Here, I'll try to motivate on group actions, and Ms. Kabiri will talk more on group theoretical aspects of classical Mechanics!

(The group theory and classical mechanics are connected in more obvious ways, I'm a little suspicious that Noether theorems are clear enough to elaborate on groups and their significance in physical contexts!)

Review of group action:

G acts on set X if there's a map $\cdot : G\times X \longrightarrow X$:

1. $e\cdot x = x \quad \forall x\in X$
2. $g_1\cdot(g_2\cdot x) = (g_1\cdot g_2)(x)$
$\forall g_1, g_2 \in G, \forall x\in X$

Example: G acts on itself by Conjugation. If $X = G$, define the map $G\times G \longrightarrow G$ by:

$(g, x) \longmapsto gxg^{-1}$

Famous examples of action are:

1) $O(n)$ on vectors.

2) $SO(1,3)$ on 4-vectors.

3) Cayley's theorem could be seen as group action of G on $S_{|G|}$ by permutations.

stabilizer or isotropy subgroup of x is $G_x = \{ g\in G \mid g\cdot x = x\}$

orbit of x is $Gx = \{g\cdot x \mid g\in G\}$

Orbit-Stabilizer theorem:

There's a bijection $\frac{G}{G_x} \longleftrightarrow Gx$

so we have $[G : G_x] = |Gx|$

Orbits partition S:

1) Covers all elements of $S \rightarrow e\cdot s = s \in Gs$
so all $s\in S$ are included in Gs.

2) If $z\in Gx \cap Gy$, then

$z = g_1 x = g_2 y \Rightarrow x = g_1^{-1} g_2 y$

$\Rightarrow$ for all $g\in G$: $gx = g g_1^{-1} g_2 y \in Gy$

$\Rightarrow Gx \subseteq Gy$ and similarly $Gy \subseteq Gx$

$\Rightarrow$ They're something like cosets!

The class equation: Let G act on itself by conjugation.

orbits of this action: $Gx = \{g.x \mid g \in G\} = \{gxg^{-1} \mid g \in G\}$

since $x \in G$, it is the definition of conjugacy class $\Longrightarrow$ orbits are conjugacy classes.

stabilizers of the action: $G_x = \{g \in G \mid g.x.g^{-1} = x\}$

$= \{g \in G \mid gx = xg\}$

It is the centralizer of $\{x\}$ set.

$G_x = C_G(\{x\})$

note: $g \in G$ lies in $Z(G)$ iff conjugacy class g is solely itself. or its centralizer is G.

since orbits partition G (in our case conjugacy classes):

$G = Gg_1 \sqcup Gg_2 \sqcup \dots \sqcup Gg_r$

g_is are representatives of distinct conj. classes of G, excluding $Z(G)$.

we find that:

$$|G| = |Z(G)| + \sum_{j=1}^{r} [G : G_{g_j}] = |Z(G)| + \sum_{j=1}^{r} [G : C_G(g_j)]$$

proof only involves finding cardinality of equation!

and using the orbit-stabilizer theorem.

Ex 1, Let G be a p-group (order of G is p^m), show that $Z(G)$ is non-trivial, or $p \mid |Z(G)|$

Utilize class equation!

$$|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)]$$

→ reps of Conjugacy class with non central elements, as above.

$[G : C_G(g_i)] > 1$ since otherwise $g_i \in Z(G)$

Since $[G : C_G(g_i)] \mid |G|$ (Lagrange) → $[G : C_G(g_i)]$ is a power of p.

$$\Rightarrow |Z(G)| = |G| - \sum_{i=1}^{r} [G : C_G(g_i)] \overset{p}{\equiv} 0$$

→ note that $|Z(G)| \neq 0$, and since $p \mid |Z(G)| \rightarrow |Z(G)| > 1$

Ex 2. If $\dfrac{G}{Z(G)}$ is cyclic $\Longrightarrow$ G is abelian!

$\dfrac{G}{Z(G)} = \langle x Z \rangle$ for some $x \in G$.

$\forall g, h \in G$ $\begin{cases} gZ = x^m Z \\ hZ = x^n Z \end{cases}$ for some $m, n \in \mathbb{Z}$.

↓

$z_1, z_2 \in Z$ $\begin{cases} g = x^m z_1 \\ h = x^n z_2 \end{cases}$ $\Longrightarrow$ $gh = x^m z_1 x^n z_2 = z_1 x^{m+n} z_2$

$= z_2 x^{m+n} z_1$

$= z_2 x^n x^m z_1$

$= x^n z_2 x^m z_1$

$= hg$ ∎

Ex 3: Any group of order p^2 is Abelian!

By Ex 1, $p \mid Z(G) \Rightarrow$ Either $Z(G) = p$ or $Z(G) = p^2$.

the latter implies G is cyclic, hence Abelian. ∎

If $|Z(G)| = p \Rightarrow \frac{G}{Z(G)}$ has order p (any odd-order group is cyclic!)

$\Rightarrow \frac{G}{Z(G)}$ is cyclic $\Rightarrow$ Ex 2 implies that G is Abelian. ∎

Sylow Theorems from the lens of group action:

Remember: A sylow p-group is a subgroup of order p^n where $p^n \mid |G|$ and $p^{n+1} \nmid |G|$

First sylow Thm: For every $p \| G| \Rightarrow$ p-sylow subgroups exist!

ASIDE

We utilize this lemma, which left as an exercise in Bonus pset!

Cauchy Thm 1: For finite Abelian G with $p \| G| \rightarrow \exists g \in G : o(g) = p$.

Cauchy theorem 1': For any finite group G, and any prime p: $p \| G|$, there is an subgroup of order p.

proof: proceed by induction. The base case $|G| = p$ $(n=1)$ is obvious!

(n) If G contains subgroup H whose $([G:H], p) = 1 \Rightarrow$

$|G| = |H|\,[G:H] \Rightarrow \{ p^n \mid |H|$

αp^n ... $\nmid$ so αp^n has no p-factor

$|H| \mid p^n$ (by Lagrange) $\Rightarrow |H| =$

so H is the desired p-sylow subgroup!

Suppose $([G:H], p) \neq 1 \overset{\text{or}}{\Longrightarrow} p \mid [G:H]$ (for any proper $H \lneq G$)

Consider the action by conjugation and write the class equation:

$$\underbrace{|G|}_{\alpha p^n} = |Z(G)| + \sum_{|G_x|>1} \underbrace{[G:G_x]}$$

G_x is subgroup of G which $p \mid [G:G_x]$

This implies $p \mid Z(G) \Longrightarrow$ Cauchy lemma gives $a \in Z(G)$: $O(a) = p$.

since $Z(G)$ is abelian $\longrightarrow \langle a \rangle \trianglelefteq Z(G)$

Since, $p^{n-1} \mid \left|\frac{G}{\langle a\rangle}\right|^{\alpha p^{n-1}}$ and $p^n \nmid \left|\frac{G}{\langle a\rangle}\right|^{\alpha p^{n-1}}$

induction gives $\tilde{K} \leq \frac{G}{\langle a\rangle}$ a p-sylow subgroup of $\frac{G}{\langle a\rangle}$.

Let $K = f^{-1}(\tilde{K})$ with $f: G \longrightarrow \frac{G}{\langle a\rangle}$ the quotient map.

we can easily check that $f(g) = g\langle a\rangle$ is homomorphism,

and $\operatorname{Ker} f = \{g \in G \mid \underbrace{g\langle a\rangle = \langle a\rangle}_{g \in \langle a\rangle}\} \longrightarrow \ker f = \langle a\rangle$

so f is p-to-1 homomorphism.

f^{-1} (map) : $\frac{G}{\langle a\rangle} \longrightarrow G$ takes $g\langle a\rangle$ to $\{g, ga, \dots, ga^{p-1}\}$

Then $K = f^{-1}(K)$ has $p^{n-1} \times p = p^n$ cardinality.

By using $\alpha \in Z(G)$ you can check that $f^{-1}(K)$ is subgroup.

So we found $K \leq G$ a p-sylow subgroup.

Sylow theorems: Let G be a finite group!

1, Every p-subgroup of G is contained in p-sylow subgroups.
2, All p-sylow subgroups of G are conjugate.
3) $n_p \overset{p}{\equiv} 1$ (number of p-sylow subgroups.)

Let S denote the set of p-sylow subgroups of G (which is $\neq \emptyset$)

G acts on S by conjugation. Let $p \in S$ be a p-sylow subgroup of G,

S_p denotes its orbit, N_p its normalizer.

$|S_p| = [G : G_p]$ is coprime to p, since $\underbrace{p \subseteq G_p}_{\text{order of } |G_p| \text{ is at least } p^n}$.

1) Let H be a non-trivial p-subgroup of G, H acts on S_p by conjugation: $|S_p| = \sum_{p'} [H : H_{p'}]$.

Like above $\longrightarrow$ $(|S_p|, p) = 1$ and $[H, H_{p'}]$ is p-power.

$\Longrightarrow$ some orbits contain only one p-sylow, or singleton.

$\exists p' : H \subset G_{p'} \longrightarrow H_{p'}$ is a subgroup.

also $P' \trianglelefteq HP'$ (why?) $\longrightarrow$ second isomorphism thom: $\frac{HP'}{P'} \cong \frac{H}{H \cap P'}$

$\frac{HP'}{P'}$ has order coprime to p, since P' is p-sylow subgroup exhausts all the factors of p.

RHS has p-powers since H is p-subgroup and $H \neq P'$.

$\Rightarrow$ so both are trivial quotients $\Rightarrow H \subset P'$

so H is contained in p-sylow subgrp P'.

2) In (1) let H be any p-sylow subgrp, $\longrightarrow$ by (1) H is a subgroup of a conjugate to P. since their orders are equal, H is conjugate to P. ⊠

3) $|S| = n_p$. Let P (a generic p-sylow) act on S by conjugation.

write $|S| = \sum_{P'} \underbrace{[P : P_{P'}]}$

each term is 1 or power of p.

orbit of $\{P\}$ is itself, so one of these term is 1.

Any other p-sylow subgroup $P' \neq P$ has nontrivial orbit!

(otherwise $P \subset N_{P'} \Rightarrow PP' \leq G \Rightarrow \underset{> p^n}{o(PP')} \mid \underset{= p^n}{o(G)}$ ⨳)

Hence, the rest of the terms are multiples of $p \Rightarrow n_p \overset{p}{\equiv} 1$

Last Exercise: p, q be two primes, G is a q-subgroup of
(For home)

$GL(n, \mathbb{Z}_p)$ $(n \geq 2)$.

show that if $q \nmid p^n - 1 \Rightarrow \exists$ non-zero $x \in \mathbb{Z}_p^n$

such that $Ax = x$ for all $A \in G$.