Aspects of Low-dimensional Quantum Gravity

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Abstract: This paper is an insightful summary of my master's thesis on quantum gravity in two and three dimensions, in which I tour from historical discoveries to recent developments. I start with JT gravity at the classical level, then continue with quantum JT gravity, specifically focusing on two and four-point correlation functions. The significance of wormhole contributions and their role in the resolution of JT gravity's spectral form factor discrepancy are briefly highlighted. Then, I include such new contributions in the JT gravity path integral formulation. The second half of this thesis is devoted to three-dimensional gravity, focusing on the asymptotic symmetries in asymptotically AdS and flat spacetimes. Finally, I give a concise account of the partition function of pure 3D gravity in asymptotically AdS spacetime and address its physical and holographic shortcomings. Several appendices have also been included to supplement the main discussions.

Keywords: Holography, AdS/CFT, JT gravity, Quantum gravity, Black hole, Gauge theory, Information paradox.

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1 Introduction

Low-dimensional quantum gravity, in two or three dimensions, holds special significance in theoretical physics mainly because it serves as a simple laboratory for understanding the universal features of quantum gravity. Even though there is little hope that these lower-dimensional quantum gravity theories will lead to a complete understanding of four-dimensional quantum gravity, they provide a suitable playground for exploring ideas and learning about quantum gravity. These models allow us to examine the complex dynamics of quantum gravity theory and study concepts like black holes, entropy, and symmetries in a more controllable and simplified framework (Applications are in information paradox and black hole evaporation [1, 2], 2D cosmology [3], and quantum chaos [4]).

My master's thesis is organized as follows.
It is well-known that two-dimensional gravity is structurally and conceptually trivial without the dilaton field [5–7]. So, we start with dilaton models in two dimensions, specifically JT gravity [8, 9]. First, we explore JT at the classical level, reviewing the motivations for its study, classical solutions, and its connection with the Schwarzian action [1]. We then proceed to the quantum level analysis of JT, beginning with a perturbative, one-loop correction of this theory, followed by a summary of the two- and four-point correlation functions and a brief discussion of their derivations [10]. Subsequently, we examine JT gravity in the presence of defects, express the diagrammatic rule of JT amplitudes, and conclude with a discussion of wormhole contributions, understanding their physical importance[11]. Here, the discussion of two-dimensional gravity ends, and we proceed to three-dimensional gravity.

The discussion of three-dimensional gravity begins at the classical level by examining its duality with the Liouville field theory [12]. Through a series of Hamiltonian reductions and redefinitions, we demonstrate that, at the classical level, three-dimensional anti-de Sitter (AdS) gravity is equivalent to a Liouville field theory on its conformal boundary. Then, we approach three-dimensional gravity from a different perspective, a gauge theoretical description. We pay close attention to the symmetries within gauge theories. In gauge theories, the asymptotic behavior of gauge fields is crucial, as it determines the model extensively. By choosing the Brown-Henneaux [13] fall-off conditions for asymptotically AdS spaces and the BMS fall-off conditions for asymptotically flat spaces, we analyze their respective asymptotic symmetries. For asymptotically AdS spacetimes, the well-known Brown-Henneaux fall-off conditions point toward the gauge/gravity duality, a result derived roughly a decade before this duality was framed in modern language [14]. Similarly,

[1]This theory is invariant under the $\mathrm{SL}(2,\mathbb{R})$ symmetry, defined by Schwarzian action in (2.17)

the gauge symmetry group for flat spaces is the centrally extended Poincaré group, known as the BMS_3 group [15, 16]. Throughout this discussion, we uncover extensive connections between these models and the geometric actions on coadjoint orbits of the Virasoro group and the $\widetilde{\mathfrak{bms}}_3$ algebra, highlighting the geometric nature of these quantum theories.

Finally, we study the partition functions of pure AdS_3 gravity [17], employing the gravitational path integral formulation in the semiclassical approximation. Additionally, we encounter several physical and holographical controversies in the spectrum of this quantum gravity theory, presenting possible proposals to define a healthy quantum gravity in asymptotically AdS_3 spacetimes. In recent years, one such proposal has resolved the issue of the negative density of states in this spectrum [18] . However, research continues to address the remaining challenges in this theory.

In Appendix A, I have provided a comprehensive list of the topics discussed throughout my thesis. For clarity and conciseness, only the topics underlined in this section are briefly addressed in this summary paper. In other appendices, I cover the connection between two-dimensional JT gravity and the SYK model at low energies (appendix B) and some prerequisites for this thesis's content, including a review of Chern-Simons and Wess-Zumino-Witten models (appendix C) and first-order formulation of Einestein-Hilbert action (appendix D.)

2 Jackiw-Teitelboim Gravity

In 1983 and 1984, Roman Jackiw [8] and Claude Teitelboim [9] independently developed the foundations for two-dimensional gravity theories, particularly JT gravity. JT is a two-dimensional model of quantum gravity that has recently attracted significant attention in studying the quantum nature of gravity and black hole physics [1, 2]. Despite its simplicity and solvability, JT gravity captures essential features of gravitational dynamics, making it a valuable tool for examining complex concepts in a more manageable framework. Furthermore, at low energies, JT gravity has a deep connection with the SYK model — a solvable interacting fermionic model— making it suitable for research in quantum chaos and holography. This connection has also settled the tension on $\text{AdS}_2/\text{CFT}_1$ interpretations.

In practical applications, JT gravity is used to investigate the dynamics of near-extremal black holes [19] and the emergence of spacetime from quantum entanglement[20]. Additionally, JT gravity's holographic description connects to random matrix theory [21], building a bridge between gravitational systems and statistical mechanics. These interrelations and applications have enriched our understanding of fundamental physics.

2.1 Overview of Two-dimensional Dilaton Models and JT gravity

Let us begin with the most general two-dimensional gravitational action that includes a dilaton field $\tilde{\Phi}$, and at most second derivatives of the dilaton field, the corresponding Euclidean action is given by:

$$I = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{g}\Big(U_1(\tilde{\Phi})R + U_2(\tilde{\Phi})g^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi} + U_3(\tilde{\Phi})\Big), \tag{2.1}$$

Using general covariance, rescaling of metric, $g^{\mu\nu}$, and redefinition of $\tilde{\Phi}$ one can remove U_1 and U_2 potentials. So, the simplified form of all dilaton 2D gravities is

$$I[g,\Phi] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{g}\Big(\Phi R + U(\Phi)\Big). \tag{2.2}$$

JT gravity is a dilaton 2D gravity with a linear choice for $U(\Phi)$ potential in 2.2, that is, $U(\Phi) = -\Lambda\Phi$.

$$I^{\Lambda}_{\mathrm{JT}}[g,\Phi] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} \sqrt{g}\Phi\big(R-\Lambda\big), \tag{2.3}$$

where Λ is the cosmological constant, where in AdS spacetime $\Lambda = -2/L^2$. For convenience, we choose $L = 1$. Additionally, we complete this action by adding the Gibbons-Hawking-York boundary term to have a well-defined variational principle and a topological term, which will be justified later on wormhole contributions.

$$\begin{aligned} I[g,\Phi] &= -S_0\chi + I_{\mathrm{JT}}[g,\Phi] \\ I_{\mathrm{JT}}[g,\Phi] &= -\frac{1}{16\pi G_N}\int_{\mathcal{M}} \sqrt{g}\Phi(R+2) - \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}} \sqrt{h}\Phi(K-1) \\ \chi &= \frac{1}{4\pi}\int_{\mathcal{M}} \sqrt{g}R + \frac{1}{2\pi}\oint_{\partial\mathcal{M}} \sqrt{h}K. \end{aligned} \tag{2.4}$$

The topological term of this action can be thought of as being produced by a shift in the dilaton field by a constant Φ_0, with $S_0 = \frac{\Phi_0}{4G_N}$. Additionally, χ is the Euler character of the spacetime manifold $\mathcal{M}$ due to the Gauss-Bonnet theorem.

2.2 Classical JT Gravity

We examine JT gravity classically, including its equations of motion, boundary conditions, and its intrinsically one-dimensional nature as Schwarzian action.

2.2.1 Classical Solutions of JT Gravity

Let us explore the classical solutions of the metric and the dilaton field. The Lorentzian action of this theory is of the form

$$S = \frac{\Phi_0}{16\pi G_N}\Big[\int \sqrt{-g}R + 2\oint \sqrt{-h}K\Big] + S_{\mathrm{JT}}[g,\Phi] + S_{\mathrm{m}}[\phi,g] \tag{2.5}$$

where Φ_0 is the background value of dilaton field, and $S_{\mathrm{m}}[\phi,g]$ refers to the matter action. Our discussion considers conformal matter coupled to JT gravity to have a controllable behavior.

To solve for metric, we vary the dilaton field. This variation results in $R(x) = -2$, which determines a hyperbolic metric on spacetime. By using a conformal parametrization of metric and solving a Liouville equation, one observes that the unique metric solution is the Poincare metric

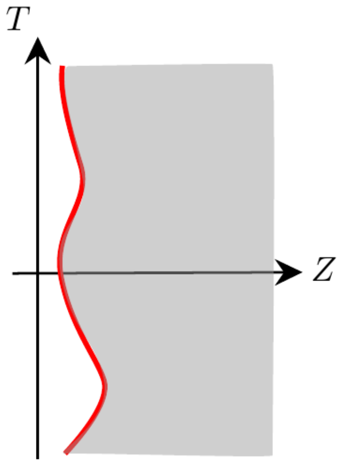


Figure 1. A patch of Euclidean AdS_2 spacetime, the red boundary curve plays a crucial dynamical role in JT gravity.

$$ds^2 = -\frac{dUdV}{(U-V)^2}, \tag{2.6}$$

where U and V are light cone coordinates of T and Z, the standard parametrization of AdS. On the other hand, the spacetime manifold is an arbitrary patch of the AdS (See figure 1 for more details.)

We find the dilaton profile by varying 2.5 with respect to the metric.

$$\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu}\nabla^2\Phi + g_{\mu\nu}\Phi = -8\pi G_N T_{\mu\nu} \tag{2.7}$$

Rewriting dilaton equations of motion in conformal gauge leads to[2]

$$\begin{aligned} -e^{2\omega}\partial_u(e^{-2\omega}\partial_u\Phi) &= 8\pi G_N T_{uu}, \\ -e^{2\omega}\partial_v(e^{-2\omega}\partial_v\Phi) &= 8\pi G_N T_{vv}, \\ 2\partial_u\partial_v\Phi + e^{2\omega}\Phi &= 16\pi G_N T_{uv}, \end{aligned} \tag{2.8}$$

where T_{ij} denotes the components of the matter stress tensor.

The conformal matter, $S_{\rm m}$, has a chiral stress tensor[3], so the differential equations in 2.8 are integrable, and the profile of dilaton is [7]

$$\Phi(u,v) = \frac{a}{U-V}\Big(1 - \frac{\mu}{a}UV - \frac{8\pi G_N}{a}(I_+(u,v) + I_-(u,v))\Big) \tag{2.9}$$

with

$$\begin{aligned} I_+(u,v) &= \int_U^\infty ds\ (s-U)(s-V)T_{UU}(s), \\ I_-(u,v) &= \int_{-\infty}^V ds\ (s-U)(s-V)T_{VV}(s) \end{aligned} \tag{2.10}$$

[2] (u,v) coordinates are associated with locally flat coordinates on spacetime manifold, in which the metric is $ds^2 = e^{-\omega(u,v)}dudv$.

[3] That is to $T_{uu}(u)$, $T_{vv}(v)$, and $T_{uv} = 0$.

Let us clarify this formulation with a physical problem.

An infalling energy pulse

An energy pulse with energy $E > 0$, falling into the Poincaré region, is modeled with the stress tensor $T_{VV}(V) = E\delta(V)$.
Substituting into equation (2.10) immediately gives

$$\begin{cases} I_+ = 0 \\ I_- = EUV\theta(V) \end{cases}$$

Thus, the profile of the dilaton field (equation (2.9)) is given by:

$$\Phi = \frac{a - 8\pi G_N EUV\theta(V)}{U - V}$$

This profile has an interesting interpretation. The vacuum state is restored after the energy pulse falls into the bulk. In this regime, the parameters are related by $\mu = 8\pi G_N E$.

2.2.2 Boundary Condition and Its Importance

After solving for classical fields, it remains to define a viable boundary condition for the metric and the dilaton field[4], which is we constrain their behavior as they approach the boundary curve. We closely follow the approach of [22] to define suitable fall-off conditions for both geometry and dilaton field. This approach is the most intuitive way to define fall-off behaviors for asymptotically AdS_2 metric and divergent dilaton field.

- **Fixing asymptotics of metric:** The geometry is fixed to asymptotically AdS_2 in Fefferman-Graham gauge, which means the following near boundary expansion for the metric
$$ds^2 = \frac{-dt^2 + dz^2}{z^2} + (\text{Subleading as } \ z \to 0). \tag{2.11}$$
Requiring this form for the asymptotic metric constrains the allowed diffeomorphism on JT gravity. The boundary curve of the gravitational patch (the red curve in 1 is specified with a function $F(t)$ in Poincare coordinates, parametrized by $(T = F(t), Z = \epsilon F'(t))$[5].

- **Fixing the asymptotics of dilaton field:** As mentioned, the dilaton field must be divergent at the holographic boundary. We fix it to have the following asymptotics.
$$\Phi = \frac{a}{2z} + (\text{Subleading as } \ z \to 0), \tag{2.12}$$
where a is a dimensionful parameter.

[4] Boundary conditions can also be addressed by the divergence of dilaton profile, starting from a uniform vanishing profile.

[5] ϵ is the holographic cutoff near the $Z = 0$ boundary.

Applying (2.12) condition on (2.9) constrains the boundary curve by the following equation[6]

$$F'(t) = 1 - \frac{8\pi G_N}{a}\Big(\int_{F(t)}^{\infty} ds\ (s - F(t))^2 T_{UU}(s) + \int_{-\infty}^{F(t)} ds\ (s - F(t))^2 T_{VV}(s)\Big) \tag{2.13}$$

This complicated equation specifies the profile of the boundary curve. We introduce a very suggestive way to rewrite this differential-integral equation in terms of holographic stress tensor.

The holographic stress tensor, in 0+1-dimensional theories, which measures the injected energy to the system in time t, is defined by [23]

$$E(t) = -\frac{a}{16\pi G_N}\{F, t\}, \qquad \{F, t\} = \frac{F'''}{F'} - \frac{3}{2}\Big(\frac{F''}{F'}\Big)^2, \tag{2.14}$$

where $\frac{16\pi G_N}{a}$ is the gravitational coupling. We also denote $C \equiv a/(16\pi G_N)$ for convenience.

By consequently applying derivatives on $F(t)$, equation 2.13 has a simple form in terms of the holographic stress tensor

$$\frac{dE(t)}{dt} = \big(T_{VV}(t) - T_{UU}(t)\big)F'^2\big|_{\partial\mathcal{M}} = \left(T_{vv}(t) - T_{uu}(t)\right)\big|_{\partial\mathcal{M}}. \tag{2.15}$$

This equation is manifestly interpreted as an energy conservation equation. The physical content of this simple relation is as follows. The boundary curve reacts to energy injection/extraction by adjusting itself. By injecting energy into the system, the boundary curve tends to the holographic boundary of AdS_2, and by extracting energy, it contracts inward (See figure 2 for a clear illustration.)

2.2.3 Dynamics of the Boundary Curve

In the absence of external sources, 2.15 simplifies to

$$\frac{d}{dt}\{F, t\} = 0. \tag{2.16}$$

This equation, known as the "Schwarizan equation," can be derived from the Schwarzian action.

$$S = -C\int \mathrm{dt}\ \{F, t\}. \tag{2.17}$$

The main property of this model is its invariance under $\mathrm{PSL}(2, \mathbb{R})$, which is under $F \to \frac{aF+b}{cF+d}$ transformation, with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Hence, it possesses three conserved charges.

This action can be motivated more geometrically[24] in the following way. By varying 2.4 with respect to the dilaton field, all the dynamics of JT will transfer to the boundary term. Parametrizing the boundary curve by $(F(\tau), Z(\tau))$, we constrain the determinant of

[6]Note that $T_{UU}(U)$ and $T_{VV}(V)$ are related to $T_{uu}(u)$ and $T_{vv}(v)$ with tensorial relations.

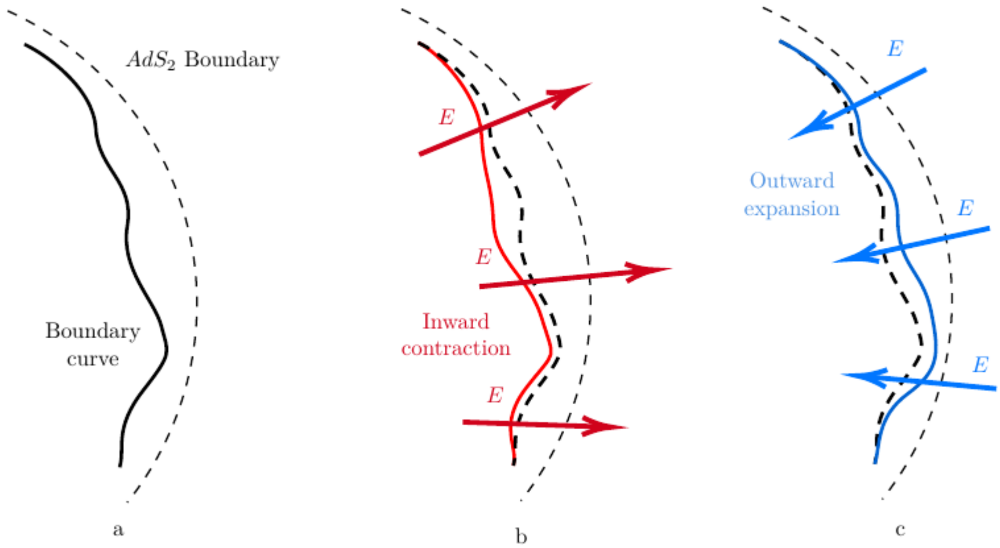


Figure 2. The effects of energy exchange on boundary curve. (a) In the initial state. (b) The boundary curve contracts inward as energy exits the system. (c) With energy injection, the boundary curve tilts toward the holographic boundary.

the boundary metric to $\sqrt{h} = 1/\epsilon$. After applying 2.11 and 2.12 boundary conditions, the boundary term of 2.4 in the first order of ϵ will become Schwarzian derivative.

$$\mathrm{I}_{\mathrm{JT}}[F] = -C \int d\tau \{F, \tau\} \tag{2.18}$$

Therefore, JT gravity is holographically equivalent to Schwarzian theory. The connection with SYK models arises since SYK is effectively the Schwarzian theory in low energies. In fact, JT and SYK are effectively equivalent in the low energy limit.

2.3 Quantum JT Gravity

In the previous part, we treated both gravity and matter classically. Here, we briefly review the quantization of JT gravity. There are several approaches to quantum JT gravity, each with advantages and weaknesses. Let us briefly mention each approach and reference them.

- **Free Particle Approach:** In this approach[25, 26], the action is transformed into a bosonic action by a redefinition of $F(t)$. Even though working with bosonic action is easy, implementing the boundary conditions 2.11 and 2.12 complicates the action, and working out the correlation functions requires unconventional techniques.

- **Limit of Liouville CFT:** By considering Liouville CFT with ZZ branes on the end of cylindrical worldsheet in a particular limit, JT action is recovered [10]. It opens the way to use the knowledge in Liouville CFT to calculate JT amplitudes. Nevertheless, the bulk picture of JT is obscured in this approach.

- **Two-diemensional Gauge Theory:** JT action has a formulation in terms of the BF topological model. The quantization of such topological models is well-known, and JT amplitudes correspond to the insertion of Wilson loops in BF dual[27]. Even though there are technical difficulties when extending this formulation with wormhole contribution.

- **Boundary Particle Approach:** Another approach which rewrites the Gibbons-Hawking-York term as a boundary particle in path integral formulation[28]. The Hamiltonian of this model resembles a non-relativistic particle in an imaginary infinite magnetic field, which has been studied in [29, 30]. Despite its physical interpretation, the symmetry structure of JT is not visible in this formulation.

Here, we just look at a one-loop result by expanding near classic extremums in the path integral and expressing the diagrammatic rule of JT gravity with a scalar bulk field.

2.3.1 One-loop Correction to Classical JT

The path integral quantization of JT, in AdS_2 black hole coordinates[7], is a sum over allowed metrics and dilaton fields with the weight of Euclidean action.

$$Z(\beta) = e^{S_0} \int [Dg_{\mu\nu}][D\Phi] \exp\Big(\frac{1}{16\pi G_N}\big[\int_{\mathcal{M}} d^2x\sqrt{g}\Phi(R+2) + 2\oint_{\partial\mathcal{M}} \sqrt{h}\Phi(K-1)\big]\Big) \tag{2.19}$$

By integrating the dilaton field and rewriting the boundary action as Schwarzian action, as mentioned in the previous chapter, 2.19 is written in this suggestive form

$$Z(\beta) = e^{S_0} \int_{\frac{\mathrm{Diff}(S^1)}{\mathrm{SL}(2,\mathbb{R})}} [Df] \exp\Big(C\int_0^\beta d\tau\{\tan\frac{\pi f(\tau)}{\beta}, \tau\}\Big), \tag{2.20}$$

where $f(\tau)$ is denotes diffeomorphism of circle, with $\mathrm{SL}(2,\mathbb{R})$ redundancy due to the symmetry of the Schwarzian action. Let us look at the one-loop correction of classical JT action. We can compute the one-loop quantum correction by expanding $f(\tau)$ around classical extremum $f_0(\tau) = \tau$. After mode expansion, working out details of I_{Sch}, and the integration measure, the one-loop correction to the partition function of JT gravity is

$$Z(\beta) = \exp^{S_0+\frac{2\pi^2 C}{\beta}} \Pi_{n\geq 2}\frac{\beta}{Cn} \xrightarrow{\zeta \text{ regularization}} \frac{1}{4\pi^2}\big(\frac{2\pi C}{\beta}\big)^{\frac{3}{2}} \exp^{S_0+\frac{2\pi^2 C}{\beta}}\,. \tag{2.21}$$

The corrected partition function 2.21 will give the logarithmic correction to the free energy of JT black holes.

$$-\beta F \equiv S_0 + \frac{2\pi^2 C}{\beta} + \frac{3}{2}\log\frac{2\pi C}{\beta} \tag{2.22}$$

Additionally, the one-loop spectrum of JT gravity is given by

$$\rho_{\mathrm{JT}}(E) = \frac{C}{2\pi^2}e^{S_0}\sinh\big(2\pi\sqrt{2CE}\big) \tag{2.23}$$

[7] Look at section 2.2 of [31] for more details about coordinates of AdS_2.

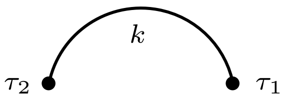
The spectrum's continuity suggests no quantum mechanical system is dual to JT gravity. The resolution is that JT is not dual to a single quantum mechanical system, but its holographic dual is an ensemble of quantum systems in certain limits, as found in [11]. We encounter similar ideas when adding wormhole contributions to quantum JT gravity.

2.3.2 Diagrammatic Rules for Quantum JT

From here, one can quantize JT gravity with a scalar matter. This detailed procedure requires advanced approaches [10]. Nevertheless, like the one used above, one-loop techniques can effectively calculate two-point functions. Let us ignore the calculation details and highlight the diagrammatic rules of JT amplitudes.

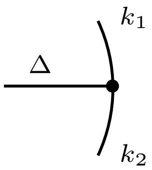
1. **Momentum variables:** Associate a momentum variable, k_i, to each closed region within the thermal circle. These momentum variables will finally be integrated with Schwarzian momentum density to yield JT amplitudes.

2. **Rule for thermal circle segments:** Write the following exponential factor for each segment of the thermal circle that encloses a region with momentum k.



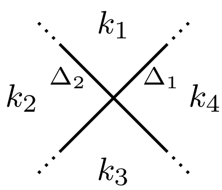
$$= e^{-\frac{k^2}{2C}(\tau_2-\tau_1)} \tag{2.24}$$

3. **Boundary vertex rule:** We have the following vertex rule for each intersection of a Schwarzian propagator with weight Δ that bisects a thermal circle to two regions with momentums k_1 and k_2.



$$\equiv \gamma_\Delta(k_1, k_2) = \sqrt{\frac{\Gamma(\Delta \pm ik_1 \pm ik_2)}{(2C)^{2\Delta}\Gamma(2\Delta)}}. \tag{2.25}$$

4. **Bulk vertex rule:** Insert the following factor for the intersection of two Schwarzian propagators.



$$= \begin{Bmatrix} \Delta_1 & k_1 & k_2 \\ \Delta_2 & k_3 & k_4 \end{Bmatrix} \tag{2.26}$$

The bracket indicates the $6j$-symbols for $\mathrm{PSL}(2,\mathbb{C})$ group, which is a complicated factor involving gamma functions and contour integration.

5. **Overall factor:** Multiply the resulting amplitude by $e^{S_0}/2$.

2.4 Wormhole Contributions to JT Gravity

In this section, I briefly explain these new contributions, how to include them in the formalism, and the resolution of the JT spectral form factor. Notice that these wormholes are spacetime wormholes, different from spatial wormholes living on a fixed timeslices.

2.4.1 Motivation

The behavior of late-time correlation functions of JT gravity does not conform with a typical quantum system as its putative holographic dual. In quantum systems, the two-point function fluctuates around an average point, whereas two-point functions of quantum JT decay with time as t^{-3}. This discrepancy can best be capsulated in spectral form factor and its evolution with time (Look at figure 3.)

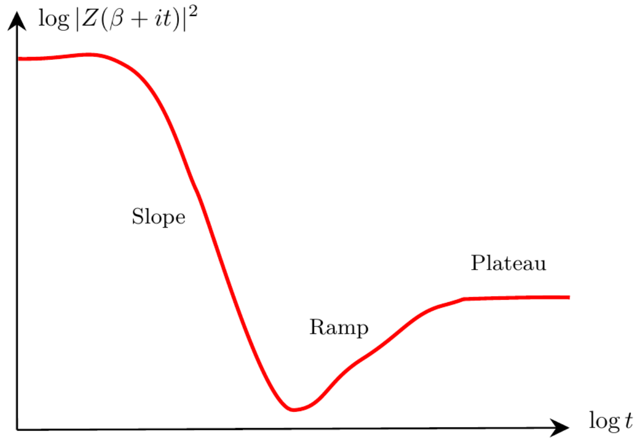


Figure 3. Evolution of the spectral form factor of a quantum system, which experiences a slope, ramp, and plateau, respectively. JT gravity's spectral form factor doesn't resemble this sketch.

The spectral form factor essentially captures the fluctuations of a system, as defined by

$$Z(\beta+it)Z(\beta-it)=\sum_{n,m}e^{-(\beta+it)E_n}e^{-(\beta-it)E_m}, \tag{2.27}$$

This is our starting motivation to include various topologies within path integral formulation.

2.4.2 JT Wormhole Amplitudes

To start the analysis, look at 2.4 once again. Adding novel topologies amounts to a sum over their genera and boundaries in the path integral relation 2.19 (A typical contributing wormhole is sketched in 4.) By $\mathcal{M}_{g,n}$, we denote a Riemann surface of genus g with n holographic boundaries, each with renormalized length β_i. Therefore, the partition function on this family of Riemann surfaces has the following genus expansion.

$$Z_{\text{grav,conn}}(\beta_1,\ldots,\beta_n)=\sum_{g=0}^{\infty}e^{S_0(2-2g-n)}Z_{g,n}(\beta_1,\ldots,\beta_n). \tag{2.28}$$

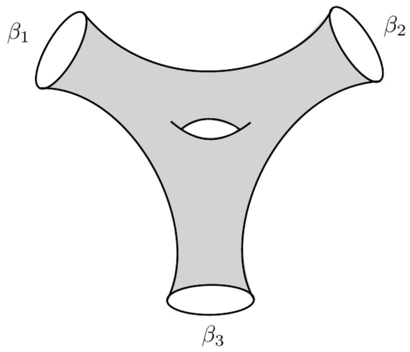


Figure 4. A spacetime wormhole with $g = 1$, and three asymptotic boundaries, $n = 3$, with β_1, β_2, and β_3 renormalized lengths for each asymptotic boundary.

Notice that complicated topologies are subdominant by increasing the handles g or holographic boundaries n.

$Z_{g,n}$ is a path integral that sums over all hyperbolic metrics with the weight of Euclidean JT action. This integration ultimately reduces the volume of the moduli space of Riemann surfaces. A few simple geometries included in this integration are

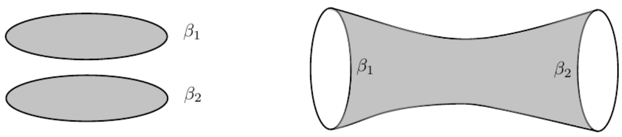


Figure 5. Disk and double trumpet topology as the simplest geometries in JT amplitudes.

- **Disk topology:** $Z_{0,1}(\beta)$, which has been considered in 2.21.
- **Trumpet topology:** $Z_{0,2}(\beta_1, \beta_2)$ that has two holographic boundaries. The final contribution of trumpet topology is

$$Z_{0,2}(\beta_1, \beta_2) = \frac{1}{2\pi} \frac{\sqrt{\beta_1 \beta_2}}{\beta_1 + \beta_2} \tag{2.29}$$

$Z_{g,n}(\beta_1, \ldots, \beta_n)$ is now calculable by breaking a Riemann surface with g handles and n boundaries into n double trumpet and then integrating on b_i (the moduli parameters of the double trumpets) and volume of Riemann surface within with genus g, and n holographic boundaries with lengths b_i. This situation is schematically depicted in 6. The volume of such moduli spaces is found by the Mirzakhani recursion relation [32], which inputs $V_{0,1}$ and $V_{0,2}$ and outputs $V_{g,n}$. This approach is comprehensively elucidated in [11].

2.4.3 Spectral Form Factor Revisited

With adding wormholes, the late-time behavior of the spectral form factor of JT changes significantly. Calculating the spectral form factor within the matrix theory formulation of JT is simpler and gives exactly the schematic figure 3 as the spectral form factor.

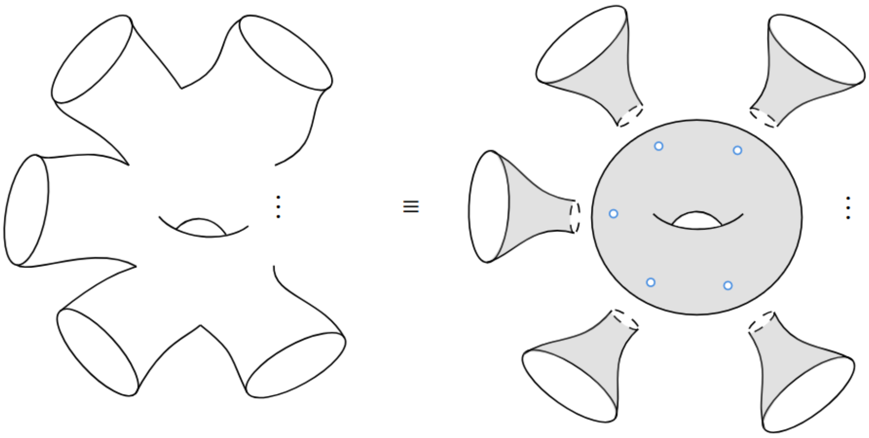


Figure 6. Riemann surfaces $\mathcal{M}_{g,n}$ can be decomposed into trumpets and a punctured Riemann surface with a certain genus.

3 Three-dimensional Gravity

This section of my thesis is devoted to several interconnected aspects of three-dimensional quantum gravity. First, I give a sketch of the 3D gravity/Liouville field theory correspondence at the classical level. Then, a correct classification scheme of symmetries is defined, which is necessary due to different fall-off conditions in gauge theory. Afterward, I will discuss the asymptotic symmetries of asymptotically AdS quantum gravity. Similar results also hold for asymptotically flat quantum gravity.

3.1 Reduction To Liouville Field Theory

Auchucarro and Townsend [33], and independently Witten [12] found that 3D gravity and its equations of motion are equivalent to a Chern-Simons theory on special gauge groups, refer to the table 3.1.

Λ	Gauge group
+ (dS)	$\mathrm{SO}(1,3)$
0(Flat)	$\mathrm{ISO}(1,2)$
-(AdS)	$\mathrm{SO}(2,2)$

Let us briefly see how this reduction takes place for asymptotically AdS spacetimes. $\mathfrak{so}(2,2)$ Lie algebra is given by the following commutators

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \epsilon_{abc} J^c, \tag{3.1}$$

with $a, b, c \in \{0, 1, 2\}$.

We define the gauge field in Chern-Simons theory[8] according to

$$A_\mu \equiv \frac{1}{\ell} e^a_\mu P_a + \omega^a_\mu J_a \tag{3.2}$$

[8]For a review of Chern-Simons and Wess-Zumino-Witten models refer to section C

By a short calculation, the Chern-Simons action will be equivalent to the Einestein-Hilbert action in the first-order formulation[9].

$$S_{\mathrm{CS}}[e,\omega] = \frac{k}{4\pi\ell}\int_{\mathcal{M}}\left(2e^a\wedge R_a[\omega] + \frac{1}{3\ell^2}\epsilon_{abc}\, e^a\wedge e^b\wedge e^c\right) \tag{3.3}$$

where $k = \frac{\ell}{4G_N}$ is the level constant of Chern-Simons theory.

Remark: Due to $\mathfrak{so}(2,2)\cong\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ isomorphism, this action can be recast into two seperate $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons theories

$$S_{\mathrm{CS}}[\Gamma] = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] \equiv S_{\mathrm{CS}}[A,\bar{A}], \tag{3.4}$$

with

$$A = (e^a/\ell + \omega^a)T_a, \quad \bar{A} = (e^a/\ell - \omega^a)T_a \tag{3.5}$$

The next step is to introduce the boundary conditions of the metric. Conventionally, Brown-Henneaux boundary conditions are chosen, in which the metric on the timelike cylinder at conformal infinity is fixed to be flat. In the Fefferman-Graham gauge, the metric has the following form

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \gamma_{ij}(r,x^k)dx^i dx^j, \tag{3.6}$$

where in $r\to\infty$ limit, γ is expanded as $\gamma_{ij} = r^2 g^{(0)}_{ij}(x^k) + \mathcal{O}(1)$.

Subsequently, a careful analysis translates this condition on metric into a condition on gauge fields on Chern-Simons theory.

Then, we solve for gauge theory using traditional methods, which involve separating variables. Through this process, we find that the Chern-Simons action transforms into the right-handed Wess-Zumino-Witten action.

$$\begin{aligned} S_{CS}[A] &= \kappa\int_{\partial\mathcal{M}} d\tau d\varphi Tr[g^{-1}\partial_\varphi g g^{-1}\partial_\tau g] + \kappa\Gamma[G] \\ &\equiv S^R_{WZW}[g] \end{aligned} \tag{3.7}$$

A Hamiltonian reduction is then necessary to transform the chiral Wess-Zumino-Witten action to standard $\mathrm{SL}(2,\mathbb{R})$ Wess-Zumino-Witten action.

Finally, by a Gauss decomposition of fields and applying the second boundary condition, the Wess-Zumino-Witten currents are constrained, and the action will take the form of the Louisville action. Look at the flowchart 7 for an overview of this process.

3.2 Rudiments of 3D Einestion Gravity

In three dimensions, quantum gravity is more controllable. This is due to the lack of propagating degrees of freedom in the bulk of spacetime. Thereby, all the dynamics of three-dimensional quantum gravity happen on its boundary. Technically, this means that three-dimensional quantum gravity is actually a topological quantum field theory. Even though physical phenomena like gravitational waves are present in this model, there are modified models that accommodate bulk degrees of freedom. Also, three-dimensional black hole solutions of gravity pave the way to test quantum gravity ideas, like the information paradox and holographic conjectures.

[9] See appendix D for the first-order formulation of Einestein gravity.

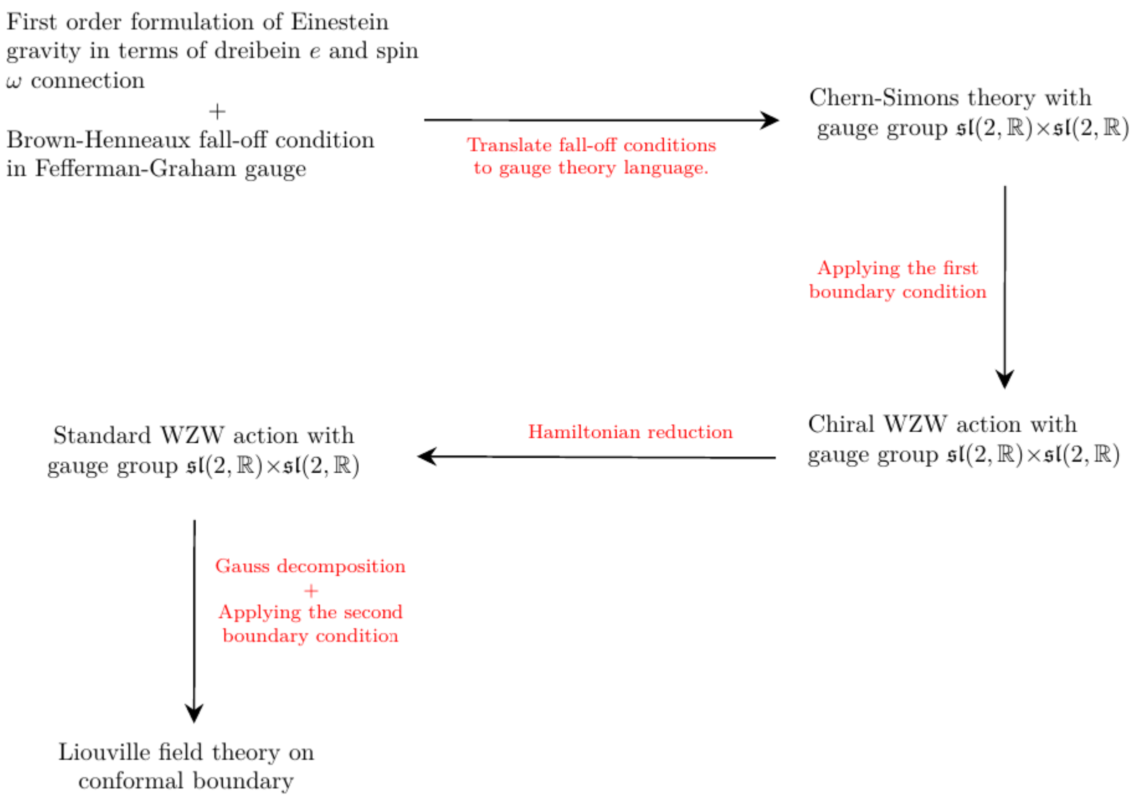


Figure 7.

3.2.1 Asymptotic Symmetries

Let us classify symmetries in a gauge theory. According to fall-off conditions in a gauge theory, one classifies symmetries into three distinct categories. This classification has two criteria: first, whether transformations will change the physical configuration of gauge fields, and second, whether transformed fields obey the fall-off condition. We have these categories (Look at figure **??** for a schematic view of symmetries.)

1. **Trivial gauge transformations:** These transformations don't change the physical configuration of gauge fields and respect fall-off conditions. Hence, they are truly gauge transformations of the gauge theory with its fall-off conditions.

2. **Non-trivial (Allowed) gauge transformations:** They change gauge fields to non-equivalent configurations, still preserving the fall-off condition.

3. **Forbidden transformations:** These transformations are no longer symmetries of the theory since they violate the specified fall-off condition.

Definition: Algebra of asymptotic symmetries of a gauge theory is the quotient algebra of allowed symmetries by the ideal of trivial symmetries.

With these concepts and definitions, let us look at three-dimensional gravity in asymptotically AdS spacetime as a gauge theory, define a suitable fall-off condition for metric, and find the algebra of its asymptotic symmetry group.

3.3 Asymptotic Symmetries of Asymptotically AdS$_3$ Gravity

The first step is to define the fall-off conditions of the metric. Intuitively, the metric must resemble the AdS metric in its asymptotics. Such metrics must include global AdS spacetime and metrics exhibiting conical deficit in their bulk[10]. We augment this set by adding any metric with similar asymptotic behavior on its conformal boundary. Working on the Fefferman-Graham gauge, we have the following notion of asymptotically AdS spacetimes.

Brown-Henneaux Fall-off Conditions: Let $\mathcal{M}$ be a pseudo-Riemannian spacetime manifold, with (r, x^a) coordinate on it $(a = 0, 1)$, and a timelike cylinder as its conformal infinity. If the metric behaves as

$$ds^2 \overset{r\to+\infty}{\sim} \frac{\ell^2}{r^2}dr^2 + \big(r^2\eta_{ab} + \mathcal{O}(1)\big)dx^a dx^b \tag{3.8}$$

in $r \to \infty$ limit, where η_{ab} is the two-dimensional Minkowski metric on the cylindrical shell, we say that $(\mathcal{M}, ds^2)$ is an asymptotically AdS spacetime in the Brown-Henneaux sense in Fefferman-Graham gauge. The cosmological constant of this spacetime is $\Lambda = -1/\ell^2$.

The next step is to find the Asymptotic Killing Vector Fields (AKVFs) and investigate their algebra. The only constraint on these AKVFs is that they have to preserve the asymptotics of metric, that is **??**. This gives the following condition on AKVFs.

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{r\pm} = 0, \qquad \mathcal{L}_\xi g_{ab} = \mathcal{O}(1) \quad (a, b = \pm) \tag{3.9}$$

By solving **??** differential equations, one finds the most general form of AKVFs to be

$$\xi = X(x^+)\partial_+ + \bar{X}(x^-)\partial_- - \frac{1}{2}\big(\partial_+ X(x^+) + \partial_-\bar{X}(x^-)\big)r\partial_r + \text{Subleading} \tag{3.10}$$

Decomposing these generators into Fourier modes

$$\ell_m \equiv \xi_{(e^{imx^+},0)}\,, \qquad \bar{\ell}_m \equiv \xi_{(0,e^{imx^-})}\,, \tag{3.11}$$

their algebra is found easily to be Witt algebra

$$i[\ell_m, \ell_n] = (m-n)\ell_{m+n}\,, \qquad i[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}\,, \qquad i[\ell_m, \bar{\ell}_n] = 0\,. \tag{3.12}$$

Theorem: The asymptotic symmetry group of AdS gravity with Brown-Henneaux boundary conditions is the direct product of two Virasoro groups, which act on the cylindrical conformal boundary as

$$(x^+, x^-) \mapsto \big(f(x^+), \bar{f}(x^-)\big) \tag{3.13}$$

which means that AKVFs act as conformal transformations of asymptotic cylinder.

[10] These metrics indicate a spacetime with a massive particle in its bulk. We expect that the presence of such particles does not change the metric's asymptotics

Then, the process continues to check wether all the above transformations are non-trivial or not. This process requires introducint the surface current algebra and checking their algebra. The answer to the previous question is positive, and asymptotic symmetries of quantum gravity in asymptotically AdS spacetimes is isomorphic to Virasoro algebra.

The same ideas similarly apply to the asymptotically flat spacetimes. First, the BMS fall-off condition is defined. Then, all AKVFs consistent with the BMS condition are classified, and one observes that they extend the Poincare algebra in some way. The asymptotic algebra of symmetries is then found to be the central extension of bms algebra, or $\widetilde{\mathfrak{bms}}_3$.

4 Partition Function of Asymptotically AdS 3D Gravity

This section of my thesis aimed to illuminate three-dimensional quantum gravity from another perspective, namely partition functions. With the holographic principle, AdS_3/CFT_2, at our disposal, it is tempting to expect these partition functions to be equivalent to those of CFT_2. Nevertheless, this is not the case, and there are multiple reasons to account for this discrepancy.

Our tool for computing partition functions is the gravitational path integral formulation in Euclidean signature, which sums all Euclidean geometries with the weight of Euclidean action.

$$\mathcal{Z} = \int [Dg_{\mu\nu}] e^{-I[g_{\mu\nu}]} \tag{4.1}$$

We expand near classical contributions to work with this formulation, equivalent to applying saddle point approximation in the path integral. So, our task is now solidified on finding such classical extremums, summing them, and then reading the partition function. Unfortunately, the final result is flawed; we comment on its shortcomings and potential resolutions.

4.1 Classical Spacetime Solutions

We utilize the uniformization method that classifies all gravitational extremums as a quotient space of a patch of Poincare half-space (Euclidean AdS_3) by a finitely generated subgroup of its automorphism group. Relevant contributions to our purpose are those with toric conformal boundary [11]. Without going into the details, all toric spacetimes are labeled by three parameters in this uniformization: τ, or the complex moduli parameter of the torus on infinity, and two integers c and d, with $c > 0$, and $(c, d) = 1$. This classical extremum has been sketched in figure **??**. Another point is that all spacetimes that share the same toric boundary can be related by a modular transformation of their toric boundary.

[11] Motivated by thermodynamical arguments on the CFT partition function, the imaginary and real parts of the moduli parameter of the torus, τ, are the temperature and angular potential of the three-dimensional spacetimes, respectively.

4.2 Partition Function on $\mathcal{M}_{0,1}$

With those classical extremums, the next step is substituting path integral to read the semi-classical partition function. A naive calculation is illuminating in the first place, which reveals a deep connection with CFT. Only substituting thermal AdS, $\mathcal{M}_{0,1}$ in the path integral, without integrating around quantum fluctuations, gives $\mathcal{Z}(\tau) = |q\bar{q}|^{-\frac{\ell}{16G_N}}$. This result resembles the vacuum contribution of a primary field to the CFT partition function.

The full partition function is more demanding and requires knowledge of all quantum states and their associated energies. Through geometric quantization of 3D gravity, one finds that the spectrum of 3D gravity is equivalent to those on torus; thereby, the partition function of quantum gravity must be that of CFT[12].

Up to this point, we computed the contribution of just $\mathcal{M}_{0,1}$ spacetime to the partition function while considering all quantum effects.

4.3 The Full Partition Function and Its Issues

The full partition function of three-dimensional gravity, which requires a sum over all classical saddles, is found by modular summation.

$$Z(\tau) = \sum_{c,d} Z_{0,1}(\gamma\tau) \tag{4.2}$$

with

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{4.3}$$

Series 4.2 is called the Poincare series and suffers from divergences. Finding correction to three-dimensional gravity's partition function involves many technical methods and mathematical tools. The final perturbative answer on $\tau = x + iy$ is

$$Z = Z_{0,1} + \frac{1}{|\eta|^2}\Big(-6 + \frac{(\pi^3 - 6\pi)(11 + 24k)}{9\zeta(3)} y^{-1} + \frac{5(53\pi^6 - 882\pi^2) + 528(\pi^6 - 90\pi^2)k + 576(\pi^6 - 90\pi^2)k^2}{2430\zeta(5)} y^{-2} + \mathcal{O}(y^{-3})\Big) \tag{4.4}$$

There are several problems with this result:

1. **Spectrum negativity:** The density of states is negative above the BTZ black hole energies, which is physically intolerable.

2. **Holographic contradiction:** We expect that all holographic partition functions, which are of the form $\mathrm{Tr}(\exp(-\beta H))$, to be expanded polynomially in $\beta \sim \mathrm{Im}\,\tau = y$. The inverse-$y$ correction contradicts these holographic expectations.

[12]One can also find the gravitational one-loop effective action, and deduce that its contributions correspond to descendent states of primary fields. Hence, the quantum gravity action terminates in one loop; in other words, 3D quantum gravity is one-loop exact.

3. **Holomorphic factorization:** Obviously, this partition function depends on $y = \text{Im}y$, and doesn't factorize holomorphically.

4. **Non-integer coefficient:** the coefficient of modular parameters were expected to be integers[13], based on partition functions of CFTs.

5. **Spectrum continuity:** This problem is hidden, but by spin decomposition, spectrum continuity is another issue with this result.

One of these issues was fixed in [18]. The idea was that considering conically singular contribution to the partition function calculation will render the spectrum positive. Other issues has not been fully resolved in the literature.

5 Conclusion

In this thesis, gravitational models in lower dimensions (two and three) have been examined to gain a deeper understanding of the quantum features of gravity. Due to their relative simplicity, these models serve as suitable platforms for testing fundamental ideas in quantum gravity. In particular, we studied the model, one of the significant and solvable models in two-dimensional gravity, allowing us to analyze some of the complex concepts of quantum gravity in simpler terms.

Moreover, asymptotic symmetries in three-dimensional gravity were explored. By studying these symmetries, we gained insights into the structure of three-dimensional space, the behavior of quantum gravity within it, and profound connections to gauge theories. These findings help us better understand how gravitational results and features transition from lower to higher dimensions.

In the future, a more detailed investigation into the connection between low-dimensional gravitational models and the features of quantum gravity in higher dimensions is particularly important. Specifically, understanding how four-dimensional phenomena such as black holes and quantum information emerge in simpler models can pave the way for a more comprehensive theory. Additionally, further exploration of entropy characteristics and the role of asymptotic symmetries in low-dimensional gravity will be key to understanding information-related issues in quantum gravity and holography.

Specifically, the following questions can be pursued:

1. Is pure three-dimensional gravity meaningful? What is the conformal dual field theory to this theory?

2. What universal features of lower-dimensional gravity can be expected to manifest in higher dimensions?

3. In lower-dimensional models, where quantum gravity theories are well-behaved and controllable, can we define a proper holographic picture for asymptotically flat and de Sitter spacetimes and use them for higher-dimensional spacetimes?

[13] Uppder bounded by the partition function of natural numbers.

A Omitted Sections and Additional Notes

To have a concise and readable summary, I have omitted most of the discussion in this section, specifically I omitted

- ✘ Motivations to study JT gravity. A primary motivation is that JT gravity describes the near-horizon physics of near-extremal black holes.
- ✘ Its duality with the topological BF model.
- ✘ The relation of dilaton field with entropy.
- ✘ Geometric derivation of Schwarzian action.
- ✘ Classical JT in the presence of quantum matter.
- ✘ The exact form of two-point and four-point functions of JT gravity.
- ✘ Insertion of defects and their physical interpretation.
- ✘ Non-perturbative effects of JT.
- ✘ How do wormhole contributions produce the correct spectral form factor?

B Review of SYK model

In this section, we explore the SYK model. This model was proposed and solved by Kitaev [34] to simplify the initial model by Sachdev - Ye [35]. It is a quantum mechanical model consisting of N Majorana fermions, all coupled via random couplings (selected from a specific distribution). We will also see how, at low energy, this model can lead to the Schwarzian action, providing a holographic description of JT gravity.

B.1 SYK Hamiltonian

This model is essentially an ensemble of simple quantum models with finite dimensions. A Hermitian Hamiltonian specifies each quantum model in this ensemble. First, using the Majorana representation (the spin-$\frac{1}{2}$ representation of the Lorentz group), we define SYK Hamiltonian. We only need Majorana representations in even dimensions; Let us briefly review these representations.

We want to find representations of the Clifford algebra

$$\{\psi_i, \psi_j\} = \delta_{ij}, \quad i, j = 1, \dots, N$$

Focusing only on even dimensions, $N = 2K$, and for the Hamiltonian to be Hermitian, we consider Hermitian representations $\psi_i^\dagger = \psi_i$. This algebra can be reduced to the fermionic algebra by defining:

$$c_i = \frac{1}{\sqrt{2}}(\psi_{2i} - i\psi_{2i+1}), \quad c_i^\dagger = \frac{1}{\sqrt{2}}(\psi_{2i} + i\psi_{2i+1}), \ \ i = 1, \dots, K$$

Using the anticommutation properties, we derive

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0, \quad \{c_i, c_j^\dagger\} = \delta_{ij}$$

These anti-commutators are the algebra of fermionic operators represented simply in terms of creation operators. Using these representations, we can construct Majorana representations.

Reminder on Fermionic Algebra Representations

We start with a vacuum annihilated by all annihilation operators:

$$c_i |0\rangle = 0, \forall i \in \{1, 2, \ldots, K\}.$$

The basis states are constructed as:

$$(c_1^\dagger)^{n_1} \ldots (c_K^\dagger)^{n_K} |0\rangle, \quad n_k = 0, 1$$

yielding $2^K = 2^{\frac{N}{2}}$ basis vectors, depending on whether a fermion occupies a specific mode. This is the only irreducible Hermitian representation of this algebra; all other representations are unitarily equivalent.
Note: For odd N, representations can be constructed by adding a γ_5 matrix to the representations of $N-1$ (even), so all representations are effectively categorized.

For Majorana fermion representations, a recursive relation exists

$$\begin{aligned}
\psi_i^{(K)} &= \psi_i^{(K-1)} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, \ldots, N-2 \\
\psi_{N-1}^{(K)} &= \frac{1}{\sqrt{2}} I_{2^{K-1}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\psi_N^{(K)} &= \frac{1}{\sqrt{2}} I_{2^{K-1}} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\end{aligned}$$

The matrices $\psi_i^{(K)}$ are $2^K \times 2^K$, requiring two initial matrices to start the recursion

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The SYK ensemble has Hamiltonians of the form

$$H = \sum_{i,j,k,l=1}^{N} J_{ijkl} \psi_i \psi_j \psi_k \psi_l,$$

where J_{ijkl} represents couplings between fermions, chosen randomly and independently from a Gaussian distribution with mean $\mu = 0$ and variance $\sigma = \frac{\sqrt{3!}J}{N^{3/2}}$.

A simple generalization of this model includes q-body interactions:

$$H = i^{q/2} \sum_{1 \le i_1 < \cdots < i_q \le N} J_{i_1 \ldots i_q} \psi_{i_1} \ldots \psi_{i_q}$$

where $J_{i_1 \ldots i_q}$ is drawn from a Gaussian distribution with mean 0 and variance $\sigma = \frac{\sqrt{(q-1)!}J}{N^{(q-1)/2}}$.

Here are some properties of this model.

1. In the large N limit, the model becomes classical. For large N, the quantum model can be understood fully by solving the classical equations of motion for G and Σ.
2. At low energy (infrared limit), a time-reparameterization symmetry $t \mapsto f(t)$ emerges, spontaneously broken in the vacuum, analogous to the Goldstone modes, with an effective action defined for them.
3. The spectrum of the model is interesting. The density of states for $q = 2$ is equivalent to integrable theories with extended low-energy tails, while for $q = 4$, the spectrum abruptly terminates.

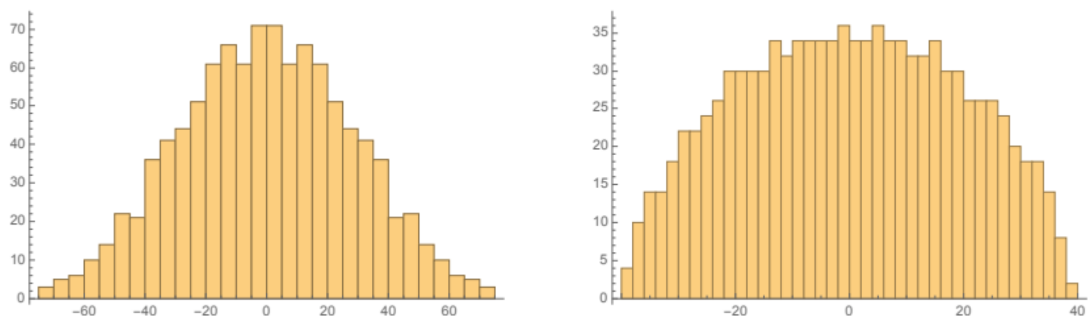


Figure 8. The left plot shows the density of states for $q = 2$, and the right plot for $q = 4$, both for $N = 20$.

B.2 Quantum-Level Analysis of SYK

Now, we begin analyzing the SYK model. We use the conventional perturbative method to study the SYK model where J is our perturbation parameter. In the limit $N \to \infty$, the summation over various Feynman diagrams can be represented as a Schwinger-Dyson series.

In Euclidean notation, the time-ordered two-point function of fermions is given by

$$G_{ij}(\tau) = \langle T\psi_i(\tau)\psi_j(0)\rangle \equiv \Theta(\tau)\langle\psi_i(\tau)\psi_j(0)\rangle - \Theta(-\tau)\langle\psi_j(0)\psi_i(\tau)\rangle,$$

where Θ is the Heaviside step function, and

$$\psi_i(\tau) = e^{\tau H}\psi_i e^{-\tau H}.$$

An important quantity derived from the two-point functions is the normalized trace of the two-point function. To calculate this, we do not need the four-fermion vertices, so by setting $J = 0$, the Hamiltonian also becomes zero, and we have $\psi_i(\tau) \equiv \psi_i$.

Using Clifford algebra, we compute the two-point functions and their normalized trace $G(\tau) = \frac{1}{N}\sum_{i=1}^{N} G_{ii}(\tau)$

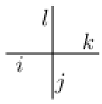
$$G_{ij}^{\text{free}}(\tau) = \frac{1}{2}\delta_{ij}\text{sgn}\tau, \quad G^{\text{free}}(\tau) = \frac{1}{N}\sum_i G_{ii}^{\text{free}} = \frac{1}{2}\text{sgn}\tau,$$

where sgn is the sign function.

In Fourier space, the two-point function is expressed as:

$$G_{ij}^{\text{free}}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} G_{ij}^{\text{free}}(\tau) = -\frac{\delta_{ij}}{i\omega}.$$

The perturbation of this theory is done by examining the four-vertex interaction and its Feynman rule.

Diagram: 

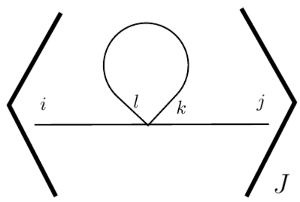
Contribution : J_{ijkl}.

First, using the vertex rules and propagators, we determine the amplitude of a diagram and then average over the couplings J. For instance, since the random couplings are selected from a Gaussian ensemble, the two-point averages J are

$$\langle J_{i_1j_1k_1l_1} J_{i_2j_2k_2l_2}\rangle_J = 3!\frac{J^2}{N^3}\delta_{i_1i_2}\delta_{j_1j_2}\delta_{k_1k_2}\delta_{l_1l_2}.$$

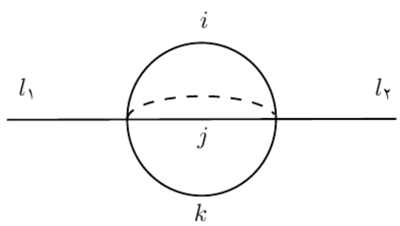
Next, we analyze the non-trivial contributions to the propagator.

1. **Tadpole contribution**



This contribution is zero because it equals $\langle J_{ijkl}\rangle_J$, which is zero since the Gaussian ensemble of couplings is chosen with zero mean.

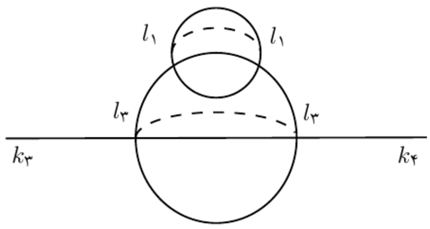
2. **Melon contribution**



Here, the dashed lines represent averages like those in the above equation, summed over repeated indices. The contribution of this diagram is

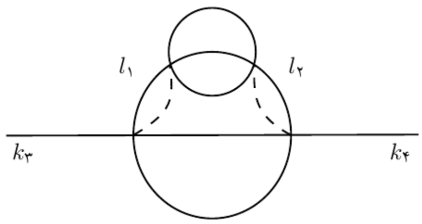
$$\mathcal{A}_{\text{melon}} = 3!\frac{J^2}{N^3} G_{ii}^{\text{free}} G_{jj}^{\text{free}} G_{kk}^{\text{free}} \delta_{l_1 l_2} = 3! J^2 (G^{\text{free}})^3 \delta_{l_1 l_2}.$$

3. **Contributions of order** J^4 Several diagrams contribute here. Two examples are shown below:



Amplitude:

$$\mathcal{A}_{(4)}^{(1)} = \frac{J^4}{N^6} \left(G_{ll}^{\text{free}} \right)^5 G_{l_1 l_3}^{\text{free}} G_{l_1 l_3}^{\text{free}} \delta_{k_3 k_4} = \frac{1}{2} J^4 \left(G^{\text{free}} \right)^6 \delta_{k_3 k_4}.$$



Amplitude:

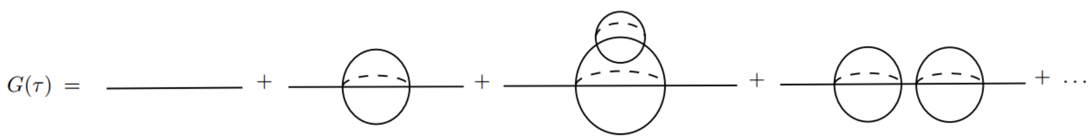
$$\mathcal{A}_{(4)}^{(2)} = \frac{J^4}{N^6} \left(G_{i_1 i_2}^{\text{free}} G_{j_1 j_2}^{\text{free}} G_{k_1 k_2}^{\text{free}} \right) \left(G_{l_1 l_1}^{\text{free}} G_{l_2 l_2}^{\text{free}} \right) \left(G_{i_1 i_2}^{\text{free}} G_{j_1 j_2}^{\text{free}} \right).$$

Simplified as

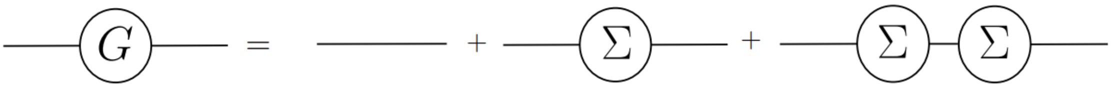
$$= \frac{1}{4N^2} J^4 \left(G^{\text{free}} \right)^4 G_{k_3 k_4}^{\text{free}}.$$

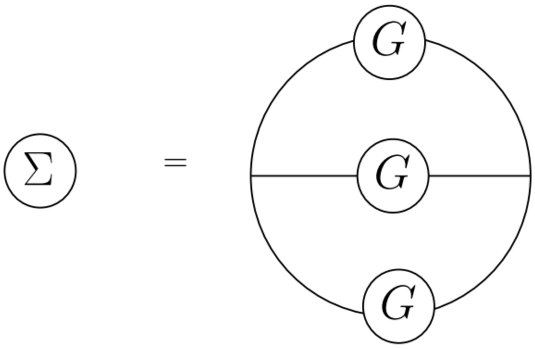
The first amplitude scales with N, while the second scales with N^{-2}. Therefore, the second diagram is negligible in the limit $N \to \infty$.

From the above calculations, we conclude that the diagrams not suppressed by powers of $1/N$ are those with only one dashed-line connection within each melon-like shape. Thus, the dominant two-point function in the $N \to \infty$ limit is:



This can be summarized as:





We can simplify these pictorial equations by introducing a matrix product representation:

$$(AB)(\tau,\tau') = \int d\tau'' A(\tau,\tau'')B(\tau'',\tau').$$

The first equation from the diagram above can be written as follows (this type of summation is referred to as the Schwinger-Dyson sum):

$$\begin{aligned} G &= G^{\text{free}} + G^{\text{free}}\Sigma G^{\text{free}} + G^{\text{free}}\Sigma G^{\text{free}}\Sigma G^{\text{free}} + \cdots \\ &= G^{\text{free}}\big[1 + \Sigma G^{\text{free}} + \Sigma G^{\text{free}}\Sigma G^{\text{free}} + \cdots\big] \\ &= G^{\text{free}}\big[1 - \Sigma G^{\text{free}}\big]^{-1} \\ &= \big[(G^{\text{free}})^{-1} - \Sigma\big]^{-1} \end{aligned} \tag{B.1}$$

Since the propagator is the inverse of the kinetic term in the Lagrangian

$$(G^{\text{free}})^{-1}(\tau,\tau') = \delta(\tau-\tau')\partial_{\tau'} \tag{B.2}$$

The following shorthand is also valid

$$G = \big[\partial_\tau - \Sigma\big]^{-1} \tag{B.3}$$

The second diagrammatic equation reduces to the following expression:

$$\Sigma(\tau,\tau') = J^2\big[G(\tau,\tau')\big]^3 \tag{B.4}$$

B.3 Low-energy Limit and Emergent Conformal Symmetry

In the low-energy limit, we examine the action of the model. Compared to the coupling of the model, which has an energy dimension, the low-energy limit implies frequencies smaller than the coupling. By rewriting (B.2) in Fourier space:

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega) \tag{B.5}$$

Neglecting $i\omega$ in the above limit, the equations (in the infrared limit) simplify to:

$$\begin{aligned}\int d\tau'' G(\tau,\tau'')\Sigma(\tau'',\tau') &= -\delta(\tau-\tau') \\ \Sigma(\tau,\tau') &= J^2 G(\tau,\tau')^{q-1}\end{aligned} \tag{B.6}$$

A key observation is that these equations exhibit additional symmetries. Under the reparameterization $\tau \to \phi(\tau)$, G and Σ transform as

$$\begin{aligned}G(\tau,\tau') &\mapsto \left[\phi'(\tau)\phi'(\tau')\right]^{\Delta} G(\phi(\tau),\phi(\tau')), \\ \Sigma(\tau,\tau') &\mapsto \left[\phi'(\tau)\phi'(\tau')\right]^{\Delta(q-1)} \Sigma(\phi(\tau),\phi(\tau'))\end{aligned} \tag{B.7}$$

This means that G and Σ transform like two-point functions in conformal field theory. For example, the first equation remains invariant under conformal transformations

$$\begin{aligned}\int d\tau'' \left[\phi'(\tau)\phi'(\tau'')\right]^{\frac{1}{q}} G(\phi(\tau),\phi(\tau'')) \left[\phi'(\tau'')\phi'(\tau')\right]^{1-\frac{1}{q}} \Sigma(\phi(\tau''),\phi(\tau')) \\ = \int d\tilde{\phi} G(\phi(\tau),\tilde{\phi})\Sigma(\tilde{\phi},\phi(\tau'))\phi'(\tau') \left[\frac{\phi'(\tau)}{\phi'(\tau')}\right]^{\frac{1}{q}} \\ = -\phi'(\tau')\delta(\phi(\tau)-\phi(\tau')) \\ = -\delta(\tau-\tau')\end{aligned} \tag{B.8}$$

B.4 Transition to the Schwarzian Action in the Limit $N \to \infty$

The study of the Schwarzian limit as $N \to \infty$ is somewhat technical and requires background field methods and path integrals. We refer to the relevant literature [36, 37].

C Introduction to Wess-Zumino-Witten and Chern-Simons theoreis

In this brief appendix, we describe the Chern-Simons and Wess-Zumino-Witten theories and highlight their properties necessary for our purposes in Section 3.1.

C.1 Chern-Simons Theory

Chern-Simons action for a compact gauge group G is given by

$$S_{\text{CS}}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \text{Tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big), \tag{C.1}$$

where k is called the level constant. A represents a Lie algebra valued 1-form with values in $\mathfrak{g}$, i.e., $A = A_\mu dx^\mu$. The term Tr denotes a non-degenerate bilinear form on the Lie algebra $\mathfrak{g}$[14].

[14] Non-degeneracy ensures that all gauge fields appearing in the action have a kinetic term. All semisimple Lie algebras possess such non-degenerate bilinear forms.

Writing $A = A^a T_a$, where T_a are the generators of the Lie algebra $\mathfrak{g}$, the first term of the action (C.1) can be expressed as:

$$\mathrm{Tr}[A \wedge dA] = \mathrm{Tr}(T_a T_b)[A^a \wedge dA^b]$$

This implies that $d_{ab} \equiv \mathrm{Tr}(T_a T_b)$ acts as the metric on the Lie algebra $\mathfrak{g}$, which must be non-degenerate.

By varying the above action and performing integration by parts, we obtain

$$\delta S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}\Big(2\delta A \wedge (dA + A \wedge A)\Big) - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \mathrm{Tr}\Big(A \wedge \delta A\Big) \tag{C.2}$$

If δA is chosen such that it vanishes on the boundary $\partial\mathcal{M}$,[15] the equation of motion is derived as

$$F \equiv dA + A \wedge A = 0 \tag{C.3}$$

where F is a Lie algebra valued 2-form valued in $\mathfrak{g}$, representing the curvature of the gauge fields A.

Locally, this equation implies

$$A = G^{-1} dG \tag{C.4}$$

indicating that the field A is a gauge transformation of the trivial configuration $A = 0$; in other words, A is a pure gauge field.[16]

This observation reveals that the Chern-Simons theory has no propagating degrees of freedom and is, in fact, a topological field theory. This is why it can be equivalent to a gravitational theory with no propagating degrees of freedom.

C.2 Wess-Zumino-Witten Model

We begin with the nonlinear sigma model and, by adding the Wess-Zumino term, resolve its conformal anomaly to arrive at the Wess-Zumino-Witten model.

C.2.1 Nonlinear Sigma Model

The Wess-Zumino-Witten model is a specific case of the nonlinear sigma model, which we will briefly introduce sigma models.

The nonlinear sigma model consists of scalar fields ϕ^i $(i = 1, \ldots, n)$, which define a map from a flat spacetime (Minkowski) to a target manifold. The target manifold is a real n-dimensional Riemannian manifold $\mathcal{M}_n$ equipped with a metric $g_{ij}(\phi)$. The scalar fields are coordinates on the Riemannian manifold, and since the metric depends on the fields ϕ, the model is nonlinear.

[15] Alternatively, we can use an improved action to ensure a well-defined variational principle even if the boundary term does not vanish.

[16] Globally, however, A may not be purely gauge due to holonomies.

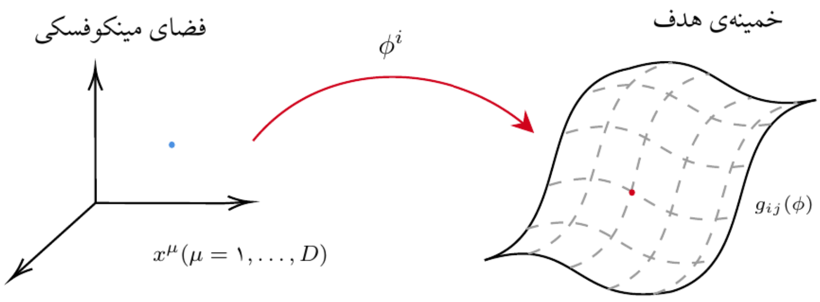


Figure 9. Fields in a nonlinear sigma model are maps from flat space to a target manifold.

The action of this model is defined as

$$S_\sigma[\phi] = \frac{1}{4a^2} \int d^D x \;\; g_{ij}(\phi) \eta^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \tag{C.5}$$

where $a^2 > 0$ is a dimensionless coupling constant.

The Wess-Zumino-Witten model (abbreviated as WZW) is a specific nonlinear sigma model. In this model, the target manifold is a semisimple Lie group G, and the fields, which are maps from flat space to the target manifold, are matrices denoted by $g(x)$.

For a two-dimensional flat space Σ with coordinates (τ, φ), the action of the nonlinear sigma model is written as:

$$S_\sigma[g] = \frac{1}{4a^2} \int_\Sigma d^2 x \;\; \mathrm{Tr}\big[\eta^{\mu\nu} \partial_\mu g \partial_\nu (g^{-1})\big] \tag{C.6}$$

Note that we must construct a scalar from the matrices to ensure the action is a scalar quantity. Therefore, we use the Killing form on the semisimple Lie group, which requires the group to be semisimple (even if not compact).

Now, we comprehensively examine this action, its equation of motion, and symmetries.

First, we derive its equation of motion. To simplify the action, we note that:

$$\partial_\nu (g^{-1}) = -g^{-1} \partial_\nu g g^{-1} \tag{C.7}$$

which directly follows from the relation $\partial_\nu (g g^{-1}) = 0$. By substituting this term into the action, we obtain:

$$S_\sigma[g] = \frac{1}{4a^2} \int_\Sigma d^2 x \;\; Tr\big[g^{-1} \partial_\mu g \; g^{-1} \partial^\mu g\big] \tag{C.8}$$

This form of the action exhibits symmetry $g \to g_L \; g \; g_R^{-1}$.

By varying the action, the equations of motion are obtained:

$$\begin{aligned} \delta S_\sigma &= \frac{1}{2a^2} \int_\Sigma d^2 x \;\; Tr\big[\big(- g^{-1} \delta g g^{-1} \partial_\mu g + g^{-1} \delta(\partial_\mu g)\big) g^{-1} \partial^\mu g\big] \\ &= \frac{1}{2a^2} \int_\Sigma d^2 x \;\; Tr\big[\partial_\mu (g^{-1} \delta g) g^{-1} \partial^\mu g\big] \\ &= -\frac{1}{2a^2} \int_\Sigma d^2 x \;\; Tr\big[g^{-1} \delta g \partial_\mu (g^{-1} \partial^\mu g)\big] \end{aligned} \tag{C.9}$$

Thus, the equation of motion is $\partial^\nu(g^{-1}\partial_\nu g) = 0$.

Using the coordinate transformation $x^\pm \equiv \tau \pm \varphi$, the equations of motion can be written as

$$\partial_+ J_+ + \partial_- J_- = 0 \tag{C.10}$$

where $J_\pm = g^{-1}\partial_\pm g$.

We observe that the left- and right-moving currents are not conserved individually. However, the symmetry of left and right multiplication should yield two conserved currents. Therefore, the action needs to be modified [17].

C.2.2 Adding the Wess-Zumino Term

To address the discussed issues, we add the following term [38],[39].

$$S = S_\sigma[g] + k\Gamma[G] \tag{C.11}$$

where k is an integer, and the Wess-Zumino action $\Gamma[G]$ is

$$\begin{aligned}\Gamma[G] &= \frac{1}{3}\int_V Tr\left[\left(G^{-1}dG\right)^3\right] \\ &= \frac{1}{3}\int_V d^3x\, \epsilon^{\mu\nu\rho} Tr\left[G^{-1}\partial_\mu G G^{-1}\partial_\nu G G^{-1}\partial_\rho G\right]\end{aligned} \tag{C.12}$$

where V is a three-dimensional manifold with a boundary Σ, and G is an extension of the fields to V satisfying $G\Big|_{\partial V} = g$ [18].

To derive the equations of motion, we compute the variation of the action under $G \to G + \delta G$, and using Stokes' theorem, we find

$$\delta\Gamma[G] = \int_\Sigma d^2x Tr\left[\epsilon^{\mu\nu}\delta g g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\right] \tag{C.13}$$

Combining this with the equation of motion derived for $S_\sigma[g]$, we obtain

$$\frac{1}{2a^2}\eta^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) - k\epsilon^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) = 0 \tag{C.14}$$

Rewriting this in light-cone coordinates ($\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_\phi)$), we get

$$(1 - 2a^2k)\partial_+(g^{-1}\partial_+ g) + (1 + 2a^2k)\partial_-(g^{-1}\partial_- g) = 0 \tag{C.15}$$

Now, by choosing the parameter k, we can ensure the conservation of currents:

- If $a^2 = -\frac{a}{2k^2}$, then $k < 0$, and the conservation of the right-moving current follows, $\partial_+ J_+ = 0$.

[17] Another reason for modifying the theory is that this model exhibits conformal symmetry classically, but upon quantization, this symmetry is broken. To restore conformal symmetry at the quantum level, new terms must be added to the action, with the Wess-Zumino term serving this purpose.

[18] Note that the extension of a manifold with its boundary is not unique, and neither is the extension of the fields g such that $G\Big|_{\partial V} = g$. Thus, the definition of Γ as given above is ambiguous. However, for the compact group $SL(2,\mathbb{R})$ and on a flat two-dimensional space, there is no issue of ambiguity. For more details, refer to [40].

- For $a^2 = \frac{a}{2k^2}$, k is necessarily positive, and we have the conservation of the left-moving current, $\partial_- J_- = 0$.

By choosing $a^2 = -\frac{a}{2k^2}$, we arrive at the WZW action, which is sometimes referred to as WZNW action

$$S_{WZW}[g] = \frac{k}{2} \int d^2x Tr\left[\eta^{\mu\nu} g^{-1}\partial_\mu g g^{-1}\partial_\nu g\right] + k\Gamma[G]. \tag{C.16}$$

The solutions to the equations of motion, $\partial_+(g^{-1}\partial_- g) = 0$, are of the form $g = \theta_+(x^+)\theta_-(x^-)$, where $\theta_+(x^+)$ and $\theta_-(x^-)$ are arbitrary functions.

Moreover, this model has two conserved currents: $J_- \equiv g^{-1}\partial_- g$ and $\bar{J}_+ \equiv -\partial_+ g g^{-1}$. It is straightforward to see that this action is invariant under the transformation $g \to \Theta_+(x^+) g \Theta_-^{-1}(x^-)$, and the conserved currents of this transformation are the two currents mentioned above. Therefore, adding the Wess-Zumino the term essentially gauges the global symmetry.

D First Order Formulation of Einstein Gravity

Instead of directly using the metric $g_{\mu\nu}$, this formulation introduces an auxiliary field [19] e^a_μ, which serves as the "square root" of the metric

$$g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x). \tag{D.1}$$

This auxiliary field is also called the *dreibein* in three dimensions[20].

The relation (D.1) can be interpreted as transforming a tensor under general coordinate transformations. The metric tensor can be brought to the Minkowski metric through transformations constructed from the vielbein field. This relation also provides valuable information about the invertibility of the frame fields.

Taking the determinant of both sides of (D.1), we find

$$e \equiv \det e^a_\mu = \sqrt{-\det g_{\mu\nu}} \neq 0. \tag{D.2}$$

Thus, the frame fields are invertible, and their inverse, e^μ_a, can be defined as

$$\begin{aligned} e^\mu_a e^a_\nu &= \delta^\mu_\nu \\ e^a_\mu e^\mu_b &= \delta^a_b \end{aligned} \tag{D.3}$$

Moreover, the frame fields are not unique; any Lorentz transformation $\Lambda \in \mathrm{SO}(2,1)$ (local or global) applied to the frame fields generates a new vielbein that still satisfies (D.1):

$$(e')^a_\mu = \Lambda^a_b(x)e^b_\mu(x) \tag{D.4}$$

[19] Sometimes called the frame fields, or vielbein in four-dimensional spacetime.

[20] Note that e^a_μ has two types of indices: Latin indices are the internal or vielbein indices, which transform under local Lorentz transformations, while Greek indices are the spacetime indices, which transform via general coordinate transformations.

Note that in the above transformations, the Lorentz transformation acts only on the vielbein indices, while general coordinate transformations act on the spacetime indices. Thus, the frame fields can be transformed in two distinct ways.

We can also use the frame fields to define a basis for the space of differential forms. The differential form e^a is defined as $e^a \equiv e^a_\mu dx^\mu$, and the Levi-Civita tensor with frame indices is defined

$$\begin{aligned} \epsilon_{\mu\nu\rho} &= e^{-1}\epsilon_{abc}e^a_\mu e^b_\nu e^c_\rho \\ \epsilon^{\mu\nu\rho} &= e\epsilon^{abc}e^\mu_a e^\nu_b e^\rho_c. \end{aligned} \tag{D.5}$$

The covariant derivative is similarly constructed from an ordinary derivative and a connection, symbolically written as $D = \partial + \Gamma$.

The connection consists of a set of 1-forms called the spin connection, denoted by $\omega^{ab} = \omega^{ab}_\mu dx^\mu$, which is antisymmetric in its internal indices, $\omega^{ab} = -\omega^{ba}$.

The relationship between the spin connection and the Levi-Civita connection is established by considering the covariant derivative of a vector field $X = X^\mu \partial_\mu$, first in the coordinate basis and then in the frame basis. The resulting relationship is

$$\omega_{\mu\ b}^{\ a} = e_\nu^{\ a} e^\lambda_{\ b} \Gamma^\nu_{\mu\lambda} - e^\lambda_{\ b}\partial_\mu e_\lambda^{\ a} \tag{D.6}$$

Analogous to the Levi-Civita connection in general relativity, which is not a tensor and has nontrivial transformation properties, the spin connection also transforms nontrivially under local Lorentz transformations. However, it allows us to construct quantities that behave well under these transformations. Under a local Lorentz transformation, the spin connection transforms as

$$\omega^a_b \to (\Lambda^{-1})^a_{\ c}\, d\Lambda^c_{\ b} + (\Lambda^{-1})^a_{\ c}\, \omega^c_{\ d}\, \Lambda^d_{\ b}, \tag{D.7}$$

and the torsion 2-form is

$$T^a \equiv de^a + \omega^a_{\ b} \wedge e^b, \tag{D.8}$$

which transforms as a vector under Lorentz transformations: $T^a \to \Lambda^a_b T^b$.

We can verify that the defined quantity corresponds to the notion of torsion in general relativity, i.e., the antisymmetric part of the Levi-Civita connection.

$$\begin{aligned} T_{\mu\nu}^{\ \lambda} &= e^\lambda_{\ a} T_{\mu\nu}^{\ a} \\ &= e^\lambda_{\ a}\big(\partial_\mu e_\nu^{\ a} - \partial_\nu e_\mu^{\ a} + \omega_{\mu\ b}^{\ a} e_\nu^{\ b} - \omega_{\nu\ b}^{\ a} e_\mu^{\ b}\big) \\ &= \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}, \end{aligned} \tag{D.9}$$

where in the second line, we used (D.8), and in the last line, we substituted (D.6) to express the spin connection in terms of the Levi-Civita connection.

After establishing these geometric preliminaries, we aim to examine general relativity in the first-order formulation. Specifically, we aim to rewrite the Einstein-Hilbert action in terms of e and ω. To this end, note that

$$d^3\sqrt{-g} = edx^0dx^1dx^2 = \frac{1}{3!}e\epsilon_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho \tag{D.10}$$

By substituting (D.5) in terms of vielbein fields and using the definition $e^a \equiv e^a_\mu dx^\mu$, we find

$$d^3\sqrt{-g} = \epsilon_{abc} e^a \wedge e^b \wedge e^c \tag{D.11}$$

The term proportional to curvature in the Einstein-Hilbert action is converted using the dual representation (possible only in three dimensions). The curvature tensor in vielbein indices is given by:

$$\begin{aligned} R^{ab} &= \frac{1}{2} R^{ab}_{\mu\nu}(x) dx^\mu \wedge dx^\nu \\ R^{\lambda\sigma}_{\mu\nu} &= e^\lambda_a e^\sigma_b R^{ab}_{\mu\nu} \end{aligned} \tag{D.12}$$

The dual representation is defined as

$$\begin{aligned} R_a &\equiv \frac{1}{2}\epsilon_{abc} R^{bc} \leftrightarrow R^{ab} \equiv -\epsilon^{abc} R_c \\ \omega_a &\equiv \frac{1}{2}\epsilon_{abc}\omega^{bc} \leftrightarrow \omega^{ab} \equiv -\epsilon^{abc}\omega_c. \end{aligned} \tag{D.13}$$

We can observe that

$$\begin{aligned} \epsilon_{abc} e^a \wedge R^{bc} &= \frac{1}{2} e \epsilon_{\mu\alpha\beta} R^{\alpha\beta}_{\nu\rho} \epsilon^{\mu\nu\rho} d^3x \\ &= d^3x\sqrt{-g}R \end{aligned} \tag{D.14}$$

Adding the two terms above, the Einstein-Hilbert action in the first-order formalism takes the following form

$$S_{\text{EH}}[e,\omega] = \frac{1}{16\pi G}\int_{\mathcal{M}} \left(2e^a \wedge R_a[\omega] - \frac{\Lambda}{3}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right). \tag{D.15}$$

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