Summaray of Master's Thesis

Aspects of Low-dimensional Quantum Gravity

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Abstract: This paper is an insightful summary of my master's thesis on quantum gravity in two and three dimensions, in which I tour from essential ingredients to recent discoveries. I start with JT gravity at the classical level, then continue with quantum JT gravity, emphasizing the correlations functions. The significance of wormhole contributions is briefly highlighted, emphasizing their role in recent advancements. The second half of this thesis is devoted to three-dimensional gravity, mainly focusing on the asymptotic symmetries in asymptotically AdS and flat spacetimes. Finally, I give a detailed account of the partition function of pure 3D gravity in asymptotically AdS spacetime and address its physical and holographic shortcomings.

Keywords: Holography, AdS/CFT, JT gravity, Quantum gravity, Black hole, Gauge theory, Information paradox.

This summary is in progress and updates continuously.

Contents

1 Introduction

Low-dimensional quantum gravity, in two or three dimensions, holds special significance in theoretical physics mainly because it serves as a simple laboratory for understanding the universal features of quantum gravity. Even though there is little hope that these lower-dimensional quantum gravity theories will lead to a complete understanding of four-dimensional quantum gravity, they provide a suitable playground for exploring ideas and learning from these theories. These models allow us to examine the complex dynamics of quantum gravity theory and study concepts like black holes, entropy, and symmetries in a more controllable and simplified framework [Applications are in information paradox and black hole evaporation [1, 2], 2D cosmology [3], and quantum chaos [4]].

This summary of my masters thesis is organized as follows.

It is well-known that two-dimensional gravity is structurally and conceptually trivial without the dilaton field. So, we start with dilaton models in two dimensions, specifically JT gravity [5, 6]. First, we explore this model at the classical level, reviewing the motivations for its study, classical solutions, and its connection with the Schwarzian quantum mechanics [1]. We then proceed to the quantum level, beginning with a perturbative, one-loop analysis of this theory, followed by a summary of the two- and four-point correlation functions and a brief discussion of their derivations [7]. Subsequently, we examine JT gravity in the presence of defects and conclude with a discussion of wormhole contributions, understanding their significance [8]. Here, we end the discussion of two-dimensional gravity and proceed to three-dimensional gravity.

Our discussion of three-dimensional gravity begins by examining its duality with the Liouville field theory [9]. Through a series of Hamiltonian reductions and redefinitions, we demonstrate that, at the classical level, three-dimensional anti-de Sitter (AdS) gravity is equivalent to a Liouville field theory on its conformal boundary. Then, we approach three-dimensional gravity from a different perspective, a gauge theoretical description. We pay close attention to the symmetries within gauge theories. In gauge theories, the asymptotic behavior of gauge fields is crucial, as it determines the model extensively. By choosing the Brown-Henneaux [10] fall-off conditions for asymptotically AdS spaces and the BMS fall-off conditions for asymptotically flat spaces, we analyze their respective asymptotic symmetries. For asymptotically AdS spacetimes, Brown-Henneaux fall-off conditions point toward the gauge/gravity duality, a result derived roughly a decade before this duality was framed in modern language [11]. Similarly, the gauge symmetry group for flat spaces is the centrally extended Poincaré group, known as the BMS_3 group [12, 13]. Throughout this discussion, we uncover extensive connections between these models and the geometric actions on coadjoint orbits of the Virasoro group and the $\widetilde{\mathfrak{bms}}_3$ algebra, highlighting the geometric nature of these quantum theories.

Finally, we study the partition functions of pure AdS gravity [14], employing the gravitational path integral formulation in the semiclassical approximation. Additionally, we enumerate potential deficiencies in the spectrum of this quantum gravity theory, presenting possible proposals to define a healthy quantum gravity in asymptotically AdS_3 spacetimes.

[1] This theory is invariant under the $\mathrm{SL}(2,\mathbb{R})$ symmetry.

In recent years, one such proposal has resolved the issue of the negative density of states in this spectrum [15] . However, research continues to address the remaining challenges in this theory.

In the appendices, we cover the mathematical preliminaries for the discussed topics, the connection between two-dimensional JT gravity and the SYK model at low energies, and some prerequisites for the content of this thesis.

2 Jackiw-Teitelboim Gravity

In 1984, Roman Jackiw and Claude Teitelboim [5, 6] independently developed the foundations for two-dimensional gravity theories, particularly JT gravity. JT is a two-dimensional theoretical model of gravity that has recently gained significant attention in quantum gravity studies and black hole physics [1, 2]. Despite its simplicity and solvability, JT gravity captures essential features of gravitational dynamics, making it a valuable tool for examining complex concepts in a more manageable framework. Furthermore, at low energies, JT gravity has a deep connection with the SYK model — a solvable interacting fermionic model— making it significant for studies in quantum chaos and holography. This connection has also driven advances in the AdS/CFT duality.

In practical applications, JT gravity is used to investigate the dynamics of near-extremal black holes [16] and the emergence of spacetime from quantum entanglement[17]. Additionally, JT gravity shares connections with random matrix theory [18], building a bridge between gravitational systems and statistical mechanics. These interrelations and applications enrich and deepen our understanding of fundamental physics.

To have a concise and readable summary, I have omitted most of the discussion in this section, specifically I omitted

- ✘ Motivations to study JT gravity. A primary motivation is that JT gravity describes the near-horizon physics of near-extremal black holes.
- ✘ Its duality with the topological BF model.
- ✘ The relation of dilaton field with entropy.
- ✘ Geometric derivation of Schwarzian action.
- ✘ Classical JT in the presence of quantum matter.
- ✘ The exact form of two-point and four-point functions of JT gravity.
- ✘ Insertion of defects and their physical interpretation.
- ✘ Non-perturbative effects of JT.
- ✘ How do wormhole contributions produce the correct spectral form factor?

2.1 Overview of 2D Dilaton Models and JT gravity

Beginning with the most general two-dimensional gravity that includes a dilaton field $\tilde{\Phi}$and at most second derivatives of the dilaton field, the corresponding Euclidean action is given by:

$$I = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{g}\Big(U_1(\tilde{\Phi})R + U_2(\tilde{\Phi})g^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi} + U_3(\tilde{\Phi})\Big), \tag{2.1}$$

Using general covariance and rescaling of metric, $g^{\mu\nu}$, one deletes U_1 and U_2. So, the simplified form of all dilaton 2D gravities is

$$I[g,\Phi] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^2x\sqrt{g}\Big(\Phi R + U(\Phi)\Big), \tag{2.2}$$

JT gravity is a dilaton 2D gravity with a linear choice for $U(\Phi)$ potential in 2.2, that is $U(\Phi) = -\Lambda\Phi$.

$$I^{\Lambda}_{\rm JT}[g,\Phi] = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} \sqrt{g}\Phi\big(R-\Lambda\big), \tag{2.3}$$

where Λ is the cosmological constant, where in AdS spacetime $\Lambda = -2\,^2$. For convenience, we choose $L = 1$. Additionally, we complete this action by adding the Gibbons-Hawking-York boundary term to have a well-defined variational principle and a topological term, which will be justified later on wormhole contributions.

$$\begin{aligned} I[g,\Phi] &= -S_0\chi + I_{\rm JT}[g,\Phi] \\ I_{\rm JT}[g,\Phi] &= -\frac{1}{16\pi G_N}\int_{\mathcal{M}}\sqrt{g}\Phi(R+2) - \frac{1}{8\pi G_N}\oint_{\partial\mathcal{M}}\sqrt{h}\Phi(K-1) \\ \chi &= \frac{1}{4\pi}\int_{\mathcal{M}}\sqrt{g}R + \frac{1}{2\pi}\oint_{\partial\mathcal{M}}\sqrt{h}K. \end{aligned} \tag{2.4}$$

2.2 Classical JT Gravity

2.2.1 Classical Solutions of JT

Let us explore the classical solutions of the metric and the dilaton field.

$$S = \frac{\Phi_0}{16\pi G_N}\Big[\int\sqrt{-g}R + 2\oint\sqrt{-h}K\Big] + S_{\rm JT}[g,\Phi] + S_{\rm m}[\phi,g] \tag{2.5}$$

In 2.5, $S_{\rm m}[\phi,g]$ refers to the matter action. Our discussion considers conformal matter coupled to JT gravity to have a well-behavior model.

To solve for metric, we vary the dilaton field. This variation results $R(x) = -2$, which determines a hyperbolic metric on spacetime. By using a conformal parametrization of metric and solving a Liouville equation, one observes that the unique metric solution is the Poincare metric, with different patches of the AdS disk model as the spacetime manifold.

$$ds^2 = -\frac{dUdV}{(U-V)^2}, \tag{2.6}$$

where U and V are light cone coordinates of t and z, the standard parametrization of AdS.

By varying 2.5 with respect to the metric, we find the profile of the dilaton.

$$\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu}\nabla^2\Phi + g_{\mu\nu}\Phi = -8\pi G_N T_{\mu\nu} \tag{2.7}$$

Rewriting them in conformal gauge leads to

$$\begin{aligned} -e^{2\omega}\partial_u(e^{-2\omega}\partial_u\Phi) &= 8\pi G_N T_{uu}, \\ -e^{2\omega}\partial_v(e^{-2\omega}\partial_v\Phi) &= 8\pi G_N T_{vv}, \\ 2\partial_u\partial_v\Phi + e^{2\omega}\Phi &= 16\pi G_N T_{uv}, \end{aligned} \tag{2.8}$$

where T_{ij} are the components of the matter stress tensor.

The conformal matter, S_{m} has a chiral stress tensor, so the differential equations in 2.8 are integrable, and the profile of dilaton is

$$\Phi(u,v) = \frac{a}{U-V}\Big(1 - \frac{\mu}{a}UV - \frac{8\pi G_N}{a}(I_+ + I_-)\Big) \tag{2.9}$$

with

$$\begin{aligned} I_+(u,v) &= \int_U^\infty ds(s-U)(s-V)T_{UU}(s), \\ I_-(u,v) &= \int_{-\infty}^V ds(s-U)(s-V)T_{VV}(s). \end{aligned} \tag{2.10}$$

2.2.2 Boundary Condition and Its Importance

After solving for classical fields, it remains to define a viable boundary condition for dilaton field[2]. We closely follow the approach of [19] to define suitable fall-off conditions for both geometry and dilaton field.

- **Fixing asymptotics of metric:** The geometry is asymptotically AdS_2, in Fefferman-Graham gauge

  $$ds^2 = \frac{-dt^2 + dz^2}{z^2} + (\text{Subleading as } \; z \to 0). \tag{2.11}$$

  Requiring this form for the asymptotic metric constrains the allowed diffeomorphism on JT gravity. The boundary curve of the gravitational patch is specified with $F(t)$ in Poincare coordinates, $(T = F(t), Z = \epsilon F'(t))$[3].

- **Fixing the dilaton profile on asymptotics:** As mentioned, the dilaton field must be divergent at the holographic boundary. We fix it to have the following asymptotics.

  $$\Phi = \frac{a}{2z} + (\text{Subleading as } \; z \to 0), \tag{2.12}$$

  where a is a dimensionful parameter.

[2]Boundary conditions can also be addressed by the divergence of dilaton profile, starting from a uniform vanishing profile.

[3]ϵ is the holographic cutoff near the $Z = 0$ boundary.

Plugging these conditions into 2.9 will give an equation for $F(t)$.

$$F'(t) = 1 - \frac{8\pi G_N}{a}\Big(\int_{F(t)}^{\infty} ds (s - F(t))^2 T_{UU}(s) + \int_{-\infty}^{F(t)} ds (s - F(t))^2 T_{VV}(s)\Big) \tag{2.13}$$

This complicated equation specifies the profile of the boundary curve. We introduce a very suggestive way to rewrite this differential-integral equation in terms of holographic stress tensor.

The holographic stress tensor, in 0+1-dimensional theories, which measures the injected energy to the system in time t, is defined by

$$E(t) = -\frac{a}{16\pi G_N}\{F, t\}, \qquad \{F, t\} = \frac{F'''}{F'} - \frac{3}{2}\Big(\frac{F''}{F'}\Big)^2, \tag{2.14}$$

where $\frac{16\pi G_N}{a}$ is gravitational coupling. We also denote $C \equiv a/(16\pi G_N)$ for convenience.

In terms of the holographic stress tensor, equation 2.13 simplifies to

$$\frac{dE(t)}{dt} = \big(T_{VV}(t) - T_{UU}(t)\big)F'^2\big|_{\partial\mathcal{M}} = (T_{vv}(t) - T_{uu}(t))\big|_{\partial\mathcal{M}} \tag{2.15}$$

This equation is correctly interpreted as an energy conservation equation. The boundary curve reacts to energy injection/extraction by adjusting itself. By injecting energy into the system, the boundary curve tends to the holographic boundary of AdS_2, and by extracting energy, it contracts inward.

2.2.3 Dynamics of the Boundary Curve

In the absence of an external source, 2.15 simplifies to

$$\frac{d}{dt}\{F, t\} = 0. \tag{2.16}$$

This equation, known as the "Schwartizan equation," can be derived from an action, the Schwartzian action.

$$S = -C\int \mathrm{dt}\ \{F, t\}. \tag{2.17}$$

Schwarzian derivative is invariant under $\mathrm{PSL}(2,\mathbb{R})$, which is under $F \to \frac{aF+b}{cF+d}$ transformation, with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

This action can be motivated more geometrically[20]. By varying 2.4 with respect to the dilaton field, all the dynamics of JT will transfer to the boundary term. Parametrizing the boundary curve by $(F(\tau), Z(\tau))$, we constrain the determinant of the boundary metric to $\sqrt{h} = 1/\epsilon$. After applying 2.11 and 2.12 boundary conditions, the boundary term of 2.4 in the first order of ϵ will become Schwartzian derivative.

$$\mathrm{I}_{\mathrm{JT}}[F] = -C\int d\tau\{F, \tau\} \tag{2.18}$$

Therefore, JT gravity is holographically equivalent to Schwartzian quantum mechanics.

2.3 Quantum JT Gravity

2.3.1 One-loop Correction to Classical JT

In the previous part, we treated both gravity and matter classically. Here, we briefly review the quantization of JT gravity. The path integral quantization of JT, in AdS thermal coordinates, is a sum over allowed metrics and dilaton fields.

$$Z(\beta) = e^{S_0} \int [Dg_{\mu\nu}][D\Phi] \exp\Big(\frac{1}{16\pi G_N}\big[\int_{\mathcal{M}} d^2x\sqrt{g}\Phi(R+2) + 2\oint_{\partial\mathcal{M}} \sqrt{h}\Phi(K-1)\big]\Big) \tag{2.19}$$

By integrating the dilaton field and rewriting the boundary action as Schwarzian action, as mentioned in the previous chapter, 2.19 is written in this suggestive form.

$$Z(\beta) = e^{S_0} \int_{\frac{\text{Diff}(S^1)}{\text{SL}(2,\mathbb{R})}} [Df] \exp\Big(C\int_0^\beta d\tau \{\tan\frac{\pi f(\tau)}{\beta}, \tau\}\Big) \tag{2.20}$$

Let us first look at a simple case. Expanding $f(\tau)$ around classical extremum $f_0(\tau) = \tau$, we can compute the one-loop quantum correction to classical JT action. After mode expansion, working out details of I_{Sch}, and the integration measure, the one-loop correction to the partition function of JT gravity is

$$Z(\beta) = \exp^{S_0 + \frac{2\pi^2 C}{\beta}} \Pi_{n\geq 2}\frac{\beta}{Cn} = \frac{1}{4\pi^2}\big(\frac{2\pi C}{\beta}\big)^{\frac{3}{2}} \exp^{S_0+\frac{2\pi^2 C}{\beta}} . \tag{2.21}$$

This will give the logarithmic correction to the entropy of black holes. The one-loop spectrum of JT gravity is given by

$$\rho_{\text{JT}}(E) = \frac{C}{2\pi^2} e^{S_0} \sinh\left(2\pi\sqrt{2CE}\right) \tag{2.22}$$

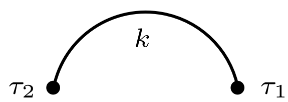
The spectrum's continuity suggests that there is no quantum mechanical system dual to JT gravity. We will return to this later when considering wormhole contributions.

2.3.2 Diagrammatic Rules for Quantum JT

From here, one can quantize JT gravity with a scalar matter. This procedure is very detailed and requires advanced approaches [7]. Nevertheless, like the one used above, one-loop techniques can be effective for calculating two-point functions. Let us ignore the calculation details and highlight the diagrammatic rules of JT amplitudes.

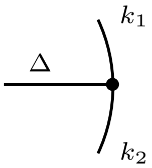
1. Associate a momentum variable to each closed region within the thermal circle, k_i. These momentum variables will finally be integrated with Schwartzian momentum density to yield JT amplitudes.

2. Write the following exponential factor for each segment of the thermal circle that encloses a region with momentum k.



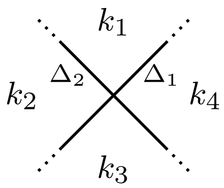
$$= \quad e^{-\frac{k^2}{2C}(\tau_2-\tau_1)} \tag{2.23}$$

3. We have the following vertex rule for each intersection of a Schwarzian propagator with weight Δ that bisects a thermal circle to two regions with momentums k_1 and k_2.



$$\equiv \quad \gamma_\Delta(k_1, k_2) \quad = \quad \sqrt{\frac{\Gamma(\Delta \pm ik_1 \pm ik_2)}{(2C)^{2\Delta}\Gamma(2\Delta)}}. \tag{2.24}$$

4. Insert the following factor for intersection of two Schwarzian propagator.



$$= \quad \begin{Bmatrix} \Delta_1 & k_1 & k_2 \\ \Delta_2 & k_3 & k_4 \end{Bmatrix} \tag{2.25}$$

The bracket indicates the $6j$-symbols for $\mathrm{PSL}(2,\mathbb{C})$ group, which is a complicated factor involving gamma functions and contour integration.

5. Multiply the resulting amplitude by $e^{S_0}/2$.

2.4 Wormhole Contributions to JT Gravity

The behavior of late-time correlation functions of JT gravity does not conform with a typical quantum system as its putative holographic dual. In quantum systems, the two-point function fluctuates around an average point, whereas two-point functions of JT decay with time as t^{-3}. This discrepancy can best be capsulated in spectral form factor and its evolution with time.

$$Z(\beta + it)Z(\beta - it) = \sum_{n,m} e^{-(\beta+it)E_n} e^{-(\beta-it)E_m}, \tag{2.26}$$

This is our starting motivation to include various topologies within path integral formulation.

To start the analysis, look at 2.4 once again. Adding novel topologies amounts to a sum over their genera in path integral relation 2.19. We also generalize the partition function to include n holographic boundaries, each with renormalized length β_i. Therefore, the partition function on this Riemann surface has the following genus expansion.

$$Z_{\text{grav,conn}}(\beta_1, \ldots, \beta_n) = \sum_{g=0}^{\infty} e^{S_0(2-2g-n)} Z_{g,n}(\beta_1, \ldots, \beta_n). \tag{2.27}$$

Notice that complicated topologies are subdominant by increasing the handles g or holographic boundaries n.

The above path integral involves a sum over all hyperbolic metrics, with the weight of Euclidean JT action. This integration ultimately reduces to the volume of the moduli space of Riemann surfaces. A few simple geometries included in this integration are

- **Disk topology:** $Z_{0,1}(\beta)$, which has been considered in 2.21.
- **Trumpet topology:** $Z_{0,2}(\beta_1, \beta_2)$ that has two holographic boundaries. The final contribution of trumpet topology is

$$Z_{0,2}(\beta_1, \beta_2) = \frac{1}{2\pi} \frac{\sqrt{\beta_1 \beta_2}}{\beta_1 + \beta_2} \tag{2.28}$$

$Z_{g,n}(\beta_1, \ldots, \beta_n)$ is now calculable by breaking a Riemann surface with g handles and n boundaries into n double trumpet and then integrating on b_i (the moduli parameters of the double trumpets) and volume of Riemann surface within with genus g, and n holographic boundaries with lengths b_i. The volume of such module spaces is found by the Mirzakhani recursion relation [21], which accepts $V_{0,1}$ and $V_{0,2}$ as input, computes $V_{g,n}$. This approach is comprehensively elucidated in [8].

3 Three-dimensional Gravity

This section of my thesis is devoted to several interconnected aspects of three-dimensional quantum gravity. First, I give a sketch of the 3D gravity/Liouville field theory correspondence at the classical level. Then, a correct classification scheme of symmetries is defined, which is necessary due to different fall-off conditions in gauge theory. Afterward, I will discuss the asymptotic symmetries of asymptotically AdS quantum gravity. Similar results also hold for asymptotically flat quantum gravity.

3.1 Reduction To Liouville Field Theory

Auchucarro and Townsend [22], and independently Witten [9] found that 3D gravity and its equations of motion are equivalent to a Chern-Simons theory on special gauge groups, refer to the table 3.1.

Λ	Gauge group
+ (dS)	$\mathrm{SO}(1,3)$
0(Flat)	$\mathrm{ISO}(1,2)$
-(AdS)	$\mathrm{SO}(2,2)$

Let us briefly see how this reduction takes place for asymptotically AdS spacetimes. $\mathfrak{so}(2,2)$ Lie algebra is given by the following commutators

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \epsilon_{abc} J^c, \tag{3.1}$$

with $a, b, c \in \{0, 1, 2\}$.

We define the gauge field in Chern-Simons theory[4] according to

$$A_\mu \equiv \frac{1}{\ell} e^a_\mu P_a + \omega^a_\mu J_a \tag{3.2}$$

By a short calculation, the Chern-Simons action will be equivalent to the Einestein-Hilbert action in the first-order formulation[5].

$$S_{\mathrm{CS}}[e,\omega] = \frac{k}{4\pi\ell} \int_{\mathcal{M}} \left(2e^a \wedge R_a[\omega] + \frac{1}{3\ell^2} \epsilon_{abc}\, e^a \wedge e^b \wedge e^c \right) \tag{3.3}$$

where $k = \frac{\ell}{4G_N}$ is the level constant of Chern-Simons theory.

Remark: Due to $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ isomorphism, this action can be recast into two seperate $\mathrm{SL}(2,\mathbb{R})$ Chern-Simons theories

$$S_{\mathrm{CS}}[\Gamma] = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] \equiv S_{\mathrm{CS}}[A,\bar{A}], \tag{3.4}$$

with

$$A = (e^a/\ell + \omega^a)T_a, \quad \bar{A} = (e^a/\ell - \omega^a)T_a \tag{3.5}$$

The next step is to introduce the boundary conditions of the metric. Conventionally, Brown-Henneaux boundary conditions are chosen, in which the metric on the timelike cylinder at conformal infinity is fixed to be flat. In the Fefferman-Graham gauge, the metric has the following form

$$ds^2 = \frac{\ell^2}{r^2} dr^2 + \gamma_{ij}(r, x^k) dx^i dx^j, \tag{3.6}$$

where in $r \to \infty$ limit, γ is expanded as $\gamma_{ij} = r^2 g^{(0)}_{ij}(x^k) + \mathcal{O}(1)$.

Subsequently, a careful analysis translates this condition on metric into a condition on gauge fields on Chern-Simons theory.

Then, we solve for gauge theory using traditional methods, which involve separating variables. Through this process, we find that the Chern-Simons action transforms into the right-handed Wess-Zumino-Witten action.

$$\begin{aligned} S_{CS}[A] &= \kappa \int_{\partial\mathcal{M}} d\tau d\varphi Tr[g^{-1}\partial_\varphi g g^{-1}\partial_\tau g] + \kappa\Gamma[G] \\ &\equiv S^R_{WZW}[g] \end{aligned} \tag{3.7}$$

A Hamiltonian reduction is then necessary to transform the chiral Wess-Zumino-Witten action to standard $\mathrm{SL}(2,\mathbb{R})$ Wess-Zumino-Witten action.

Finally, by a Gauss decomposition of fields and applying the second boundary condition, the Wess-Zumino-Witten currents are constrained, and the action will take the form of the Louisville action. Look at the flowchart 1 for an overview of this process.

[4]For a review of Chern-Simons and Wess-Zumino-Witten models refer to section B

[5]See appendix C for the first-order formulation of Einestein gravity.

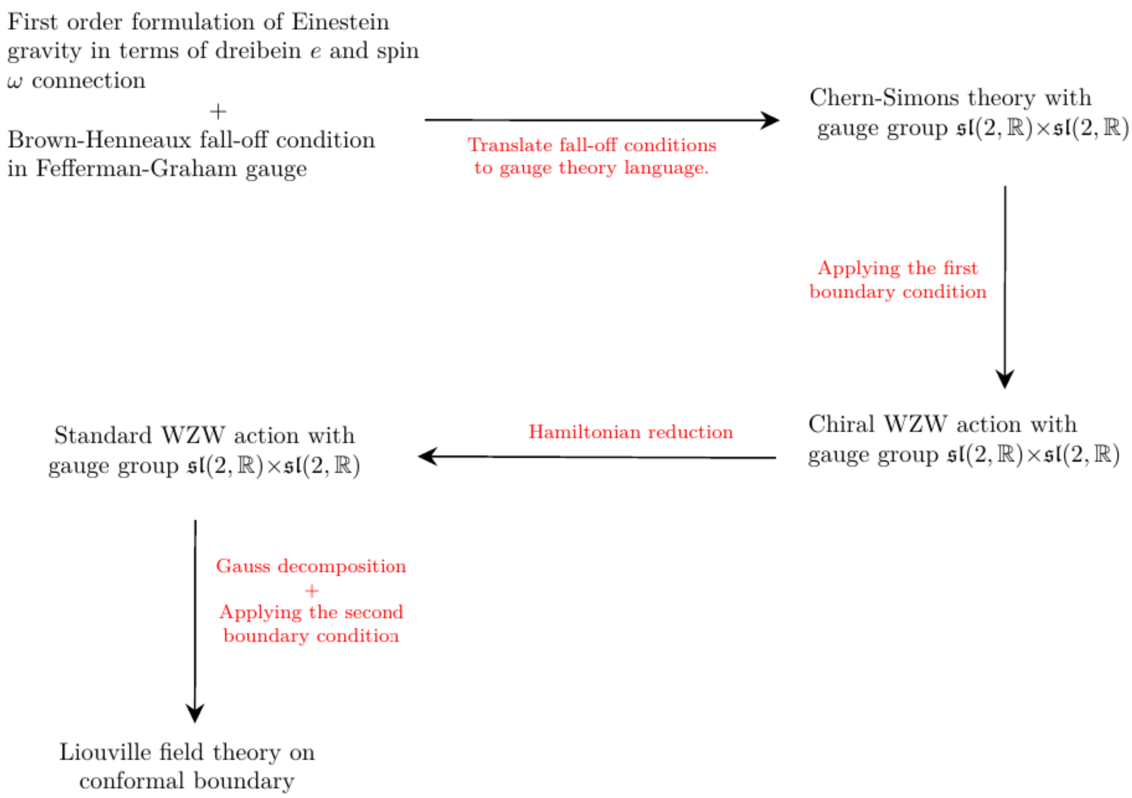


Figure 1.

3.2 Overview of 3D Einestion Gravity

In three dimensions, quantum gravity is more controllable. This is due to the lack of propagating degrees of freedom in the bulk of spacetime. Thereby, all the dynamics of three-dimensional quantum gravity happen on its boundary. Technically, this means that three-dimensional quantum gravity is actually a topological quantum field theory. Even though physical phenomena like gravitational waves are present in this model, there are modified models that accommodate bulk degrees of freedom. Also, three-dimensional black hole solutions of gravity pave the way to test quantum gravity ideas, like the information paradox and holographic conjectures.

Let us classify symmetries in a gauge theory. According to fall-off conditions in a gauge theory, one classifies symmetries into three distinct categories. This classification has two criteria: first, whether transformations will change the physical configuration of gauge fields, and second, whether transformed fields obey the fall-off condition. We have these categories (Look at figure **??** for a schematic view of symmetries.)

1. **Trivial gauge transformations:** These transformations don't change the physical configuration of gauge fields and respect fall-off conditions. Hence, they are truly gauge transformations of the gauge theory with its fall-off conditions.

2. **Non-trivial (Allowed) gauge transformations:** They change gauge fields to non-equivalent configurations, still preserving the fall-off condition.

3. **Forbidden transformations:** These transformations are no longer symmetries of the theory since they violate the specified fall-off condition.

Definition: Algebra of asymptotic symmetries of a gauge theory is the quotient algebra of allowed symmetries by the ideal of trivial symmetries.

With these concepts and definitions, let us look at three-dimensional gravity in asymptotically AdS spacetime as a gauge theory, define a suitable fall-off condition for metric, and find the algebra of its asymptotic symmetry group.

3.3 Asymptotic Symmetries of Asymptotically AdS Gravity

The first step is to define the fall-off conditions of the metric. Intuitively, the metric must resemble the AdS metric in its asymptotics. Such metrics must include global AdS spacetime and metrics exhibiting conical deficit in their bulk[6]. We augment this set by adding any metric with similar asymptotic behavior on its conformal boundary. Working on the Fefferman-Graham gauge, we have the following notion of asymptotically AdS spacetimes.

Brown-Henneaux Fall-off Conditions: Let $\mathcal{M}$ be a pseudo-Riemannian spacetime manifold, with (r, x^a) coordinate on it $(a = 0, 1)$, and a timelike cylinder as its conformal infinity. If the metric behaves as

$$ds^2 \overset{r\to+\infty}{\sim} \frac{\ell^2}{r^2}dr^2 + \big(r^2\eta_{ab} + \mathcal{O}(1)\big)dx^a dx^b \tag{3.8}$$

in $r \to \infty$ limit, where η_{ab} is the two-dimensional Minkowski metric on the cylindrical shell, we say that $(\mathcal{M}, ds^2)$ is an asymptotically AdS spacetime in the Brown-Henneaux sense in Fefferman-Graham gauge. The cosmological constant of this spacetime is $\Lambda = -1/\ell^2$.

The next step is to find the Asymptotic Killing Vector Fields (AKVFs) and investigate their algebra. The only constraint on these AKVFs is that they have to preserve the asymptotics of metric, that is **??**. This gives the following condition on AKVFs.

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{r\pm} = 0, \qquad \mathcal{L}_\xi g_{ab} = \mathcal{O}(1) \quad (a, b = \pm) \tag{3.9}$$

By solving **??** differential equations, one finds the most general form of AKVFs to be

$$\xi = X(x^+)\partial_+ + \bar{X}(x^-)\partial_- - \frac{1}{2}\big(\partial_+ X(x^+) + \partial_- \bar{X}(x^-)\big)r\partial_r + \text{Subleading} \tag{3.10}$$

Decomposing these generators into Fourier modes

$$\ell_m \equiv \xi_{(e^{imx^+},0)}\,, \qquad \bar{\ell}_m \equiv \xi_{(0,e^{imx^-})}\,, \tag{3.11}$$

their algebra is found easily to be Witt algebra

$$i[\ell_m, \ell_n] = (m-n)\ell_{m+n}\,, \qquad i[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}\,, \qquad i[\ell_m, \bar{\ell}_n] = 0\,. \tag{3.12}$$

[6]These metrics indicate a spacetime with a massive particle in its bulk. We expect that the presence of such particles does not change the metric's asymptotics

Theorem: The asymptotic symmetry group of AdS gravity with Brown-Henneaux boundary conditions is the direct product of two Virasoro groups, which act on the cylindrical conformal boundary as

$$(x^+, x^-) \mapsto \big(f(x^+), \bar{f}(x^-)\big) \tag{3.13}$$

which means that AKVFs act as conformal transformations of asymptotic cylinder.

Then, the process continues to check wether all the above transformations are non-trivial or not. This process requires introducint the surface current algebra and checking their algebra. The answer to the previous question is positive, and asymptotic symmetries of quantum gravity in asymptotically AdS spacetimes is isomorphic to Virasoro algebra.

The same ideas similarly apply to the asymptotically flat spacetimes. First, the BMS fall-off condition is defined. Then, all AKVFs consistent with the BMS condition are classified, and one observes that they extend the Poincare algebra in some way. The asymptotic algebra of symmetries is then found to be the central extension of bms algebra, or $\widetilde{\mathfrak{bms}}_3$.

4 Partition Function of Asymptotically AdS 3D Gravity

4.1 Classical Solutions of Spacetime

4.2 Partition Function on $\mathcal{M}_{c,d}$

4.3 Summing Over all Contributions

4.4 Possible Scenarios

5 Conclusion

A Review of SYK model

In this section, we explore the SYK model. This model was proposed and solved by Kitaev [23] to simplify the initial model by Sachdev - Ye [24]. It is a quantum mechanical model consisting of N Majorana fermions, all coupled via random couplings (selected from a specific distribution). We will also see how, at low energy, this model can lead to the Schwarzian action, providing a holographic description of gravity.

A.1 SYK Hamiltonian

This model is essentially an ensemble of simple quantum models with finite dimensions. A Hermitian Hamiltonian specifies each quantum model in this ensemble. First, using the Majorana representation (the spin-$\frac{1}{2}$ representation of the Lorentz group), we define SYK Hamiltonian. We only need Majorana representations in even dimensions; Let us briefly review these representations.

We want to find representations of the Clifford algebra

$$\{\psi_i, \psi_j\} = \delta_{ij}, \quad i, j = 1, \ldots, N$$

Focusing only on even dimensions, $N = 2K$, and for the Hamiltonian to be Hermitian, we consider Hermitian representations $\psi_i^\dagger = \psi_i$. This algebra can be reduced to the fermionic algebra by defining:

$$c_i = \frac{1}{\sqrt{2}}(\psi_{2i} - i\psi_{2i+1}), \quad c_i^\dagger = \frac{1}{\sqrt{2}}(\psi_{2i} + i\psi_{2i+1}), \;\; i = 1, \dots, K$$

Using the anticommutation properties, we derive

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0, \quad \{c_i, c_j^\dagger\} = \delta_{ij}$$

These anti-commutators are the algebra of fermionic operators represented simply in terms of creation operators. Using these representations, we can construct Majorana representations.

Reminder on Fermionic Algebra Representations

We start with a vacuum annihilated by all annihilation operators:

$$c_i |0\rangle = 0, \forall i \in \{1, 2, \dots, K\}.$$

The basis states are constructed as:

$$(c_1^\dagger)^{n_1} \dots (c_K^\dagger)^{n_K} |0\rangle, \quad n_k = 0, 1$$

yielding $2^K = 2^{\frac{N}{2}}$ basis vectors, depending on whether a fermion occupies a specific mode. This is the only irreducible Hermitian representation of this algebra; all other representations are unitarily equivalent.
Note: For odd N, representations can be constructed by adding a γ_5 matrix to the representations of $N - 1$ (even), so all representations are effectively categorized.

For Majorana fermion representations, a recursive relation exists

$$\begin{aligned}
\psi_i^{(K)} &= \psi_i^{(K-1)} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, \dots, N-2 \\
\psi_{N-1}^{(K)} &= \frac{1}{\sqrt{2}} I_{2^{K-1}} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\psi_N^{(K)} &= \frac{1}{\sqrt{2}} I_{2^{K-1}} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\end{aligned}$$

The matrices $\psi_i^{(K)}$ are $2^K \times 2^K$, requiring two initial matrices to start the recursion

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The SYK ensemble has Hamiltonians of the form

$$H = \sum_{i,j,k,l=1}^{N} J_{ijkl} \psi_i \psi_j \psi_k \psi_l,$$

where J_{ijkl} represents couplings between fermions, chosen randomly and independently from a Gaussian distribution with mean $\mu = 0$ and variance $\sigma = \frac{\sqrt{3!}J}{N^{3/2}}$.

A simple generalization of this model includes q-body interactions:

$$H = i^{q/2} \sum_{1 \leq i_1 < \cdots < i_q \leq N} J_{i_1 \ldots i_q} \psi_{i_1} \ldots \psi_{i_q}$$

where $J_{i_1 \ldots i_q}$ is drawn from a Gaussian distribution with mean 0 and variance $\sigma = \frac{\sqrt{(q-1)!}J}{N^{(q-1)/2}}$.

Here are some properties of this model.

1. In the large N limit, the model becomes classical. For large N, the quantum model can be understood fully by solving the classical equations of motion for G and Σ.

2. At low energy (infrared limit), a time-reparameterization symmetry $t \mapsto f(t)$ emerges, spontaneously broken in the vacuum, analogous to the Goldstone modes, with an effective action defined for them.

3. The spectrum of the model is interesting. The density of states for $q = 2$ is equivalent to integrable theories with extended low-energy tails, while for $q = 4$, the spectrum abruptly terminates.

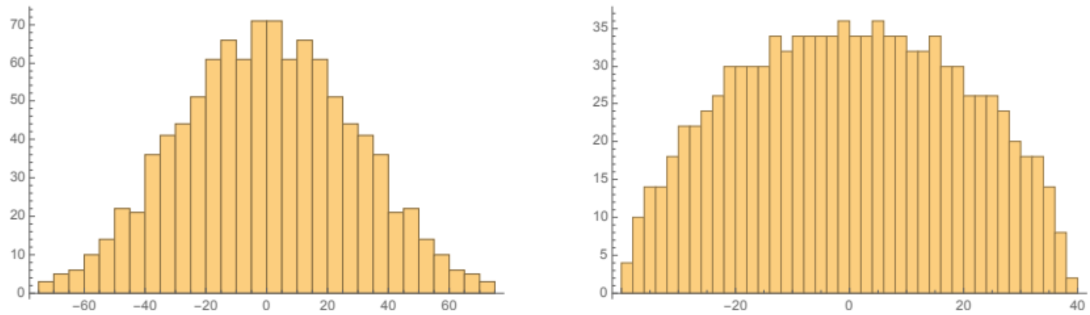


Figure 2. The left plot shows the density of states for $q = 2$, and the right plot for $q = 4$, both for $N = 20$.

A.2 Quantum-Level Analysis of the Model

Now, we begin analyzing the SYK model. We use the conventional perturbative method to study the SYK model where J is our perturbation parameter. In the limit $N \to \infty$, the summation over various Feynman diagrams can be represented as a Schwinger-Dyson series.

In Euclidean notation, the time-ordered two-point function of fermions is given by

$$G_{ij}(\tau) = \langle T\psi_i(\tau)\psi_j(0)\rangle \equiv \Theta(\tau)\langle\psi_i(\tau)\psi_j(0)\rangle - \Theta(-\tau)\langle\psi_j(0)\psi_i(\tau)\rangle,$$

where Θ is the Heaviside step function, and

$$\psi_i(\tau) = e^{\tau H} \psi_i e^{-\tau H}.$$

An important quantity derived from the two-point functions is the normalized trace of the two-point function. To calculate this, we do not need the four-fermion vertices, so by setting $J = 0$, the Hamiltonian also becomes zero, and we have $\psi_i(\tau) \equiv \psi_i$.

Using Clifford algebra, we compute the two-point functions and their normalized trace $G(\tau) = \frac{1}{N} \sum_{i=1}^{N} G_{ii}(\tau)$

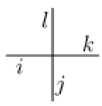
$$G_{ij}^{\text{free}}(\tau) = \frac{1}{2}\delta_{ij}\text{sgn}\tau, \quad G^{\text{free}}(\tau) = \frac{1}{N}\sum_i G_{ii}^{\text{free}} = \frac{1}{2}\text{sgn}\tau,$$

where sgn is the sign function.

In Fourier space, the two-point function is expressed as:

$$G_{ij}^{\text{free}}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} G_{ij}^{\text{free}}(\tau) = -\frac{\delta_{ij}}{i\omega}.$$

The perturbation of this theory is done by examining the four-vertex interaction and its Feynman rule.

Diagram: 

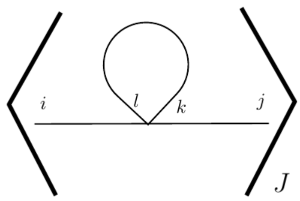
Contribution : J_{ijkl}.

First, using the vertex rules and propagators, we determine the amplitude of a diagram and then average over the couplings J. For instance, since the random couplings are selected from a Gaussian ensemble, the two-point averages J are

$$\langle J_{i_1j_1k_1l_1} J_{i_2j_2k_2l_2} \rangle_J = 3!\frac{J^2}{N^3}\delta_{i_1i_2}\delta_{j_1j_2}\delta_{k_1k_2}\delta_{l_1l_2}.$$

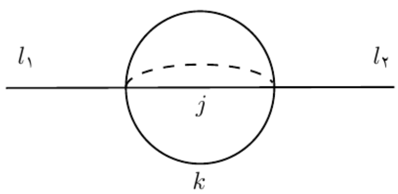
Next, we analyze the non-trivial contributions to the propagator.

1. **Tadpole contribution**



   This contribution is zero because it equals $\langle J_{ijkl} \rangle_J$, which is zero since the Gaussian ensemble of couplings is chosen with zero mean.

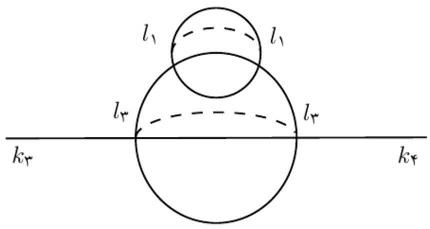
2. **Melon contribution**



Here, the dashed lines represent averages like those in the above equation, summed over repeated indices. The contribution of this diagram is

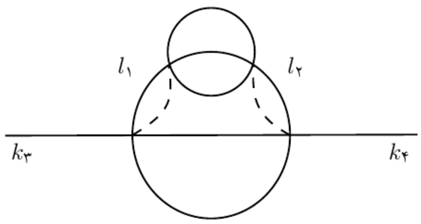
$$\mathcal{A}_{\text{melon}} = 3!\frac{J^2}{N^3} G^{\text{free}}_{ii} G^{\text{free}}_{jj} G^{\text{free}}_{kk} \delta_{l_1 l_2} = 3! J^2 (G^{\text{free}})^3 \delta_{l_1 l_2}.$$

3. **Contributions of order** J^4 Several diagrams contribute here. Two examples are shown below:



Amplitude:

$$\mathcal{A}^{(1)}_{(4)} = \frac{J^4}{N^6} \left(G^{\text{free}}_{ll} \right)^5 G^{\text{free}}_{l_1 l_3} G^{\text{free}}_{l_1 l_3} \delta_{k_3 k_4} = \frac{1}{2} J^4 \left(G^{\text{free}} \right)^6 \delta_{k_3 k_4}.$$



Amplitude:

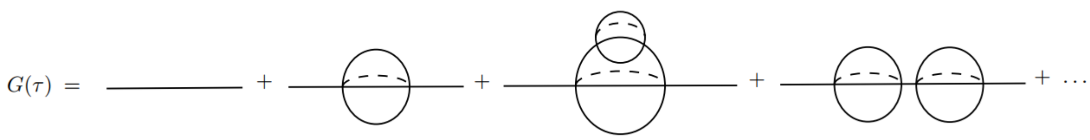
$$\mathcal{A}^{(2)}_{(4)} = \frac{J^4}{N^6} \left(G^{\text{free}}_{i_1 i_2} G^{\text{free}}_{j_1 j_2} G^{\text{free}}_{k_1 k_2} \right) \left(G^{\text{free}}_{l_1 l_1} G^{\text{free}}_{l_2 l_2} \right) \left(G^{\text{free}}_{i_1 i_2} G^{\text{free}}_{j_1 j_2} \right).$$

Simplified as

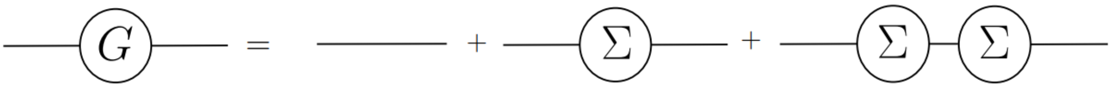
$$= \frac{1}{4N^2} J^4 \left(G^{\text{free}} \right)^4 G^{\text{free}}_{k_3 k_4}.$$

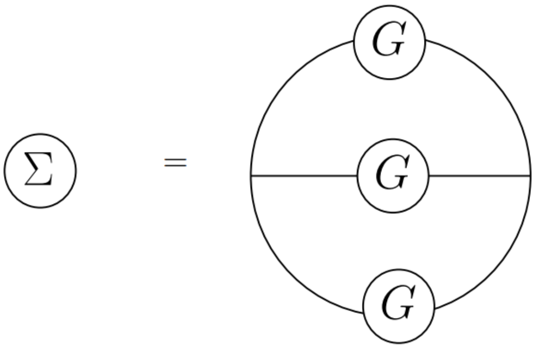
The first amplitude scales with N, while the second scales with N^{-2}. Therefore, the second diagram is negligible in the limit $N \to \infty$.

From the above calculations, we conclude that the diagrams not suppressed by powers of $1/N$ are those with only one dashed-line connection within each melon-like shape. Thus, the dominant two-point function in the $N \to \infty$ limit is:



This can be summarized as:





We can simplify these pictorial equations by introducing a matrix product representation:

$$(AB)(\tau,\tau') = \int d\tau'' A(\tau,\tau'')B(\tau'',\tau').$$

The first equation from the diagram above can be written as follows (this type of summation is referred to as the Schwinger-Dyson sum):

$$\begin{aligned} G &= G^{\text{free}} + G^{\text{free}}\Sigma G^{\text{free}} + G^{\text{free}}\Sigma G^{\text{free}}\Sigma G^{\text{free}} + \cdots \\ &= G^{\text{free}}\left[1 + \Sigma G^{\text{free}} + \Sigma G^{\text{free}}\Sigma G^{\text{free}} + \cdots\right] \\ &= G^{\text{free}}\left[1 - \Sigma G^{\text{free}}\right]^{-1} \\ &= \left[(G^{\text{free}})^{-1} - \Sigma\right]^{-1} \end{aligned} \tag{A.1}$$

Since the propagator is the inverse of the kinetic term in the Lagrangian

$$(G^{\text{free}})^{-1}(\tau,\tau') = \delta(\tau-\tau')\partial_{\tau'} \tag{A.2}$$

The following shorthand is also valid

$$G = \left[\partial_\tau - \Sigma\right]^{-1} \tag{A.3}$$

The second diagrammatic equation reduces to the following expression:

$$\Sigma(\tau,\tau') = J^2\left[G(\tau,\tau')\right]^3 \tag{A.4}$$

A.3 Low-energy Limit and Emergent Conformal Symmetry

In the low-energy limit, we examine the action of the model. Compared to the coupling of the model, which has an energy dimension, the low-energy limit implies frequencies smaller than the coupling. By rewriting (A.2) in Fourier space:

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega) \tag{A.5}$$

Neglecting $i\omega$ in the above limit, the equations (in the infrared limit) simplify to:

$$\begin{aligned} \int d\tau'' G(\tau,\tau'')\Sigma(\tau'',\tau') &= -\delta(\tau-\tau') \\ \Sigma(\tau,\tau') &= J^2 G(\tau,\tau')^{q-1} \end{aligned} \tag{A.6}$$

A key observation is that these equations exhibit additional symmetries. Under the reparameterization $\tau \to \phi(\tau)$, G and Σ transform as

$$\begin{aligned} G(\tau,\tau') &\mapsto \left[\phi'(\tau)\phi'(\tau')\right]^{\Delta} G(\phi(\tau),\phi(\tau')), \\ \Sigma(\tau,\tau') &\mapsto \left[\phi'(\tau)\phi'(\tau')\right]^{\Delta(q-1)} \Sigma(\phi(\tau),\phi(\tau')) \end{aligned} \tag{A.7}$$

This means that G and Σ transform like two-point functions in conformal field theory. For example, the first equation remains invariant under conformal transformations

$$\begin{aligned} &\int d\tau'' \left[\phi'(\tau)\phi'(\tau'')\right]^{\frac{1}{q}} G(\phi(\tau),\phi(\tau'')) \left[\phi'(\tau'')\phi'(\tau')\right]^{1-\frac{1}{q}} \Sigma(\phi(\tau''),\phi(\tau')) \\ &\qquad = \int d\tilde{\phi}\, G(\phi(\tau),\tilde{\phi})\Sigma(\tilde{\phi},\phi(\tau'))\phi'(\tau') \left[\frac{\phi'(\tau)}{\phi'(\tau')}\right]^{\frac{1}{q}} \\ &\qquad = -\phi'(\tau')\delta(\phi(\tau)-\phi(\tau')) \\ &\qquad = -\delta(\tau-\tau') \end{aligned} \tag{A.8}$$

A.4 Transition to the Schwarzian Action in the Limit $N \to \infty$

The study of the Schwarzian limit as $N \to \infty$ is somewhat technical and requires background field methods and path integrals. We refer to the relevant literature [25, 26].

B Introduction to Wess-Zumino-Witten and Chern-Simons theoreis

In this brief appendix, we describe the Chern-Simons and Wess-Zumino-Witten theories and highlight their properties necessary for our purposes in Section 3.1.

B.1 Chern-Simons Theory

Chern-Simons action for a compact gauge group G is given by

$$S_{\text{CS}}[A] = \frac{k}{4\pi}\int_{\mathcal{M}} \text{Tr}\Big(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\Big), \tag{B.1}$$

where k is called the level constant. A represents a Lie algebra valued 1-form with values in $\mathfrak{g}$, i.e., $A = A_\mu dx^\mu$. The term Tr denotes a non-degenerate bilinear form on the Lie algebra $\mathfrak{g}$[7].

[7] Non-degeneracy ensures that all gauge fields appearing in the action have a kinetic term. All semisimple Lie algebras possess such non-degenerate bilinear forms.

Writing $A = A^a T_a$, where T_a are the generators of the Lie algebra $\mathfrak{g}$, the first term of the action (B.1) can be expressed as:

$$\mathrm{Tr}[A \wedge dA] = \mathrm{Tr}(T_a T_b)[A^a \wedge dA^b]$$

This implies that $d_{ab} \equiv \mathrm{Tr}(T_a T_b)$ acts as the metric on the Lie algebra $\mathfrak{g}$, which must be non-degenerate.

By varying the above action and performing integration by parts, we obtain

$$\delta S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}\Big(2\delta A \wedge (dA + A \wedge A)\Big) - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \mathrm{Tr}\Big(A \wedge \delta A\Big) \tag{B.2}$$

If δA is chosen such that it vanishes on the boundary $\partial\mathcal{M}$,[8] the equation of motion is derived as

$$F \equiv dA + A \wedge A = 0 \tag{B.3}$$

where F is a Lie algebra valued 2-form valued in $\mathfrak{g}$, representing the curvature of the gauge fields A.

Locally, this equation implies

$$A = G^{-1} dG \tag{B.4}$$

indicating that the field A is a gauge transformation of the trivial configuration $A = 0$; in other words, A is a pure gauge field.[9]

This observation reveals that the Chern-Simons theory has no propagating degrees of freedom and is, in fact, a topological field theory. This is why it can be equivalent to a gravitational theory with no propagating degrees of freedom.

B.2 Wess-Zumino-Witten Model

We begin with the nonlinear sigma model and, by adding the Wess-Zumino term, resolve its conformal anomaly to arrive at the Wess-Zumino-Witten model.

B.2.1 Nonlinear Sigma Model

The Wess-Zumino-Witten model is a specific case of the nonlinear sigma model, which we will briefly introduce sigma models.

The nonlinear sigma model consists of scalar fields ϕ^i $(i = 1, \ldots, n)$, which define a map from a flat spacetime (Minkowski) to a target manifold. The target manifold is a real n-dimensional Riemannian manifold $\mathcal{M}_n$ equipped with a metric $g_{ij}(\phi)$. The scalar fields are coordinates on the Riemannian manifold, and since the metric depends on the fields ϕ, the model is nonlinear.

[8] Alternatively, we can use an improved action to ensure a well-defined variational principle even if the boundary term does not vanish.

[9] Globally, however, A may not be purely gauge due to holonomies.

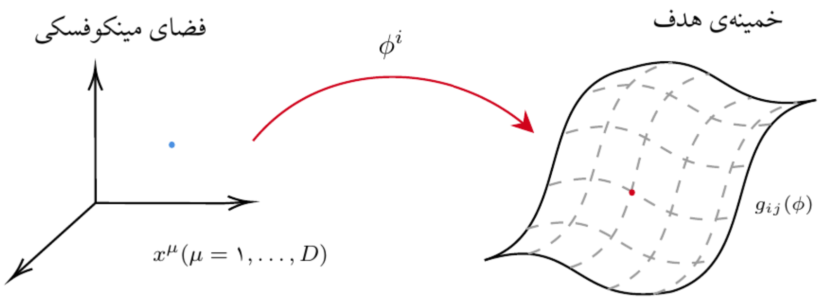


Figure 3. Fields in a nonlinear sigma model are maps from flat space to a target manifold.

The action of this model is defined as

$$S_\sigma[\phi] = \frac{1}{4a^2}\int d^D x \;\; g_{ij}(\phi)\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j \tag{B.5}$$

where $a^2 > 0$ is a dimensionless coupling constant.

The Wess-Zumino-Witten model (abbreviated as WZW) is a specific nonlinear sigma model. In this model, the target manifold is a semisimple Lie group G, and the fields, which are maps from flat space to the target manifold, are matrices denoted by $g(x)$.

For a two-dimensional flat space Σ with coordinates (τ, φ), the action of the nonlinear sigma model is written as:

$$S_\sigma[g] = \frac{1}{4a^2}\int_\Sigma d^2x \;\; \mathrm{Tr}\big[\eta^{\mu\nu}\partial_\mu g\partial_\nu(g^{-1})\big] \tag{B.6}$$

Note that we must construct a scalar from the matrices to ensure the action is a scalar quantity. Therefore, we use the Killing form on the semisimple Lie group, which requires the group to be semisimple (even if not compact).

Now, we comprehensively examine this action, its equation of motion, and symmetries.

First, we derive its equation of motion. To simplify the action, we note that:

$$\partial_\nu(g^{-1}) = -g^{-1}\partial_\nu g g^{-1} \tag{B.7}$$

which directly follows from the relation $\partial_\nu(gg^{-1}) = 0$. By substituting this term into the action, we obtain:

$$S_\sigma[g] = \frac{1}{4a^2}\int_\Sigma d^2x \;\; Tr\big[g^{-1}\partial_\mu g \; g^{-1}\partial^\mu g\big] \tag{B.8}$$

This form of the action exhibits symmetry $g \to g_L \, g \, g_R^{-1}$.

By varying the action, the equations of motion are obtained:

$$\begin{aligned}\delta S_\sigma &= \frac{1}{2a^2}\int_\Sigma d^2x \;\; Tr\big[\big(-g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\delta(\partial_\mu g)\big)g^{-1}\partial^\mu g\big] \\ &= \frac{1}{2a^2}\int_\Sigma d^2x \;\; Tr\big[\partial_\mu(g^{-1}\delta g)g^{-1}\partial^\mu g\big] \\ &= -\frac{1}{2a^2}\int_\Sigma d^2x \;\; Tr\big[g^{-1}\delta g\partial_\mu(g^{-1}\partial^\mu g)\big]\end{aligned} \tag{B.9}$$

Thus, the equation of motion is $\partial^\nu(g^{-1}\partial_\nu g) = 0$.

Using the coordinate transformation $x^\pm \equiv \tau \pm \varphi$, the equations of motion can be written as

$$\partial_+ J_+ + \partial_- J_- = 0 \tag{B.10}$$

where $J_\pm = g^{-1}\partial_\pm g$.

We observe that the left- and right-moving currents are not conserved individually. However, the symmetry of left and right multiplication should yield two conserved currents. Therefore, the action needs to be modified [10].

B.3 Adding the Wess-Zumino Term

To address the discussed issues, we add the following term [27],[28].

$$S = S_\sigma[g] + k\Gamma[G] \tag{B.11}$$

where k is an integer, and the Wess-Zumino action $\Gamma[G]$ is

$$\begin{aligned}\Gamma[G] &= \frac{1}{3}\int_V Tr\Big[\Big(G^{-1}dG\Big)^3\Big] \\ &= \frac{1}{3}\int_V d^3x\, \epsilon^{\mu\nu\rho} Tr\Big[G^{-1}\partial_\mu G G^{-1}\partial_\nu G G^{-1}\partial_\rho G\Big]\end{aligned} \tag{B.12}$$

where V is a three-dimensional manifold with a boundary Σ, and G is an extension of the fields to V satisfying $G\Big|_{\partial V} = g$ [11].

To derive the equations of motion, we compute the variation of the action under $G \to G + \delta G$, and using Stokes' theorem, we find

$$\delta\Gamma[G] = \int_\Sigma d^2x Tr\Big[\epsilon^{\mu\nu}\delta g g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\Big] \tag{B.13}$$

Combining this with the equation of motion derived for $S_\sigma[g]$, we obtain

$$\frac{1}{2a^2}\eta^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) - k\epsilon^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) = 0 \tag{B.14}$$

Rewriting this in light-cone coordinates ($\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_\phi)$), we get

$$(1 - 2a^2k)\partial_+(g^{-1}\partial_+ g) + (1 + 2a^2k)\partial_-(g^{-1}\partial_- g) = 0 \tag{B.15}$$

Now, by choosing the parameter k, we can ensure the conservation of currents:

- If $a^2 = -\frac{a}{2k^2}$, then $k < 0$, and the conservation of the right-moving current follows, $\partial_+ J_+ = 0$.

[10] Another reason for modifying the theory is that this model exhibits conformal symmetry classically, but upon quantization, this symmetry is broken. To restore conformal symmetry at the quantum level, new terms must be added to the action, with the Wess-Zumino term serving this purpose.

[11] Note that the extension of a manifold with its boundary is not unique, and neither is the extension of the fields g such that $G\Big|_{\partial V} = g$. Thus, the definition of Γ as given above is ambiguous. However, for the compact group $SL(2,\mathbb{R})$ and on a flat two-dimensional space, there is no issue of ambiguity. For more details, refer to [29].

- For $a^2 = \frac{a}{2k^2}$, k is necessarily positive, and we have the conservation of the left-moving current, $\partial_- J_- = 0$.

By choosing $a^2 = -\frac{a}{2k^2}$, we arrive at the WZW action, which is sometimes referred to as WZNW action

$$S_{WZW}[g] = \frac{k}{2} \int d^2x Tr \left[\eta^{\mu\nu} g^{-1}\partial_\mu g g^{-1}\partial_\nu g\right] + k\Gamma[G]. \tag{B.16}$$

The solutions to the equations of motion, $\partial_+(g^{-1}\partial_- g) = 0$, are of the form $g = \theta_+(x^+)\theta_-(x^-)$, where $\theta_+(x^+)$ and $\theta_-(x^-)$ are arbitrary functions.

Moreover, this model has two conserved currents: $J_- \equiv g^{-1}\partial_- g$ and $\bar{J}_+ \equiv -\partial_+ g g^{-1}$. It is straightforward to see that this action is invariant under the transformation $g \to \Theta_+(x^+) g \Theta_-^{-1}(x^-)$, and the conserved currents of this transformation are the two currents mentioned above. Therefore, adding the Wess-Zumino the term essentially gauges the global symmetry.

C First Order Formulation of Einstein Gravity

Instead of directly using the metric $g_{\mu\nu}$, this formulation introduces an auxiliary field [12] e^a_μ, which serves as the "square root" of the metric

$$g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x). \tag{C.1}$$

This auxiliary field is also called the *dreibein* in three dimensions[13].

The relation (C.1) can be interpreted as transforming a tensor under general coordinate transformations. The metric tensor can be brought to the Minkowski metric through transformations constructed from the vielbein field. This relation also provides valuable information about the invertibility of the frame fields.

Taking the determinant of both sides of (C.1), we find

$$e \equiv \det e^a_\mu = \sqrt{-\det g_{\mu\nu}} \neq 0. \tag{C.2}$$

Thus, the frame fields are invertible, and their inverse, e^μ_a, can be defined as

$$\begin{aligned} e^\mu_a e^a_\nu &= \delta^\mu_\nu \\ e^a_\mu e^\mu_b &= \delta^a_b \end{aligned} \tag{C.3}$$

Moreover, the frame fields are not unique; any Lorentz transformation $\Lambda \in \mathrm{SO}(2,1)$ (local or global) applied to the frame fields generates a new vielbein that still satisfies (C.1):

$$(e')^a_\mu = \Lambda^a_b(x)e^b_\mu(x) \tag{C.4}$$

[12] Sometimes called the frame fields, or vielbein in four-dimensional spacetime.

[13] Note that e^a_μ has two types of indices: Latin indices are the internal or vielbein indices, which transform under local Lorentz transformations, while Greek indices are the spacetime indices, which transform via general coordinate transformations.

Note that in the above transformations, the Lorentz transformation acts only on the vielbein indices, while general coordinate transformations act on the spacetime indices. Thus, the frame fields can be transformed in two distinct ways.

We can also use the frame fields to define a basis for the space of differential forms. The differential form e^a is defined as $e^a \equiv e^a_\mu dx^\mu$, and the Levi-Civita tensor with frame indices is defined

$$\begin{aligned}\epsilon_{\mu\nu\rho} &= e^{-1}\epsilon_{abc}e^a_\mu e^b_\nu e^c_\rho \\ \epsilon^{\mu\nu\rho} &= e\epsilon^{abc}e^\mu_a e^\nu_b e^\rho_c.\end{aligned} \tag{C.5}$$

The covariant derivative is similarly constructed from an ordinary derivative and a connection, symbolically written as $D = \partial + \Gamma$.

The connection consists of a set of 1-forms called the spin connection, denoted by $\omega^{ab} = \omega^{ab}_\mu dx^\mu$, which is antisymmetric in its internal indices, $\omega^{ab} = -\omega^{ba}$.

The relationship between the spin connection and the Levi-Civita connection is established by considering the covariant derivative of a vector field $X = X^\mu\partial_\mu$, first in the coordinate basis and then in the frame basis. The resulting relationship is

$$\omega_{\mu\ b}^{\ a} = e_\nu^{\ a} e^\lambda_{\ b}\Gamma^\nu_{\mu\lambda} - e^\lambda_{\ b}\partial_\mu e_\lambda^{\ a} \tag{C.6}$$

Analogous to the Levi-Civita connection in general relativity, which is not a tensor and has nontrivial transformation properties, the spin connection also transforms nontrivially under local Lorentz transformations. However, it allows us to construct quantities that behave well under these transformations. Under a local Lorentz transformation, the spin connection transforms as

$$\omega^a_b \to (\Lambda^{-1})^a_{\ c}\, d\Lambda^c_{\ b} + (\Lambda^{-1})^a_{\ c}\, \omega^c_{\ d}\, \Lambda^d_{\ b}, \tag{C.7}$$

and the torsion 2-form is

$$T^a \equiv de^a + \omega^a_{\ b} \wedge e^b, \tag{C.8}$$

which transforms as a vector under Lorentz transformations: $T^a \to \Lambda^a_{\ b} T^b$.

We can verify that the defined quantity corresponds to the notion of torsion in general relativity, i.e., the antisymmetric part of the Levi-Civita connection.

$$\begin{aligned}T_{\mu\nu}^{\ \lambda} &= e^\lambda_{\ a} T_{\mu\nu}^{\ a} \\ &= e^\lambda_{\ a}\big(\partial_\mu e_\nu^{\ a} - \partial_\nu e_\mu^{\ a} + \omega_{\mu\ b}^{\ a} e_\nu^{\ b} - \omega_{\nu\ b}^{\ a} e_\mu^{\ b}\big) \\ &= \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu},\end{aligned} \tag{C.9}$$

where in the second line, we used (C.8), and in the last line, we substituted (C.6) to express the spin connection in terms of the Levi-Civita connection.

After establishing these geometric preliminaries, we aim to examine general relativity in the first-order formulation. Specifically, we aim to rewrite the Einstein-Hilbert action in terms of e and ω. To this end, note that

$$d^3\sqrt{-g} = edx^0 dx^1 dx^2 = \frac{1}{3!}e\epsilon_{\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho \tag{C.10}$$

By substituting (C.5) in terms of vielbein fields and using the definition $e^a \equiv e^a_\mu dx^\mu$, we find

$$d^3\sqrt{-g} = \epsilon_{abc} e^a \wedge e^b \wedge e^c \tag{C.11}$$

The term proportional to curvature in the Einstein-Hilbert action is converted using the dual representation (possible only in three dimensions). The curvature tensor in vielbein indices is given by:

$$\begin{aligned} R^{ab} &= \frac{1}{2} R^{ab}_{\mu\nu}(x) dx^\mu \wedge dx^\nu \\ R^{\lambda\sigma}_{\mu\nu} &= e^\lambda_a e^\sigma_b R^{ab}_{\mu\nu} \end{aligned} \tag{C.12}$$

The dual representation is defined as

$$\begin{aligned} R_a &\equiv \frac{1}{2}\epsilon_{abc} R^{bc} \leftrightarrow R^{ab} \equiv -\epsilon^{abc} R_c \\ \omega_a &\equiv \frac{1}{2}\epsilon_{abc}\omega^{bc} \leftrightarrow \omega^{ab} \equiv -\epsilon^{abc}\omega_c. \end{aligned} \tag{C.13}$$

We can observe that

$$\begin{aligned} \epsilon_{abc} e^a \wedge R^{bc} &= \frac{1}{2} e \epsilon_{\mu\alpha\beta} R^{\alpha\beta}_{\nu\rho} \epsilon^{\mu\nu\rho} d^3x \\ &= d^3x \sqrt{-g} R \end{aligned} \tag{C.14}$$

Adding the two terms above, the Einstein-Hilbert action in the first-order formalism takes the following form

$$S_{\mathrm{EH}}[e,\omega] = \frac{1}{16\pi G}\int_{\mathcal{M}} \left(2e^a \wedge R_a[\omega] - \frac{\Lambda}{3}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right). \tag{C.15}$$

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