Introduction to General Relativity - HW4 - 401208729

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Q1.

$$\nabla_\beta g_{\alpha\sigma} \overset{?}{=} 0$$

By definition of covariant derivative on forms:

$$\nabla_\beta g_{\alpha\sigma} = \partial_\beta g_{\alpha\sigma} - \Gamma_{\beta\alpha}^{\lambda} g_{\lambda\sigma} - \Gamma_{\beta\sigma}^{\lambda} g_{\alpha\lambda}$$

we find the second term, the third term is the same with $\alpha \longleftrightarrow \sigma$ exchange. since both $\Gamma^{\rho}_{\mu\nu}$ and $g_{\mu\nu}$ are symmetric under $\mu \longleftrightarrow \nu$.

$$\Gamma^{\lambda}_{\beta\alpha} g_{\lambda\sigma} \overset{\text{By}}{\underset{\text{def}}{=\!=}} g_{\lambda\sigma} \times \frac{1}{2} g^{\lambda\rho} (\partial_\beta g_{\alpha\rho} + \partial_\alpha g_{\rho\beta} - \partial_\rho g_{\alpha\beta})$$

By $g_{\lambda\sigma} g^{\lambda\rho} = g_{\sigma\lambda} g^{\lambda\rho} = \delta_\sigma^{\ \rho}$ identity

(since $g^{\mu\nu}$ is inverse of $g_{\mu\nu}$ ...)

$$\Gamma^{\lambda}_{\beta\alpha} g_{\lambda\sigma} = \frac{1}{2} \delta_\sigma^{\ \rho} (\partial_\beta g_{\alpha\rho} + \partial_\alpha g_{\rho\beta} - \partial_\rho g_{\alpha\beta})$$

Now since $\partial_\gamma(\delta^\alpha_\beta) = 0$ (since it's constant tensor) we can conclude that

$$\Gamma^\lambda_{\beta\alpha} g_{\lambda\sigma} = \frac{1}{2}\Big\{\partial_\beta(\underbrace{\delta^\rho_\sigma g_{\alpha\rho}}_{}) + \partial_\alpha(\underbrace{\delta^\rho_\sigma g_{\rho\beta}}_{}) - \underbrace{\delta^\rho_\sigma \partial_\rho}_{\partial_\sigma}(g_{\alpha\beta})\Big\}$$

By contraction $= g_{\alpha\sigma}$ $\quad g_{\sigma\beta}$

$$= \frac{1}{2}(\partial_\beta g_{\alpha\sigma} + \partial_\alpha g_{\sigma\beta} - \partial_\sigma g_{\alpha\beta})$$

Now $\Gamma^\lambda_{\beta\sigma} g_{\alpha\lambda}$ is likewise (just $\alpha \longleftrightarrow \sigma$ exchange is necessary.)

$$\Gamma^\lambda_{\beta\sigma} g_{\alpha\lambda} = \frac{1}{2}(\partial_\beta g_{\sigma\alpha} + \partial_\sigma g_{\alpha\beta} - \partial_\alpha g_{\beta\sigma})$$

Adding them all results:

$$\nabla_\beta g_{\alpha\sigma} = \partial_\beta g_{\alpha\sigma}$$

$$- \frac{1}{2}(\underbrace{\partial_\beta g_{\alpha\sigma}}_{3} + \underbrace{\partial_\alpha g_{\sigma\beta}}_{1} - \overbrace{\partial_\sigma g_{\alpha\beta}}^{2})$$

$$- \frac{1}{2}(\underbrace{\partial_\beta g_{\sigma\alpha}}_{3} + \underbrace{\partial_\sigma g_{\alpha\beta}}_{2} - \underbrace{\partial_\alpha g_{\beta\sigma}}_{1})$$

① since $g_{\beta\sigma} = g_{\sigma\beta}$, these terms cancel.

② similarly for these tems.

③ they add up to $\partial_\beta g_{\alpha\sigma}$ which exactly cancels the first line $\Rightarrow$

$$\nabla_\beta g_{\alpha\sigma} = 0 \qquad \blacksquare$$

Q2. By transformation rules of vectors in T_pM and T_p^*M (according to Carroll's Convention.)

$$e_\nu \longrightarrow e'_\nu = \frac{\partial x^\lambda}{\partial x'^\nu} e_\lambda$$

$$e^\mu \longrightarrow e'^\mu = \frac{\partial x'^\mu}{\partial x^\rho} e^\rho$$

$$\Gamma'^\mu_{\nu\sigma} = e'^\mu \cdot \frac{\partial}{\partial x'^\sigma} e'_\nu \longrightarrow ?$$

plug those relations into def of Γ - in terms of einbein (or tetrad):

$$\left(\frac{\partial x'^\mu}{\partial x^\rho} e^\rho\right)\left(\frac{\partial}{\partial x'^\sigma}\right)\left(\frac{\partial x^\lambda}{\partial x'^\nu} e_\lambda\right)$$

$$= \frac{\partial x'^\mu}{\partial x^\rho} e^\rho \left(\frac{\partial^2 x^\lambda}{\partial x'^\sigma \partial x'^\nu} e_\lambda + \frac{\partial e_\lambda}{\partial x'^\sigma} \times \frac{\partial x^\lambda}{\partial x'^\nu}\right)$$

chain rule

$$= \frac{\partial x'^\mu}{\partial x^\rho} e^\rho \left(\frac{\partial^2 x^\lambda}{\partial x'^\sigma \partial x'^\nu} e_\lambda + \frac{\partial x^\xi}{\partial x'^\sigma} \frac{\partial e_\lambda}{\partial x^\xi} \frac{\partial x^\lambda}{\partial x'^\nu}\right)$$

$$= \frac{\partial x'^{\gamma}}{\partial x^{\rho}} \frac{\partial x^{\xi}}{\partial x'^{\sigma}} \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \underbrace{(e^{\rho} \partial_{\xi} e_{\lambda})}_{\text{This is } \Gamma^{\rho}_{\xi\lambda}}$$

$$+ \frac{\partial x'^{\gamma}}{\partial x^{\rho}} \frac{\partial^2 x^{\lambda}}{\partial x'^{\sigma} \partial x'^{\nu}} e^{\rho} e_{\lambda}$$

As you see the first term is the tensorial transformation of $\Gamma^{\gamma}_{\nu\sigma}$, while the second part is the excessive part that doesn't vanish necessarily and prohibits Γ from being a tensor.

only for linear transformation when $\frac{\partial^2 x^{\lambda}}{\partial x'^{\sigma} \partial x'^{\nu}} = 0$ Γ is a tensor (since linear transformation correspond to poincare group, so Γ is tensor under poincare transformation.)

Q3. Consider $\begin{cases} x = u+v \\ y = u-v \\ z = 2uv - w \end{cases}$ in 3D flat space ($\mathbb{R}^3$).

In usual Cartesian Coordinates, tangent vectors are

$\partial_x, \partial_y, \partial_z$. By chain rule:

$$\partial_u = \frac{\partial}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial}{\partial z}\frac{\partial z}{\partial u}$$

so
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} u \\ v \\ w \end{array} & \begin{bmatrix} 1 & 1 & 2v \\ 1 & -1 & 2u \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

$$\Longrightarrow \begin{cases} \partial_u = \partial_x + \partial_y + 2v\,\partial_z \\ \partial_v = \partial_x - \partial_y + 2u\,\partial_z \\ \partial_w = -\partial_z \end{cases}$$

Inverse transformation $\begin{cases} u = \frac{1}{2}(x+y) \\ v = \frac{1}{2}(x-y) \end{cases}$

$$\Longrightarrow \begin{cases} \partial_u = \partial_x + \partial_y + (x-y)\,\partial_z \\ \partial_v = \partial_x - \partial_y + (x+y)\,\partial_z \\ \partial_w = -\partial_z \end{cases}$$

Replace $(\partial_u, \partial_v, \partial_w)$ by (e_u, e_v, e_w) and $(\partial_x, \partial_y, \partial_z)$ by (e_x, e_y, e_z) to find the relation in the questions convention.

For dual basis :

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

So we need inverse transformations.

As we saw $\begin{cases} u = \frac{1}{2}(x+y) \\ v = \frac{1}{2}(x-y) \end{cases}$

And $z = 2uv - w \Longrightarrow$

$$w = 2 \times \frac{1}{4}(x-y)(x+y) - z$$

$$\Longrightarrow w(x,y,z) = \frac{1}{2}(x^2 - y^2) - z$$

so $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{array}{c} \\ x \rightarrow \\ y \rightarrow \\ z \rightarrow \end{array} \begin{array}{c} \begin{array}{ccc} \ \ u \downarrow & \quad v \downarrow & \quad w \downarrow \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \overset{=u+v}{x} \\ \frac{1}{2} & -\frac{1}{2} & \underset{(v-u)}{-y} \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

$$\Rightarrow \begin{cases} du = \frac{1}{2}(dx + dy) \\ dv = \frac{1}{2}(dx - dy) \\ dw = x\,dx - y\,dy - dz \end{cases}$$

you can check that $du^\alpha(\partial_\beta) = \delta^\alpha_\beta$.

when α, β are in $\{u, v, w\}$, so the

vectors & forms are normalized :) .

Q4. use $\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)$

let's look at $\Gamma^{\theta}_{ij} = \frac{1}{2} g^{\theta\alpha}\left(g_{\alpha i,j} + g_{\alpha j,i} - \partial_\alpha g_{ij}\right)$

since the metric is diagonal $\Longrightarrow \alpha = \theta$

and since non of metric's components depend on ϕ explicitly, the calculation is very easy!

$\Gamma^{\theta}_{ij} = \frac{1}{2} \times \underbrace{g^{\theta\theta}}_{1}$

- $i=j=\theta \rightarrow$ zero since all derivatives of $g_{\theta\theta}=1$ vanish.
- $i=j=\phi \rightarrow$ just $-\partial_\theta g_{\phi\phi} = -2\sin\theta\cos\theta$ survives
- $i=\theta, j=\phi$ or vice versa $\longrightarrow$ zero since metric is diagonal and $\partial_\phi g_{\theta\theta}=0$

$$\left(\Gamma^{\theta}\right)_{ij} = \begin{array}{c c} & \begin{array}{cc} \theta & \phi \end{array} \\ \begin{array}{c} \theta \\ \phi \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & -\sin\theta\cos\theta \end{pmatrix} \end{array}$$

For Γ^{ϕ}_{ij}, since $g^{\phi\phi} = \frac{1}{\sin^2\theta}$

$$\Gamma^{\phi}{}_{ij} = \frac{1}{2} g^{\phi\alpha} \left(g_{\alpha i,j} + g_{\alpha j,i} - g_{ij,\alpha} \right)$$

$$\Gamma^{\phi}{}_{ij} = \frac{1}{2} \sin^2\theta$$

- $i = j = \theta \to$ all terms vanish
- $i = j = \phi \to$ since all derivatives are w.r.t ϕ, they vanish.
- $i = \theta, j = \phi$ or vice versa $\to$ just $\partial_\theta g_{\phi\phi} = 2\sin\theta\cos\theta$ contributes

$$(\Gamma^{\phi})_{ij} = \begin{array}{c|cc} & \theta & \phi \\ \hline \theta & 0 & \cot\theta \\ \phi & \cot\theta & 0 \end{array}$$

Geodesic eq : $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}{}_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$

Along $\theta \Rightarrow \frac{d^2 x^\theta}{d\lambda^2} - \sin\theta\cos\theta \left(\frac{dx^\phi}{d\lambda} \right)^2 = 0$

" $\phi \to \frac{d^2 x^\phi}{d\lambda^2} + 2\cot\theta \frac{dx^\theta}{d\lambda} \frac{dx^\phi}{d\lambda} = 0$

Notice that $x^\mu = (x^\theta(\lambda), x^\phi(\lambda))$ are geodesics on the S^2-sphere.

(b) so simple! $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

$$S = \int_{\lambda_0}^{\lambda_1} d\lambda \left(\left(\frac{d\theta(\lambda)}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\phi(\lambda)}{d\lambda}\right)^2 \right)^{1/2}$$

when $x^\mu(\lambda) = (\theta(\lambda), \phi(\lambda))$ is a curve parametrized by $\lambda \in (\lambda_0, \lambda_1)$.

To exterimize the lengh, we variate $\theta(\lambda), \phi(\lambda)$ by

$$\begin{cases} \theta(\lambda) \longrightarrow \theta(\lambda) + \delta\theta(\lambda) & \text{with} \quad \delta\theta(\lambda)\big|_{\lambda_0, \lambda_1} = 0 \\ \phi(\lambda) \longrightarrow \phi(\lambda) + \delta\phi(\lambda) & \text{with} \quad \delta\phi(\lambda)\big|_{\lambda_0, \lambda_1} = 0 \end{cases}$$

when at the end, the geodesic path results from usual Euler-Lagrange equations.

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \left(\frac{d}{d\lambda} \begin{smallmatrix}\theta \\ \phi\end{smallmatrix}\right)} = \frac{\partial \mathcal{L}}{\partial \begin{smallmatrix}\theta \\ \phi\end{smallmatrix}}$$ for both θ, ϕ components.

I elaborate on the reason why not taking its

$\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\phi}{d\lambda}\right)^2 \right)$ is like the Lagrajian, and λ parameter is like time. So the situation is quite similar to Classical Mechanics.

Square root into consider later at the end of this exercise.

$$\frac{d}{d\lambda}\frac{\partial L}{\partial(\frac{d}{d\lambda}\theta)} = \frac{d}{d\lambda}\left(2\frac{d\theta}{d\lambda}\right) = 2\frac{d^2\theta}{d\lambda^2}$$

$$\frac{\partial L}{\partial \theta} = 2\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2$$

$$\Rightarrow \frac{d^2\theta(\lambda)}{d\lambda^2} - \sin\theta\cos\theta\left(\frac{d\phi(\lambda)}{d\lambda}\right)^2 = 0 \quad \blacksquare$$

For ϕ- compoment of E-L equation:

$$\frac{d}{d\lambda}\frac{\partial L}{\partial(\frac{d}{d\lambda}\phi)} = 2\frac{d}{d\lambda}\left(\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)\right)$$

since $\frac{d}{d\lambda}$ is total derivative we have to be CAREFUL, using chain rule:

$$\frac{d}{d\lambda} = \frac{\partial}{\partial\theta}\frac{d\theta}{d\lambda} + \frac{\partial}{\partial\phi}\frac{d\phi}{d\lambda}$$

so $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial(\frac{d\phi}{d\lambda})}\right) = 2\left(\frac{d}{d\lambda}\sin^2\theta\right)\frac{d\phi}{d\lambda} +$

$$2\sin^2\theta\frac{d^2\phi}{d\lambda^2}$$

when $\frac{d}{d\lambda} \sin^2\theta = \frac{d\theta}{d\lambda}\frac{\partial}{\partial\theta}(\sin^2\theta) + \cancel{\frac{d\phi}{d\lambda}\frac{\partial}{\partial\phi}(\sin^2\theta)}^{\,0}$

$$= 2\sin\theta\cos\theta \times \frac{d\theta}{d\lambda}$$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{d\phi}{d\lambda}\right)}\right) = \left(4\sin\theta\cos\theta\right)\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} + 2\sin^2\theta\,\frac{d^2\phi}{d\lambda^2}$$

And $\frac{\partial \mathcal{L}}{\partial\phi}$ also vanishes, therefore

$$4\sin\theta\cos\theta\,\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} + 2\sin^2\theta\,\frac{d^2\phi}{d\lambda^2} = 0$$

Divide by $2\sin^2\theta$ to get:

$$\frac{d^2\phi}{d\lambda^2} + 2\cot\theta\,\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} = 0 \qquad \boxtimes$$

Note: Lagrangian was $\mathcal{L} = \left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right)^{1/2}$

actually, but since zeros of $8\sqrt{\#}$ are equal to zeros of $\frac{8\#}{2\sqrt{\#}}$ so we doesn't need to carry square root with ourselves!