Some questions and notes :

Prove that if H, K are finite subgrps of G whose orders are relatively prime, then $H \cap K = \{e\}$.

- We know that $H \cap K$ is a subgroup of H & K.

$\rightarrow \; O(H \cap K) \mid O(H)$

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since $(O(K), O(H)) = 1 \implies O(H \cap K) = 1 \rightarrow H \cap K = \{e\}$ $\blacksquare$

If A is an Abelian group with $B \leq A$, sh that A/B is Abelian.

$\forall x, y \in A \rightarrow xB, yB \in A/B \rightarrow (xB)(yB) = (xy)B$

$= (yx)B = (yB)(xB)$ $\blacksquare$

Various Defs of normal subgroup ($H \leq G$)

① $\begin{array}{l} xH = x'H \\ yH = y'H \end{array} \implies xyH = x'y'H$

② $Ha = aH \quad \forall a \in G$

③ $\forall g \in G \quad gHg^{-1} = H$

④ $\forall g \in G \quad gHg^{-1} \subseteq H$

→ Weak Commutativity → $\forall g \in G$ $\forall h \in H$ $\exists h' \in H$: ⑤ $gh = h'g$ ⑤' $hg = gh'$

⑥ Any left Coset is a right Coset.

⑦ $\forall a, b \in H$: $(aH)(bH) = (ab)H$

ضرب زدوژی

Independence of representative :

suppose $a \neq c, b \neq d$: $aH = cH$, $bH = dH$ → $c^{-1}a \in H$, $d^{-1}b \in H$

To show $abH = cdH$ it suffices to show $(cd)^{-1}(ab) \in H$.

→ $(cd)^{-1}ab = d^{-1}c^{-1}ab = d^{-1}(\underbrace{(c c^{-1}a)}_{\in H}\underbrace{d)(d^{-1}}_{1=e} b)^{\in H}$

since $d^{-1}((c^{-1}a)d) \in H$ for $H \trianglelefteq G$ and $d^{-1}b \in H$ → $(cd)^{-1}ab \in H$ ☒

$2 \longleftrightarrow 3$ is obvious $\blacksquare$

$4 \longrightarrow 3$ (i) $gHg^{-1} \subseteq H \quad \forall g:$

so for $g^{-1}H(g^{-1})^{-1} = g^{-1}Hg \subseteq H \xrightarrow{gx \; xg^{-1}}$

$\overset{(ii)}{\Longrightarrow} H \subseteq gHg^{-1} \Longrightarrow$ so (i), (ii) $\longrightarrow gHg^{-1} = H$

$3 \longrightarrow 4$ obvious!

$4 \longrightarrow 5$ $ghg^{-1} \in gHg^{-1} \subseteq H$

$\Longrightarrow ghg^{-1} = h' \longrightarrow gh = h'g$ $\blacksquare$

$5 \longrightarrow 4$ $\forall h \in H \quad \exists h' \in H: \quad gh = h'g \longrightarrow ghg^{-1} = h'$

$\Longrightarrow gHg^{-1} \subseteq H$ $\blacksquare$

$2 \longrightarrow 1$

$xyH = x(yH) = x(Hy) \overset{\text{فرض}}{=} x(Hy') = (xH)y'$

$= (x'H)y' = (x'y'H)$ $\blacksquare$

$1 \longrightarrow 4$ obviously $gH = ghH \quad \forall h \in H$

$\hookrightarrow g^2H = (gH)(gH) = (ghH)(gH)$

$$= (ghg)H \xrightarrow{g^{-1}\times}$$

$$gH = (hg)H \xrightarrow{g^{-1}\times} H = g^{-1}hgH$$

$$\Longrightarrow g^{-1}hg \in H \Longrightarrow g^{-1}Hg \subseteq H$$

$\therefore$ this proceeds ✓✓

Example of $\frac{G}{H} \not\leq$ subgps of G.

Take $Q = \{\pm 1, \pm i, \pm j, \pm k\}$

and $H = \{\pm 1\}$ First $H \trianglelefteq G$ $\left\{ \begin{array}{l} \text{since } q \times 1 \times q = q^2 \overset{\text{or}}{=} \pm 1 \text{ for } q \in Q. \\ q(-1)q = -q^2 = \pm 1 \end{array} \right.$

$$\frac{Q}{H} = \{ qH \mid q \in Q \}$$

has cosets : $\{\{1,-1\}, \{i,-i\}, \{j,-j\}, \{k,-k\}\}$

with Cayley table :

	$\{-1,1\}$	$\{i,-i\}$	$\{j,-j\}$	$\{k,-k\}$
$\{1,-1\}$	$\{-1,1\}$	$\{i,-i\}$	$\{j,-j\}$	$\{k,-k\}$
$\{i,-i\}$	$\{i,-i\}$	$\{1,-1\}$	$\{k,-k\}$	$\{j,-j\}$
$\{j,-j\}$	$\{j,-j\}$	$\{k,-k\}$	$\{1,-1\}$	$\{i,-i\}$

$\{k, -k\}$ $\{1, -1\}$ $\{i, -j\}$ $\{j, -i\}$ $\{1, -1\}$

This is klein 4-group!

so $\frac{Q}{H} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

But order 4 subgrps of Q are:

$\{1, -1, i, -i\}$
$\{1, -1, j, -j\}$
$\{1, -1, k, -k\}$

$C_4 \to$ cyclic. $\langle i \rangle$ $\langle -i \rangle$ $\langle j \rangle$ $\langle -j \rangle$ $\langle k \rangle$ $\langle -k \rangle$

✓

Find all normal subgrps of S_4!

Just $\{e\}$, A_4, S_4!

(a_1, a_2)	6
$(a_1 a_2)(a_3, a_4)$	3
$(a_1\ a_2\ a_3)$	8
$(a_1 a_2 a_3\ a_4)$	6
trivial	1

By $\sigma(i_1 \dots i_r)\sigma^{-1}$
$= (\sigma i_1 \dots \sigma i_r)$

if any of these elements are in $S_4 \Rightarrow$ it must contain all of the same structure by $\sigma h \sigma^{-1} \in H$

S_4 24

$+ \quad O(H) \mid 24$

اگر عنصر مرتبه 2 باشد ← کل S_n را می‌سازد!

عنصر مرتبه 3 ← فقط جایگشت‌های زوج؟

$(e) + \overset{3}{(--)(--)} + \overset{8}{(---)}$

$= 1 + 8 + 3 = 12 \longrightarrow$ This is A_4.

دورهای به طول 4 فقط ←

$(e) + \{4 \text{ مرتبه}\} + (2 \text{ مرتبه}) = S_n$

ترکیب

پس تنها این‌ها هستند.