varphi \circ \psi \xrightarrow{\text{push frwd}}$$