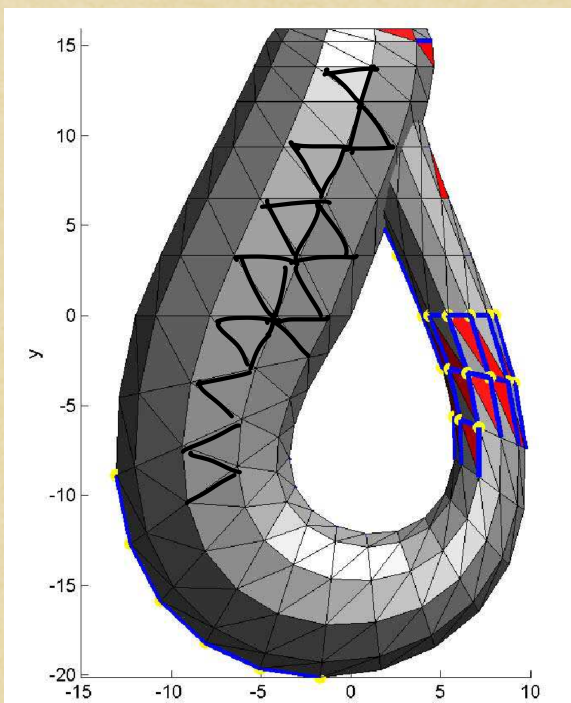
$\pi_1(\mathbb{R}) =$ $\mathbb{R}$

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

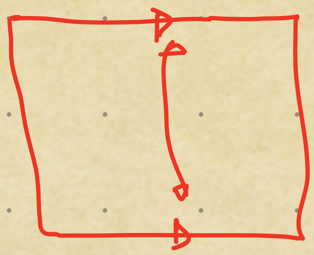


Homology S^2: $H_0(S^2)$, $H_1(S^2)$, $H_2(S^2)$

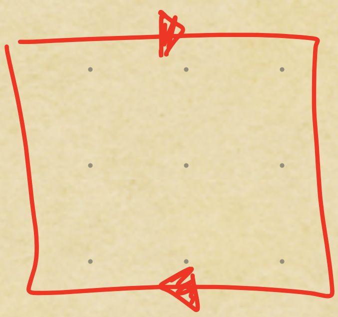
$$H_n(S^2) = \{e\} \quad n \geq 3$$

T^2, نوار موبیوس, ... ✓

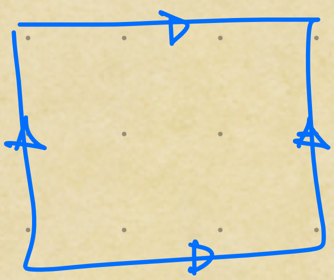
Exactly 5 polygon ...



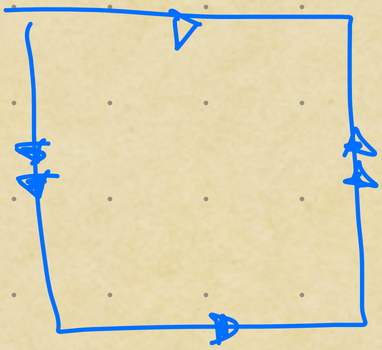
$\xrightarrow{?}$ cylinder



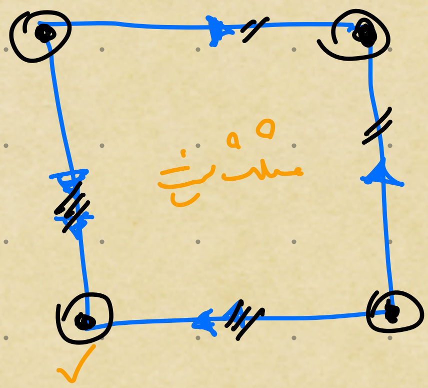
$\rightarrow$ ✓ نوار موبیوس



$\rightarrow$ Torus $\cong T^2$

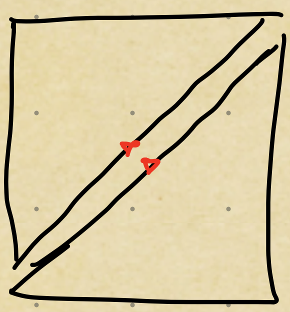


→ Klein Bottle

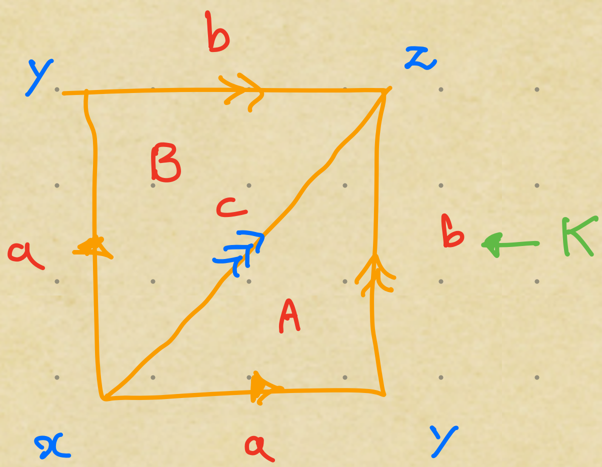


→ S^2

Triangularization of S^2 :



→



Chain Complex :

3-simplex

$$0 \xleftrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} \underbrace{C_0(K)} \xrightarrow{\partial_0} 0$$

$\langle A, B \rangle$ $\qquad$ $\underline{\langle a, b, c \rangle}$ $\qquad$ $\langle x, y, z \rangle$

For $H_0(K) = \dfrac{Z_0(K)}{B_0(K)}$

$i, j, k \in \mathbb{Z}$

$\partial_0(ix + jy + kz) = ? \equiv 0$

$Z_0 = \ker \partial_0 =$ $\quad \partial_0(x) = 0$

$Z_0 = \langle x, y, z \rangle \equiv \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ $\quad \partial_1((p_0 p_1)) = p_1 - p_0$

$\partial_2((p_0 p_1 p_2)) =$

$(p_1 p_2) - (p_0 p_2) + (p_0 p_1)$

$B_0 = \mathrm{Im}(\partial_1) = \{ c \in C_1 \mid \exists x \in C_2 : c = \partial_1 x \}$

$C_1 \xrightarrow{\partial_1} C_0$

$\partial_1(la + mb + nc) \xrightarrow{\text{تعریف؟}}$

$\partial_1 a =$

$\partial_1 b =$

$\partial_1 c =$

$l(y - x) + m(z - y) + n(z - x) =$

$(\mathbb{Z} \oplus \mathbb{Z})$ $\qquad \mathrm{Im}(\partial_1)$

$-(l+n)x + (l-m)y + (m+n)z \in C_0(K)$

$f : C_0(K) \cong Z_0(K) \longrightarrow \mathbb{Z}$

$f(lx + my + nz) = l + m + n$

$\ker f = \{ x \in C_1 \mid f(x) = 0 \}$ $\qquad \ker f = B_0(K)$

Fundamental theorem on group homomorphism:

$$\phi : G_1 \longrightarrow G_2 \qquad \frac{G_1}{\ker f} \cong \mathrm{Im}(f)$$

$$\ker f \trianglelefteq G_1$$

$$H_0(K) = \frac{Z_0(K)}{B_0(K)} \cong \mathbb{Z}$$

$$H_0(S^2) = \mathbb{Z}$$

$$H_0(\text{space}) = \bigoplus_{\#\text{comp}} \mathbb{Z}$$

$$H_0(\ ) = \mathbb{Z} \oplus \mathbb{Z} + \mathbb{Z}$$

$$\partial_1(la + mb + nc) = l(y-x) + m(z-y) + n(z-x)$$

$$= -(l+n)x + (l-m)y + (m+n)z$$

$$\partial_1 : \begin{pmatrix} l \\ m \\ n \end{pmatrix} \longmapsto \begin{pmatrix} -l-n \\ l-m \\ m+n \end{pmatrix}$$

$$[\partial_1]_{\text{std}} = \begin{array}{ccc} l & m & n \\ \hline \end{array}\!\!\! \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$B_0(K) = \mathrm{Im}(\partial_1) \qquad (l, m, n) \in \mathbb{Z}^3 \xrightarrow{?} \begin{matrix} \mathbb{Z}^2 \\ \mathbb{Z}^3 \\ \mathbb{Z}^1 \end{matrix}$$

The image of $[\partial_1]$ is in a ...

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & \boxed{0} & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow$$

ناصفر
2 تا سطر بدست آمد

$$\begin{array}{c} x \\ y \\ z \end{array} \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle \longrightarrow \text{generates } \operatorname{im}(\partial_1) = B_0(K)$$

$$\langle (y-x), (z-y) \rangle = \operatorname{Im}(\partial_1) = B_0(K)$$

Free Abelian group + manipulation $\longleftrightarrow$ Vector space
linear operator.

$$H_0 = \frac{Z_0}{B_0} = \frac{\langle x, y, z \rangle}{\langle \underbrace{y-x}_{y=x}, z-y \rangle} \simeq \frac{\langle x, x, z \rangle}{\langle 0, z-y \rangle} = \frac{\langle x, x, x \rangle = \langle x \rangle}{\underbrace{\langle 0, 0 \rangle}_{\{0\}}}$$

$$\cong \frac{\langle x \rangle}{\{0\}} = \langle x \rangle \cong \mathbb{Z} \quad ✓$$

$H \trianglelefteq G$

$\dfrac{G}{H}$, $[g]$

$g_1, g_2 \in G$: $[g_1] = [g_2] \longleftrightarrow g_1 g_2^{-1} \in H$

$$\frac{\langle x, y, z \rangle}{\ldots} = \qquad [ax + by + cz] = [\tilde{a}x + \tilde{b}y + \tilde{c}z]$$

$\langle y-x \rangle$

$\in \langle y-x \rangle$

$(a-\tilde{a})x + (b-\tilde{b})y + (c-\tilde{c})z = k(y-x)$

$x + y + z$

$2x + \quad + 2$

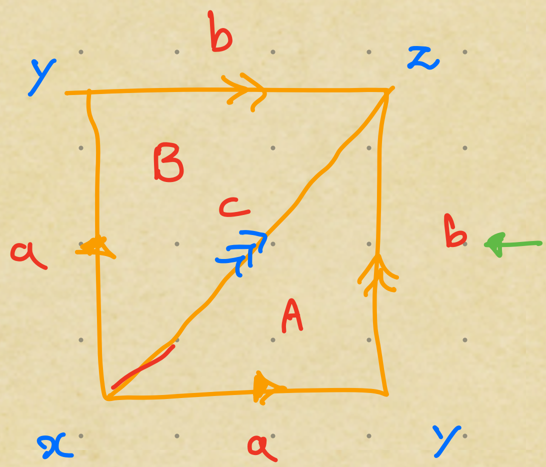
$\overline{-x + y + 0} \in \langle y-x \rangle$ [illegible]

$H_1 = \dfrac{Z_1}{B_1}$ $\qquad Z_1 = \ker \partial_1$

$$C_1 \xrightarrow{\partial_1} C_0$$

$$C_1 = \{ \underbrace{A\vec{a} + Bb + Cc}\ |\ A, B, C \in \mathbb{Z} \}$$

$\partial_1(\qquad) = ?$



$A(y - \underline{x}) + B(z - y) + \underline{C}(z - x) = 0$

$x(-A-C) + y(A-B) + z(B+C)$

$A = B = -C$

$A = B = -\alpha$

$C = \alpha$

$\alpha(a) - \alpha b + \alpha c = \alpha(-a - b + c) \cong \mathbb{Z}$

$= \langle -a - b + c \rangle$

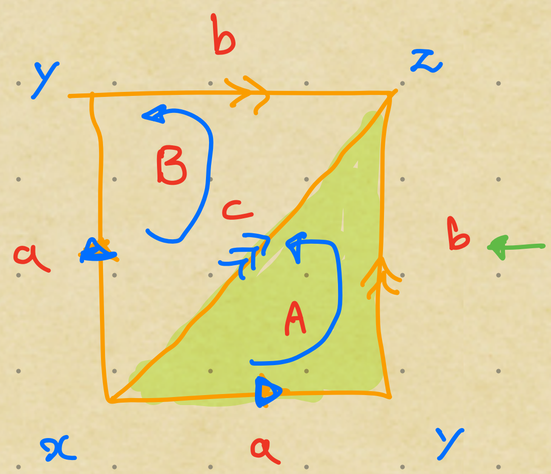
$$\begin{array}{c|ccc} & a & b & c \\ \hline x & -1 & 0 & -1 \\ y & 1 & -1 & 0 \\ & 1 & & \end{array}$$

$[\partial_1]_{\mathbb{Z}} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ \cdots \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

2

$Z_1(K) = \langle -a-b+c \rangle \cong \mathbb{Z}$

$B_1(K) = \operatorname{Im} \partial_2 \qquad C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to$



$C_2 = \{ \underbrace{mA + nB} \mid m,n \in \mathbb{Z} \}$

$\partial_2(mA + nB) =$

$m(a+b-c) + n(-a-b+c)$

$= \underbrace{(m+n)}\,a + (m+n)\,b + (-m-n)\,c$

$$[\partial_2]_{3\times 2} = \begin{array}{c|cc} & A & B \\ \hline a & 1 & 1 \\ b & 1 & 1 \\ c & -1 & -1 \end{array} \longrightarrow \text{Row reduce.}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \underline{1}$$

$$\begin{array}{c} a \\ b \\ c \end{array}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \longrightarrow \langle a+b-c \rangle = B_1(K)$$

$$\frac{\langle -a-b+c\rangle}{\langle a+b-c\rangle} = \frac{Z_1}{B_1} = H_1 \cong \frac{G}{G} \cong \{e\}$$

$H_1(S^2)=\{e\}$ ✓

$\langle x\rangle = \langle -x\rangle$

$\langle 1\rangle = \{0,1,-1,1,-2,\dots\}$

$\langle -1\rangle = \{0,-1,1,\dots\}$

$H_2 = \frac{Z_2}{B_2}$ ✓ $\quad Z_2 = \ker \partial_2 \;:\; \partial_2(\underbrace{mA+nB}) = 0$

$\{\}$

$(m+n)a + (m+n)b - (m+n)c$

$m=-n$

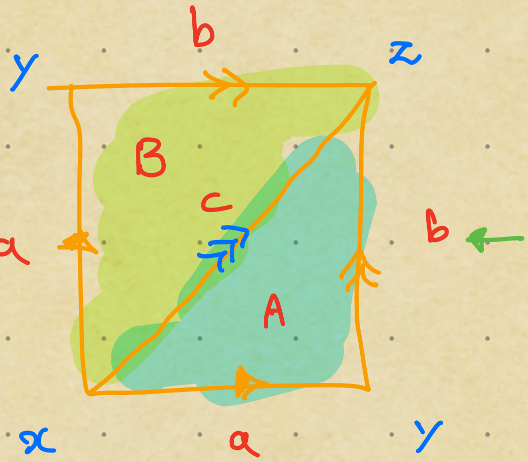
$B_2(K) = \{\sigma \in Z_2(K) \mid \sigma = \partial_3\sigma', \underbrace{\sigma'\in Z_3}\}$

$m(A-B) \longrightarrow Z_2 = \langle A-B\rangle$

$\underline{\underline{B_2 = \{e\}}}$ $\quad 0 \xrightarrow{\partial_3} Z_3$

Returns $\to Im = \phi$

$H_2 = \frac{\langle A-B\rangle}{\langle e\rangle} = \langle A-B\rangle \cong \mathbb{Z}$ ✓



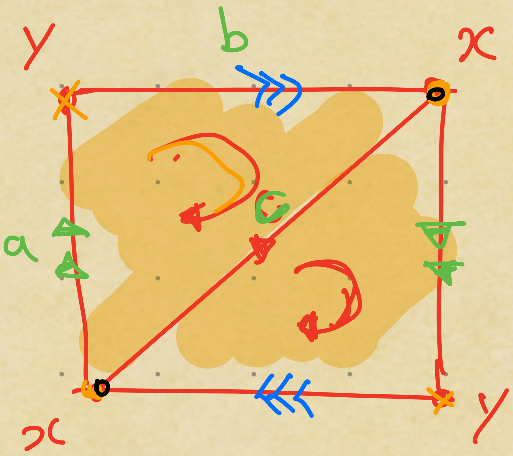
$H_0(S^2) = H_2(S^2) = \mathbb{Z}$

$H_1(S^2) = \{e\}$ ✓

$\mathbb{Z}_2 = \{0,1\}$

$RP^2 \to H_1\ \mathbb{Z}_2$

$H_1(\mathbb{R}P^2)$



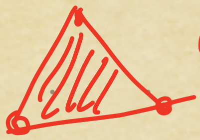
$Z_1 = \ker \partial_1 \qquad C_1 \xrightarrow{\partial_1} C_0$

$\partial_1(ia + jb + kc)$

$= \cdots = i(y-x) + j(x-y) + k\cancel{(x-x)}$

$= (i-j)(x-y) =$

$i=j$



$\in K$

$\in K$

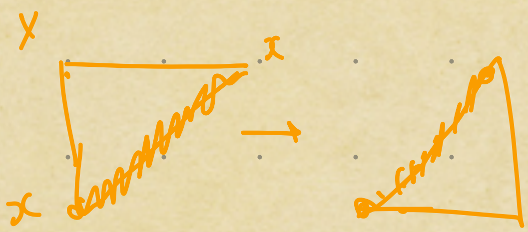
$\in K$

$\sigma, \sigma' \in K$

$\sigma \cap \sigma' = \phi$

$\sigma\cap\sigma' \leq \sigma$

$\sigma\cap\sigma' \leq \sigma'$



$Z_1 = \{\, i(a+b) + kc \,\} = \mathbb{Z}\oplus\mathbb{Z}$

$= \langle a+b, c\rangle$

$\operatorname{Im}\partial_2 = B_1 \longrightarrow mA + nB \xrightarrow{\partial_2} (m+n)a + (m+n)b + (n-m)c$

$B_1(K) = \langle a+b-c,\ a+b+c\rangle \longleftarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & +1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$

$$\frac{\langle a+b,\ c\rangle}{\langle \underbrace{a+b}-c,\ a+b+c\rangle} = \frac{\langle c, c\rangle}{\langle 0, 2c\rangle} = \frac{\langle c\rangle}{\langle 2c\rangle} \overset{?}{=} \mathbb{Z}_2$$

$x \cdot m \cdot e$

$H_1(\mathbb{RP}^1)$

$$\frac{\mathbb{Z}}{\underbrace{2\mathbb{Z}}} \cong \qquad x, y \in \mathbb{Z} \qquad [x] \cong [y]$$

$x - y \in 2\mathbb{Z}$

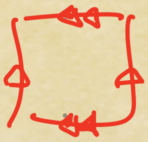
$f,\ Z_0(K)$

$\downarrow$

$\mathbb{Z}$

$$\frac{\mathbb{Z}}{2\mathbb{Z}} = \{[0], [1]\} \cong \mathbb{Z}_2 \ \checkmark$$

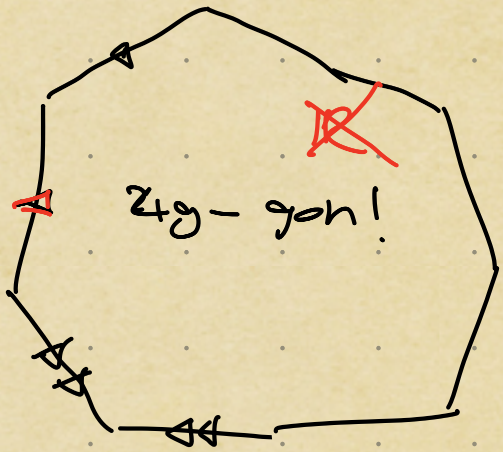
$\checkmark$



$g = 1$

genus g

$g \geq 2$



UHP $\left(\frac{1}{y^2}(dx^2 + dy^2)\right)$

$$\frac{|dz|^2}{\mathrm{Im}(z)^2}$$

$H_1(K) \cong \pi_1^{ab}(|K|)$ $\checkmark$

Hurewicz Theorem:

For any space X, (path-connected), $n \in \mathbb{N}$

$h_*: \pi_n(X) \longrightarrow H_n(X)$

$n=1$ (h_* is not isomorphism) $\longrightarrow$

$\tilde{h}_* = \dfrac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]} \longrightarrow H_1(X)$

$\pi_1^{ab}(X) \longrightarrow H_1(X)$ isomorphism

$\{0, 1, 2, 3\} \;+\; \underline{\underline{4}}$

$\{1, -1, i, -i\} \;\times$

$n \geqslant 2$

$\tilde{h}$ is isomorphism ($\underbrace{X \text{ is } (n-1)\text{-connected}}$).

h

sm $(n-1)$-

$(n-1)$ connected تعریف ($\underline{\pi_i(X)} = 0$, $i < n$)

$(T^n) \times \qquad \pi_n(T^n) \neq H_n(T^n)$

$n=2$

$$\underbrace{H_k(S^n, \mathbb{Z})}_{\text{ضرب}} = \begin{cases} \mathbb{Z} & k = 0, n \\ \underbrace{\{e\}} & \text{else.} \end{cases}$$

$\pi_n(X) \qquad I = [0,1]$

$f : I^n \longrightarrow S$

$n=1$

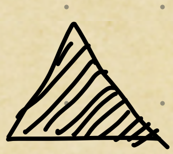
$[0,1] \longrightarrow S$ لوپ

$[0,1] \times [0,1] \longrightarrow S$

$\mathbb{R}^3$

$I^3 : \underbrace{[0,1] \times [0,1] \times [0,1]}_{3} \longrightarrow S^2$

$n=3$



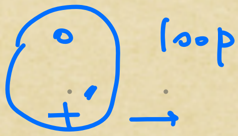
$J : \pi_k(\underline{\underline{SO(n)}}) \longrightarrow \pi_{n+k}(S^n)$

Relates homotopy.

Mostly based spectral / filtred sequen

homological algebra.

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$\pi_{n\geq 1}(S)$ 

... π_n آبلی

There's only 5 solid polygon.