Aspects of Low-dimensional Quantum Gravity

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ABSTRACT: This paper is an insightful summary of my master's thesis on quantum gravity in two and three dimensions, in which I tour from essential ingredients to recent discoveries. I start with JT gravity at the classical level, then continue with quantum JT gravity, emphasizing the correlations functions. The significance of wormhole contributions is briefly highlighted, emphasizing their role in recent advancements. The second half of this thesis is devoted to three-dimensional gravity, mainly focusing on the asymptotic symmetries in asymptotically AdS and flat spacetimes. Finally, I give a detailed account of the partition function of pure 3D gravity in asymptotically AdS spacetime and address its physical and holographic shortcomings.

Keywords: Holography, AdS/CFT, JT gravity, Quantum gravity, black hole.

This summary is in progress and updates continuously.

Contents

1 Introduction

Low-dimensional quantum gravity, in two or three dimensions, holds special significance in theoretical physics mainly because it serves as a simple laboratory for understanding the universal features of quantum gravity. Even though there is little hope that these lower-dimensional quantum gravity theories will lead to a complete understanding of four-dimensional quantum gravity, they provide a suitable playground for exploring ideas and learning from these theories. These models allow us to examine the complex dynamics of quantum gravity theory and study concepts like black holes, entropy, and symmetries in a more controllable and simplified framework [Applications are in information paradox and black hole evaporation [1, 2], 2D cosmology [3], and quantum chaos [4]].

This summary of my masters thesis is organized as follows.

It is well-known that two-dimensional gravity is structurally and conceptually trivial without the dilaton field. So, we start with dilaton models in two dimensions, specifically JT gravity [5, 6]. First, we explore this model at the classical level, reviewing the motivations for its study, classical solutions, and its connection with the Schwarzian quantum mechanics [1]. We then proceed to the quantum level, beginning with a perturbative, one-loop analysis of this theory, followed by a summary of the two- and four-point correlation functions and a brief discussion of their derivations [7]. Subsequently, we examine JT gravity in the presence of defects and conclude with a discussion of wormhole contributions, understanding their significance [8]. Here, we end the discussion of two-dimensional gravity and proceed to three-dimensional gravity.

Our discussion of three-dimensional gravity begins by examining its duality with the Liouville field theory [9]. Through a series of Hamiltonian reductions and redefinitions, we demonstrate that, at the classical level, three-dimensional anti-de Sitter (AdS) gravity is equivalent to a Liouville field theory on its conformal boundary. Then, we approach three-dimensional gravity from a different perspective, a gauge theoretical description. We pay close attention to the symmetries within gauge theories. In gauge theories, the asymptotic behavior of gauge fields is crucial, as it determines the model extensively. By choosing the Brown-Henneaux [10] fall-off conditions for asymptotically AdS spaces and the BMS fall-off conditions for asymptotically flat spaces, we analyze their respective asymptotic symmetries. For asymptotically AdS spacetimes, Brown-Henneaux fall-off conditions point toward the gauge/gravity duality, a result derived roughly a decade before this duality was framed in modern language [11]. Similarly, the gauge symmetry group for flat spaces is the centrally extended Poincaré group, known as the BMS_3 group [12, 13]. Throughout this discussion, we uncover extensive connections between these models and the geometric actions on coadjoint orbits of the Virasoro group and the $\widehat{\mathfrak{bms}}_3$ algebra, highlighting the geometric nature of these quantum theories.

Finally, we study the partition functions of pure AdS gravity [14], employing the gravitational path integral formulation in the semiclassical approximation. Additionally, we enumerate potential deficiencies in the spectrum of this quantum gravity theory, presenting possible proposals to define a healthy quantum gravity in asymptotically AdS_3 spacetimes.

[1] This theory is invariant under the $\mathrm{SL}(2, \mathbb{R})$ symmetry.

In recent years, one such proposal has resolved the issue of the negative density of states in this spectrum [15] . However, research continues to address the remaining challenges in this theory.

In the appendices, we cover the mathematical preliminaries for the discussed topics, the connection between two-dimensional JT gravity and the SYK model at low energies, and some prerequisites for the content of this thesis.

2 Jackiw-Teitelboim Gravity

In 1984, Roman Jackiw and Claude Teitelboim [5, 6] independently developed the foundations for two-dimensional gravity theories, particularly JT gravity. JT is a two-dimensional theoretical model of gravity that has recently gained significant attention in quantum gravity studies and black hole physics [1, 2]. Despite its simplicity and solvability, JT gravity captures essential features of gravitational dynamics, making it a valuable tool for examining complex concepts in a more manageable framework. Furthermore, at low energies, JT gravity has a deep connection with the SYK model — a solvable interacting fermionic model— making it significant for studies in quantum chaos and holography. This connection has also driven advances in the AdS/CFT duality.

In practical applications, JT gravity is used to investigate the dynamics of near-extremal black holes [16] and the emergence of spacetime from quantum entanglement[17]. Additionally, JT gravity shares connections with random matrix theory [18], building a bridge between gravitational systems and statistical mechanics. These interrelations and applications enrich and deepen our understanding of fundamental physics.

To have a concise and readable summary, I have omitted most of the discussion in this section, specifically I omitted

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2.1 Overview of 2D Dilaton Models and JT gravity

Beginning with the most general two-dimensional gravity that includes a dilaton field $\tilde{\Phi}$and at most second derivatives of the dilaton field, the corresponding Euclidean action is given by:

$$I = -\frac{1}{16\pi G_N} \int_{\mathcal{M}} d^2x \sqrt{g} \Big(U_1(\tilde{\Phi}) R + U_2(\tilde{\Phi}) g^{\mu\nu} \partial_\mu \tilde{\Phi} \partial_\nu \tilde{\Phi} + U_3(\tilde{\Phi}) \Big), \tag{2.1}$$

Using general covariance and rescaling of metric, $g^{\mu\nu}$, one deletes U_1 and U_2. So, the simplified form of all dilaton 2D gravities is

$$I[g,\Phi] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{g}\Big(\Phi R + U(\Phi)\Big), \tag{2.2}$$

JT gravity is a dilaton 2D gravity with a linear choice for $U(\Phi)$ potential in 2.2, that is $U(\Phi) = -\Lambda\Phi$.

$$I^{\Lambda}_{\text{JT}}[g,\Phi] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} \sqrt{g}\Phi\big(R-\Lambda\big), \tag{2.3}$$

where Λ is the cosmological constant, where in AdS spacetime $\Lambda = -2\,^2$. For convenience, we choose $L = 1$. Additionally, we complete this action by adding the Gibbons-Hawking-York boundary term to have a well-defined variational principle and a topological term, which will be justified later on wormhole contributions.

$$\begin{aligned}
I[g,\Phi] &= -S_0\chi + I_{\text{JT}}[g,\Phi] \\
I_{\text{JT}}[g,\Phi] &= -\frac{1}{16\pi G_N}\int_{\mathcal{M}}\sqrt{g}\Phi(R+2) - \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}}\sqrt{h}\Phi(K-1) \\
\chi &= \frac{1}{4\pi}\int_{\mathcal{M}}\sqrt{g}R + \frac{1}{2\pi}\oint_{\partial\mathcal{M}}\sqrt{h}K.
\end{aligned} \tag{2.4}$$

2.2 Classical JT Gravity

We address ... here.

2.2.1 Classical Solutions of JT

Let us explore the classical solutions of the metric and the dilaton field.

$$S = \frac{\Phi_0}{16\pi G_N}\Big[\int\sqrt{-g}R + 2\oint\sqrt{-h}K\Big] + S_{\text{JT}}[g,\Phi] + S_{\text{m}}[\phi,g] \tag{2.5}$$

In 2.5, $S_{\text{m}}[\phi,g]$ refers to the matter action. Our discussion considers conformal matter coupled to JT gravity to have a well-behavior model.

To solve for metric, we vary the dilaton field. This variation results $R(x) = -2$, which determines a hyperbolic metric on spacetime. By using a conformal parametrization of metric and solving a Liouville equation, one observes that the unique metric solution is the Poincare metric, with different patches of the AdS disk model as the spacetime manifold.

$$ds^2 = -\frac{dUdV}{(U-V)^2}, \tag{2.6}$$

where U and V are light cone coordinates of t and z, the standard parametrization of AdS.

By varying 2.5 with respect to the metric, we find the profile of the dilaton.

$$\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^2\Phi + g_{\mu\nu}\Phi = -8\pi G_N T_{\mu\nu} \tag{2.7}$$

Rewriting them in conformal gauge leads to

$$\begin{aligned}
-e^{2\omega}\partial_u(e^{-2\omega}\partial_u\Phi) &= 8\pi G_N T_{uu}, \\
-e^{2\omega}\partial_v(e^{-2\omega}\partial_v\Phi) &= 8\pi G_N T_{vv}, \\
2\partial_u\partial_v\Phi + e^{2\omega}\Phi &= 16\pi G_N T_{uv},
\end{aligned} \tag{2.8}$$

where T_{ij} are the components of the matter stress tensor.

The conformal matter, $S_{\rm m}$ has a chiral stress tensor, so the differential equations in 2.8 are integrable, and the profile of dilaton is

$$\Phi(u,v) = \frac{a}{U-V}\left(1 - \frac{\mu}{a}UV - \frac{8\pi G_N}{a}(I_+ + I_-)\right) \tag{2.9}$$

with

$$\begin{aligned} I_+(u,v) &= \int_U^\infty ds(s-U)(s-V)T_{UU}(s), \\ I_-(u,v) &= \int_{-\infty}^V ds(s-U)(s-V)T_{VV}(s). \end{aligned} \tag{2.10}$$

2.2.2 Boundary Condition and Its Importance

2.2.3 Dynamics of the Boundary Curve

2.3 Quantum JT Gravity

2.4 Wormhole Contributions to JT Gravity

3 3D Gravity

3.1 Reduction To Liouville Field Theory

3.2 Overview of 3D Einestion Gravity

3.3 Asymptotic Symmetries of AdS Gravity

3.4 Asymptotic Symmetries of Flat Gravity

4 Partition Function of Asymptotically AdS 3D Gravity

4.1 Classical Solutions of Spacetime

4.2 Partition Function on $\mathcal{M}_{c,d}$

4.3 Summing Over all Contributions

4.4 Possible Scenarios

5 Conclusion

A Mathematical Preliminaries

B Review of SYK model

C Introduction to WZW and CS theoreis

D First Order Formulation of Einstein Gravity

E GHY Boundary Terms in Einstein Gravity

References

[1] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, Journal of High Energy Physics **2019** (2019) .

[2] K. Goto, T. Hartman and A. Tajdini, *Replica wormholes for an evaporating 2d black hole*, *Journal of High Energy Physics* **2021** (2021) .

[3] J. Cotler, K. Jensen and A. Maloney, *Low-dimensional de sitter quantum gravity*, *Journal of High Energy Physics* **2020** (2020) .

[4] D. Marolf and H. Maxfield, *Transcending the ensemble: baby universes, spacetime wormholes, and the order and disorder of black hole information*, *Journal of High Energy Physics* **2020** (2020) .

[5] R. Jackiw, *Lower dimensional gravity*, *Nuclear Physics B* **252** (1985) 343.

[6] C. Teitelboim, *Gravitation and hamiltonian structure in two spacetime dimensions*, *Physics Letters B* **126** (1983) 41.

[7] T.G. Mertens, G.J. Turiaci and H.L. Verlinde, *Solving the schwarzian via the conformal bootstrap*, *Journal of High Energy Physics* **2017** (2017) .

[8] P. Saad, S.H. Shenker and D. Stanford, "JT gravity as a matrix integral."

[9] E. Witten, *2 + 1 dimensional gravity as an exactly soluble system*, *Nuclear Physics B* **311** (1988) 46.

[10] J.D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207.

[11] J. Maldacena*International Journal of Theoretical Physics* **38** (1999) 11131133.

[12] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, *Gravitational waves in general relativity. 7. waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond. A* **269** (1962) 21.

[13] R. Sachs, *Asymptotic symmetries in gravitational theory*, *Phys. Rev.* **128** (1962) 2851.

[14] A. Maloney and E. Witten, *Quantum gravity partition functions in three dimensions*, *Journal of High Energy Physics* **2010** (2010) .

[15] N. Benjamin, S. Collier and A. Maloney, *Pure gravity and conical defects*, 2020.

[16] L.V. Iliesiu and G.J. Turiaci, *The statistical mechanics of near-extremal black holes*, *Journal of High Energy Physics* **2021** (2021) .

[17] G. Penington, *Entanglement wedge reconstruction and the information paradox*, 2020.

[18] P. Saad, S.H. Shenker and D. Stanford, *Jt gravity as a matrix integral*, 2019.