Advanced Control Systems (I)	Faculty of Mechanical Engineering Dynamics & Control

Instructor:	Final Project 1	Student:
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I. Developing the Dynamic Model

In this section we write the dynamic modelling equation to find the tank and jacket temperature. We make the following assumptions:

- The volumes and liquids are constant with constant density and heat capacity
- Perfect mixing is assumed in both the tank and jacket
- The tank inlet flowrate, jacket flowrate, tank inlet temperature and jacket inlet temperature may change.(these are the inputs)
- The rate of heat transfer from the jacket to the tank is governed by the equation $Q=UA(T_j-T)$, where U is overall heat transfer coefficients and A is the area for heat transfer.

the notation that we used is present in TABLE I.

TABLE I: System Parameters & Variables.

Variables	**Subscripts**
A	area of heat transfer
c_p	heat capacity
F	volumetric flowrate
ρ	density
T	temperature
t	time
Q	rate of heat transfer
U	heat transfer coefficient
V	volume
i	inlet
j	jacket
ji	jacket inlet
s	steady-state

A. Material Balance Around Tank

The first step is to write a material balance around the tank assuming constant density:

$$\frac{d\rho V}{dt} = \rho F_i - \rho F \tag{1}$$

Also, assuming constant volume ($dV/dt=0$) we find:

$$F = F_i$$

B. Energy Balance Around Tank

The next step is to write an energy balance around the tank.

accumulation = in by flow – out by flow + in by heat transfer + work done on system

$$\frac{dTE}{dt} = \rho F\bar{T}\bar{E}_i - \rho F\bar{T}\bar{E} + Q + W_T$$

The next step is to neglect the kinetic and potential energy and write the total work done on the system as a combination of the shaft work and energy added to the system to get the fluid into the the tank and the energy that the system performs on the surrounding to force fluid out. This allows us to write the energy balance as:

$$\frac{dU}{dt} = \rho F\left(\bar{U} + \frac{p_i}{\rho_i}\right) - \rho F\left(\bar{U} + \frac{p}{\rho}\right) + Q + W_s$$

and since $H = U+pV$ we can rewrite the energy balance as:

$$H = U + pV$$

$$\frac{dH}{dt} - \frac{dpV}{dt} = \rho F\bar{H}_i - \rho F\bar{H} + Q + W_s$$

Note that

$$\frac{dpV}{dt} = p\frac{dV}{dt} + V\frac{dp}{dt} = 0$$

and the volume is constant. Also, so the mean pressure change can be neglected since the density is constant:

$$\frac{dH}{dt} = \rho F\bar{H}_i - \rho F\bar{H} + Q + W_s$$

Neglecting the work done by the mixing impeller and assuming single-phase and a constant heat capacity, we find:

$$\rho V c_p \frac{dT}{dt} = \rho F c_p \left(T_i - T\right) + Q$$

or

$$\frac{dT}{dt} = \frac{F}{V}\left(T_i - T\right) - \frac{Q}{\rho V c_p} \qquad (1)$$

we must also perform a material and energy balance around jacket and use the connecting relationship for heat transfer between the jacket and the tank.

C. Material Balance Around Jacket

The material balance around the jacket is (assuming constant density):

$$\frac{d\rho_j V_j}{dt} = \rho_j F_{ji} - \rho_j F_j$$

Assuming constant volume (*dV/dt=0*) we find:

$$F_j = F_{ji}$$

D. Energy Balance Around Jacket

Next, we write and energy balance around the jacket. Making the same assumption as around the tank:

$$\frac{dT_j}{dt} = \frac{F_j}{V_j}\left(T_{ji} - T_j\right) - \frac{Q}{\rho_j V_j c_{pj}} \qquad (2)$$

We also have the relationship for heat transfer from the jacket to the tank:

$$Q = AU\left(T_j - T\right) \qquad (3)$$

substituting (3) into (1) and (2) yields the two modelling equations for this system:

$$\begin{cases} \dfrac{dT}{dt} = \dfrac{F}{V}\left(T_i - T\right) + \dfrac{UA\left(T_j - T\right)}{\rho V c_p} \\ \dfrac{dT_j}{dt} = \dfrac{F_j}{V_j}\left(T_{ji} - T_j\right) - \dfrac{UA\left(T_j - T\right)}{\rho_j V_j c_{pj}} \end{cases} \tag{4\&5}$$

II. STEADY-STATE CONDITIONS

Before linearizing the nonlinear model to find the state-space form, we must find state variable values at at steady state. The steady-state is obtained by solving the dynamic equations for the steady-state values of the system variables and some parameters for this process are given below:

A. Parameters and Steady-State Values

$F_s = 1\ \dfrac{\text{ft}^3}{\text{min}}$	$\rho c_p = 61.3\ \dfrac{\text{Btu}}{{}^\circ\text{F ft}}$	$\rho_j c_{pj} = 61.3\ \dfrac{\text{Btu}}{{}^\circ\text{F ft}}$
$T_{is} = 50\,{}^\circ\text{F}$	$T_s = 125\,{}^\circ\text{F}$	$V = 10\,\text{ft}^3$
$T_{jis} = 200\,{}^\circ\text{F}$	$T_{js} = 150\,{}^\circ\text{F}$	$V_j = 1\,\text{ft}^3$

Noticed that the values of UA and F_{js} have not been specified. These values can be obtained by solving the two dynamic equations and at steady state:

$$\begin{cases} \dfrac{F_s}{V}\left(T_{is} - T_s\right) + \dfrac{UA\left(T_{js} - T_s\right)}{\rho V c_p} = 0 \\ \dfrac{F_{js}}{V_j}\left(T_{jis} - T_{js}\right) - \dfrac{UA\left(T_{js} - T_s\right)}{\rho_j V_j c_{pj}} = 0 \end{cases} \qquad \text{steady-state}$$

By substituting the known values in the above equations, we find:

$$\begin{cases} \dfrac{1}{10}(50-125) + \dfrac{UA(150-125)}{(61.3)(10)} = 0 \\ \dfrac{F_{js}}{1}(200-150) - \dfrac{UA(150-125)}{(61.3)(1)} = 0 \end{cases} \rightarrow \begin{cases} UA = 183.9\ \dfrac{\text{Btu}}{{}^\circ\text{F min}} \\ F_{js} = 1.5\ \dfrac{\text{ft}^3}{\text{min}} \end{cases}$$

III. STATE-SPACE MODELING

Here we linearize the nonlinear modeling equations to find state-space form:

$$\begin{cases} \mathbf{x} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} + D\mathbf{u} \end{cases}$$

where the state, input and output vectors are in division form:

$$x_1 = x = T - T_s \qquad x_2 = x_j = T_j - T_{js}$$

$$u_1 = u_{F_j} = F_j - F_{js} \qquad u_2 = u_F = F - F_s \qquad u_3 = u_{T_i} = T_i - T_{is} \qquad u_4 = u_{T_{ji}} = T_{ji} - T_{jis}$$

Substituting state variables in dynamic equations of system:

$$\begin{cases} \dfrac{d\left(x_1+T_s\right)}{dt}=\dfrac{\left(u_2+F_s\right)}{V}\left(\left(u_3+T_{is}\right)-\left(x_1+T_s\right)\right)+\dfrac{UA\left(\left(x_2+T_{js}\right)-\left(x_1+T_s\right)\right)}{\rho V c_p} \\ \dfrac{d\left(x_2+T_{js}\right)}{dt}=\dfrac{\left(u_1+F_{js}\right)}{V_j}\left(\left(u_4+T_{jis}\right)-\left(x_2+T_{js}\right)\right)-\dfrac{UA\left(\left(x_2+T_{js}\right)-\left(x_1+T_s\right)\right)}{\rho_j V_j c_{pj}} \end{cases}$$

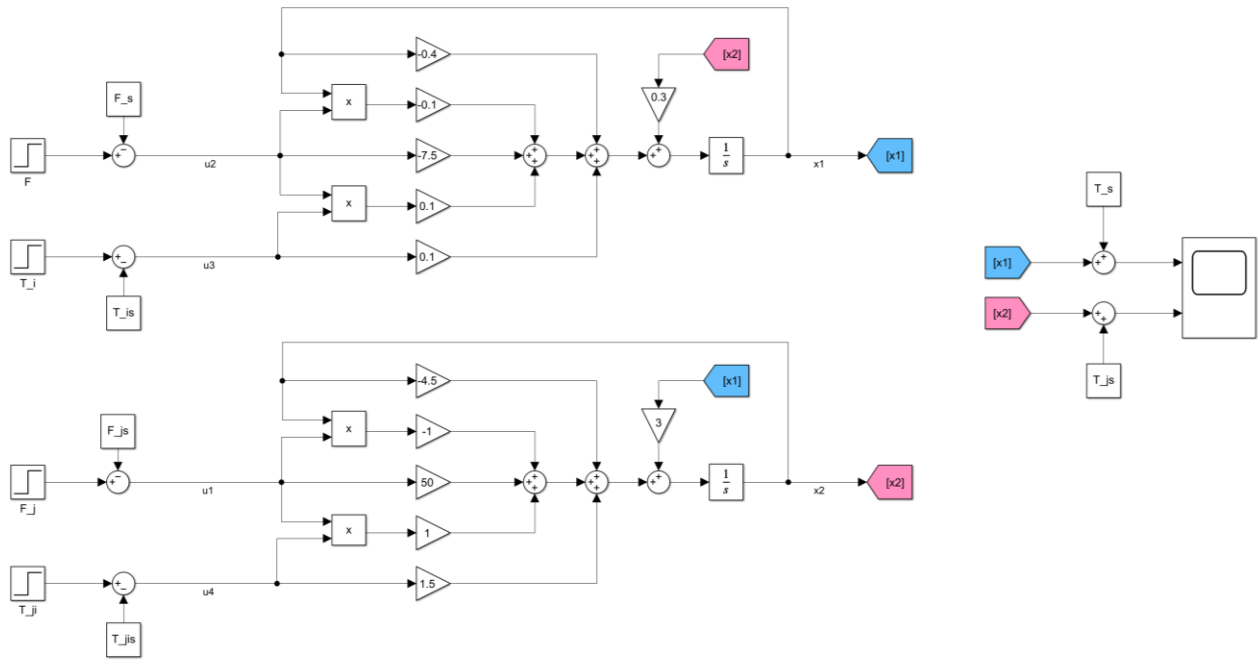
$$\begin{cases} \dfrac{dx_1}{dt}=\dfrac{\left(u_2+1\right)}{10}\left(\left(u_3+50\right)-\left(x_1+125\right)\right)+\dfrac{(183.9)\left(\left(x_2+150\right)-\left(x_1+125\right)\right)}{613} \\ \dfrac{dx_2}{dt}=\dfrac{\left(u_1+1.5\right)}{1}\left(\left(u_4+200\right)-\left(x_2+150\right)\right)-\dfrac{(183.9)\left(\left(x_2+150\right)-\left(x_1+125\right)\right)}{61.3} \end{cases}$$

Finally, the differential equation of the nonlinear system is obtained as follows:

$$NLDEs:\begin{cases} \dot{x}_1=-0.4x_1+0.3x_2-0.1x_1u_2+0.1u_2u_3+0.1u_3-7.5u_2 \\ \dot{x}_2=3x_1-4.5x_2-x_2u_1+u_1u_4+1.5u_4+50u_1 \end{cases}$$

A. Nonlinear Model Simulation

In this section, we implement and simulate the dynamic equations of the nonlinear system using MATLAB's Simulink software.



We obtain the simulation results of the nonlinear system for step and impulse inputs for small and large changes in the input and then discuss each plot in detail.

Small change		
input	working point	change
F = 1.2	F_s = 1	0.2
F_j = 1.7	F_{js} = 1.5	0.2
T_i = 55	T_{is} = 50	5
T_{ji} = 210	T_{jis} = 200	10

Large change		
input	working point	change
F = 1.6	F_s = 1	0.6
F_j = 1.9	F_{js} = 1.5	0.4
T_i = 70	T_{is} = 50	20
T_{ji} = 240	T_{jis} = 200	40

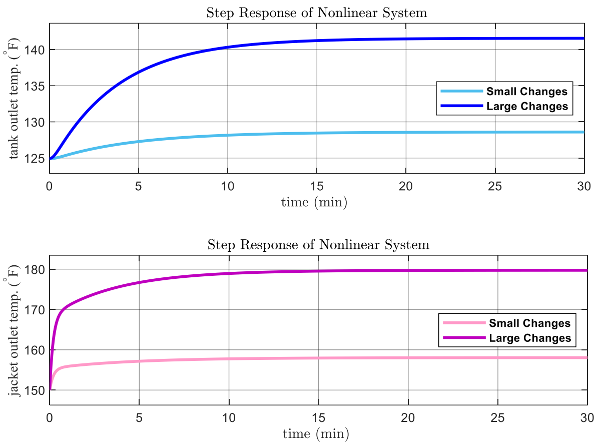


Fig 1&2: Response of nonlinear system to step inputs

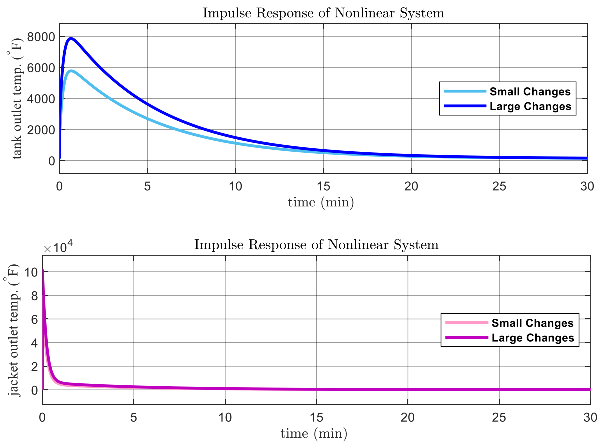


Fig 3&4: Response of nonlinear system to impulse inputs

Figure 1: This diagram is the response of the nonlinear system to the step input. In this diagram, which shows the changes in the tank outlet temperature for small changes in the inputs, the temperature changes are done slowly(compared to the other state variable), so that we see a temperature change of 15°F within 15 minutes for large changes and change of 3°F within 15 minutes for small changes .

Figure 2: This diagram shows the response of the nonlinear system to the step input caused by small changes in the inputs. According to the diagram, it can be concluded that temperature changes occur in a short period of time. Considering that this diagram is related to jacket fluid temperature changes, we expected that the temperature changes of the fluid inside the jacket would be faster so that the heat transfer to the fluid inside the tank would be more favorable.

Figure 3: This diagram shows the system's impulse response. according to the result, there is an extreme changes in the tank outlet temperature, which is an unrealistic result, this response does not happen in reality in this way and the initial temperature changes is not as intense as you can see in the figure.

Figure 4: As we said about the impulse response of the nonlinear system related to the tank temperature, the shock response to the jacket temperature also has an unrealistic result, but compared to the shock response related to the tank temperature, it shows that this system returns to its original state faster. And this is due to the goal of the system designer to quickly transfer heat from the jacket fluid to the fluid inside the tank.

B. Linearization Of Nonlinear Model

In order to linearize the nonlinear model of the system, We expand the system differential equations around the working point using the two-variable Taylor series:

$$f\left(\mathbf{x},\mathbf{u}\right):\begin{cases}\dot{x}_1=-0.4x_1+0.3x_2-0.1x_1u_2+0.1u_2u_3+0.1u_3-7.5u_2\\ \dot{x}_2=3x_1-4.5x_2-x_2u_1+u_1u_4+1.5u_4+50u_1\end{cases}$$

$$f(\mathbf{x},\mathbf{u})=f(\mathbf{x}_0,\mathbf{u}_0)+\frac{1}{1!}\left[\frac{\partial f(\mathbf{x}_0,\mathbf{u}_0)}{\partial \mathbf{x}}(\mathbf{x}-\mathbf{x}_0)+\frac{\partial f(\mathbf{x}_0,\mathbf{u}_0)}{\partial \mathbf{u}}(\mathbf{u}-\mathbf{u}_0)\right]+\cdots$$

Considering that the working point of the system is the steady-state point, so we will have for state variables and inputs:

$$\mathbf{x}_0=\mathbf{x}_s=0 \qquad\qquad \mathbf{u}_0=\mathbf{u}_s=0$$

Finally, by placing the known values as well as the working point conditions in the Taylor expansion, we obtain the dynamic equations of the linear system as follows:

$$f(\mathbf{x},\mathbf{u})=\begin{bmatrix}0\\0\end{bmatrix}+\frac{1}{1!}\left[\begin{bmatrix}-0.4-0.1u_2 & 0.3\\ 3 & -4.5-u_1\end{bmatrix}_{(\mathbf{0},\mathbf{0})}\mathbf{x}+\begin{bmatrix}0 & -7.5 & 0.1u_2+0.1 & 0\\ -x_2+u_4+50 & 0 & 0 & u_1+1.5\end{bmatrix}_{(\mathbf{0},\mathbf{0})}\mathbf{u}\right]+\cdots$$

$$f(\mathbf{x},\mathbf{u})=\begin{bmatrix}-0.4 & 0.3\\ 3 & -4.5\end{bmatrix}\mathbf{x}+\begin{bmatrix}0 & -7.5 & 0.1 & 0\\ 50 & 0 & 0 & 1.5\end{bmatrix}\mathbf{u}$$

$$A=\begin{bmatrix}-0.4 & 0.3\\ 3 & -4.5\end{bmatrix}\qquad B=\begin{bmatrix}0 & -7.5 & 0.1 & 0\\ 50 & 0 & 0 & 1.5\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\qquad D=\mathbf{0}$$

$$LDEs:\begin{cases}\dot{x}_1=-0.4x_1+0.3x_2-7.5u_2+0.1u_3\\ \dot{x}_2=3x_1-4.5x_2+50u_1+1.5u_4\end{cases}$$

C. Tarnsfer Function

To obtain the transfer function of the system, which is a multi-input and multi-output system, we proceed as follows:

$$G(s)=C(sI-A)^{-1}B+D$$

$$G(s)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}-0.4 & 0.3\\ 3 & -4.5\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & -7.5 & 0.1 & 0\\ 50 & 0 & 0 & 1.5\end{bmatrix}+\mathbf{0}$$

$$G(s)=\frac{1}{s^2+4.9s+0.9}\begin{bmatrix}15 & -7.5s-33.75 & 0.1s+0.45 & 0.45\\ 50s+20 & -22.5 & 0.3 & 1.5s+0.6\end{bmatrix}$$

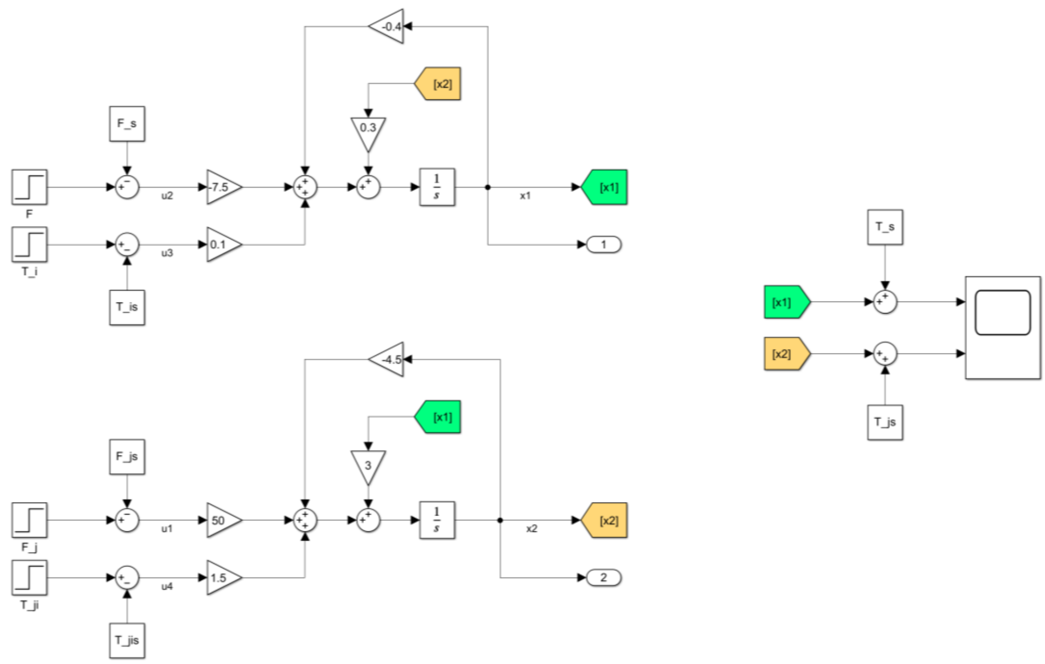
The system has two outputs and four inputs, so we will have a total of eight transfer functions as follows:

$$G_{11}=\frac{15}{s^2+4.9s+0.9}\quad G_{12}=\frac{-7.5s-33.75}{s^2+4.9s+0.9}\quad G_{13}=\frac{0.1s+0.45}{s^2+4.9s+0.9}\quad G_{14}=\frac{0.45}{s^2+4.9s+0.9}$$

$$G_{21}=\frac{50s+20}{s^2+4.9s+0.9}\quad G_{22}=\frac{-22.5}{s^2+4.9s+0.9}\quad G_{23}=\frac{0.3}{s^2+4.9s+0.9}\quad G_{24}=\frac{1.5s+0.6}{s^2+4.9s+0.9}$$

D. Linear Model Vs. Nonlinear Model Simulation

In this section, we implement and simulate the dynamic equations of the linear system vs. nonlinear system.



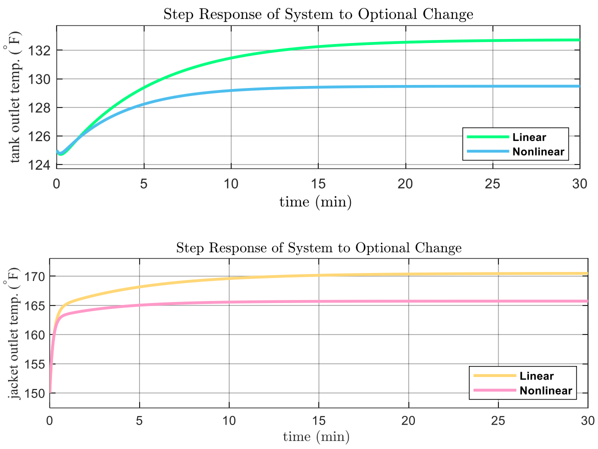


Fig 1.5&1.6: Response of linear vs nonlinear system to step inputs

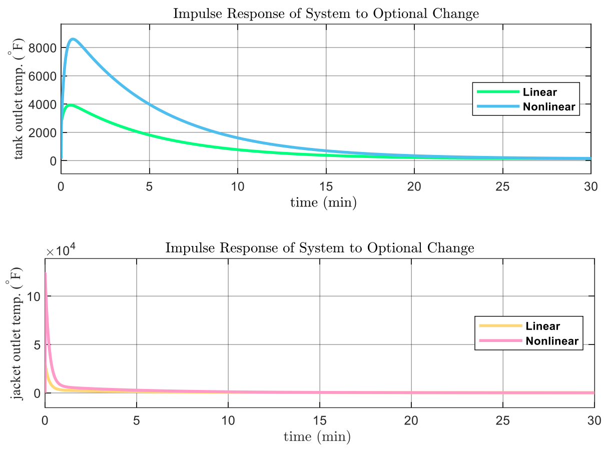


Fig 1.7&1.8 : Response of linear vs nonlinear system to step inputs

Figure 1.5: This diagram is the response of the nonlinear & linear system to the step input. In this diagram, which shows the changes in the tank outlet temperature for optional changes in the inputs, the temperature changes are done slowly(compared to the other state variable), so that we see a temperature change of 7°F within 15 minutes for linear system and change of 4°F within 15 minutes for nonlinear system.

Figure 1.6: This diagram shows the response of the nonlinear & linear system to the step input caused by optional changes in the inputs. According to the diagram, it can be concluded that temperature changes occur in a short period of time. Considering that this diagram is related to jacket fluid temperature changes, we expected that the temperature changes of the fluid inside the jacket would be faster so that the heat transfer to the fluid inside the tank would be more favorable.

Figure 1.7: This diagram shows the system's impulse response. according to the result, there is an extreme changes in the tank outlet temperature, which is an unrealistic result, this response does not happen in reality in this way and the initial temperature changes is not as intense as you can see in the figure.

Figure 1.8: As we said about the shock response of the nonlinear system related to the tank temperature, the shock response to the jacket temperature also has an unrealistic result, but compared to the shock response related to the tank temperature, it shows that this system returns to its original state faster. And this is due to the goal of the system designer to quickly transfer heat from the jacket fluid to the fluid inside the tank

D. Controllability & Observablity

$$P = \begin{bmatrix} B & AB \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -7.5 & 0.1 & 0 & 15 & 0.3 & -0.04 & 0.45 \\ 50 & 0 & 0 & 1.5 & -225 & -22.5 & 0.3 & -6.75 \end{bmatrix}$$

```
P = [0 -7.5 0.1 0 15 0.3 -0.04 0.45;
     50 0 0 1.5 -225 -22.5 0.3 -6.75];
rank(P)    %n=2
ans = 2
```

$$rank(P) = n = 2 \qquad \rightarrow \qquad \text{Controllable}$$

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -0.4 & 0.3 \\ 3 & -4.5 \end{bmatrix}$$

```
Q = [1   0;
     0   1;
    -0.4 0.3;
     3  -4.5];
rank(Q)    %n=2
ans = 2
```

$$rank(Q) = n = 2 \qquad \rightarrow \qquad \text{Observable}$$