properties-of-inverse-trigonometric-functions

Total clusters: 4, Total questions: 36

Cluster Summary

- Cluster 0: 3 question(s)
- Cluster 1: 3 question(s)
- Cluster 2: 13 question(s)
- Noise: 17 question(s)

Cluster 0 (3)

Q1.

Let m and M respectively be the minimum and the maximum values of $f(x) = \sin^{-1}2x + \sin 2x + \cos^{-1}2x + \cos 2x,\ x \in \left[0, \frac{\pi}{8}\right]$. Then m + M is equal to :

A. $1 + \sqrt{2} + \pi$
B. $\left(1 + \sqrt{2}\right)\pi$
C. $\pi + \sqrt{2}$
D. $1 + \pi$

$f(x) = \sin^{-1}(2x) + \sin 2x + \cos^{-1}(2x) + \cos 2x$

$= \sin^{-1}(2x) + \cos^{-1}(2x) + \sin 2x + \cos 2x$

$= \frac{\pi}{2} + \sqrt{2}\left(\frac{1}{\sqrt{2}} \sin 2x + \frac{1}{\sqrt{2}} \cos 2x\right)$

$= \frac{\pi}{2} + \sqrt{2}\left(\cos \frac{\pi}{4} \sin 2x + \sin \frac{\pi}{4} \cos 2x\right)$

$= \frac{\pi}{2} + \sqrt{2} . \sin\left(2x + \frac{\pi}{4}\right)$

f(x) is maximum when $\sin\left(2x + \frac{\pi}{4}\right)$ is maximum means $x = \frac{\pi}{8}$ or $\sin\left(2 \times \frac{\pi}{8} + \frac{\pi}{4}\right) = \sin \frac{\pi}{2} = 1$

$\therefore [f(x)]_{\max} = \frac{\pi}{2} + \sqrt{2} . 1 = \frac{\pi}{2} + \sqrt{2} = M$

f(x) is minimum when $\sin\left(2x + \frac{\pi}{4}\right)$ is minimum means $x = 0$ or $\sin\left(2 \times 0 + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\therefore [f(x)]_{\min} = \frac{\pi}{2} + \sqrt{2} . \frac{1}{\sqrt{2}} = \frac{\pi}{2} + 1 = m$

$\therefore m+M=\frac{\pi}{2}+\sqrt{2}+\frac{\pi}{2}+1=\pi+\sqrt{2}+1$

Q2.

The sum of the absolute maximum and absolute minimum values of the function $f(x)=\tan^{-1}(\sin x-\cos x)$ in the interval $[0,\pi]$ is :

A. 0
B. $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\frac{\pi}{4}$
C. $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)-\frac{\pi}{4}$
D. $\frac{-\pi}{12}$

$f(x)=\tan^{-1}(\sin x-\cos x), \quad [0,\pi]$

Let $g(x)=\sin x-\cos x$

$=\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$ and $x-\frac{\pi}{4}\in\left[\frac{-\pi}{4},\frac{3\pi}{4}\right]$

$\therefore g(x)\in\left[-1,\sqrt{2}\right]$

and $\tan^{-1}x$ is an increasing function

$\therefore f(x)\in\left[\tan^{-1}(-1),\tan^{-1}\sqrt{2}\right]$

$\in\left[-\frac{\pi}{4},\tan^{-1}\sqrt{2}\right]$

$\therefore$ Sum of $f_{\max}$ and $f_{\min}=\tan^{-1}\sqrt{2}-\frac{\pi}{4}$

$=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)-\frac{\pi}{4}$

Q3.

Let M and m respectively be the maximum and minimum values of the function
f(x) = $\tan^{-1}$ (sin x + cos x) in $\left[0,\frac{\pi}{2}\right]$, then the value of tan(M − m) is equal to :

A. $2+\sqrt{3}$
B. $2-\sqrt{3}$
C. $3+2\sqrt{2}$
D. $3-2\sqrt{2}$

Let g(x) = sin x + cos x = $\sqrt{2}$ sin$\left(x+\frac{\pi}{4}\right)$

$g(x) \in \left[1, \sqrt{2}\right]$ for $x \in [0, \pi/2]$

$f(x) = \tan^{-1}(\sin x + \cos x) \in \left[\frac{\pi}{4}, \tan^{-1}\sqrt{2}\right]$

$\tan(\tan^{-1}\sqrt{2} - \frac{\pi}{4}) = \frac{\sqrt{2}-1}{1+\sqrt{2}} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = 3 - 2\sqrt{2}$

Answer Key

1. Q1: A
2. Q2: C
3. Q3: D

Explanations

1. Q1:

$$f(x) = \sin^{-1}(2x) + \sin 2x + \cos^{-1}(2x) + \cos 2x$$

$$= \sin^{-1}(2x) + \cos^{-1}(2x) + \sin 2x + \cos 2x$$

$$= \frac{\pi}{2} + \sqrt{2}\left(\frac{1}{\sqrt{2}} \sin 2x + \frac{1}{\sqrt{2}} \cos 2x\right)$$

$$= \frac{\pi}{2} + \sqrt{2}\left(\cos \frac{\pi}{4} \sin 2x + \sin \frac{\pi}{4} \cos 2x\right)$$

$$= \frac{\pi}{2} + \sqrt{2} . \sin\left(2x + \frac{\pi}{4}\right)$$

f(x) is maximum when $\sin\left(2x + \frac{\pi}{4}\right)$ is maximum means $x = \frac{\pi}{8}$ or $\sin\left(2 \times \frac{\pi}{8} + \frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1$

$$\therefore [f(x)]_{\max} = \frac{\pi}{2} + \sqrt{2} . 1 = \frac{\pi}{2} + \sqrt{2} = M$$

f(x) is minimum when $\sin\left(2x + \frac{\pi}{4}\right)$ is minimum means $x = 0$ or $\sin\left(2 \times 0 + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$$\therefore [f(x)]_{\min} = \frac{\pi}{2} + \sqrt{2} . \frac{1}{\sqrt{2}} = \frac{\pi}{2} + 1 = m$$

$$\therefore m + M = \frac{\pi}{2} + \sqrt{2} + \frac{\pi}{2} + 1 = \pi + \sqrt{2} + 1$$

2. Q2:

$$f(x) = \tan^{-1}(\sin x - \cos x), \quad [0, \pi]$$

Let $g(x) = \sin x - \cos x$

$= \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$ and $x - \frac{\pi}{4} \in \left[\frac{-\pi}{4}, \frac{3\pi}{4}\right]$

$\therefore g(x) \in \left[-1, \sqrt{2}\right]$

and $\tan^{-1}x$ is an increasing function

$\therefore f(x) \in \left[\tan^{-1}(-1), \tan^{-1}\sqrt{2}\right]$

$\in \left[-\frac{\pi}{4}, \tan^{-1}\sqrt{2}\right]$

$\therefore$ Sum of $f_{\max}$ and $f_{\min} = \tan^{-1}\sqrt{2} - \frac{\pi}{4}$

$= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) - \frac{\pi}{4}$

3. Q3: Let g(x) = sin x + cos x = $\sqrt{2}$ sin$\left(x + \frac{\pi}{4}\right)$

g(x)∈ $\left[1, \sqrt{2}\right]$ for x∈ [0, π/2]

f(x) = tan^{-1} (sin x + cos x) $\in \left[\frac{\pi}{4}, \tan^{-1}\sqrt{2}\right]$

tan$\left(\tan^{-1}\sqrt{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}-1}{1+\sqrt{2}} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = 3 - 2\sqrt{2}$

Cluster 1 (3)

Q1.

Let S = $\left\{x : \cos^{-1}x = \pi + \sin^{-1}x + \sin^{-1}[2x+1]\right\}$. Then $\sum_{x \in S}(2x-1)^2$ is equal to _______.

$\cos^{-1} x = \pi + \sin^{-1} x + \sin^{-1}(2x+1)$

$2\cos^{-1} x - \sin^{-1}(2x+1) = \dfrac{3\pi}{2}$

$2\alpha - \beta = \dfrac{3\pi}{2}$ where $\cos^{-1} x = \alpha, \sin^{-1}(2x+1) = \beta$

$2\alpha = \dfrac{3\pi}{2} + \beta$

$\cos 2\alpha = \sin\beta$

$2\cos^2\alpha - 1 = \sin\beta$

$2x^2 - 1 = 2x + 1$

$x^2 - x - 1 = 0$

$\Rightarrow x = \frac{1-\sqrt{5}}{2}, \left\{x = \frac{1+\sqrt{5}}{2} \text{ rejected }\right\}$

$\therefore 4x^2 - 4x = 4$

$(2x-1)^2 = 5$

Q2.

Let S be the set of all solutions of the equation $\cos^{-1}(2x) - 2\cos^{-1}\left(\sqrt{1-x^2}\right) = \pi, x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Then $\sum\limits_{x \in S} 2\sin^{-1}\left(x^2-1\right)$ is equal to :

A. $\pi - 2\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

B. $\pi - \sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

C. $\frac{-2\pi}{3}$

D. None

$\cos^{-1}(2x) = \pi + 2\cos^{-1}\sqrt{1-x^2}$

Since $\cos^{-1}(2x) \in [0, \pi]$

R.H.S. $\geq \pi$

$\pi + 2\cos^{-1}\sqrt{1-x^2} = \pi$

$\Rightarrow \cos^{-1}\sqrt{1-x^2} = 0$

$\Rightarrow \sqrt{1-x^2} = 1$

$\Rightarrow x = 0$

but at $x = 0$

$\cos^{-1}(2x) = \cos^{-1}(0) = \dfrac{\pi}{2}$

$\therefore$ No solution possible for given equation.

$x \in \phi$

Q3.

If $S = \left\{x \in \mathbb{R} : \sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}\right\}$, then $\sum\limits_{x \in s}\left(\sin\left(\left(x^2+x+5\right)\frac{\pi}{2}\right) - \cos\left(\left(x^2+x+5\right)\pi\right)\right)$ is equal to

________________.

Given equation is

$\sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}$.

Let's denote:

$A = \sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right)$,

$B = \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right)$.

So, we have the equation $A - B = \frac{\pi}{4}$.

We can also write this as $A = B + \frac{\pi}{4}$.

This gives us

$$\sin(A) = \sin\left(B + \frac{\pi}{4}\right).$$

We can use the identity $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and rewrite this equation as:

$$\frac{x+1}{\sqrt{(x+1)^2+1}} = \frac{x}{\sqrt{x^2+1}} \cos\left(\frac{\pi}{4}\right) + \sqrt{1 - \left(\frac{x}{\sqrt{x^2+1}}\right)^2} \sin\left(\frac{\pi}{4}\right).$$

After simplifying, we get:

$$\frac{x+1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+1}} + \sqrt{1 - \frac{x^2}{x^2+1}}\right).$$

Let's square both sides to remove the square roots:

On the left side, squaring gives:

$$\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right)^2 = \frac{(x+1)^2}{x^2+2x+2}.$$

On the right side, squaring gives:

$$\left(\frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+1}} + \sqrt{1 - \frac{x^2}{x^2+1}}\right)\right)^2 = \frac{1}{2}\left(\frac{x^2}{x^2+1} + 2\frac{x}{\sqrt{x^2+1}}\sqrt{1 - \frac{x^2}{x^2+1}} + 1 - \frac{x^2}{x^2+1}\right).$$

$$\therefore \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(\frac{x^2}{x^2+1} + 2\frac{x}{\sqrt{x^2+1}}\sqrt{1 - \frac{x^2}{x^2+1}} + 1 - \frac{x^2}{x^2+1}\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(2 \times \frac{x}{\sqrt{x^2+1}}\sqrt{\frac{x^2+1-x^2}{x^2+1}} + 1\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(2 \times \frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}} + 1\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(2 \times \frac{x}{x^2+1} + 1\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(\frac{2x+x^2+1}{x^2+1}\right)$$

$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(\frac{(x+1)^2}{x^2+1}\right)$

$$\Rightarrow \frac{x+1}{\sqrt{x^2+2x+2}} = \frac{x+1}{\sqrt{2}\sqrt{x^2+1}}$$

$$\Rightarrow x = -1 \text{ OR } \sqrt{x^2+2x+2} = \sqrt{2}\cdot\sqrt{x^2+1}$$

$$\Rightarrow x = 0, x = 2 \text{ (Rejected)}$$

$$S = \{0, -1\}$$

$$\sum_{x\in R}\left(\sin\left(\left(x^2+x+5\right)\frac{\pi}{2}\right) - \cos\left(\left(x^2+x+5\right)\pi\right)\right)$$

$$= \left[\sin\left(\frac{5\pi}{2}\right) - \cos(5\pi)\right] + \left[\sin\left(\frac{5\pi}{2}\right) - \cos(5\pi)\right]$$

$$= (1-(-1)) + (1-(-1))$$

$$= 2 + 2$$

$$= 4$$

Answer Key

1. Q1: -1
2. Q2: D
3. Q3: -1

Explanations

1. Q1:

$\cos^{-1} x = \pi + \sin^{-1} x + \sin^{-1}(2x+1)$
$2\cos^{-1} x - \sin^{-1}(2x+1) = \frac{3\pi}{2}$
$2\alpha - \beta = \frac{3\pi}{2}$ where $\cos^{-1} x = \alpha, \sin^{-1}(2x+1) = \beta$
$2\alpha = \frac{3\pi}{2} + \beta$
$\cos 2\alpha = \sin\beta$
$2\cos^2\alpha - 1 = \sin\beta$
$2x^2 - 1 = 2x + 1$
$x^2 - x - 1 = 0$
$\Rightarrow x = \frac{1-\sqrt{5}}{2}, \left\{x = \frac{1+\sqrt{5}}{2} \text{ rejected } \right\}$
$\therefore 4x^2 - 4x = 4$
$(2x-1)^2 = 5$

$$\cos^{-1}(2x) = \pi + 2\cos^{-1}\sqrt{1-x^2}$$

Since $\cos^{-1}(2x) \in [0, \pi]$

R.H.S. $\geq \pi$

$$\pi + 2\cos^{-1}\sqrt{1-x^2} = \pi$$

2. Q2: $\Rightarrow \cos^{-1}\sqrt{1-x^2} = 0$

$$\Rightarrow \sqrt{1-x^2} = 1$$

$$\Rightarrow x = 0$$

but at $x = 0$

$$\cos^{-1}(2x) = \cos^{-1}(0) = \frac{\pi}{2}$$

$\therefore$ No solution possible for given equation.

$x \in \phi$

3. Q3: Given equation is

$$\sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}.$$

Let's denote:

$$A = \sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right),$$

$$B = \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right).$$

So, we have the equation $A - B = \frac{\pi}{4}$.

We can also write this as $A = B + \frac{\pi}{4}$.

This gives us

$$\sin(A) = \sin\left(B + \frac{\pi}{4}\right).$$

We can use the identity $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and rewrite this equation as:

$$\frac{x+1}{\sqrt{(x+1)^2+1}} = \frac{x}{\sqrt{x^2+1}} \cos\left(\frac{\pi}{4}\right) + \sqrt{1 - \left(\frac{x}{\sqrt{x^2+1}}\right)^2} \sin\left(\frac{\pi}{4}\right).$$

After simplifying, we get:

$$\frac{x+1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+1}} + \sqrt{1 - \frac{x^2}{x^2+1}}\right).$$

Let's square both sides to remove the square roots:

On the left side, squaring gives:

$$\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right)^2 = \frac{(x+1)^2}{x^2+2x+2}.$$

On the right side, squaring gives:

$$\left(\frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+1}} + \sqrt{1 - \frac{x^2}{x^2+1}}\right)\right)^2 = \frac{1}{2}\left(\frac{x^2}{x^2+1} + 2\frac{x}{\sqrt{x^2+1}}\sqrt{1 - \frac{x^2}{x^2+1}} + 1 - \frac{x^2}{x^2+1}\right).$$

$$\therefore \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(\frac{x^2}{x^2+1} + 2\frac{x}{\sqrt{x^2+1}}\sqrt{1 - \frac{x^2}{x^2+1}} + 1 - \frac{x^2}{x^2+1}\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(2 \times \frac{x}{\sqrt{x^2+1}}\sqrt{\frac{x^2+1-x^2}{x^2+1}} + 1\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(2 \times \frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}} + 1\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(2 \times \frac{x}{x^2+1} + 1\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(\frac{2x+x^2+1}{x^2+1}\right)$$

$$\Rightarrow \frac{(x+1)^2}{x^2+2x+2} = \frac{1}{2}\left(\frac{(x+1)^2}{x^2+1}\right)$$

$$\Rightarrow \frac{x+1}{\sqrt{x^2+2x+2}} = \frac{x+1}{\sqrt{2}\sqrt{x^2+1}}$$

$$\Rightarrow x = -1 \text{ OR } \sqrt{x^2+2x+2} = \sqrt{2}\cdot\sqrt{x^2+1}$$

$$\Rightarrow x = 0, x = 2 \text{ (Rejected)}$$

$$S = \{0, -1\}$$

$$\sum_{x \in R}\left(\sin\left(\left(x^2+x+5\right)\frac{\pi}{2}\right) - \cos\left(\left(x^2+x+5\right)\pi\right)\right)$$

$$= \left[\sin\left(\frac{5\pi}{2}\right) - \cos(5\pi)\right] + \left[\sin\left(\frac{5\pi}{2}\right) - \cos(5\pi)\right]$$

$$= (1-(-1)) + (1-(-1))$$

$$= 2 + 2$$

$$= 4$$

Cluster 2 (13)

Q1.

If $\frac{\pi}{2} \le x \le \frac{3\pi}{4}$, then $\cos^{-1}\left(\frac{12}{13}\cos x + \frac{5}{13}\sin x\right)$ is equal to

A. $x + \tan^{-1}\frac{5}{12}$
B. $x - \tan^{-1}\frac{4}{3}$
C. $x + \tan^{-1}\frac{4}{5}$
D. $x - \tan^{-1}\frac{5}{12}$

$$\frac{\pi}{2} \le x \le \frac{3\pi}{4}$$

$$\cos^{-1}\left(\frac{12}{13}\cos x + \frac{5}{12}\sin x\right)$$

$$\cos^{-1}(\cos x \cos\alpha + \sin x \sin\alpha)$$

$$\cos^{-1}(\cos(x-\alpha))$$

$$\Rightarrow x - \alpha \text{ because } x - \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow x - \tan^{-1}\frac{5}{12}$$

Q2.

$\cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{33}{65}\right)$ is equal to:

A. $\frac{33}{65}$
B. 1
C. $\frac{32}{65}$
D. 0

$$\cos\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{33}{65}\right)$$

$$\cos\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{33}{56}\right)$$

$$\cos\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{5}{12}}{1+\frac{3}{4}\cdot\frac{5}{12}}\right) + \tan^{-1}\frac{33}{56}\right)$$

$$\cos\left(\tan^{-1}\frac{56}{33} + \cot^{-1}\frac{56}{33}\right)$$

$$\cos\left(\frac{\pi}{2}\right) = 0$$

Q3.

The value of $\cot^{-1}\left(\frac{\sqrt{1+\tan^2(2)}-1}{\tan(2)}\right) - \cot^{-1}\left(\frac{\sqrt{1+\tan^2\left(\frac{1}{2}\right)}+1}{\tan\left(\frac{1}{2}\right)}\right)$ is equal to

A. $\pi - \frac{3}{2}$

B. $\pi + \frac{5}{2}$

C. $\pi - \frac{5}{4}$

D. $\pi + \frac{3}{2}$

$\cot^{-1}\left(\frac{|\sec 2|-1}{\tan 2}\right)-\cot^{-1}\left(\frac{|\sec \frac{1}{2}|+1}{\tan \frac{1}{2}}\right)$

$$=\cot^{-1}\left(\frac{-1-\cos 2}{\sin 2}\right)-\cot^{-1}\left(\frac{1+\cos \frac{1}{2}}{\sin \frac{1}{2}}\right)$$

$$=\pi-\cot^{-1}(\cot 1)-\cot^{-1}\left(\cot \frac{1}{4}\right)$$

$$=\pi-1-\frac{1}{4}=\pi-\frac{5}{4}$$

Q4.

If $y=\cos\left(\frac{\pi}{3}+\cos^{-1}\frac{x}{2}\right)$, then $(x-y)^2+3y^2$ is equal to

$$y=\cos\left(\frac{\pi}{3}+\cos^{-1}\frac{x}{2}\right)$$

$$=\cos\left(\frac{\pi}{3}\right)\cos\left(\cos^{-1}\left(\frac{x}{2}\right)\right)-\sin\left(\frac{\pi}{3.}\right)\sin\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$$

$$=\frac{1}{2}\cdot\frac{x}{2}-\frac{\sqrt{3}}{2}\cdot\sqrt{1-\frac{x^2}{4}}$$

$$\Rightarrow 4y=x-\sqrt{3}\sqrt{4-x^2}$$

$$\Rightarrow (4y-x)^2=3\left(4-x^2\right)$$

$$\Rightarrow 16y^2+x^2-8xy=12-3x^2$$

$$x^2+4y^2-2xy=3$$

$$(x-y)^2+3y^2=3$$

Q5.

For $\alpha,\beta,\gamma\neq 0$, if $\sin^{-1}\alpha+\sin^{-1}\beta+\sin^{-1}\gamma=\pi$ and $(\alpha+\beta+\gamma)(\alpha-\gamma+\beta)=3\alpha\beta$, then γ equals

A. $\sqrt{3}$

B. $\frac{\sqrt{3}}{2}$

C. $\frac{1}{\sqrt{2}}$

D. $\frac{\sqrt{3}-1}{2\sqrt{2}}$

Let $\sin^{-1}\alpha=A,\sin^{-1}\beta=B,\sin^{-1}\gamma=C$

$A + B + C = \pi$

$(\alpha + \beta)^2 - \gamma^2 = 3\alpha\beta$

$\alpha^2 + \beta^2 - \gamma^2 = \alpha\beta$

$$\frac{\alpha^2 + \beta^2 - \gamma^2}{2\alpha\beta} = \frac{1}{2}$$

$$\Rightarrow \cos C = \frac{1}{2}$$

$\sin C = \gamma$

$$\cos C = \sqrt{1 - \gamma^2} = \frac{1}{2}$$

$$\gamma = \frac{\sqrt{3}}{2}$$

Q6.

If $\sin^{-1}\frac{\alpha}{17} + \cos^{-1}\frac{4}{5} - \tan^{-1}\frac{77}{36} = 0, 0 < \alpha < 13$, then $\sin^{-1}(\sin\alpha) + \cos^{-1}(\cos\alpha)$ is equal to :

A. 16

B. π

C. 16 $- 5\pi$

D. 0

$\sin^{-1}\left(\frac{\alpha}{17}\right) = -\cos^4\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{77}{36}\right)$

Let $\cos^{-1}\left(\frac{4}{5}\right) = p$ and $\tan^{-1}\left(\frac{77}{36}\right) = q$

$\Rightarrow \sin\left(\sin^{-1}\frac{\alpha}{17}\right) = \sin(q - p)$

$= \sin q \cdot \cos p - \cos q \cdot \sin p$

$\Rightarrow \frac{\alpha}{17} = \frac{77}{85} \cdot \frac{4}{5} - \frac{36}{85} \cdot \frac{3}{5}$

$\Rightarrow \alpha = \frac{200}{25} = 8$

$\sin^{-1}\sin 8 + \cos^{-1}\cos 8$

$= -8 + 3\pi + 8 - 2\pi$

$= \pi$

Q7.

If the sum of all the solutions of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}, -1<x<1, x\neq 0$, is $\alpha-\frac{4}{\sqrt{3}}$, then α is equal to ______________.

Case-I

$-1<x<0$

$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$

$\tan^{-1}\frac{2x}{1-x^2}=\frac{-\pi}{3}$

$2\tan^{-1}x=\frac{-\pi}{3}$

$\tan^{-1}x=\frac{-\pi}{6}$

$x=\frac{-1}{\sqrt{3}}$

Case-II

$0<x<1$

$\tan^{-1}\frac{2x}{1-x^2}+\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$

$$\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{6}$$

$$2\tan^{-1}x=\frac{\pi}{6}$$

$$\tan^{-1}x=\frac{\pi}{12}$$

$$x=2-\sqrt{3}$$

$$\text{Sum }=\frac{-1}{\sqrt{3}}+2-\sqrt{3}=2-\frac{4}{\sqrt{3}}$$

$$\Rightarrow\alpha=2$$

Q8.

The value of $\tan^{-1}\left(\frac{\cos\left(\frac{15\pi}{4}\right)-1}{\sin\left(\frac{\pi}{4}\right)}\right)$ is equal to :

A. $-\frac{\pi}{4}$
B. $-\frac{\pi}{8}$
C. $-\frac{5\pi}{12}$
D. $-\frac{4\pi}{9}$

$\tan^{-1}\left(\frac{\cos\left(\frac{15\pi}{4}\right)-1}{\sin\frac{\pi}{4}}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}-1}{\frac{1}{\sqrt{2}}}\right)$

$= \tan^{-1}(1-\sqrt{2}) = -\tan^{-1}(\sqrt{2}-1)$

$= -\frac{\pi}{8}$

Q9.

Let $x = \sin(2\tan^{-1}\alpha)$ and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$. If $S = \{a \in R : y^2 = 1 - x\}$, then $\sum\limits_{\alpha \in S} 16\alpha^3$ is equal to

________________.

$\because x = \sin\left(2\tan^{-1}\alpha\right) = \frac{2\alpha}{1+\alpha^2}$ ...... (i)

and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right) = \sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$

Now, $y^2 = 1 - x$

$\frac{1}{5} = 1 - \frac{2\alpha}{1+\alpha^2}$

$\Rightarrow 1 + \alpha^2 = 5 + 5\alpha^2 - 10\alpha$

$\Rightarrow 2\alpha^2 - 5\alpha + 2 = 0$

$\therefore \alpha = 2, \frac{1}{2}$

$\therefore \sum\limits_{\alpha \in S} 16\alpha^3 = 16 \times 2^3 + 16 \times \frac{1}{2^3} = 130$

Q10.

$\tan\left(2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{\sqrt{5}}{2} + 2\tan^{-1}\frac{1}{8}\right)$ is equal to :

A. 1

B. 2
C. $\frac{1}{4}$
D. $\frac{5}{4}$

$\tan\left(2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{\sqrt{5}}{2}+2\tan^{-1}\frac{1}{8}\right)$

$=\tan\left(2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\cdot\frac{1}{8}}\right)+\sec^{-1}\frac{\sqrt{5}}{2}\right)$

$=\tan\left[2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}\right]$

$=\tan\left[\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{2}\right]$

$=\tan\left[\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{2}\right]$

$=\tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{8}}\right]=\tan\left[\tan^{-1}\frac{\frac{5}{4}}{\frac{5}{8}}\right]$

$=\tan[\tan^{-1}2]=2$

Q11.

If $0<x<\frac{1}{\sqrt{2}}$ and $\frac{\sin^{-1}x}{\alpha}=\frac{\cos^{-1}x}{\beta}$, then the value of $\sin\left(\frac{2\pi\alpha}{\alpha+\beta}\right)$ is
:

A. $4\sqrt{(1-x^2)}\left(1-2x^2\right)$
B. $4x\sqrt{(1-x^2)}\left(1-2x^2\right)$
C. $2x\sqrt{(1-x^2)}\left(1-4x^2\right)$
D. $4\sqrt{(1-x^2)}\left(1-4x^2\right)$

Let $\frac{\sin^{-1}x}{\alpha}=\frac{\cos^{-1}x}{\beta}=k\Rightarrow\sin^{-1}x+\cos^{-1}x=k(\alpha+\beta)$

$\Rightarrow\alpha+\beta=\frac{\pi}{2k}$

Now, $\frac{2\pi\,\alpha}{\alpha+\beta}=\frac{2\pi\,\alpha}{\frac{\pi}{2k}}=4k\alpha=4\sin^{-1}x$

Here $\sin\left(\frac{2\pi\,\alpha}{\alpha+\beta}\right)=\sin(4\sin^{-1}x)$

Let $\sin^{-1}x=\theta$

$\because x\in\left(0,\frac{1}{\sqrt{2}}\right)\Rightarrow\theta\in\left(0,\frac{\pi}{4}\right)$

$\Rightarrow x=\sin\theta$

$\Rightarrow\cos\theta=\sqrt{1-x^2}$

$\Rightarrow \sin 2\theta = 2x \, . \, \sqrt{1 - x^2}$

$\Rightarrow \cos 2\theta = \sqrt{1 - 4x^2(1 - x^2)} = \sqrt{(2x^2 - 1)^2} = 1 - 2x^2$

$\because \left(\cos 2\theta > 0 \text{ as } 2\theta \in \left(0, \frac{\pi}{2}\right)\right)$

$\Rightarrow \sin 4\theta = 2 \, . \, 2x\sqrt{1 - x^2}(1 - 2x^2)$

$= 4x\sqrt{1 - x^2}(1 - 2x^2)$

Q12.

If $\frac{\sin^{1} x}{a} = \frac{\cos^{-1} x}{b} = \frac{\tan^{-1} y}{c}$; $0 < x < 1$,
then the value of $\cos\left(\frac{\pi c}{a+b}\right)$ is :

A. $\frac{1-y^2}{2y}$
B. $\frac{1-y^2}{y\sqrt{y}}$
C. $1 - y^2$
D. $\frac{1-y^2}{1+y^2}$

$\frac{\sin^{-1} x}{a} = \frac{\cos^{-1} x}{b} = \frac{\tan^{-1} y}{c}$

$\frac{\sin^{-1} x}{a} = \frac{\cos^{-1} x}{b} = \frac{\sin^{-1} x + \cos^{-1} x}{a+b} = \frac{\pi}{2(a+b)}$

Now, $\frac{\tan^{-1} y}{c} = \frac{\pi}{2(a+b)}$

$2\tan^{-1} y = \frac{\pi c}{a+b}$

$\Rightarrow \cos\left(\frac{\pi c}{a+b}\right) = \cos(2\tan^{-1} y) = \frac{1-y^2}{1+y^2}$

Q13.

The value of $\tan\left(2\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right)\right)$ is equal to :

A. $\frac{-181}{69}$
B. $\frac{220}{21}$
C. $\frac{-291}{76}$
D. $\frac{151}{63}$

$2\tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{6/5}{1 - \frac{9}{25}}\right) = \tan^{-1}\left(\frac{\frac{6}{5}}{\frac{16}{25}}\right) = \tan^{-1}\frac{15}{8}$

$\therefore 2\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{15}{8}\right) + \tan^{-1}\left(\frac{5}{12}\right)$

$$= \tan^{-1}\left(\frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{15}{8},\frac{5}{12}}\right)$$

$$= \tan^{-1}\left(\frac{180+40}{21}\right) = \tan^{-1}\left(\frac{220}{21}\right)$$

Answer Key

1. Q1: D
2. Q2: D
3. Q3: C
4. Q4: 3
5. Q5: B
6. Q6: B
7. Q7: -1
8. Q8: B
9. Q9: 2
10. Q10: B
11. Q11: B
12. Q12: D
13. Q13: B

Explanations

1. Q1:
   $$\frac{\pi}{2} \leq x \leq \frac{3\pi}{4}$$
   $$\cos^{-1}\left(\frac{12}{13}\cos x + \frac{5}{12}\sin x\right)$$
   $$\cos^{-1}(\cos x \cos \alpha + \sin x \sin \alpha)$$
   $$\cos^{-1}(\cos(x-\alpha))$$
   $$\Rightarrow x - \alpha \text{ because } x - \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
   $$\Rightarrow x - \tan^{-1}\frac{5}{12}$$
2. Q2:

$$\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{33}{65}\right)$$
$$\cos\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{33}{56}\right)$$
$$\cos\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{5}{12}}{1+\frac{3}{4}\cdot\frac{5}{12}}\right)+\tan^{-1}\frac{33}{56}\right)$$
$$\cos\left(\tan^{-1}\frac{56}{33}+\cot^{-1}\frac{56}{33}\right)$$
$$\cos\left(\frac{\pi}{2}\right)=0$$

3. Q3:

$$\cot^{-1}\left(\frac{|\sec 2|-1}{\tan 2}\right)-\cot^{-1}\left(\frac{\left|\sec\frac{1}{2}\right|+1}{\tan\frac{1}{2}}\right)$$
$$=\cot^{-1}\left(\frac{-1-\cos 2}{\sin 2}\right)-\cot^{-1}\left(\frac{1+\cos\frac{1}{2}}{\sin\frac{1}{2}}\right)$$
$$=\pi-\cot^{-1}(\cot 1)-\cot^{-1}\left(\cot\frac{1}{4}\right)$$
$$=\pi-1-\frac{1}{4}=\pi-\frac{5}{4}$$

4. Q4:

$$y=\cos\left(\frac{\pi}{3}+\cos^{-1}\frac{x}{2}\right)$$
$$=\cos\left(\frac{\pi}{3}\right)\cos\left(\cos^{-1}\left(\frac{x}{2}\right)\right)-\sin\left(\frac{\pi}{3.}\right)\sin\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$$
$$=\frac{1}{2}\cdot\frac{x}{2}-\frac{\sqrt{3}}{2}\cdot\sqrt{1-\frac{x^2}{4}}$$
$$\Rightarrow 4y=x-\sqrt{3}\sqrt{4-x^2}$$
$$\Rightarrow (4y-x)^2=3\left(4-x^2\right)$$
$$\Rightarrow 16y^2+x^2-8xy=12-3x^2$$
$$x^2+4y^2-2xy=3$$
$$(x-y)^2+3y^2=3$$

5. Q5:

Let $\sin^{-1}\alpha=A,\sin^{-1}\beta=B,\sin^{-1}\gamma=C$

$$\mathrm{A}+\mathrm{B}+\mathrm{C}=\pi$$
$$(\alpha+\beta)^2-\gamma^2=3\alpha\beta$$
$$\alpha^2+\beta^2-\gamma^2=\alpha\beta$$
$$\frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta}=\frac{1}{2}$$
$$\Rightarrow \cos \mathrm{C}=\frac{1}{2}$$
$$\sin \mathrm{C}=\gamma$$
$$\cos \mathrm{C}=\sqrt{1-\gamma^2}=\frac{1}{2}$$
$$\gamma=\frac{\sqrt{3}}{2}$$

6. Q6: $\sin^{-1}\left(\frac{\alpha}{17}\right)=-\cos^4\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{77}{36}\right)$

   Let $\cos^{-1}\left(\frac{4}{5}\right)=p$ and $\tan^{-1}\left(\frac{77}{36}\right)=q$

   $$\Rightarrow \sin\left(\sin^{-1}\frac{\alpha}{17}\right)=\sin(q-p)$$

   $$=\sin q\cdot\cos p-\cos q\cdot\sin p$$

   $$\Rightarrow \frac{\alpha}{17}=\frac{77}{85}\cdot\frac{4}{5}-\frac{36}{85}\cdot\frac{3}{5}$$

   $$\Rightarrow \alpha=\frac{200}{25}=8$$

   $$\sin^{-1}\sin 8+\cos^{-1}\cos 8$$

   $$=-8+3\pi+8-2\pi$$

   $$=\pi$$

7. Q7: Case-I

   $$-1<x<0$$

   $$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$

   $$\tan^{-1}\frac{2x}{1-x^2}=\frac{-\pi}{3}$$

   $$2\tan^{-1}x=\frac{-\pi}{3}$$

   $$\tan^{-1}x=\frac{-\pi}{6}$$

$x=\frac{-1}{\sqrt{3}}$

Case-II

$0<x<1$

$\tan^{-1}\frac{2x}{1-x^2}+\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$

$$\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{6}$$

$$2\tan^{-1}x=\frac{\pi}{6}$$

$$\tan^{-1}x=\frac{\pi}{12}$$

$x=2-\sqrt{3}$

$$\text{Sum }=\frac{-1}{\sqrt{3}}+2-\sqrt{3}=2-\frac{4}{\sqrt{3}}$$

$\Rightarrow\alpha=2$

8. Q8:

$\tan^{-1}\left(\frac{\cos\left(\frac{15\pi}{4}\right)-1}{\sin\frac{\pi}{4}}\right)$

$=\tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}-1}{\frac{1}{\sqrt{2}}}\right)$

$=\tan^{-1}(1-\sqrt{2})=-\tan^{-1}(\sqrt{2}-1)$

$=-\frac{\pi}{8}$

9. Q9:

$\because x=\sin\left(2\tan^{-1}\alpha\right)=\frac{2\alpha}{1+\alpha^2}$ ...... (i)

and $y=\sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)=\sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right)=\frac{1}{\sqrt{5}}$

Now, $y^2=1-x$

$\frac{1}{5}=1-\frac{2\alpha}{1+\alpha^2}$

$\Rightarrow 1+\alpha^2=5+5\alpha^2-10\alpha$

$\Rightarrow 2\alpha^2-5\alpha+2=0$

$\therefore \alpha = 2, \frac{1}{2}$

$\therefore \sum_{\alpha \in S} 16\alpha^3 = 16 \times 2^3 + 16 \times \frac{1}{2^3} = 130$

10. Q10:

$\tan\left(2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{\sqrt{5}}{2} + 2\tan^{-1}\frac{1}{8}\right)$

$= \tan\left(2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\cdot\frac{1}{8}}\right) + \sec^{-1}\frac{\sqrt{5}}{2}\right)$

$= \tan\left[2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right]$

$= \tan\left[\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}} + \tan^{-1}\frac{1}{2}\right]$

$= \tan\left[\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2}\right]$

$= \tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{8}}\right] = \tan\left[\tan^{-1}\frac{\frac{5}{4}}{\frac{5}{8}}\right]$

$= \tan[\tan^{-1}2] = 2$

11. Q11:

Let $\frac{\sin^{-1}x}{\alpha} = \frac{\cos^{-1}x}{\beta} = k \Rightarrow \sin^{-1}x + \cos^{-1}x = k(\alpha+\beta)$

$\Rightarrow \alpha + \beta = \frac{\pi}{2k}$

Now, $\frac{2\pi\,\alpha}{\alpha+\beta} = \frac{2\pi\,\alpha}{\frac{\pi}{2k}} = 4k\alpha = 4\sin^{-1}x$

Here $\sin\left(\frac{2\pi\,\alpha}{\alpha+\beta}\right) = \sin(4\sin^{-1}x)$

Let $\sin^{-1}x = \theta$

$\because x \in \left(0, \frac{1}{\sqrt{2}}\right) \Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow x = \sin\theta$

$\Rightarrow \cos\theta = \sqrt{1-x^2}$

$\Rightarrow \sin 2\theta = 2x \,.\, \sqrt{1-x^2}$

$\Rightarrow \cos 2\theta = \sqrt{1-4x^2(1-x^2)} = \sqrt{(2x^2-1)^2} = 1-2x^2$

$\because \left(\cos 2\theta > 0 \text{ as } 2\theta \in \left(0, \frac{\pi}{2}\right)\right)$

$\Rightarrow \sin 4\theta = 2\,.\, 2x\sqrt{1-x^2}(1-2x^2)$

$= 4x\sqrt{1-x^2}(1-2x^2)$

12. Q12: $\frac{\sin^{-1}x}{a}=\frac{\cos^{-1}x}{b}=\frac{\tan^{-1}y}{c}$

$\frac{\sin^{-1}x}{a}=\frac{\cos^{-1}x}{b}=\frac{\sin^{-1}x+\cos^{-1}x}{a+b}=\frac{\pi}{2(a+b)}$

Now, $\frac{\tan^{-1}y}{c}=\frac{\pi}{2(a+b)}$

$2\tan^{-1}y=\frac{\pi c}{a+b}$

$\Rightarrow \cos\left(\frac{\pi c}{a+b}\right)=\cos(2\tan^{-1}y)=\frac{1-y^2}{1+y^2}$

13. Q13: $2\tan^{-1}\left(\frac{3}{5}\right)=\tan^{-1}\left(\frac{6/5}{1-\frac{9}{25}}\right)=\tan^{-1}\left(\frac{\frac{6}{5}}{\frac{16}{25}}\right)=\tan^{-1}\frac{15}{8}$

$\therefore 2\tan^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)=\tan^{-1}\left(\frac{15}{8}\right)+\tan^{-1}\left(\frac{5}{12}\right)$

$=\tan^{-1}\left(\frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{15}{8},\frac{5}{12}}\right)$

$=\tan^{-1}\left(\frac{180+40}{21}\right)=\tan^{-1}\left(\frac{220}{21}\right)$

Noise (17)

Q1.

If for some $\alpha,\beta;\alpha\leq\beta,\alpha+\beta=8$ and $\sec^2\left(\tan^{-1}\alpha\right)+\operatorname{cosec}^2\left(\cot^{-1}\beta\right)=36$, then $\alpha^2+\beta$ is

____

If $\left(\tan\left(\tan^{-1}(\alpha)\right)+1\left(\cot\left(\cot^{-1}\beta\right)\right)^2=36\right.$

$\alpha^2+\beta^2=34$

$\alpha\beta=15$

$\alpha=3,\beta=5$

$\therefore \alpha^2+\beta=9+5=14$

Q2.

If $\alpha>\beta>\gamma>0$, then the expression

$\cot^{-1}\left\{\beta+\frac{(1+\beta^2)}{(\alpha-\beta)}\right\}+\cot^{-1}\left\{\gamma+\frac{(1+\gamma^2)}{(\beta-\gamma)}\right\}+\cot^{-1}\left\{\alpha+\frac{(1+\alpha^2)}{(\gamma-\alpha)}\right\}$

is equal to :

A. 3π

B. $\frac{\pi}{2}-(\alpha+\beta+\gamma)$
C. π
D. 0

$$\Rightarrow \cot^{-1}\left(\frac{\alpha\beta+1}{\alpha-\beta}\right)+\cot^{-1}\left(\frac{\beta\gamma+1}{\beta-\gamma}\right)+\cot^{-1}\left(\frac{\alpha\gamma+1}{\gamma-\alpha}\right)$$
$$\Rightarrow \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\beta}\right)+\tan^{-1}\left(\frac{\beta-\gamma}{1+\beta\gamma}\right)+\pi+\tan^{-1}\left(\frac{\gamma-\alpha}{1+\gamma\alpha}\right)$$
$$\Rightarrow \left(\tan^{-1}\alpha-\tan^{-1}\beta\right)+\left(\tan^{-1}\beta-\tan^{-1}\gamma\right)+\left(\pi+\tan^{-1}\gamma-\tan^{-1}\alpha\right)$$
$$\Rightarrow \pi$$

Q3.

Let $x=\frac{m}{n}$ (m, n are co-prime natural numbers) be a solution of the equation $\cos\left(2\sin^{-1}x\right)=\frac{1}{9}$ and let $\alpha, \beta(\alpha>\beta)$ be the roots of the equation $mx^2-nx-m+n=0$. Then the point (α, β) lies on the line

A. $3x-2y=-2$
B. $3x+2y=2$
C. $5x+8y=9$
D. $5x-8y=-9$

Assume $\sin^{-1}x=\theta$

$$\cos(2\theta)=\frac{1}{9}$$
$$\sin\theta=\pm\frac{2}{3}$$

as m and n are co-prime natural numbers,

$\mathrm{x}=\frac{2}{3}$

i.e. $m=2, n=3$

So, the quadratic equation becomes $2x^2-3x+1=0$ whose roots are $\alpha=1, \beta=\frac{1}{2}$

$\left(1, \frac{1}{2}\right)$ lies on $5x+8y=9$

Q4.

For $n\in\mathrm{N}$, if $\cot^{-1}3+\cot^{-1}4+\cot^{-1}5+\cot^{-1}n=\frac{\pi}{4}$, then n is equal to ________.

For $n \in \mathbb{N}$, if $\cot^{-1} 3 + \cot^{-1} 4 + \cot^{-1} 5 + \cot^{-1} n = \frac{\pi}{4}$, then n is equal to .

Given the equation:

$$\cot^{-1} 3 + \cot^{-1} 4 + \cot^{-1} 5 + \cot^{-1}(n) = \frac{\pi}{4}$$

we can use the identity for the sum of inverse cotangents. Starting with the first two terms:

$$\cot^{-1} 3 + \cot^{-1} 4 = \cot^{-1}\left(\frac{3\times4-1}{3+4}\right) = \cot^{-1}\left(\frac{11}{7}\right)$$

Now, adding the third term:

$$\cot^{-1}\left(\frac{11}{7}\right) + \cot^{-1}(5)$$

we apply the identity again:

$$\cot^{-1}\left(\frac{11}{7}\right) + \cot^{-1}\left(\frac{n\times5-1}{5+n}\right) = \frac{\pi}{4}$$

Rewriting this to isolate the sum of the terms, we proceed as follows:

$$\cot^{-1}\left(\frac{\left(\frac{11}{7}\times\frac{5n-1}{5+n}-1\right)}{\left(\frac{11}{7}+\frac{5n-1}{5+n}\right)}\right) = \frac{\pi}{4}$$

This simplifies to:

$$\frac{11}{7}\left(\frac{5n-1}{5+n}\right) - 1 = \frac{11}{7} + \frac{5n-1}{5+n}$$

Solving the equation:

$$\frac{55n-11}{5+n} - 1 = \frac{11}{7} + \frac{5n-1}{5+n}$$

Further simplification yields:

$$55n - 11 - 35 - 7n = 55 + 11n + 35n - 7$$

Bringing the terms together, we get:

$$48n - 46 = 48$$

Therefore:

$$2n = 94$$

So finally:

$$n = 47$$

Q5.

Let $S = \left\{x \in R : 0 < x < 1 \text{ and } 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$.

If $\mathrm{n(S)}$ denotes the number of elements in S then :

A. $\mathrm{n(S)} = 0$
B. $\mathrm{n(S)} = 1$ and only one element in S is less than $\frac{1}{2}$.
C. $\mathrm{n(S)} = 1$ and the elements in S is more than $\frac{1}{2}$.
D. $\mathrm{n(S)} = 1$ and the element in S is less than $\frac{1}{2}$.

$$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\text{Put } x = \tan\theta \quad \theta \in \left(0, \frac{\pi}{4}\right)$$

$$2\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$2\tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right] = \cos^{-1}[\cos(2\theta)]$$

$$\Rightarrow 2\left(\frac{\pi}{4} - \theta\right) = 2\theta \Rightarrow \theta = \frac{\pi}{8}$$

$$\Rightarrow x = \tan\frac{\pi}{8} = \sqrt{2} - 1 \simeq 0.414$$

Q6.

Let (a, b) $\subset (0, 2\pi)$ be the largest interval for which $\sin^{-1}(\sin\theta) - \cos^{-1}(\sin\theta) > 0, \theta \in (0, 2\pi)$, holds.

If $\alpha x^2 + \beta x + \sin^{-1}\left(x^2 - 6x + 10\right) + \cos^{-1}\left(x^2 - 6x + 10\right) = 0$ and $\alpha - \beta = b - a$, then α is equal to :

A. $\frac{\pi}{16}$
B. $\frac{\pi}{48}$
C. $\frac{\pi}{8}$
D. $\frac{\pi}{12}$

$$\sin^{-1}\sin\theta - \left(\frac{\pi}{2} - \sin^{-1}\sin\theta\right) > 0$$

$$\Rightarrow \sin^{-1}\sin\theta > \frac{\pi}{4}$$

$$\Rightarrow \sin\theta > \frac{1}{\sqrt{2}}$$

So, $\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

$\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) = (\mathrm{a}, \mathrm{b})$

$b - a = \frac{\pi}{2} = \alpha - \beta$

$\Rightarrow \beta = \alpha - \frac{\pi}{2}$

$\Rightarrow \alpha x^2 + \beta \mathrm{x} + \sin^{-1}\left[(\mathrm{x}-3)^2 + 1\right] + \cos^{-1}\left[(\mathrm{x}-3)^2 + 1\right] = 0$

$x = 3, 9\alpha + 3\beta + \frac{\pi}{2} + 0 = 0$

$$\Rightarrow 9\alpha + 3\left(\alpha - \frac{\pi}{2}\right) + \frac{\pi}{2} = 0$$

$\Rightarrow 12\alpha - \pi = 0$

$$\alpha = \frac{\pi}{12}$$

Q7.

Let $a_1 = 1, a_2, a_3, a_4, \ldots..$ be consecutive natural numbers.

Then $\tan^{-1}\left(\frac{1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{1}{1+a_2a_3}\right) + \ldots.. + \tan^{-1}\left(\frac{1}{1+a_{2021}a_{2022}}\right)$ is equal to :

A. $\frac{\pi}{4} - \cot^{-1}(2022)$
B. $\frac{\pi}{4} - \tan^{-1}(2022)$
C. $\cot^{-1}(2022) - \frac{\pi}{4}$
D. $\tan^{-1}(2022) - \frac{\pi}{4}$

$a_1 = 1, a_2, a_3, a_4, \ldots..$ be consecutive natural numbers.

$$\therefore \quad a_2 = 2, a_3 = 3, \ldots, a_{2021} = 2021, a_{2022} = 2022$$

$$\tan^{-1}\left[\frac{1}{1+a_1a_2}\right] = \tan^{-1}\left[\frac{1}{1+1\cdot 2}\right] = \tan^{-1}(1) - \tan^{-1}\left(\frac{1}{2}\right)$$

$$\tan^{-1}\left[\frac{1}{1+a_2a_3}\right] = \tan^{-1}\left[\frac{1}{1+2\cdot 3}\right] = \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right)$$

.
.
.
.

$$\tan^{-1}\left[\frac{1}{1+a_{2021}a_{2022}}\right] = \tan^{-1}\left[\frac{1}{1+2021\cdot 2022}\right]$$

$$= \tan^{-1}\left(\frac{1}{2021}\right) - \tan^{-1}\left(\frac{1}{2022}\right)$$

$$\therefore \tan^{-1}\left(\frac{1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{1}{1+a_2a_3}\right) + \ldots\ldots + \tan^{-1}\left(\frac{1}{1+a_{2021}a_{2022}}\right)$$

$$= \tan^{-1}(1) - \tan^{-1}\left(\frac{1}{2022}\right) = \frac{\pi}{4} - \cot^{-1}(2022)$$

$$= \frac{\pi}{4} - \left(\frac{\pi}{2} - \tan^{-1}(2022)\right) = \tan^{-1}(2022) - \frac{\pi}{4}$$

Q8.

For $x \in (-1, 1]$, the number of solutions of the equation $\sin^{-1} x = 2\tan^{-1} x$ is equal to __________.

We're given the equation $\sin^{-1} x = 2\tan^{-1} x$ for x in the interval $(-1, 1]$. We want to find the number of solutions.

Step 1: Apply the sine and tangent functions to both sides :

We can rewrite the equation by applying the sine function to both sides :

$\sin(\sin^{-1} x) = \sin(2\tan^{-1} x)$.

This simplifies to:

$x = \sin(2\tan^{-1} x)$.

Step 2: Use the double-angle identity for sine :

Recall that $\sin(2y) = 2\sin(y)\cos(y)$. Applying this identity to the right-hand side gives :

$x = 2\sin(\tan^{-1} x)\cos(\tan^{-1} x)$.

Step 3: Use the identities for sine and cosine of an inverse tangent :

Recall that $\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$ and $\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$.

Substituting these into the equation gives :

$$x = 2 \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}.$$

This simplifies to :

$$x = \frac{2x}{1+x^2}.$$

Step 4: Solve for x :

We have :

$$x = \frac{2x}{1+x^2}.$$

Cross-multiplying gives :

$$x(1+x^2) = 2x.$$

This simplifies to :

$$x^3 + x - 2x = 0.$$

Rearranging terms gives :

$$x^3 - x = 0.$$

This factors to:

$$x(x^2 - 1) = 0.$$

Setting each factor equal to zero gives the solutions $x = 0$, $x = -1$, and $x = 1$.

However, we are given that $x \in (-1, 1]$. Therefore, the only solutions in this interval are $x = 0$ and $x = 1$.

So there are 2 solutions to the equation $\sin^{-1} x = 2\tan^{-1} x$ in the interval $x \in (-1, 1]$.

Q9.

$$50 \tan\left(3\tan^{-1}\left(\frac{1}{2}\right) + 2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) + 4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}(2\sqrt{2})\right)$$

is equal to ____________.

$$50\tan\left(\tan^{-1}\frac{1}{2} + 2\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}(2)\right)$$
$$+4\sqrt{2}\tan\left(\frac{\tan^{-1}}{2}(2\sqrt{2})\right)$$

$\Rightarrow 50\tan\left(\pi+\tan^{-1}\left(\frac{1}{2}\right)\right)+4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}2\sqrt{2}\right)$

$\Rightarrow \quad 50\left(\frac{1}{2}\right)+4\sqrt{2}\tan\alpha$

Where $2\alpha=\tan^{-1}2\sqrt{2}$

$\Rightarrow \frac{2\tan\alpha}{1-\tan^2\alpha}=2\sqrt{2}$ .. (i)

$\Rightarrow \quad 2\sqrt{2}\tan^2\alpha+2\tan\alpha-2\sqrt{2}=0$

$\Rightarrow \quad 2\sqrt{2}\tan^2\alpha+4\tan\alpha-2\tan\alpha-2\sqrt{2}=0$

$\Rightarrow (2\sqrt{2}\tan\alpha-2)(\tan\alpha-\sqrt{2})=0$

$\Rightarrow \tan\alpha=\sqrt{2}$ or $\frac{1}{\sqrt{2}}$

$\Rightarrow \tan\alpha=\frac{1}{\sqrt{2}}$

$(\tan\alpha=\sqrt{2}$ doesn't satisfy (i))

$\Rightarrow \quad 25+4\sqrt{2}\frac{1}{\sqrt{2}}=29$

Q10.

The value of $\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$ is :

A. $\frac{26}{25}$
B. $\frac{25}{26}$
C. $\frac{50}{51}$
D. $\frac{52}{51}$

$\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$

$=\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)\right)$

$=\cot\left(\sum_{n=1}^{50}(\tan^{-1}(n+1)-\tan^{-1}n\right)$

$=\cot(\tan^{-1}51-\tan^{-1}1)$

$= \cot\left(\tan^{-1}\left(\frac{51-1}{1+51}\right)\right)$

$= \cot\left(\cot^{-1}\left(\frac{52}{50}\right)\right)$

$= \frac{26}{25}$

Q11.

Let $x * y = x^2 + y^3$ and $(x * 1) * 1 = x * (1 * 1)$.

Then a value of $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ is :

A. $\frac{\pi}{4}$
B. $\frac{\pi}{3}$
C. $\frac{\pi}{2}$
D. $\frac{\pi}{6}$

The star "*" in this context represents a binary operation, similar to addition (+), subtraction (-), multiplication (×), and division (÷). It is a custom operation defined by the problem statement, and the specific rules of the operation are provided in the problem.

In this case, the operation "*" is defined by the equation $x * y = x^2 + y^3$, which means if you have two numbers x and y, then the result of applying the "*" operation to them is $x^2 + y^3$.

The problem also specifies an additional rule for this operation: $(x * 1) * 1 = x * (1 * 1)$, which needs to be taken into account when solving the problem. This is a type of "associativity" condition.
Given,

$x * y = x^2 + y^3$

$\therefore x * 1 = x^2 + 1^3 = x^2 + 1$

Now, $(x * 1) * 1 = (x^2 + 1) * 1$

$\Rightarrow (x * 1) * 1 = (x^2 + 1)^2 + 1^3$

$\Rightarrow (x * 1) * 1 = x^4 + 1 + 2x^2 + 1$

Also, $x * (1 * 1)$

$= x * (1^2 + 1^3)$

$= x * 2$

$= x^2 + 2^3$

$= x^2 + 8$

Given that,

$(x * 1) * 1 = x * (1 * 1)$

$\therefore x^4 + 1 + 2x^2 + 1 = x^2 + 8$

$\Rightarrow x^4 + x^2 - 6 = 0$

$\Rightarrow x^4 + 3x^2 - 2x^2 - 6 = 0$

$\Rightarrow x^2(x^2 + 3) - 2(x^3 + 3) = 0$

$\Rightarrow (x^2 + 3)(x^2 - 2) = 0$

$\Rightarrow x^2 = 2, -3$

[$x^2 = -3$ not possible as square of anything should be always positive]

$\therefore x^2 = 2$

$\therefore$ Now,

$2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$

$= 2\sin^{-1}\left(\frac{2^2+2-2}{2^2+2+2}\right)$

$= 2\sin^{-1}\left(\frac{4}{8}\right)$

$= 2\sin^{-1}\left(\frac{1}{2}\right)$

$= 2 \times \frac{\pi}{6}$

$= \frac{\pi}{3}$

Q12.

The set of all values of k for which

$(\tan^{-1}x)^3 + (\cot^{-1}x)^3 = k\pi^3$, $x \in R$, is the interval :

A. $\left[\frac{1}{32}, \frac{7}{8}\right)$
B. $\left(\frac{1}{24}, \frac{13}{16}\right)$
C. $\left[\frac{1}{48}, \frac{13}{16}\right]$

D. $\left[\frac{1}{32}, \frac{9}{8}\right)$

$(\tan^{-1}x)^3 + (\cot^{-1}x)^3 = k\pi^3$

Let $f(t) = t^3 + \left(\frac{\pi}{2} - t\right)^3$

Where $t = \tan^{-1}x$; $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= t^3 + \left(\frac{\pi}{2}\right)^3 - \frac{3\pi^2 t}{4} + \frac{3\pi}{2}t^2 - t^3$

$f(t) = \frac{3\pi}{2}t^2 - \frac{3\pi^2}{4} . t + \frac{\pi^3}{8}$

This is a quadratic equation of t.

Here, coefficient of t^2 term is $\frac{3\pi}{2}$ which is > 0.

$\therefore$ It is a upward parabola.

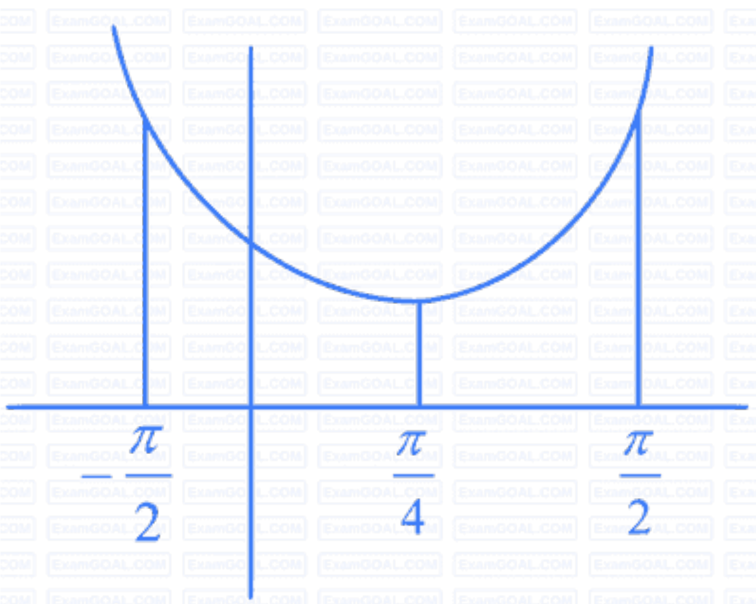
Now, $f'(t) = 3\pi t - \frac{3\pi^2}{4}$

$f''(t) = 3\pi > 0$

$\therefore 3\pi t - \frac{3\pi^2}{4} = 0$

$\Rightarrow t = \frac{\pi}{4}$ (minima)

$\therefore$ vertex of graph at $\frac{\pi}{4}$



$\therefore$ Minimum value at $\frac{\pi}{4}$ and maximum value at $-\frac{\pi}{2}$.

$\therefore f\left(\frac{\pi}{4}\right) = \frac{\pi^3}{64} + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)^3 = \frac{\pi^3}{32}$

$f\left(-\frac{\pi}{2}\right) = -\frac{\pi^3}{8} + \pi^3$

$= \frac{7\pi^3}{8}$

$\therefore k\pi^3 \in \left[\frac{\pi^3}{32}, \frac{7\pi^3}{8}\right)$

$\Rightarrow k \in \left[\frac{1}{32}, \frac{7}{8}\right)$

Q13.

A possible value of $\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$ is :

A. $\sqrt{7}-1$

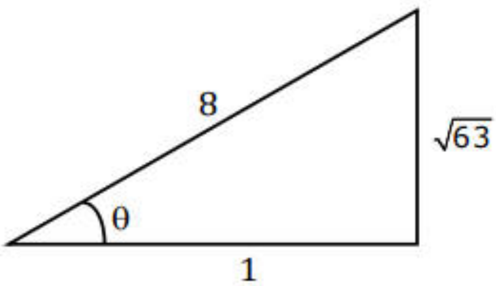
B. $\frac{1}{\sqrt{7}}$

C. $2\sqrt{2}-1$

D. $\frac{1}{2\sqrt{2}}$

$\tan\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

$\sin^{-1}\left(\frac{\sqrt{63}}{8}\right) = \theta \sin\theta = \frac{\sqrt{63}}{8}$



$\cos\theta = \frac{1}{8}$

$2\cos^2\frac{\theta}{2} - 1 = \frac{1}{8}$

$\cos^2\frac{\theta}{2} = \frac{9}{16}$

$\cos\frac{\theta}{2} = \frac{3}{4}$

$\frac{1-\tan^2\frac{\theta}{4}}{1+\tan^2\frac{\theta}{4}} = \frac{3}{4}$

$\tan\frac{\theta}{4} = \frac{1}{\sqrt{7}}$

Q14.

cosec$\left[2\cot^{-1}(5)+\cos^{-1}\left(\frac{4}{5}\right)\right]$ is equal to :

A. $\frac{75}{56}$
B. $\frac{65}{56}$
C. $\frac{56}{33}$
D. $\frac{65}{33}$

$$\cos ec\left(2\cot^{-1}(5)+\cos^{-1}\left(\frac{4}{5}\right)\right)$$

$$\cos ec\left(2\tan^{-1}\left(\frac{1}{5}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$$

$$=\cos ec\left(\tan^{-1}\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$$

$$=\cos ec\left(\tan^{-1}\left(\frac{5}{12}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$$

Let $\tan^{-1}(5/12)=\theta\Rightarrow\sin\theta=\frac{5}{13},\cos\theta=\frac{12}{13}$

and $\cos^{-1}\left(\frac{4}{5}\right)=\phi\Rightarrow\cos\phi=\frac{4}{5}$ and $\sin\phi=\frac{3}{5}$

$$=\cos ec(\theta+\phi)$$

$$=\frac{1}{\sin\theta\cos\phi+\cos\theta\sin\phi}$$

$$=\frac{1}{\frac{5}{13}\cdot\frac{4}{5}+\frac{12}{13}\cdot\frac{3}{5}}=\frac{65}{56}$$

Q15.

If 0 < a, b < 1, and $\tan^{-1}a+\tan^{-1}b=\frac{\pi}{4}$, then the value of

$(a+b)-\left(\frac{a^2+b^2}{2}\right)+\left(\frac{a^3+b^3}{3}\right)-\left(\frac{a^4+b^4}{4}\right)+\ldots..$ is :

A. $\log_e 2$
B. e
C. $\log_e\left(\frac{e}{2}\right)$
D. $e^2=1$

$\tan^{-1}a+\tan^{-1}b=\frac{\pi}{4}$ 0 < a, b < 1

$$\Rightarrow\frac{a+b}{1-ab}=1$$

a + b = 1 − ab

(a + 1)(b + 1) = 2

Now $\left[a-\frac{a^2}{2}+\frac{a^3}{3}+\ldots\right]+\left[b-\frac{b^2}{2}+\frac{b^3}{3}+\ldots\right]$

$=\log_e(1+a)+\log_e(1+b)$

($\because$ expansion of $\log_e(1 + x)$)

$=\log_e[(1+a)(1+b)]$

$=\log_e 2$

Q16.

The sum of possible values of x for

$\tan^{-1}(x+1)+\cot^{-1}\left(\frac{1}{x-1}\right)=\tan^{-1}\left(\frac{8}{31}\right)$ is :

A. $-\frac{32}{4}$
B. $-\frac{33}{4}$
C. $-\frac{31}{4}$
D. $-\frac{30}{4}$

$\tan^{-1}(x+1)+\cot^{-1}\left(\frac{1}{x-1}\right)=\tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \tan^{-1}\left(\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right)=\tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \frac{(1+x)+(x-1)}{1-(1+x)(x-1)}=\frac{8}{31}$

$\Rightarrow \frac{2x}{2-x^2}=\frac{8}{31}$

$\Rightarrow 4x^2+31x-8=0$

$\Rightarrow x=-8, \frac{1}{4}$

but at $x=\frac{1}{4}$

$LHS>\frac{\pi}{2}$ and $RHS<\frac{\pi}{2}$

So, only solution is $x = -8 = -\frac{32}{4}$

Q17.

If $(\sin^{-1}x)^2 - (\cos^{-1}x)^2 = a$; 0 < x < 1, $a \neq 0$, then the value of $2x^2 - 1$ is :

A. $\cos\left(\frac{4a}{\pi}\right)$
B. $\sin\left(\frac{2a}{\pi}\right)$
C. $\cos\left(\frac{2a}{\pi}\right)$
D. $\sin\left(\frac{4a}{\pi}\right)$

Given $a = (\sin^{-1}x)^2 - (\cos^{-1}x)^2$

$= (\sin^{-1}x + \cos^{-1}x)(\sin^{-1}x - \cos^{-1}x)$

$= \frac{\pi}{2}\left(\frac{\pi}{2} - 2\cos^{-1}x\right)$

$\Rightarrow 2\cos^{-1}x = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow \cos^{-1}(2x^2 - 1) = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow 2x^2 - 1 = \cos\left(\frac{\pi}{2} - \frac{2a}{\pi}\right)$

Answer Key

1. Q1: -1
2. Q2: C
3. Q3: C
4. Q4: -1
5. Q5: D
6. Q6: D
7. Q7: A, D
8. Q8: 39
9. Q9: 50
10. Q10: A
11. Q11: B
12. Q12: A
13. Q13: B
14. Q14: B
15. Q15: A
16. Q16: A
17. Q17: B

Explanations

1. Q1:

$$\text{If } \left(\tan\left(\tan^{-1}(\alpha)\right) + 1\left(\cot\left(\cot^{-1}\beta\right)\right)\right)^2 = 36$$
$$\alpha^2 + \beta^2 = 34$$
$$\alpha\beta = 15$$
$$\alpha = 3, \beta = 5$$
$$\therefore \alpha^2 + \beta = 9 + 5 = 14$$

2. Q2:

$$\Rightarrow \cot^{-1}\left(\frac{\alpha\beta+1}{\alpha-\beta}\right) + \cot^{-1}\left(\frac{\beta\gamma+1}{\beta-\gamma}\right) + \cot^{-1}\left(\frac{\alpha\gamma+1}{\gamma-\alpha}\right)$$
$$\Rightarrow \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\beta}\right) + \tan^{-1}\left(\frac{\beta-\gamma}{1+\beta\gamma}\right) + \pi + \tan^{-1}\left(\frac{\gamma-\alpha}{1+\gamma\alpha}\right)$$
$$\Rightarrow \left(\tan^{-1}\alpha - \tan^{-1}\beta\right) + \left(\tan^{-1}\beta - \tan^{-1}\gamma\right) + \left(\pi + \tan^{-1}\gamma - \tan^{-1}\alpha\right)$$
$$\Rightarrow \pi$$

3. Q3:

Assume $\sin^{-1}x = \theta$

$$\cos(2\theta) = \frac{1}{9}$$
$$\sin\theta = \pm\frac{2}{3}$$

as m and n are co-prime natural numbers,

$\mathrm{x} = \frac{2}{3}$

i.e. $m = 2, n = 3$

So, the quadratic equation becomes $2x^2 - 3x + 1 = 0$ whose roots are $\alpha = 1, \beta = \frac{1}{2}$

$\left(1, \frac{1}{2}\right)$ lies on $5x + 8y = 9$

4. Q4:

For $n \in \mathbb{N}$, if $\cot^{-1}3 + \cot^{-1}4 + \cot^{-1}5 + \cot^{-1}n = \frac{\pi}{4}$, then n is equal to .

Given the equation:

$\cot^{-1}3 + \cot^{-1}4 + \cot^{-1}5 + \cot^{-1}(n) = \frac{\pi}{4}$

we can use the identity for the sum of inverse cotangents. Starting with the first two terms:

$$\cot^{-1} 3 + \cot^{-1} 4 = \cot^{-1}\left(\frac{3\times4-1}{3+4}\right) = \cot^{-1}\left(\frac{11}{7}\right)$$

Now, adding the third term:

$$\cot^{-1}\left(\frac{11}{7}\right) + \cot^{-1}(5)$$

we apply the identity again:

$$\cot^{-1}\left(\frac{11}{7}\right) + \cot^{-1}\left(\frac{n\times5-1}{5+n}\right) = \frac{\pi}{4}$$

Rewriting this to isolate the sum of the terms, we proceed as follows:

$$\cot^{-1}\left(\frac{\left(\frac{11}{7}\times\frac{5n-1}{5+n}-1\right)}{\left(\frac{11}{7}+\frac{5n-1}{5+n}\right)}\right) = \frac{\pi}{4}$$

This simplifies to:

$$\frac{11}{7}\left(\frac{5n-1}{5+n}\right) - 1 = \frac{11}{7} + \frac{5n-1}{5+n}$$

Solving the equation:

$$\frac{55n-11}{5+n} - 1 = \frac{11}{7} + \frac{5n-1}{5+n}$$

Further simplification yields:

$$55n - 11 - 35 - 7n = 55 + 11n + 35n - 7$$

Bringing the terms together, we get:

$$48n - 46 = 48$$

Therefore:

$$2n = 94$$

So finally:

$$n = 47$$

5. Q5: $2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x=\tan\theta \quad \theta\in\left(0,\dfrac{\pi}{4}\right)$

$$2\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$2\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right]=\cos^{-1}[\cos(2\theta)]$$

$$\Rightarrow 2\left(\frac{\pi}{4}-\theta\right)=2\theta\Rightarrow\theta=\frac{\pi}{8}$$

$$\Rightarrow x=\tan\frac{\pi}{8}=\sqrt{2}-1\simeq 0.414$$

6. Q6: $\sin^{-1}\sin\theta-\left(\frac{\pi}{2}-\sin^{-1}\sin\theta\right)>0$

$\Rightarrow \sin^{-1}\sin\theta>\frac{\pi}{4}$

$\Rightarrow \sin\theta>\frac{1}{\sqrt{2}}$

So, $\theta\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$

$\theta\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)=(\mathrm{a},\mathrm{b})$

$b-a=\frac{\pi}{2}=\alpha-\beta$

$\Rightarrow \beta=\alpha-\frac{\pi}{2}$

$\Rightarrow \alpha x^2+\beta\mathrm{x}+\sin^{-1}\left[(\mathrm{x}-3)^2+1\right]+\cos^{-1}\left[(\mathrm{x}-3)^2+1\right]=0$

$x=3, 9\alpha+3\beta+\frac{\pi}{2}+0=0$

$$\Rightarrow 9\alpha+3\left(\alpha-\frac{\pi}{2}\right)+\frac{\pi}{2}=0$$

$$\Rightarrow 12\alpha-\pi=0$$

$$\alpha=\frac{\pi}{12}$$

7. Q7: $a_1=1, a_2, a_3, a_4, \ldots\ldots$ be consecutive natural numbers.

$\therefore \quad a_2 = 2, a_3 = 3, \ldots., a_{2021} = 2021, a_{2022} = 2022$

$$\tan^{-1}\left[\frac{1}{1+a_1 a_2}\right] = \tan^{-1}\left[\frac{1}{1+1\cdot 2}\right] = \tan^{-1}(1) - \tan^{-1}\left(\frac{1}{2}\right)$$

$$\tan^{-1}\left[\frac{1}{1+a_2 a_3}\right] = \tan^{-1}\left[\frac{1}{1+2\cdot 3}\right] = \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right)$$

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$$\tan^{-1}\left[\frac{1}{1+a_{2021} a_{2022}}\right] = \tan^{-1}\left[\frac{1}{1+2021\cdot 2022}\right]$$

$$= \tan^{-1}\left(\frac{1}{2021}\right) - \tan^{-1}\left(\frac{1}{2022}\right)$$

$\therefore \tan^{-1}\left(\frac{1}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{1}{1+a_2a_3}\right) + \ldots.. + \tan^{-1}\left(\frac{1}{1+a_{2021}a_{2022}}\right)$

$= \tan^{-1}(1) - \tan^{-1}\left(\frac{1}{2022}\right) = \frac{\pi}{4} - \cot^{-1}(2022)$

$= \frac{\pi}{4} - \left(\frac{\pi}{2} - \tan^{-1}(2022)\right) = \tan^{-1}(2022) - \frac{\pi}{4}$

8. Q8:

We're given the equation $\sin^{-1} x = 2\tan^{-1} x$ for x in the interval $(-1, 1]$. We want to find the number of solutions.

Step 1: Apply the sine and tangent functions to both sides :

We can rewrite the equation by applying the sine function to both sides :

$\sin(\sin^{-1} x) = \sin(2\tan^{-1} x)$.

This simplifies to:

$x = \sin(2\tan^{-1} x)$.

Step 2: Use the double-angle identity for sine :

Recall that $\sin(2y) = 2\sin(y)\cos(y)$. Applying this identity to the right-hand side gives :

$x = 2\sin(\tan^{-1} x)\cos(\tan^{-1} x)$.

Step 3: Use the identities for sine and cosine of an inverse tangent :

Recall that $\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$ and $\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$.
Substituting these into the equation gives :

$x = 2 \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}$.

This simplifies to :

$x = \frac{2x}{1+x^2}$.

Step 4: Solve for x :

We have :

$x = \frac{2x}{1+x^2}$.

Cross-multiplying gives :

$x(1+x^2) = 2x$.

This simplifies to :

$x^3 + x - 2x = 0$.

Rearranging terms gives :

$x^3 - x = 0$.

This factors to:

$x(x^2 - 1) = 0$.

Setting each factor equal to zero gives the solutions $x = 0$, $x = -1$, and $x = 1$.

However, we are given that $x \in (-1, 1]$. Therefore, the only solutions in this interval are $x = 0$ and $x = 1$.

So there are 2 solutions to the equation $\sin^{-1} x = 2\tan^{-1} x$ in the interval $x \in (-1, 1]$.

9. Q9: $50 \tan\left(\tan^{-1} \frac{1}{2} + 2\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}(2)\right)$
$+4\sqrt{2} \tan\left(\frac{\tan^{-1}}{2}(2\sqrt{2})\right)$

$\Rightarrow 50 \tan\left(\pi + \tan^{-1}\left(\frac{1}{2}\right)\right) + 4\sqrt{2} \tan\left(\frac{1}{2} \tan^{-1} 2\sqrt{2}\right)$

$\Rightarrow \quad 50\left(\frac{1}{2}\right)+4\sqrt{2}\tan\alpha$

Where $2\alpha=\tan^{-1}2\sqrt{2}$

$\Rightarrow \frac{2\tan\alpha}{1-\tan^2\alpha}=2\sqrt{2}$ .. (i)

$\Rightarrow \quad 2\sqrt{2}\tan^2\alpha+2\tan\alpha-2\sqrt{2}=0$

$\Rightarrow \quad 2\sqrt{2}\tan^2\alpha+4\tan\alpha-2\tan\alpha-2\sqrt{2}=0$

$\Rightarrow (2\sqrt{2}\tan\alpha-2)(\tan\alpha-\sqrt{2})=0$

$\Rightarrow \tan\alpha=\sqrt{2}$ or $\frac{1}{\sqrt{2}}$

$\Rightarrow \tan\alpha=\frac{1}{\sqrt{2}}$

($\tan\alpha=\sqrt{2}$ doesn't satisfy (i))

$\Rightarrow \quad 25+4\sqrt{2}\frac{1}{\sqrt{2}}=29$

10. Q10:

$$\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)\right)$$

$$=\cot\left(\sum_{n=1}^{50}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)\right)$$

$$=\cot\left(\sum_{n=1}^{50}(\tan^{-1}(n+1)-\tan^{-1}n\right)$$

$$=\cot(\tan^{-1}51-\tan^{-1}1)$$

$$=\cot\left(\tan^{-1}\left(\frac{51-1}{1+51}\right)\right)$$

$$=\cot\left(\cot^{-1}\left(\frac{52}{50}\right)\right)$$

$$=\frac{26}{25}$$

11. Q11: The star "*" in this context represents a binary operation, similar to addition (+), subtraction (-), multiplication (×), and division (÷). It is a custom operation defined by the problem statement, and the specific rules of the operation are provided in the problem.

    In this case, the operation "*" is defined by the equation

$x * y = x^2 + y^3$, which means if you have two numbers x and y, then the result of applying the "*" operation to them is $x^2 + y^3$.

The problem also specifies an additional rule for this operation: $(x * 1) * 1 = x * (1 * 1)$, which needs to be taken into account when solving the problem. This is a type of "associativity" condition.
Given,

$x \, * \, y = x^2 + y^3$

$\therefore x \, * \, 1 = x^2 + 1^3 = x^2 + 1$

Now, $(x \, * \, 1) \, * \, 1 = (x^2 + 1) \, * \, 1$

$\Rightarrow (x \, * \, 1) \, * \, 1 = (x^2 + 1)^2 + 1^3$

$\Rightarrow (x \, * \, 1) \, * \, 1 = x^4 + 1 + 2x^2 + 1$

Also, $x \, * \, (1 \, * \, 1)$

$= x \, * \, (1^2 + 1^3)$

$= x \, * \, 2$

$= x^2 + 2^3$

$= x^2 + 8$

Given that,

$(x \, * \, 1) \, * \, 1 = x \, * \, (1 \, * \, 1)$

$\therefore x^4 + 1 + 2x^2 + 1 = x^2 + 8$

$\Rightarrow x^4 + x^2 - 6 = 0$

$\Rightarrow x^4 + 3x^2 - 2x^2 - 6 = 0$

$\Rightarrow x^2(x^2 + 3) - 2(x^3 + 3) = 0$

$\Rightarrow (x^2 + 3)(x^2 - 2) = 0$

$\Rightarrow x^2 = 2, \, -3$

[$x^2 = -3$ not possible as square of anything should be always positive]

$\therefore x^2 = 2$

$\therefore$ Now,

$2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$

$= 2\sin^{-1}\left(\frac{2^2+2-2}{2^2+2+2}\right)$

$= 2\sin^{-1}\left(\frac{4}{8}\right)$

$= 2\sin^{-1}\left(\frac{1}{2}\right)$

$= 2 \times \frac{\pi}{6}$

$= \frac{\pi}{3}$

12. Q12:

$(\tan^{-1}x)^3 + (\cot^{-1}x)^3 = k\pi^3$

Let $f(t) = t^3 + \left(\frac{\pi}{2} - t\right)^3$

Where $t = \tan^{-1}x$; $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= t^3 + \left(\frac{\pi}{2}\right)^3 - \frac{3\pi^2 t}{4} + \frac{3\pi}{2}t^2 - t^3$

$f(t) = \frac{3\pi}{2}t^2 - \frac{3\pi^2}{4} . t + \frac{\pi^3}{8}$

This is a quadratic equation of t.

Here, coefficient of t^2 term is $\frac{3\pi}{2}$ which is > 0.

$\therefore$ It is a upward parabola.

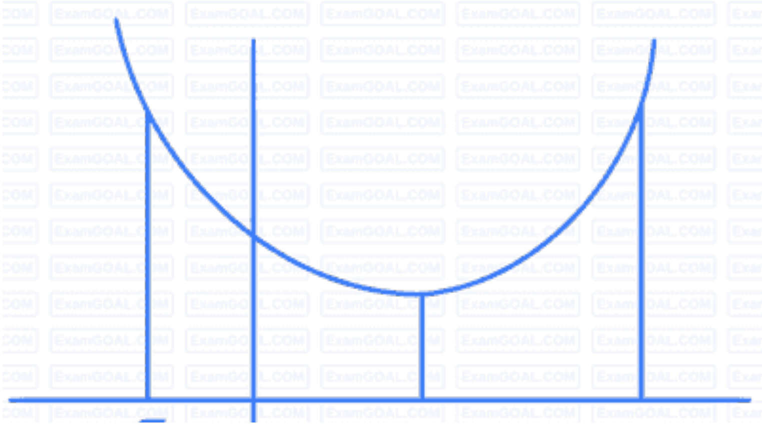
Now, $f'(t) = 3\pi t - \frac{3\pi^2}{4}$

$f''(t) = 3\pi > 0$

$\therefore 3\pi t - \frac{3\pi^2}{4} = 0$

$\Rightarrow t = \frac{\pi}{4}$ (minima)

$\therefore$ vertex of graph at $\frac{\pi}{4}$



$\therefore$ Minimum value at $\frac{\pi}{4}$ and maximum value at $-\frac{\pi}{2}$.

$$\therefore f\left(\frac{\pi}{4}\right)=\frac{\pi^3}{64}+\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^3=\frac{\pi^3}{32}$$

$$f\left(-\frac{\pi}{2}\right)=-\frac{\pi^3}{8}+\pi^3$$

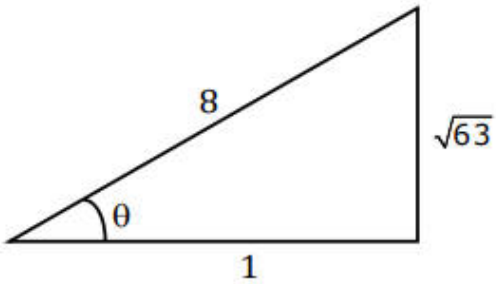
$$=\frac{7\pi^3}{8}$$

$$\therefore k\pi^3 \in\left[\frac{\pi^3}{32}, \frac{7\pi^3}{8}\right)$$

$$\Rightarrow k \in\left[\frac{1}{32}, \frac{7}{8}\right)$$

13. Q13: $\tan\left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8}\right)$

$$\sin^{-1}\left(\frac{\sqrt{63}}{8}\right)=\theta \sin\theta=\frac{\sqrt{63}}{8}$$



$$\cos\theta=\frac{1}{8}$$

$$2\cos^2\frac{\theta}{2}-1=\frac{1}{8}$$

$$\cos^2\frac{\theta}{2}=\frac{9}{16}$$

$$\cos\frac{\theta}{2}=\frac{3}{4}$$

$$\frac{1-\tan^2\frac{\theta}{4}}{1+\tan^2\frac{\theta}{4}}=\frac{3}{4}$$

$$\tan\frac{\theta}{4}=\frac{1}{\sqrt{7}}$$

14. Q14: $\cos ec\left(2\cot^{-1}(5)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

$\cos ec\left(2\tan^{-1}\left(\frac{1}{5}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

$= \cos ec\left(\tan^{-1}\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

$= \cos ec\left(\tan^{-1}\left(\frac{5}{12}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

Let $\tan^{-1}(5/12)=\theta \Rightarrow \sin\theta=\frac{5}{13}, \cos\theta=\frac{12}{13}$

and $\cos^{-1}\left(\frac{4}{5}\right)=\phi \Rightarrow \cos\phi=\frac{4}{5}$ and $\sin\phi=\frac{3}{5}$

$= \cos ec(\theta+\phi)$

$= \frac{1}{\sin\theta\cos\phi+\cos\theta\sin\phi}$

$= \frac{1}{\frac{5}{13}\cdot\frac{4}{5}+\frac{12}{13}\cdot\frac{3}{5}} = \frac{65}{56}$

15. Q15: $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$ $0 < a, b < 1$

$\Rightarrow \frac{a+b}{1-ab} = 1$

$a + b = 1 - ab$

$(a + 1)(b + 1) = 2$

Now $\left[a-\frac{a^2}{2}+\frac{a^3}{3}+\ldots.\right]+\left[b-\frac{b^2}{2}+\frac{b^3}{3}+\ldots.\right]$

$= \log_e(1+a)+\log_e(1+b)$

($\because$ expansion of $\log_e(1 + x)$)

$= \log_e[(1+a)(1+b)]$

$= \log_e 2$

16. Q16: $\tan^{-1}(x + 1) + \cot^{-1}\left(\frac{1}{x-1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \tan^{-1}\left(\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right) = \tan^{-1}\left(\frac{8}{31}\right)$

$\Rightarrow \frac{(1+x)+(x-1)}{1-(1+x)(x-1)} = \frac{8}{31}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{8}{31}$

$\Rightarrow 4x^2 + 31x - 8 = 0$

$\Rightarrow x = -8, \frac{1}{4}$

but at $x = \frac{1}{4}$

$LHS > \frac{\pi}{2}$ and $RHS < \frac{\pi}{2}$

So, only solution is x = $-$ 8 = $-\frac{32}{4}$

17. Q17: Given $a = (\sin^{-1}x)^2 - (\cos^{-1}x)^2$

$= (\sin^{-1}x + \cos^{-1}x)(\sin^{-1}x - \cos^{-1}x)$

$= \frac{\pi}{2}\left(\frac{\pi}{2} - 2\cos^{-1}x\right)$

$\Rightarrow 2\cos^{-1}x = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow \cos^{-1}(2x^2 - 1) = \frac{\pi}{2} - \frac{2a}{\pi}$

$\Rightarrow 2x^2 - 1 = \cos\left(\frac{\pi}{2} - \frac{2a}{\pi}\right)$