Introduction to General Relativity - HW6 - 401208729

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Q1. For diagonal $g_{\mu\nu}$, things get much easier.

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma} \right)$$

Case I: $\Gamma^{\lambda}_{\alpha\beta}$ and $\lambda \neq \alpha \neq \beta$.

Since $g_{\mu\nu}$ is diagonal $\Rightarrow$ $g^{\lambda\sigma}$ only contributes for $\sigma = \lambda$, then the paranthesis becomes $\left(g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda} \right)$ and since $\alpha \neq \beta \neq \lambda$, all these three terms which are off-diagonal, will vanish $\Rightarrow$ $\Gamma^{\lambda}_{\alpha\beta} = 0$

Case II: $\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\mu\sigma,\mu} + g_{\mu\sigma,\mu} - g_{\mu\mu,\sigma} \right)$

In this case, again $\sigma = \lambda$ contributes and only the last term in paranthesis is non-zero, since all the other terms are off-diagonal.

$$\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2} g^{\lambda\lambda} \partial_{\lambda} g_{\mu\mu}$$ (λ is not-summed over.)

And since elements of $g^{\mu\nu}$ are inverse of $g_{\mu\nu}$ (

Because metric is diagonal.)

$$\Rightarrow \Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu}$$

Case III: $\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2} g^{\lambda\sigma}(g_{\mu\sigma,\lambda} + g_{\lambda\sigma,\mu} - g_{\mu\lambda,\sigma})$

Again, σ has to be λ $(\sigma = \lambda)$ $\Rightarrow$

off-digonal

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}\underbrace{g^{\lambda\lambda}}_{=(g_{\lambda\lambda})^{-1}}(\cancel{g_{\mu\lambda,\lambda}} + g_{\lambda\lambda,\mu} - \cancel{g_{\mu\lambda,\lambda}})$$

$$= \frac{1}{2}\frac{\partial_\mu g_{\lambda\lambda}}{g_{\lambda\lambda}}$$

Remember that $\frac{\partial}{\partial x}\ln u = \frac{u'}{u}$, therefore,

$\frac{\partial_\mu g_{\lambda\lambda}}{g_{\lambda\lambda}}$ is $\partial_\mu \ln g_{\lambda\lambda}$.

To avoid minus sign inside logarithm, add an absolute value

$$\longrightarrow \Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2}\partial_\mu \ln|g_{\lambda\lambda}|$$

Finally use $(a \ln x = \ln x^a)$ relation.

$$\longrightarrow \Gamma^{\lambda}_{\mu\lambda} = \partial_\mu \ln\left(|g_{\lambda\lambda}|^{1/2}\right) = \partial_\mu \ln\left(\sqrt{|g_{\lambda\lambda}|}\right)$$ ▨

Case IV : $\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2} g^{\lambda\sigma}\left(g_{\lambda\sigma,\lambda} + g_{\lambda\sigma,\lambda} - g_{\lambda\lambda,\sigma}\right)$

$\rightarrow$ since $\lambda = \sigma \Rightarrow \Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2} g^{\lambda\lambda}\left(g_{\lambda\lambda,\lambda}\right)$

By the argument on the Case III ,

$$\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2}\left(g_{\lambda\lambda}\right)^{-1}\partial_\lambda\left(g_{\lambda\lambda}\right) = \frac{1}{2}\partial_\lambda \ln|g_{\lambda\lambda}|$$

$$= \frac{1}{2}\partial_\lambda \ln\left(\sqrt{|g_{\lambda\lambda}|}\right)$$ ▨

Q2. By definition:

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\mu}_{\nu\alpha} T^{\alpha\nu} + \Gamma^{\nu}_{\nu\beta} T^{\mu\beta}$$

we prove this relation: $\Gamma^{\mu}_{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\lambda \sqrt{|g|}$

we've observed in the classroom (when variating the Einstein Hilbert action) that

$$\delta g = +g\, g^{\mu\nu}\,\delta g_{\mu\nu} \qquad (4.68 \text{ Carroll})$$

so $\partial_\lambda \sqrt{|g|} = \frac{\partial_\lambda |g|}{2\sqrt{|g|}} = \frac{1}{2\sqrt{|g|}} \times |g|\, g^{\mu\nu}\,\delta g_{\mu\nu}$

$$\Rightarrow \frac{1}{\sqrt{|g|}}\partial_\lambda \sqrt{|g|} = +\frac{1}{2}\, g^{\mu\nu}\,\partial_\lambda g_{\mu\nu}$$

Note that $\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2}\underbrace{g^{\mu\nu}}_{\text{symmetric under } \mu\leftrightarrow\sigma}\left(g_{\mu\nu,\lambda} + \underbrace{g_{\lambda\nu,\mu} - g_{\mu\lambda;\nu}}_{\text{anti-symmetric under } \mu\leftrightarrow\nu}\right)$

$=$ they vanish!

$$= \frac{1}{2}\, g^{\mu\nu}\,\partial_\lambda g_{\mu\nu}.$$

So the expressions for $\Gamma^{\mu}_{\mu\lambda}$ and $\frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|})$ are equal. $\blacksquare$

Rewrite $\nabla_\nu T^{\mu\nu}$ as:

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\mu}_{\nu\alpha} T^{\alpha\nu} + \frac{1}{\sqrt{|g|}} \partial_\beta(\sqrt{|g|})\, T^{\mu\beta}$$

↳ factor $\frac{1}{\sqrt{|g|}}$ $\longrightarrow$

$$= \frac{1}{\sqrt{|g|}} \left\{ \sqrt{|g|}\, \partial_\nu T^{\mu\nu} + \sqrt{|g|}\, \Gamma^{\mu}_{\nu\alpha} T^{\alpha\nu} + \partial_\beta(\sqrt{|g|})\, T^{\mu\beta} \right\}$$

these terms are product derivative

dummy index to ν

$$= \frac{1}{\sqrt{|g|}} \partial_\nu\left(\sqrt{|g|}\, T^{\mu\nu}\right) + \Gamma^{\mu}_{\nu\alpha} T^{\alpha\nu} \quad \blacksquare$$

Q3. we provide two proofs, first one is an exact version of Carroll's book, the second one involves Jacobi identity and is a new proof.

proof 1.

Let's first work out the form of the $R_{\mu\nu\alpha\beta}$ in normal Coordinates.

In normal Coordinates, we have $\partial_\sigma g_{\mu\nu} = 0 \Rightarrow$ all Christoffel symbols vanish $\Rightarrow \Gamma^\mu_{\alpha\beta} = 0$

since $\Gamma = 0$

and $R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \cancel{\Gamma^\theta_{\beta\nu}\Gamma^\alpha_{\theta\mu}} - \cancel{\Gamma^\theta_{\beta\mu}\Gamma^\alpha_{\theta\nu}}$

Now $R_{\alpha\beta\mu\nu} = g_{\alpha\sigma} R^\sigma{}_{\beta\mu\nu} = g_{\alpha\sigma}(\partial_\mu \Gamma^\sigma_{\beta\nu} - \partial_\nu \Gamma^\sigma_{\beta\mu})$

since

$$g_{\alpha\sigma}\partial_\mu \Gamma^\sigma_{\beta\nu} = \frac{1}{2}\underbrace{g_{\alpha\sigma} g^{\sigma\gamma}}_{\delta_\alpha^\gamma}\left(\partial_\mu\partial_\beta g_{\nu\gamma} + \partial_\mu \partial_\nu g_{\gamma\beta} - \partial_\mu\partial_\gamma g_{\beta\nu}\right)$$

$$= \frac{1}{2}\left(\partial_\mu\partial_\beta g_{\nu\alpha} + \partial_\mu\partial_\nu g_{\alpha\beta} - \partial_\mu\partial_\alpha g_{\beta\nu}\right)$$

They cancel.

By $\mu \leftrightarrow \nu$ interchange

$$g_{\alpha\sigma}\, \partial_\nu \Gamma^{\sigma}_{\beta\mu} = \tfrac{1}{2}\left(\partial_\nu \partial_\beta g_{\mu\alpha} + \underbrace{\partial_\nu \partial_\mu g_{\alpha\beta}} - \partial_\nu \partial_\alpha g_{\beta\mu}\right)$$

$\Rightarrow R_{\alpha\beta\mu\nu}$ in normal coordinates will become:

$$R_{\alpha\beta\mu\nu} = \tfrac{1}{2}\left(\partial_\mu \partial_\beta g_{\nu\alpha} + \partial_\nu \partial_\alpha g_{\beta\mu} - \partial_\nu \partial_\beta g_{\mu\alpha} - \partial_\mu \partial_\alpha g_{\beta\nu}\right)$$

Now, let's work out the $\nabla_\sigma R_{\alpha\beta\mu\nu}$ in normal coordinates. As you know, covariant derivative of it is of the form $\partial_\sigma R_{\alpha\beta\mu\nu} + \Gamma R + \Gamma R + \Gamma R + \Gamma R$

since $\Gamma = 0$ in normal coordinates $\Rightarrow \nabla_\sigma R_{\alpha\beta\mu\nu} = \partial_\sigma R_{\alpha\beta\mu\nu}$

$$\Rightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} = \tfrac{1}{2}\left(\cancel{\partial_\lambda \partial_\mu \partial_\sigma g_{\rho\nu}} + \cancel{\partial_\lambda \partial_\nu \partial_\rho g_{\mu\sigma}} - \cancel{\partial_\lambda \partial_\mu \partial_\rho g_{\nu\sigma}} - \cancel{\partial_\lambda \partial_\nu \partial_\sigma g_{\mu\rho}}\right)$$

$$\nabla_\rho R_{\sigma\lambda\mu\nu} = \frac{1}{2}\left(\cancel{\partial_\rho \partial_\mu \partial_\lambda g_{\sigma\nu}} + \cancel{\partial_\rho \partial_\nu \partial_\sigma g_{\mu\lambda}} - \cancel{\partial_\rho \partial_\mu \partial_\sigma g_{\nu\lambda}} - \cancel{\partial_\rho \partial_\nu \partial_\lambda g_{\sigma\mu}} \right)$$

$$\nabla_\sigma R_{\lambda\rho\mu\nu} = \frac{1}{2}\left(\cancel{\partial_\sigma \partial_\mu \partial_\rho g_{\lambda\nu}} + \cancel{\partial_\sigma \partial_\nu \partial_\lambda g_{\mu\rho}} - \cancel{\partial_\sigma \partial_\mu \partial_\lambda g_{\nu\rho}} - \cancel{\partial_\sigma \partial_\nu \partial_\rho g_{\lambda\mu}} \right)$$

And adding them up gives:

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$$

And using symmetries of Riemann tensor, it's obvious that $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

But wait! It is just for normal Coordinates!
Is it valid for any Coordinate?

Of Course!

To switch between different Coordinates, we use tensorial transformation relation:

$$\underbrace{R_{\rho'\sigma'\mu'\nu'}}_{\text{arbitrary Coordinates.}} = \frac{\partial x^{\rho}}{\partial x^{\rho'}} \frac{\partial x^{\sigma}}{\partial x^{\sigma'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \underbrace{R_{\rho\sigma\mu\nu}}_{\text{normal Coordinates!}}$$

Especially, $B_{\lambda\rho\sigma\mu\nu} = \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}$

is a tensor & trasforms like that.

But all the Componts of $B_{\lambda\rho\sigma\mu\nu}$ are zero!

Hence, by plugging into trasformations $B_{\lambda'\rho'\sigma'\mu'\nu'} = 0$

$\Rightarrow$ The identity holds in any Cordinates!

Second proof:

It involves using Jacobi identity for Covariant derivatives:

$$[\nabla_\mu, [\nabla_\nu, \nabla_\kappa]] + [\nabla_\nu, [\nabla_\kappa, \nabla_\mu]] + [\nabla_\kappa, [\nabla_\mu, \nabla_\nu]] = 0$$

Take a vector field X^ρ (arbitrary), and act one of these:

$$[\nabla_\kappa, [\nabla_\mu, \nabla_\nu]] X^\rho = \nabla_\kappa [\nabla_\mu, \nabla_\nu] X^\rho - [\nabla_\mu, \nabla_\nu] \nabla_\kappa X^\rho$$

By definition of $[\nabla_\mu, \nabla_\nu]$ in terms of Riemann tensor an torsion-free $g_{\mu\nu}$.

$$= \underbrace{\nabla_\kappa \left(R^\rho{}_{\sigma\mu\nu} X^\sigma \right)}_{(\nabla_\kappa R^\rho{}_{\sigma\mu\nu}) X^\sigma + \cancel{R^\rho{}_{\sigma\mu\nu} \nabla_\kappa X^\sigma}} - \cancel{R^\rho{}_{\sigma\mu\nu} \nabla_\kappa X^\sigma} + R^\sigma{}_{\kappa\mu\nu} \nabla_\sigma R^\rho$$

$$= (\nabla_\kappa R^\rho{}_{\sigma\mu\nu}) X^\sigma + R^\sigma{}_{\kappa\mu\nu} \nabla_\sigma X^\rho$$

Now write this into Jacobi identity:

$$0 = \left(\nabla_\kappa R^\rho{}_{\sigma\mu\nu} + \text{other cyclic permutations of } \kappa, \mu, \nu \right) X^\sigma$$

$$+ \underbrace{\left(R^\sigma{}_{\kappa\mu\nu} + \text{other cyclic permutations of } \kappa, \mu, \nu \right)} \nabla_\sigma X^\rho$$

This vanish since we've verified that $R_{\mu[\kappa\rho\nu]} = 0$ in previous exercise.

And since X^σ was arbitrary $\Rightarrow$

$$\nabla_\kappa R^\rho{}_{\sigma\mu\nu} + \nabla_\mu R^\rho{}_{\sigma\nu\kappa} + \nabla_\nu R^\rho{}_{\sigma\kappa\mu} = 0$$

lower ρ to down and use $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$ symmetry.

$$\nabla_\kappa R_{\mu\nu\rho\sigma} + \nabla_\mu R_{\nu\kappa\rho\sigma} + \nabla_\nu R_{\kappa\mu\rho\sigma} = 0 \quad \blacksquare$$

Finally, a simple contraction would lead to Einstein tensor.

Ricci tensor

Ricci scalar

$$0 = \underbrace{g^{\nu\sigma} g^{\mu\lambda}}\left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho \underbrace{R_{\sigma\lambda\mu\nu}} + \nabla_\sigma R_{\lambda\rho\mu\nu}\right)$$

Ricci scalar

$$= \nabla^\mu R_{\rho\mu} - \nabla_\rho R + \underbrace{\nabla^\nu R_{\rho\nu}} \overset{\text{by Bianchi}}{=} 0$$

ν is a dummy index

$\Rightarrow$ change it with μ.

$\downarrow$

$$2\nabla^\mu R_{\rho\mu} = \nabla_\rho R \rightarrow \nabla^\mu R_{\rho\mu} - \tfrac{1}{2}\nabla_\rho R = 0$$

To make it a total derivative, write ∇_ρ as $g_{\rho\mu} \nabla^\mu$.

$$\nabla^\mu \underbrace{\left(R_{\rho\mu} - \tfrac{1}{2} g_{\rho\mu} R \right)}_{} = 0$$

$G_{\rho\mu}$, is symmetric ✓

Now by field equations $\longrightarrow$ $G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$.

since $\nabla_\mu G^{\mu\nu} = 0$ $\Longrightarrow$ by the equation

$$\nabla_\mu T^{\mu\nu} = 0$$

Q4. Geodesic equation on $\theta = \pi/2$ hypersurface can be found via variation of the length or via standard method!

we pursue the standard method.

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\phi^2$$

using problem (1) results:

$$\Gamma^t_{r\phi} = \Gamma^r_{t\phi} = \Gamma^\phi_{rt} = 0$$

$$\Gamma^t_{rr} = -\frac{1}{2}(g_{tt})^{-1}\underbrace{\partial_t g_{rr}}_{\text{صفر}} = 0 \qquad \Gamma^t_{\phi\phi} = 0 \text{ (similarly)}$$

$$\Gamma^r_{tt} = -\frac{1}{2}(g_{rr})^{-1}\partial_r(g_{tt}) = -\frac{1}{2}\left(1-\frac{r_s}{r}\right)\left(-\frac{r_s}{r^2}\right)$$

$$= \frac{r_s}{2r^3}(r-r_s)$$

$$\Gamma^r_{\phi\phi} = -\frac{1}{2}(g_{rr})^{-1}\partial_r(g_{\phi\phi}) = -\frac{1}{2}\left(1-\frac{r_s}{r}\right)2r = -(r-r_s)$$

$$\Gamma^t_{rt} = \partial_r\left(\ln\sqrt{g_{tt}}\right) = \frac{1}{2}\frac{\partial_r g_{tt}}{g_{tt}} = \frac{1}{2}\left(-\frac{r_s}{r^2}\right)\times\frac{-1}{1-\frac{r_s}{r}}$$

$$= \frac{r_s}{2r}\times\frac{1}{r-r_s} \longrightarrow \text{similarly } \Gamma^t_{tr}$$

$$\Gamma^\phi_{r\phi} = \frac{1}{2}\frac{\partial_r g_{\phi\phi}}{g_{\phi\phi}} = \frac{1}{2}\frac{2r}{r^2} = \frac{1}{r} \qquad \Gamma^\phi_{\phi r} = \frac{1}{r}$$

$$\Gamma^r_{rr} = \frac{1}{2}\,\frac{\partial_r g_{rr}}{g_{rr}} = \frac{1}{2} \times \partial_r \left(1-\frac{r_s}{r}\right)^{-1} \left(1-\frac{r_s}{r}\right)$$

$$= -\tfrac{1}{2} \times \frac{-r_s/r^2}{\left(1-\frac{r_s}{r}\right)^{\cancel{2}}} \times \cancel{\left(1-\frac{r_s}{r}\right)} = +\frac{r_s}{2r^2} \times \frac{1}{\left(1-\frac{r}{r_s}\right)}$$

$$= \frac{r_s}{2r} \times \frac{1}{r_s - r}$$

so, geodesic eqution is :

$$\frac{d^2x^r}{d\lambda^2} + \Gamma^r_{\alpha\beta}\,\frac{dx^\alpha}{d\lambda}\,\frac{dx^\beta}{d\lambda} = 0$$

t-Compont :

$$\frac{d^2x^t}{d\lambda^2} + \Gamma^t_{\alpha\beta}\,\frac{dx^\alpha}{dt}\,\frac{dx^\beta}{dt} = 0$$

$$\downarrow$$

$$\frac{d^2x^t}{d\lambda^2} + \frac{r_s}{r} \times \frac{1}{r-r_s}\,\frac{dx^t}{d\lambda}\,\frac{dx^r}{d\lambda} = 0$$

r_Compont : $\displaystyle \frac{d^2x^r}{d\lambda^2} + \Gamma^r_{\alpha\beta}\,\frac{dx^\alpha}{d\lambda}\,\frac{dx^\beta}{d\lambda} = 0$

$$\frac{d^2x^r}{d\lambda^2} + \frac{r_s}{2r^3}(r-r_s)\,\frac{dx^t}{d\lambda}\,\frac{dx^t}{d\lambda}$$

$$+ \frac{r_s}{2r(r_s-r)}\,\frac{dx^r}{d\lambda}\,\frac{dx^r}{d\lambda} + (r_s - r)\,\frac{dx^\phi}{d\lambda}\,\frac{dx^\phi}{d\lambda} = 0$$

ϕ _ Component:
$$\frac{d^2x^\phi}{d\lambda^2} + \Gamma^\phi_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

$$\frac{d^2x^\phi}{d\lambda^2} + \frac{2}{r}\frac{dx^r}{d\lambda}\frac{dx^\phi}{d\lambda} = 0$$

when $x^\mu(\lambda) = (x^t(\lambda), x^r(\lambda), x^\phi(\lambda))$ is the trajectory of the particle, parametrized by λ.

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b) start from $U^\mu = \frac{dx^\mu}{d\tau}$ which is normalized properly

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$$

$$\downarrow$$

$$\underbrace{-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2}_{\oplus} + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{r^2\left(\frac{d\theta}{d\tau}\right)^2}_{\oplus}$$

$$\underbrace{+r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2}_{\oplus} = -1$$

since $r < 2GM \implies 1 - \frac{2GM}{r} < 0$.

so we can drop underlined terms to get following inequality:

$$\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \leq -1$$

$$\Downarrow \qquad \times\left(1-\frac{2GM}{r}\right) \longrightarrow \text{since its negative the sign changes.}$$

$$\left(\frac{dr}{d\tau}\right)^2 \geq \frac{2GM}{r} - 1$$

$$\Rightarrow \left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}$$

For maximum life time, integrate this relation:

$$\tau_{max} = \int_{2GM}^{0} \left|\frac{d\tau}{dr}\right|_{max} = -\int_{2GM}^{0} \frac{dr}{\sqrt{\frac{2GM}{r} - 1}} =$$

$$\int_{0}^{2GM} \frac{dr}{\sqrt{\frac{2GM}{r} - 1}} = GM\pi$$

The result of this integral is given by Mathematica. You can look it at Mathematica notebook include.

For Solar BH, to make the correct scale, we recover

$$\frac{1}{c^3} \Rightarrow \tau_{max} = \frac{\pi GM}{c^3}$$

$$\tau_{max}^{\odot} = \frac{\pi \times (6.67\times10^{-11}) \times (1.9\times10^{30})}{(3\times10^{8})^{3}} \sim 1.5\times10^{-5} \text{ seconds (!)}$$

Maximal time happens when all positive contributions add to zero:

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2 = 0$$

For freely falling particle, ϕ, θ are constant and we're left with

$$\underbrace{-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2}_{} = 0$$

This is nothing but $(U^0)^2$, and multiplying by rest mass (m_0) it gives the energy E.

$\Downarrow$

$$E = m_0 \sqrt{\frac{2GM}{r} - 1}\left(\frac{dt}{d\tau}\right) = 0$$

Q5. This problem has been solved for Schwarzschild metric in the Carroll's text book, we quote the results of Γ, $R^{\gamma}{}_{\nu\alpha\beta}$, $R_{\mu\nu}$, R from the textbook and check them by mathematica code.

Start from: $ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$

(After some consideration on spherical metric, static, independent of θ,ϕ and redefining variables!)

The trace reversed equation:

$$R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \longrightarrow \mathrm{Tr}$$

$$R - 2R + 4\Lambda = 0 \longrightarrow R = 4\Lambda$$

plug it back in above:

$$R_{\mu\nu} - 2\Lambda g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \longrightarrow R_{\mu\nu} = \Lambda g_{\mu\nu}$$

$$\begin{cases} R_{tt} = -\Lambda e^{2\alpha(r)} \\ R_{rr} = \Lambda e^{2\beta(r)} \\ R_{\theta\theta} = \Lambda r^2 \end{cases}$$

As Carroll suggests in 5.16 for the above metric:

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = \frac{2}{r}(\partial_r \alpha + \partial_r \beta)$$

from above results

$$-e^{2(\beta-\alpha)} \Lambda e^{2\alpha} + \Lambda e^{2\beta}$$

$$= -\Lambda e^{2\beta} + \Lambda e^{2\beta} = 0 \, !$$

$$\partial_r \alpha + \partial_r \beta = 0 \implies (\alpha = -\beta - \tilde{C})$$ when $\tilde{C}$ is constant.

From 5.14 relations for Ricci tensors:

$$R_{\theta\theta} = e^{-2\beta}\left(r(\partial_r \beta - \partial_r \alpha) - 1\right) + 1$$

$$\implies R_{\theta\theta} = \Lambda r^2$$

utilizing $\beta + \tilde{C} = -\alpha$

$$e^{2(\alpha+\tilde{C})} \underbrace{\left(+2r\partial_r \alpha + 1\right)}_{\text{This is total derivative.}} + \Lambda r^2 = 1 \implies$$

$$\implies e^{2\tilde{C}} \partial_r\left(r e^{2\alpha} + \frac{\Lambda r^3}{3}\right) = 1 \implies r e^{2\alpha} + \frac{\Lambda r^3}{3} = r e^{-2\tilde{C}} + C$$

$$\implies e^{2\alpha} = \left(1 + \frac{C}{r} - \frac{\Lambda r^2}{3}\right) e^{-2\tilde{C}}$$

$$ds^2 = -e^{2\tilde{C}}\left(1+\frac{C}{r}-\frac{\Lambda r^2}{3}\right)dt^2 + e^{2\tilde{C}}\left(1+\frac{C}{r}-\frac{\Lambda r^2}{3}\right)^{-1}dr^2 + r^2 d\Omega^2$$

By requiring that:

(when $\Lambda = 0$, the space is Minkowskian at $r \to \infty$,

the $e^{2\tilde{C}} = 1 \to \tilde{C} = 0$)

(when $\Lambda = 0$, in weak field approximation, we have to

recover $\phi_N = -\frac{GM}{r}$, which done in the classroom

and resulted $C = -2GM$)

$$ds^2 = -\left(1-\frac{2GM}{r}-\frac{\Lambda r^2}{3}\right)dt^2 + \left(1-\frac{2GM}{r}-\frac{\Lambda r^2}{3}\right)^{-1}dr^2 + r^2 d\Omega^2$$

which reduces to usual Schwarzschild metric when $\Lambda = 0$.

Q6.

$$\Box \xi^{\alpha} = -R^{\alpha}{}_{\beta}\, \xi^{\beta}$$

Utilizing Killing equation ($\nabla_{(\mu}\xi_{\nu)} = 0$) and definition of $[\nabla_\mu, \nabla_\nu] X^\rho$, we get:

$$\nabla_{(\mu}\xi_{\nu)} = 0 \longrightarrow \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \rightarrow$$ Contract with ∇^μ

$$\Rightarrow \Box \xi_\nu + \nabla_\mu \nabla_\nu \xi^\mu = 0$$

At this stage, it's good to interchange $\nabla_\mu \nabla_\nu$ with $\nabla_\nu \nabla_\mu$, this is done via $[\nabla_\mu, \nabla_\nu]$:

$$[\nabla_\mu, \nabla_\nu]\xi^\beta = R^{\beta}{}_{\alpha\mu\nu}\, \xi^\alpha$$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\xi^\beta = R^{\beta}{}_{\alpha\mu\nu}\, \xi^\alpha$$

change the index

$$\nabla_\mu \nabla_\nu \xi^{\cancel{\beta}\,\mu} = \nabla_\nu \nabla_\mu \xi^{\cancel{\beta}\,\mu} + \underbrace{R^{\cancel{\beta}\,\mu}{}_{\alpha\mu\nu}}_{\text{Ricci } R_{\alpha\nu}} \xi^\alpha$$

substitute the result in:

$$\Box \xi_\nu + \nabla_\nu \nabla_\mu \xi^\mu + R_{\alpha\nu}\, \xi^\alpha = 0$$

$$\Box \xi^\nu + \underbrace{\nabla^\nu (\nabla_\mu \xi^\mu)}_{} + R^{\nu}{}_{\alpha}\, \xi^\alpha = 0$$

we show that this term vanishes,

therefore $\Box \xi^\nu = -R^{\nu}{}_{\beta}\, \xi^\beta$. ∎

Killing equation : $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ $\longrightarrow$

Contract indices $\Longrightarrow$ $g^{\mu\nu}(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = 0$

using metric Compatibility $\Longrightarrow$ $\nabla_\mu \xi^\mu + \nabla_\mu \xi^\mu = 2 \nabla_\mu \xi^\mu = 0$

$\Longrightarrow$ $\nabla_\mu \xi^\mu = 0$ $\varnothing$