Classification of groups of small orders :

Our aim is to do some trial and error to find work out order 1, 2, 3, 4 groups.

There is just one group of order :

$$\boxed{1 \longrightarrow G = \{e\}}$$

$$\boxed{2 \longrightarrow G = \{e, g\} \quad \text{and} \quad g^2 = e}$$

e.g $\{0, 1\}$ with sum modulo 2.

Of order 3 : suppose $\{e, g_1, g_2\}$ be our group :

If $g_2 = g_1^{-1}$

then $g_1 \cdot g_1 \xrightarrow{?} \begin{cases} e & \times \\ g_1 & \times \\ g_2 & \checkmark \end{cases}$

And $\rightarrow g_2^2 \xrightarrow{?} \begin{cases} e & \times \\ g_1 & \checkmark \\ g_2 & \times \end{cases}$

so the group is $\{e, g_1, g_2 = g_1^2\}$

e.g $\rightarrow \{1, e^{\frac{1\pi i}{3}}, e^{\frac{2\pi i}{3}}\}$ with multiplication

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Of order 4 :

① $C_4 \to \{e, g, g^2, g^3\}$ like $\{1, -1, i, -i\}$ with usual multiplication

cyclic

self_inverse

② or $\{e, g_1, g_2, g_3\}$

suppose $g_1^2 = e$; $g_2 = g_3^{-1}$;

$$g_1 \times g_2 \xrightarrow{?} \begin{cases} e & \times \\ g_1 & \rightarrow \checkmark \\ g_2 & \times \\ g_3 & \rightarrow \checkmark \end{cases}$$

$$g_1 g_2 = g_1 \xrightarrow{\times g_1} g_2 = g_1^2 = e \;!$$

The other way: $g_1^2 = e$ & $g_1 g_2 = g_3$

$g_2 = g_1 g_3$

what about g_2^{-1} ?
$$\begin{cases} e & \times \\ g_1 & \times \\ g_2 & \checkmark \end{cases}$$
$\underline{g_3 \quad ?}$ suppose this's

Show this gives C_4

with $g_1 = (-1)$

with $g_2 = i$

$g_3 = -i$

$$\begin{cases} g_2^2 = e \\ g_3^2 = e \end{cases} \xrightarrow{\text{so}} \quad g_1 g_2 \overset{?}{=} \text{ just } = g_3 \longrightarrow \begin{cases} g_2 = g_1 g_3 \\ g_1 = g_3 g_2 \end{cases}$$

$$\text{but } (g_2 g_1) = g_2^{-1} g_1 = (g_1^{-1} g_2)^{-1}$$

$$= (g_1 g_2)^{-1} = g_3^{-1} = g_3$$

so $g_2 g_1 = g_3$ and ... commutative ✓

this is nothing but $\mathbb{Z}_2 \times \mathbb{Z}_2$ grp (Klein grp.)

$$\{(0,0), (0,1), (1,0), (1,1)\}$$

Componentwise sum modulo 2.

About $O(G) \geq 5$ things get complicated, one can just proceed by checking all the assumptions...

Theorem: Any group of prime order is cyclic, hence Abelian.

About order 5 group? $\xrightarrow{!}$ just cyclic!

One possible scenario is: $\{e, g_1, g_2, g_3, g_4\}$

$g_1^2 = g_2^2 = g_3^2 = g_4^2 = e$

Define:

$g_1 g_2 = g_3$; $g_2 = g_1 g_3$; $g_1 = g_3 g_2$

$g_2 g_1 = g_2^{-1} g_1 = (g_1^{-1} g_2)^{-1} = (g_1 g_2)^{-1} = g_3^{-1} = g_3$

So its commutative. ✓

Similarly $g_1 g_4 \stackrel{?}{=} g_2$; $g_4 = g_1 g_2$; $g_1 = g_2 g_4$

since $\rightarrow g_1 = g_2 g_4 = g_1 g_3 g_4 \xrightarrow{\times g_1}$

$e = g_3 g_4$ $\qquad g_3 = g_4^{-1}$

since $g_4^{-1} = g_4 \rightarrow g_3 = g_4$!

so it can't be the case!