Locality Sensitive Hashing

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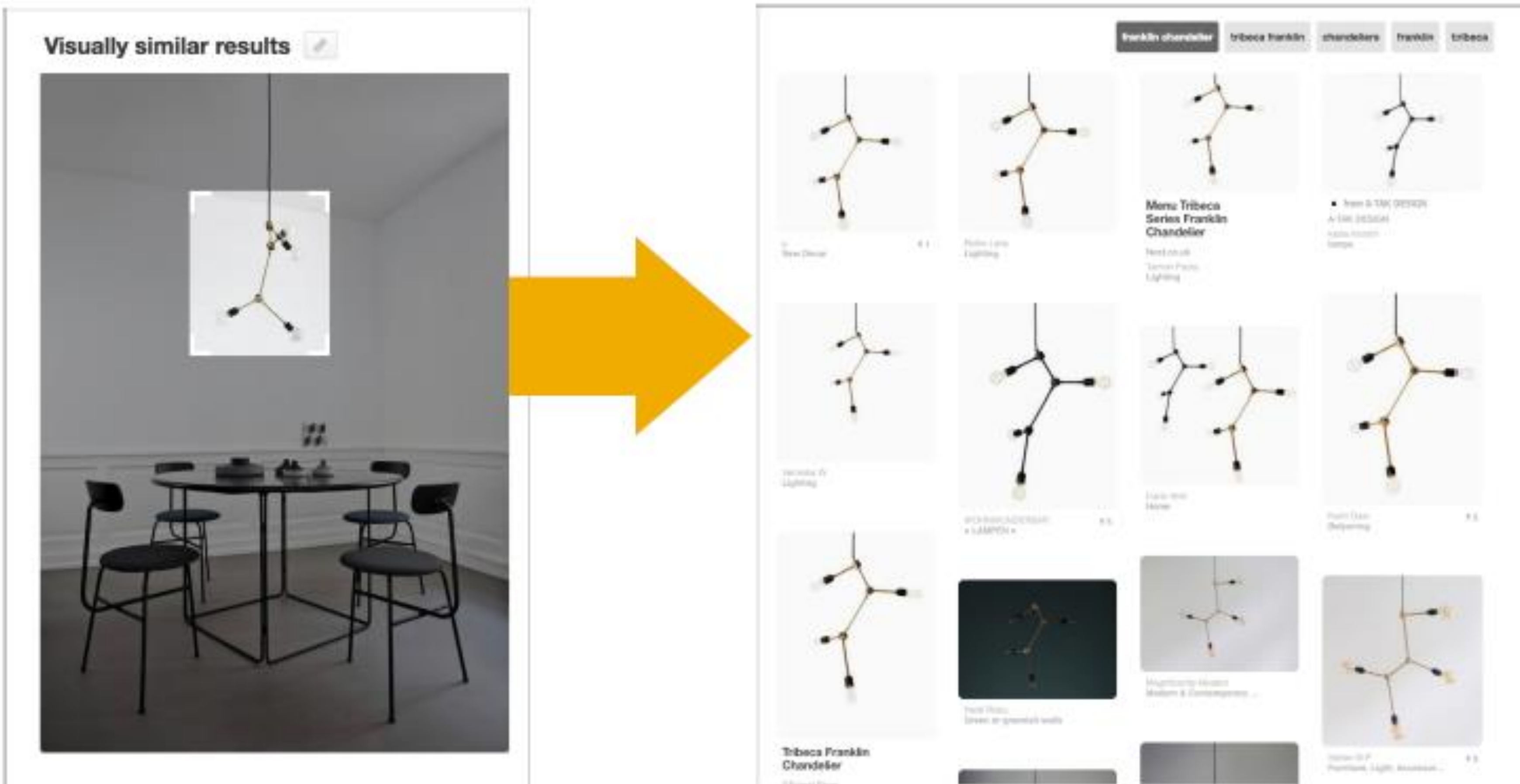
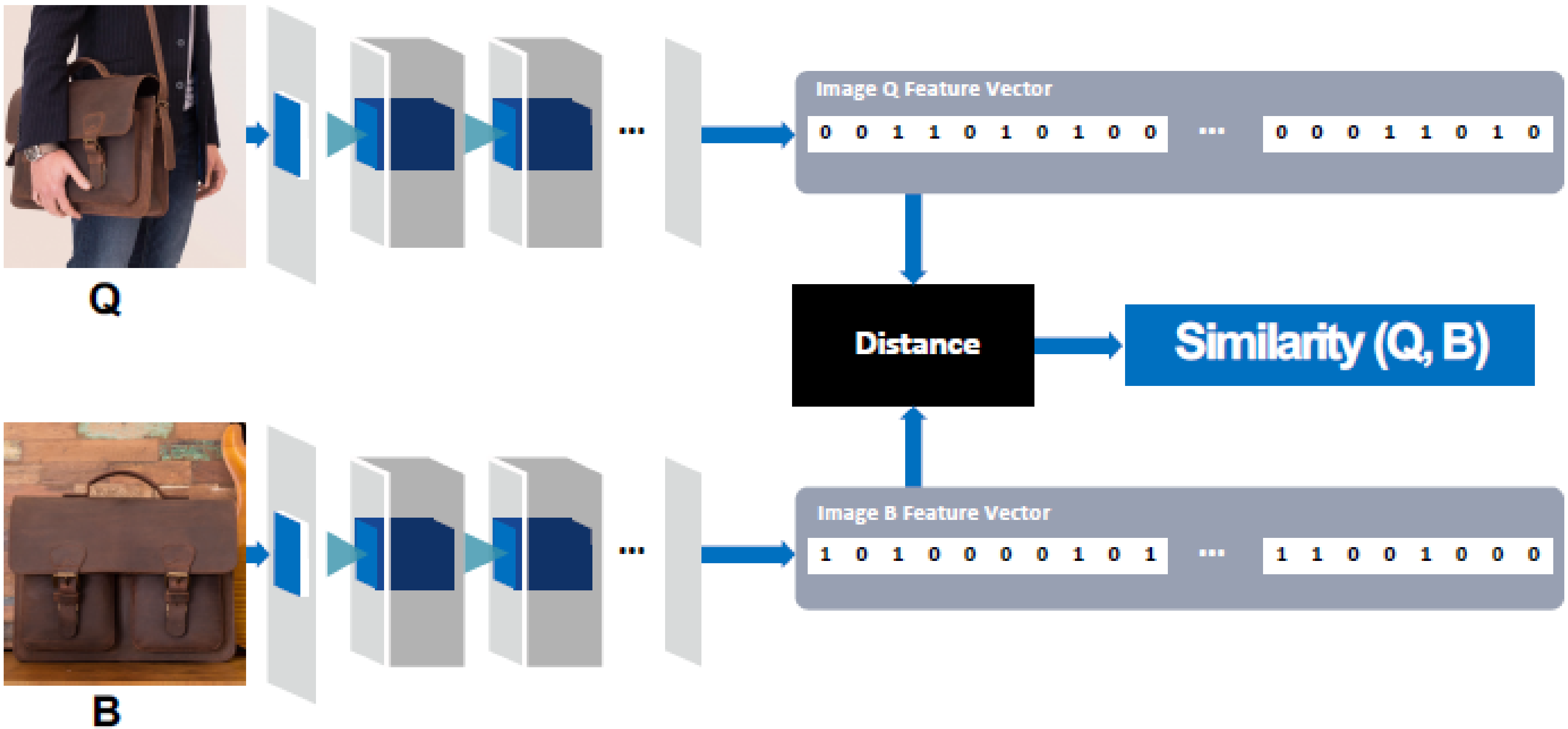
Pinterest visual search

Given a **query image patch**, find similar images

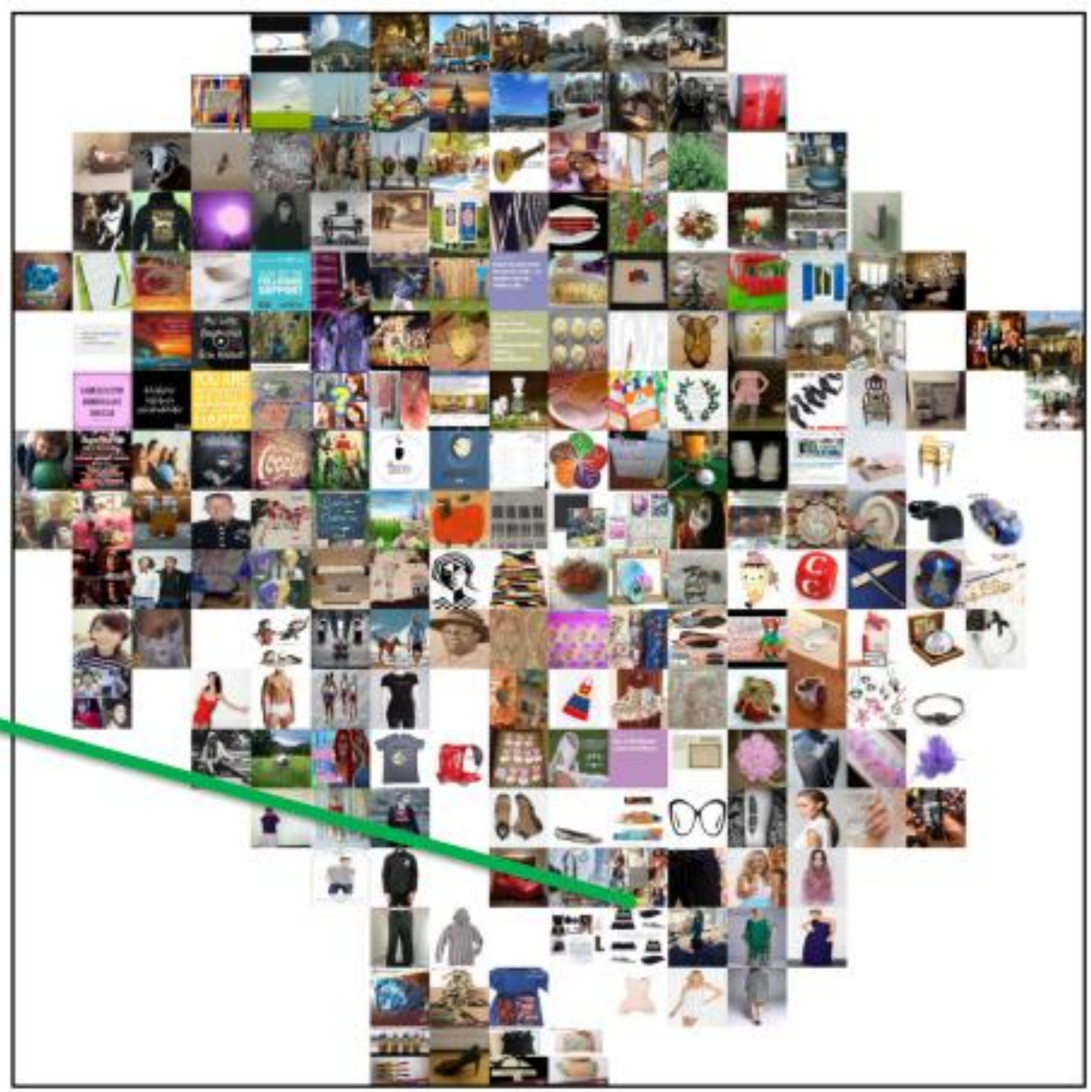


How does it work?



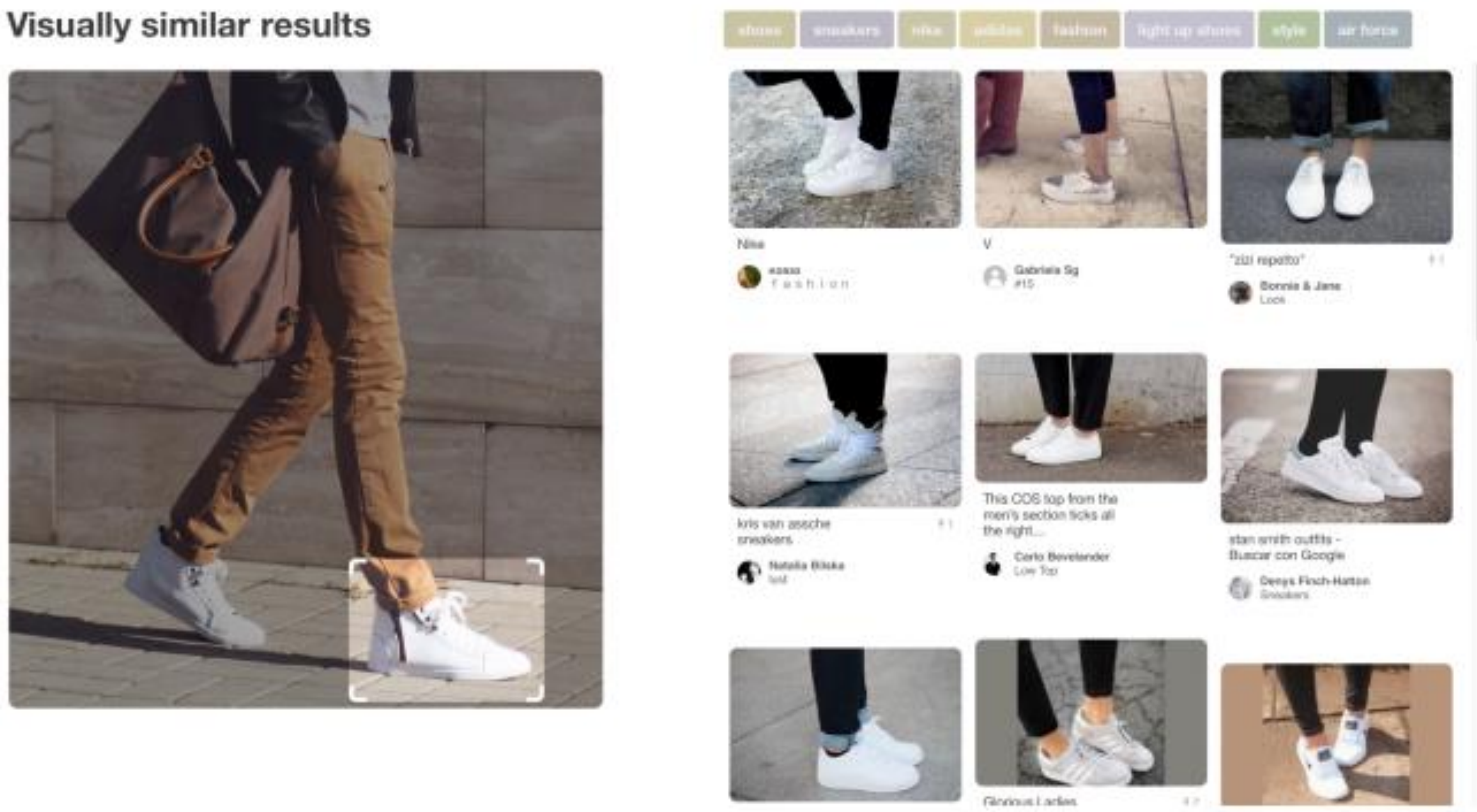
- Collect billions of images
- Determine feature vector for each image (4k dim)
- **Given a query Q, find nearest neighbors FAST**

How does it work?



Nearest neighbor query
in the embedding space

Application : visual search



A common metaphor

- Many problems can be expressed as finding "similar" sets:
  - Find near-neighbors in high-dimensional space
- Examples:
  - Pages with similar words
    - For duplicate detection, classification by topic
  - Customers who purchased similar products
    - Products with similar customer sets
  - Images with similar features
    - Image completion
  - Recommendations and search

Problem

- Given: High dimensional data points $x_1, x_2, \ldots$
  - **For example:** Image is a long vector of pixel colors
- And some distance function $d(x_1, x_2)$
  - which quantifies the "distance" between x_1 and x_2
- **Goal:** Find all pairs of data points (x_i, x_j) that are within distance threshold $d(x_i, x_j) \le s$
- **Note:** Naïve solution would take $O(N^2)$ where N is the number of data points
- MAGIC: This can be done in $O(N)$!! How??

LSH

- LSH is really a family of related techniques
- In general, one throws items into buckets using several different "hash functions"
- You examine only those pairs of items that share a bucket for at least one of these hashings
- **Upside:** Designed correctly, only a small fraction of pairs are ever examined
- **Downside:** There are *false negatives* – pairs of similar items that never even get considered

Finding similar DOCs

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Motivation for Min-Hash/LSH

- **Suppose we need to find near-duplicate documents among $N = 1$ million documents**
  - Naïvely, we would have to compute pairwise similarities for **every pair of docs**
    - $N(N-1)/2 \approx 5*10^{11}$ comparisons
    - At 10^5 secs/day and 10^6 comparisons/sec, it would take **5 days**
  - For $N = 10$ million, it takes more than a year…
- Similarly, we have a dataset of 10m images, quickly find the most similar to query image **Q**

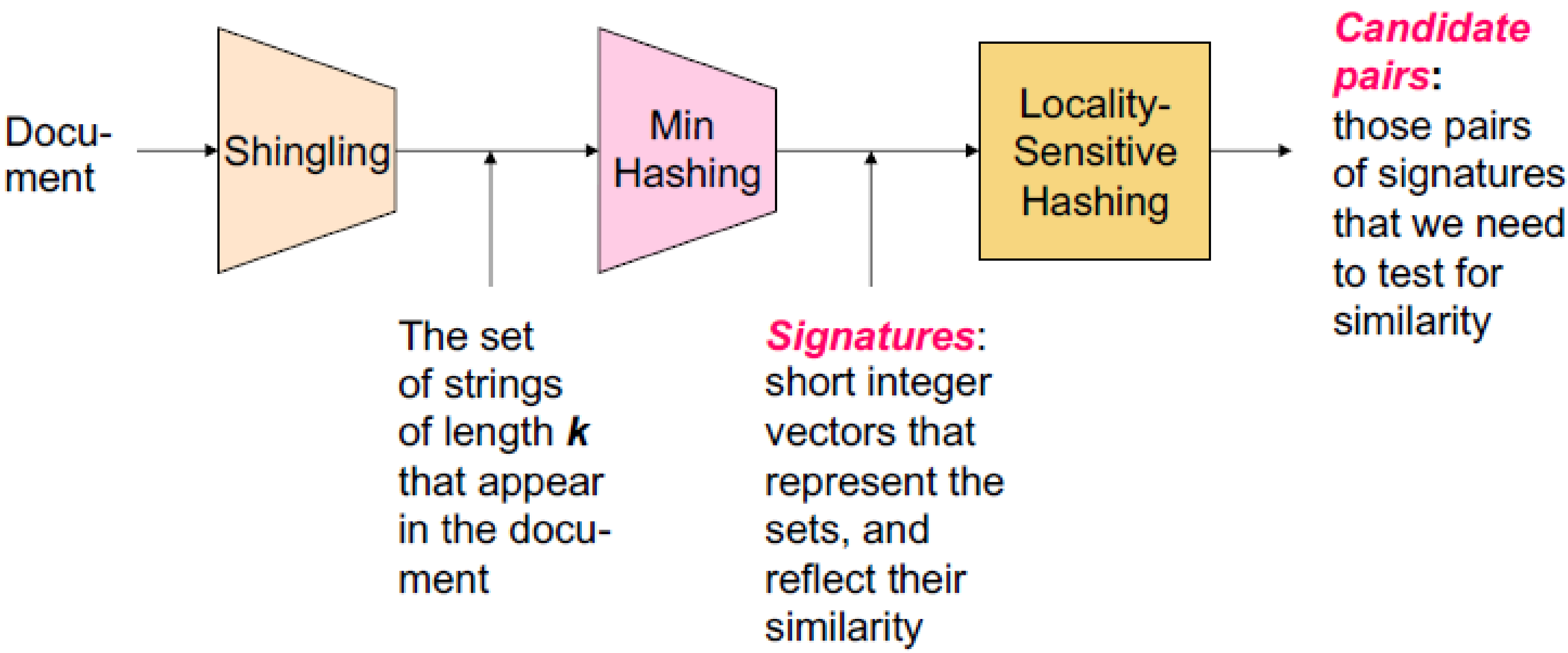
3 Essential Steps for Similar Docs

1. ***Shingling:*** Converts a document into a set representation (Boolean vector)

2. ***Min-Hashing:*** Convert large sets to short signatures, while preserving similarity

3. ***Locality-Sensitive Hashing:*** Focus on pairs of signatures likely to be from similar documents
   - **Candidate pairs!**

The Big Picture



Documents as High Dim data

Step 1: *Shingling:* Converts a document into a set

- A ***k*-shingle** (or ***k*-gram**) for a document is a sequence of *k* tokens that appears in the doc
  - Tokens can be characters, words or something else, depending on the application
  - Assume tokens = characters for examples
- To **compress long shingles**, we can **hash** them to (say) 4 bytes
- **Represent a document by the set of hash values of its *k*-shingles**

Compressing Shingles

- **Example: k=2**; document D_1= abcab
  Set of 2-shingles: $S(D_1) = \{ab, bc, ca\}$
  Hash the shingles: $h(D_1) = \{1, 5, 7\}$
- k = 8, 9, or 10 is often used in practice

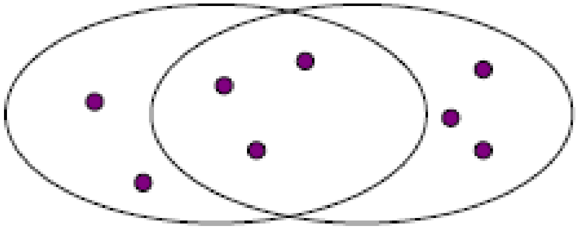
- **Benefits of shingles:**
  - Documents that are intuitively similar will have many shingles in common
  - Changing a word only affects k-shingles within distance k-1 from the word

Similarity Metric for Shingles

- Document D_1 is represented by a set of its k-shingles $C_1=S(D_1)$
- A natural similarity measure is the **Jaccard similarity:**

$$sim(D_1, D_2) = |C_1 \cap C_2|/|C_1 \cup C_2|$$

Jaccard distance: $d(C_1, C_2) = 1 - |C_1 \cap C_2|/|C_1 \cup C_2|$



3 in intersection.
8 in union.
Jaccard similarity
= 3/8

From Sets to Boolean Matices

Encode sets using 0/1 (bit, Boolean) vectors

- **Rows** = elements (shingles)
- **Columns** = sets (documents)
  - 1 in row ***e*** and column ***s*** if and only if ***e*** is a member of ***s***
  - Column similarity is the Jaccard similarity of the corresponding sets (rows with value *1)*
  - **Typical matrix is sparse!**
- **Each document is a column:**
  - **Example: sim(C_1 ,C_2) = ?**
    - Size of intersection = 3; size of union = 6, Jaccard similarity (not distance) = 3/6
    - **d(C_1,C_2) = 1 – (Jaccard similarity) = 3/6**

Documents

Shingles

1	1	1	0
1	1	0	1
0	1	0	1
0	0	0	1
1	0	0	1
1	1	1	0
1	0	1	0

We don't really construct the matrix; just imagine it exists

Outline review

- **So far:**
  - Documents $\rightarrow$ Sets of shingles
  - Represent sets as Boolean vectors in a matrix
- **Next goal: Find similar columns while computing small signatures**
  - **Similarity of columns == similarity of signatures**
- **Warnings:**
  - Comparing all pairs takes too much time: **Job for LSH**
    - These methods can produce false negatives, and even false positives (if the optional check is not made)

Min-Hashing : hashing columns

- **Key idea:** "hash" each column C to a small ***signature*** $h(C)$, such that:
  - $sim(C_1, C_2)$ is the same as the "similarity" of signatures $h(C_1)$ and $h(C_2)$

- **Goal: Find a hash function $h(\cdot)$ such that:**
  - If $sim(C_1,C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$
  - If $sim(C_1,C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$

- **Idea: Hash docs into buckets. Expect that "most" pairs of near duplicate docs hash into the same bucket!**

Min-Hashing Goal

- **Goal: Find a hash function *h(·)* such that:**
  - if $sim(C_1,C_2)$ is high, then with high prob. $h(C_1) = h(C_2)$
  - if $sim(C_1,C_2)$ is low, then with high prob. $h(C_1) \neq h(C_2)$
- **Clearly, the hash function depends on the similarity metric:**
  - Not all similarity metrics have a suitable hash function
- **There is a suitable hash function for the Jaccard similarity:** It is called **Min-Hashing**

Min-Hashing overview

- Permute the rows of the Boolean matrix using some permutation π
  - Thought experiment – not real
- Define minhash function for this permutation π, **h_π(C) =** the number of the first (in the permuted order) row in which column *C* has value 1.
  - Denoted this as: $h_\pi(C) = \min_\pi \pi(C)$
- Apply, to all columns, several randomly chosen permutations π to create a signature for each column
- **Result is a signature matrix:** Columns = sets, Rows = minhash values for each permutation π

Min-Hashing Example

2nd element of the permutation (row 1) is the first to map to a 1

$h_\pi(C) = \min_\pi \pi(C)$

Permutation π

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

$h_2(3)=1$ (permutation 2, column 3)

4th element of the permutation (row 1) is the first to map to a 1

Similarity for Signatures

- We know: $\Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$
- Now generalize to multiple hash functions
- **The *similarity of two signatures* is the fraction of the hash functions in which they agree**
- Thus, the expected similarity of two signatures equals the Jaccard similarity of the columns or sets that the signatures represent
  - And the longer the signatures, the smaller will be the expected error

Min-Hashing Example

Permutation π

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

Similarities:

	1-3	2-4	1-2	3-4
Col/Col	0.75	0.75	0	0
Sig/Sig	0.67	1.00	0	0

Locality Sensitive Hashing

- **Goal:** Find documents with Jaccard similarity at least ***s*** (for some similarity threshold, e.g., ***s***=0.8)

- **LSH – General idea:** Use a hash function that tells whether ***x*** and ***y*** is a ***candidate pair:*** a pair of elements whose similarity must be evaluated

- **For Min-Hash matrices:**
  - Hash columns of signature matrix ***M*** to many buckets
  - Each pair of documents that hashes into the same bucket is a **candidate pair**

2	1	4	1
1	2	1	2
2	1	2	1

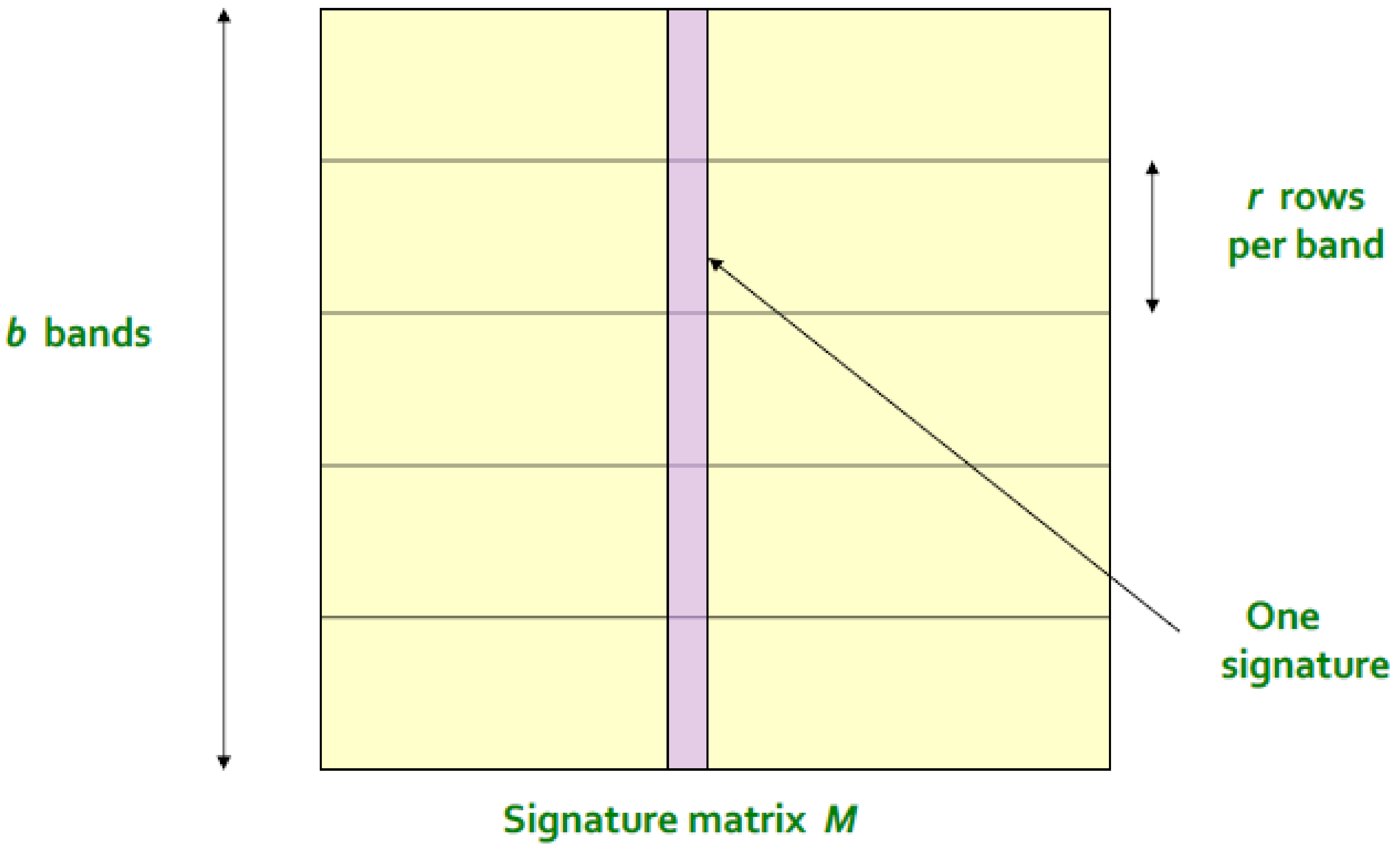
Locality Sensitive Hashing

- **Pick a similarity threshold s (0 < s < 1)**
- Columns $\boldsymbol{x}$ and $\boldsymbol{y}$ of $\boldsymbol{M}$ are a **candidate pair** if their signatures agree on at least fraction $\boldsymbol{s}$ of their rows:
  $\boldsymbol{M(i, x) = M(i, y)}$ for at least frac. $\boldsymbol{s}$ values of $\boldsymbol{i}$
  - We expect documents $\boldsymbol{x}$ and $\boldsymbol{y}$ to have the same (Jaccard) similarity as their signatures

LSH for Min-Hash

- **Big idea:** Hash columns of signature matrix *M* several times
- Arrange that (only) **similar columns** are likely to **hash to the same bucket**, with high probability
- Candidate pairs are those that hash to the same bucket

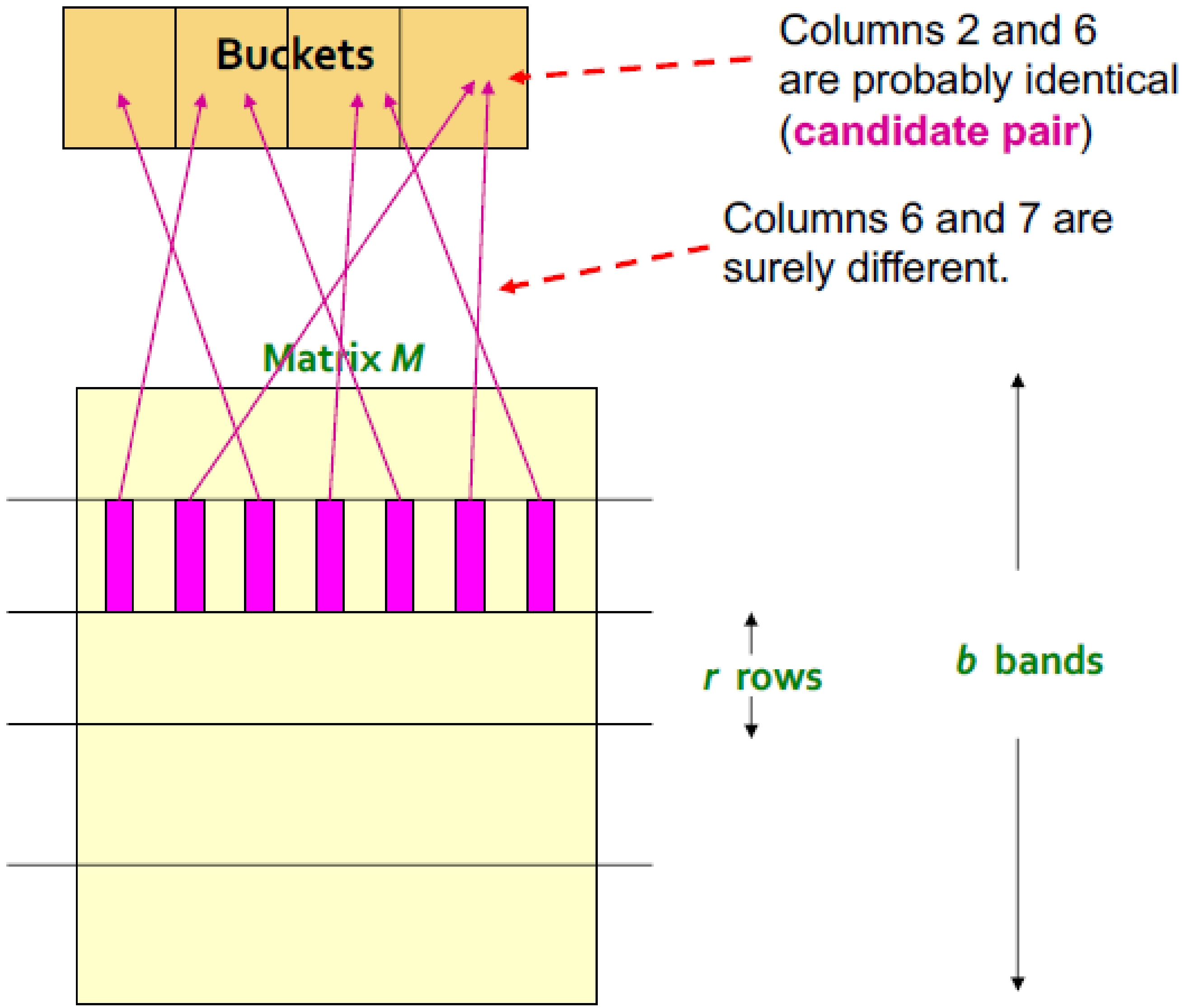
Partition M into b Bands



Partition M into b Bands

- Divide matrix M into b bands of r rows
- For each band, hash its portion of each column to a hash table with k buckets
  - Make k as large as possible
- *Candidate* column pairs are those that hash to the same bucket for $\geq$ **1** band
- Tune b and r to catch most similar pairs, but few non-similar pairs

Hashing bands



Simplifying Assumption

- There are **enough buckets** that columns are unlikely to hash to the same bucket unless they are **identical** in a particular band
- Hereafter, we assume that "**same bucket**" means "**identical in that band**"
- Assumption needed only to simplify analysis, not for correctness of algorithm

Example of bands

Assume the following case:

- Suppose 100,000 columns of ***M*** (100k docs)
- Signatures of 100 integers (rows)
- Therefore, signatures take 40MB
- **Goal:** Find pairs of documents that are at least $\boldsymbol{s} = 0.8$ similar
- Choose $\boldsymbol{b} = 20$ bands of $\boldsymbol{r} = 5$ integers/band

C1,C2 are 80% Similar

- **Find pairs of** $\geq$ ***s***=0.8 similarity, set **b**=20, **r**=5
- **Assume:** $\mathrm{sim}(C_1, C_2) = 0.8$
  - Since $\mathrm{sim}(C_1, C_2) \geq \mathbf{s}$, we want C_1, C_2 to be a **candidate pair**: We want them to hash to at **least 1 common bucket** (at least one band is identical)
- **Probability C_1, C_2 identical in one particular band:** $(0.8)^5 = 0.328$
- Probability C_1, C_2 are ***not*** similar in all of the 20 bands: $(1\text{-}0.328)^{20} = 0.00035$
  - i.e., about 1/3000th of the 80%-similar column pairs are **false negatives** (we miss them)
  - **We would find 99.965% pairs of truly similar documents**

C1,C2 are 30% Similar

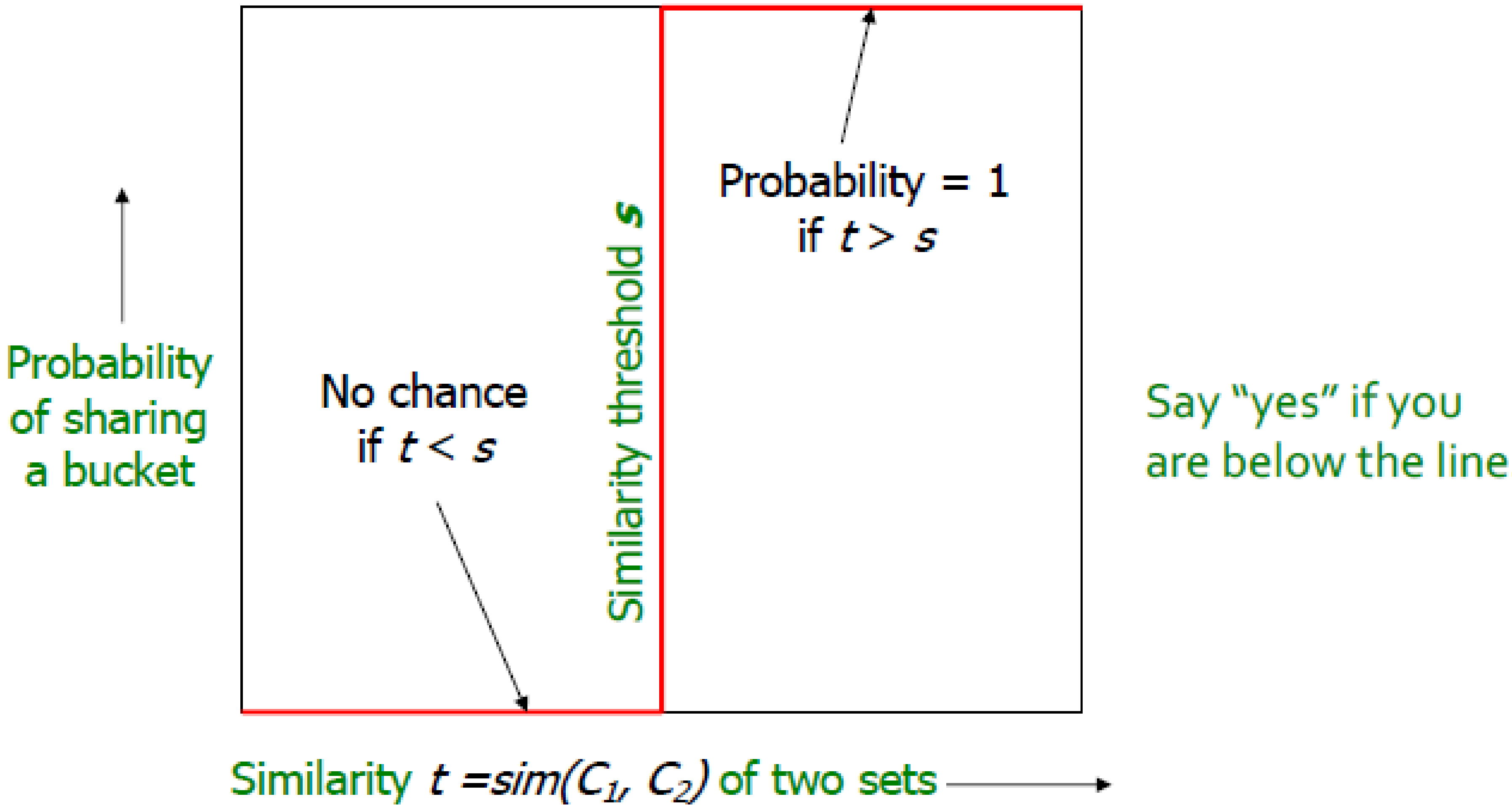
- **Find pairs of ≥ *s*=0.8 similarity, set b=20, r=5**
- **Assume:** sim(C_1, C_2) = 0.3
  - Since sim(C_1, C_2) < **s** we want C_1, C_2 to hash to **NO common buckets** (all bands should be different)
- **Probability C_1, C_2 identical in one particular band:** $(0.3)^5 = 0.00243$
- Probability C_1, C_2 identical in at least 1 of 20 bands: $1 - (1 - 0.00243)^{20} = 0.0474$
  - In other words, approximately 4.74% pairs of docs with similarity 0.3 end up becoming **candidate pairs**
    - They are **false positives** since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold s

LSH Involves a Tradeoff

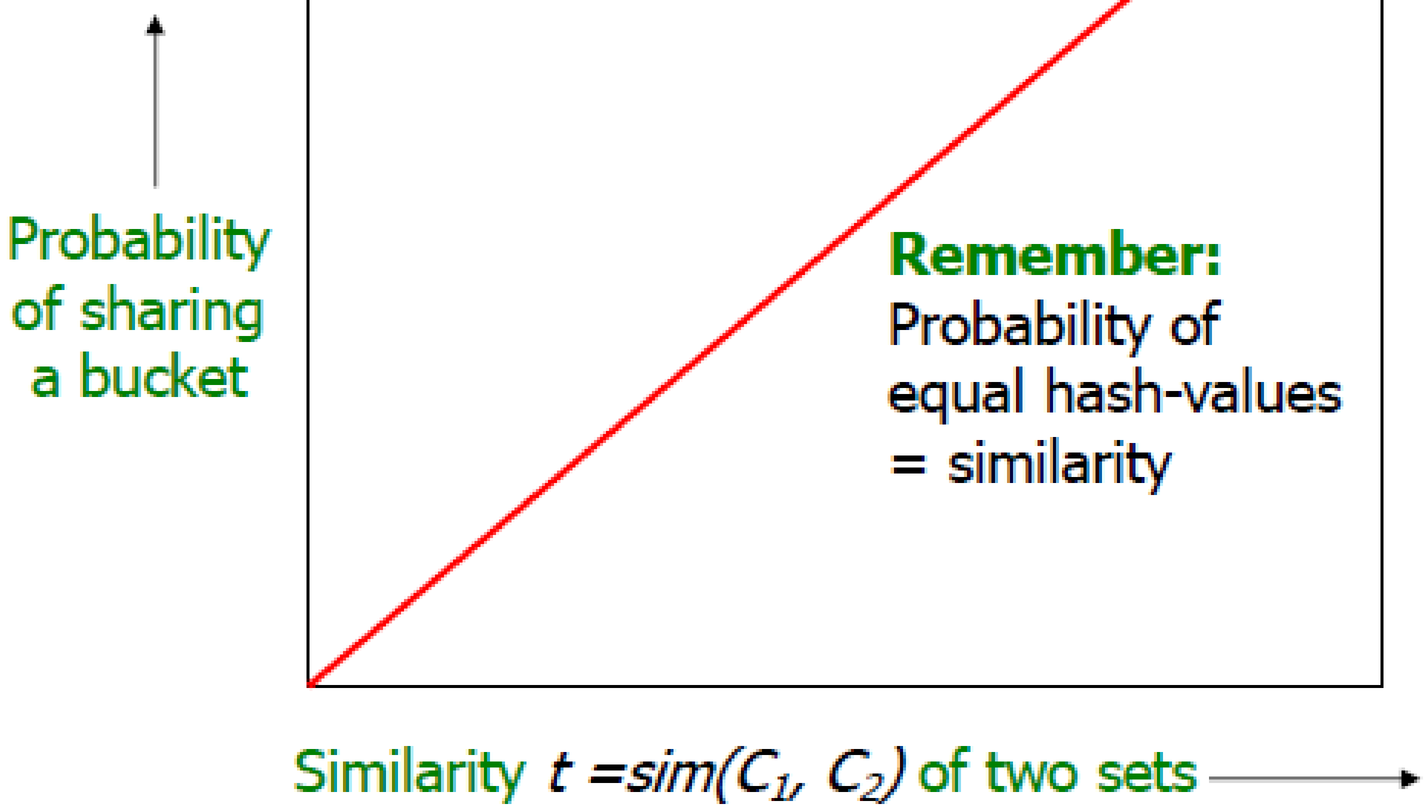
- **Pick:**
  - The number of Min-Hashes (rows of ***M***)
  - The number of bands ***b***, and
  - The number of rows ***r*** per band
    to balance false positives/negatives
    - Note, M=b*r

- **Example:** If we had only 10 bands of 10 rows, the number of false positives would go down, but the number of false negatives would go up

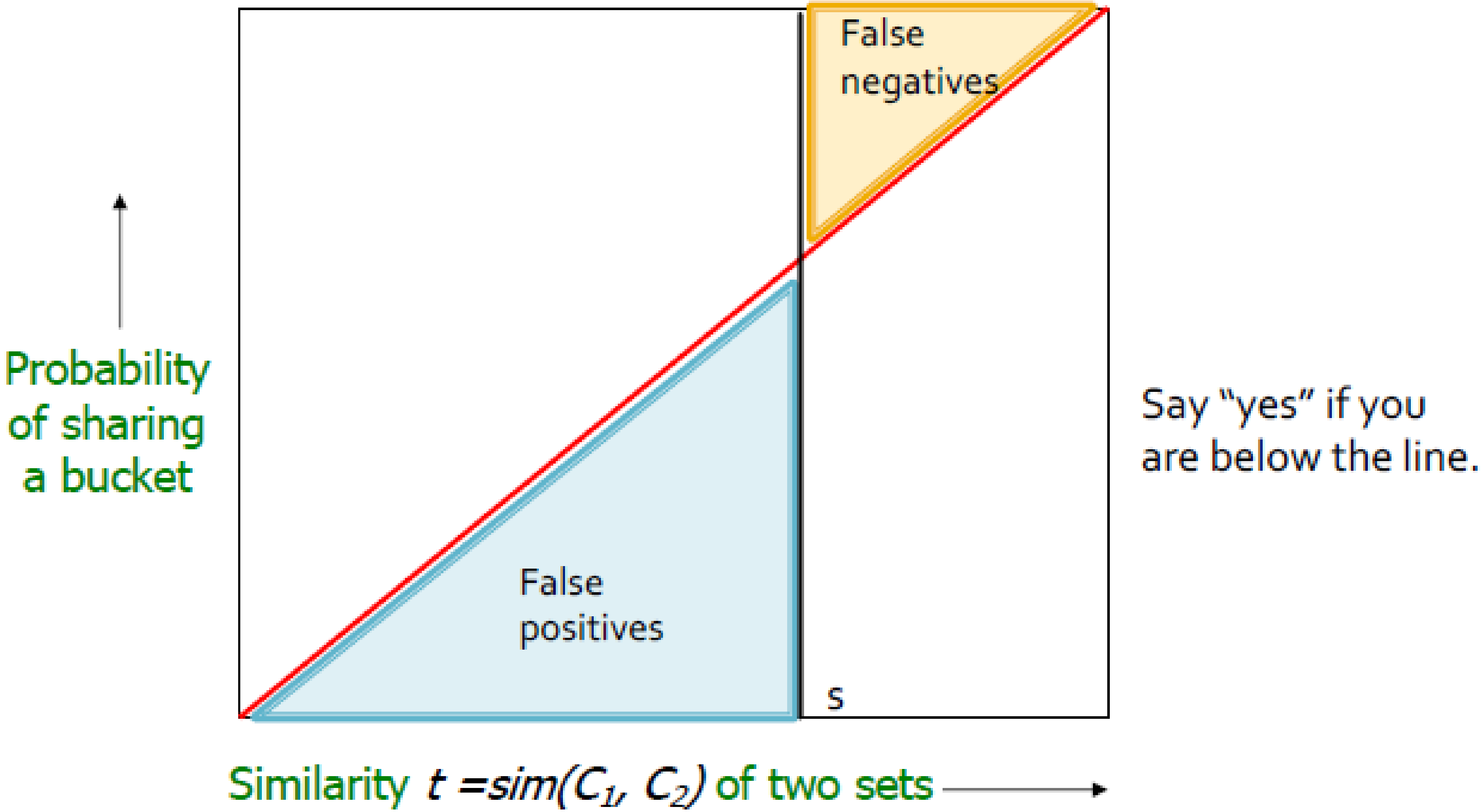
Analysis of LSH



What 1 Band of 1 Row Gives You



What 1 Band of 1 Row Gives You

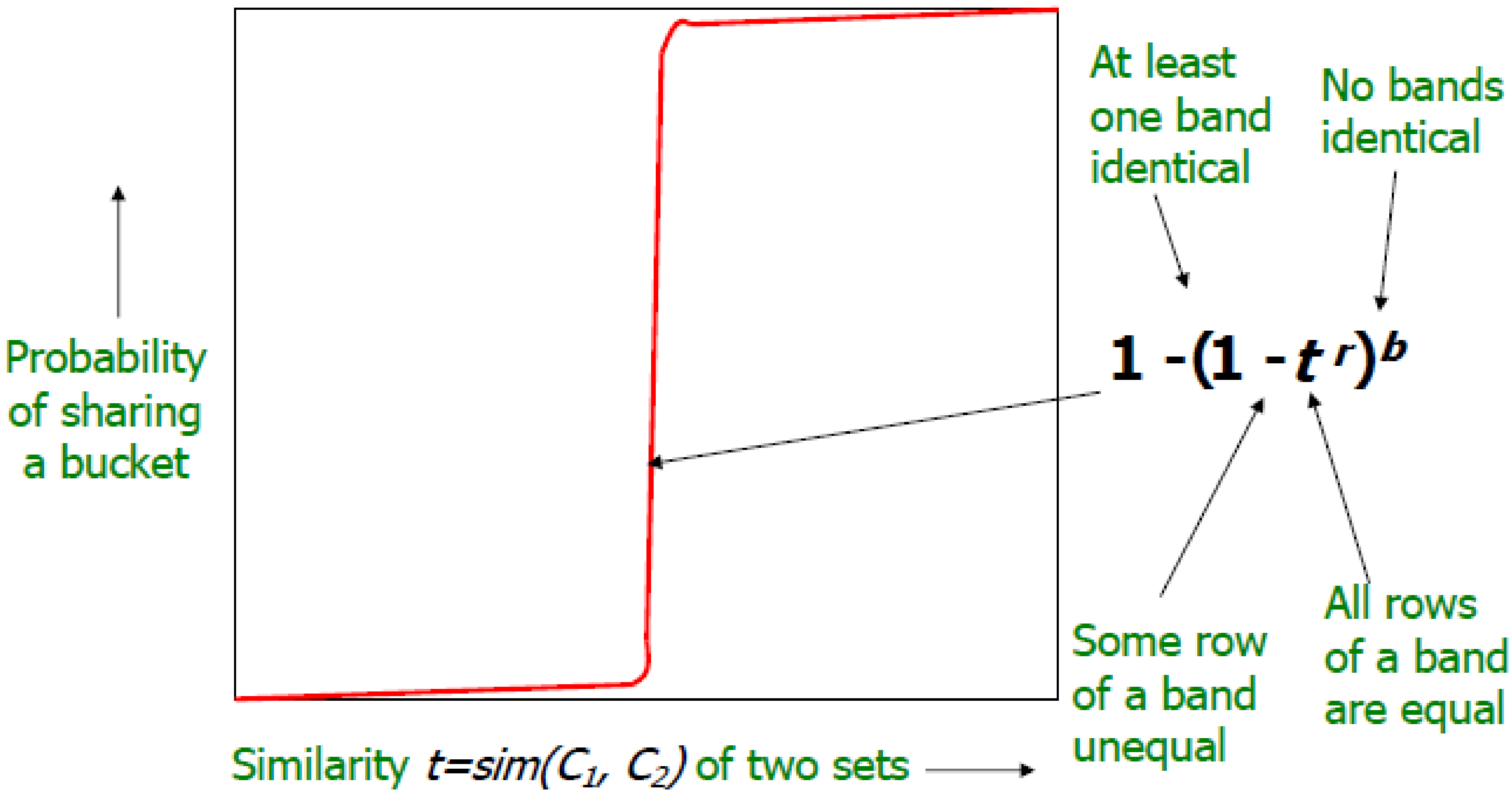


B Bands, r rows/band

- Say columns C_1 and C_2 have similarity t
- Pick any band (r rows)
  - Prob. that all rows in band equal = t^r
  - Prob. that some row in band unequal = $1 - t^r$
- Prob. that no band identical = $(1 - t^r)^b$
- Prob. that at least 1 band identical =

$$1 - (1 - t^r)^b$$

What b Bands of r Rows Gives You



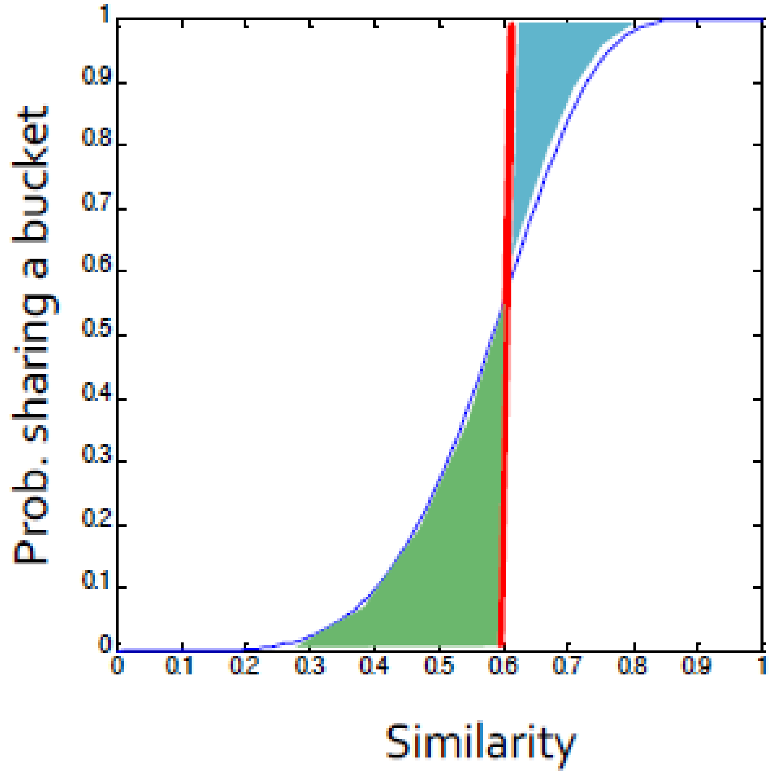
Example : b=20; r=5

- Similarity threshold s
- Prob. that at least 1 band is identical:

s	$1-(1-s^r)^b$
0.2	0.006
0.3	0.047
0.4	0.186
0.5	0.470
0.6	0.802
0.7	0.975
0.8	0.9996

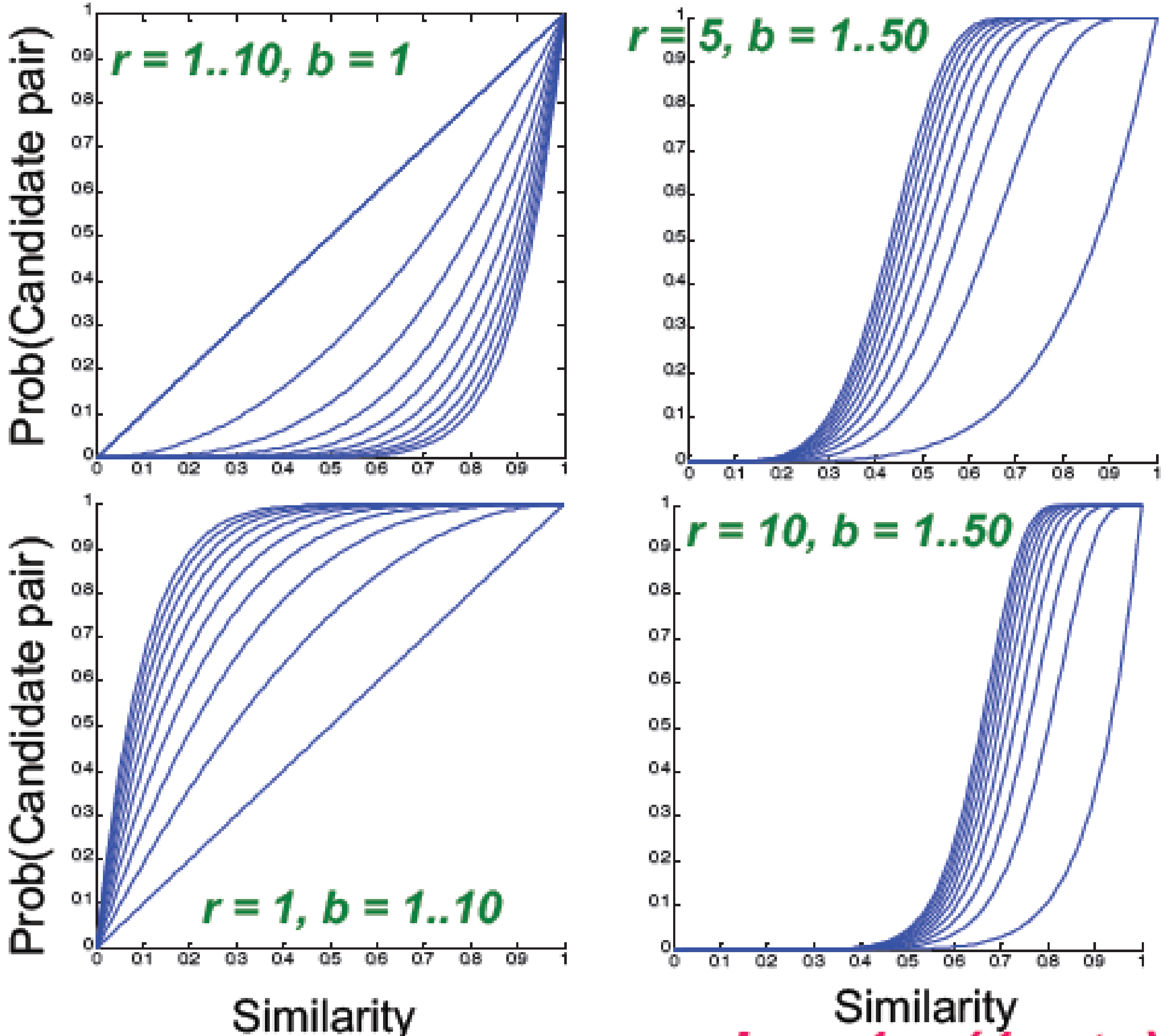
Picking r and b : The S-Curve

- **Picking *r* and *b* to get the best S-curve**
  - 50 hash-functions (r=5, b=10)



Blue area: False Negative rate
Green area: False Positive rate

The S-Curve



$$prob = 1 - (1 - t^{r})^{b}$$

LSH summary

- Tune ***M*, *b*, *r*** to get almost all pairs with similar signatures, but eliminate most pairs that do not have similar signatures
- Check in main memory that **candidate pairs** really do have **similar signatures**
- **Optional:** In another pass through data, check that the remaining candidate pairs really represent similar documents

LSH summary

- **Shingling:** Convert documents to set representation
  - We used hashing to assign each shingle an ID
- **Min-Hashing:** Convert large sets to short signatures, while preserving similarity
  - We used **similarity preserving hashing** to generate signatures with property $\mathbf{Pr}[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$
  - We used hashing to get around generating random permutations
- **Locality-Sensitive Hashing:** Focus on pairs of signatures likely to be from similar documents
  - We used hashing to find **candidate pairs** of similarity $\geq$ **s**