Toward more reliable printed stereo

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The printing and displaying of pictures as stereo pairs has been widespread for a long time, and yet it is still a common experience that published stereo pairs can be difficult to fuse, can cause eye-strain, can give a false or confusing sense of depth, and are sometimes even front-to-back inverted. This short note is intended to describe simply how to minimize or eliminate these common problems. The method is to match the focus and the convergence cues at the viewer's eyes, and to calculate the geometry reliably and accurately to better than the crude approximation now almost universally in use. Experience has shown that stereo so printed is acceptable to many people who normally cannot use the technique, and that a more secure sense of positioning and depth can be achieved.

Keywords: stereo, printing

INTRODUCTION

The basic techniques of drawing pictures in stereo were outlined by Rule[1] in 1938 in a formalism necessarily restricted because it was necessary to work by hand at that time. Since then, computer-generated stereo diagrams have become commonplace, especially for visualizing molecular structures, and also in some engineering and design applications. However, despite the long history of the technique, many people are still plagued by an inability to fuse the majority of published and displayed stereo images, or else can do so only with considerable discomfort. There are several reasons for this, which all have more to do with the stereo printed than with the unfortunate viewer. The most obvious to users of lens stereoscopes is a consequence of the almost universal acceptance of the recommendation by Rule (1941)[2] that the strength of the lenses be calculated to allow the eyes to relax to infinity. Eyes relaxed to infinity naturally point straight ahead, and objects viewed at infinity can display no parallax. Thus any geometry that results either in the eyes having to converge or in the left- and right-images appearing different must involve contradictory signals; these tend at best to reduce the credibility of the fused image, and cause discomfort to all but the most practiced viewers. Cheap commercial lens stereoscopes often also induce a high degree of chromatic aberration, which viewers with good vision can find disturbing.

A second reason is the common use of rotational transformations followed by orthogonal projection rather than a proper perspective calculation. These improper calculations necessarily induce a keystone distortion relative to the true geometry, to which the eye is notably intolerant. Small errors result in incorrectly curved fused images, while larger ones make the images very difficult to fuse. Indeed, an extra even more serious error is often observed: the already inappropriate rotations are calculated in the wrong direction, which hand-inverts the fused image. Even when the approximating rotations are calculated properly, if they do not agree with the relative positioning of the two images on the paper (which often appears to be a matter of guesswork), then a false sense of depth obtains. This effect is, indeed, the rule rather than the exception nowadays. Saunders[3] also noted that rotations are not really the correct way to perform the calculations (unless the paper corotates), and asserted that a lateral shift (equal to the gap between the two eyes) is formally required. This contradicted Johnson,[4] who had claimed (somewhat contentiously) that the visible difference is hardly discernible, and used this as a rather bizarre justification for the use of rotations, which are actually harder to compute (cf., Thomas[5]). The present author inclines towards Saunders' opinion, on the ground that the more accurate the printing, the more likely it is to be successful.

In this paper the entire calculation of stereo printing is reworked from first principles, with the deliberate intent of reducing to a minimum every avoidable error, and establishing calculations that can only be reliable. The original motivation for this arose from a project in which it was essential to be able to judge angles correctly in a simplified three-dimensional model of proteins.[6] A happy consequence was that several colleagues who were previously unable to utilize stereo images now have the facility.

This discussion is concerned mainly with improving printed stereo, and it should be noted that different techniques are available for electronically displayed images, where the possibility of changing the viewpoint dynamically can be exploited with spectacular success, even for people with only one eye.[7] The "simplified" homogeneous equations used with graphics hardware are also quite different from the more general ones developed here.[8]

The full calculations require some understanding of geometrical optics, and are sufficiently involved that it seems best to start with the more elementary calculation of true perspective without intermediary optics, which is, indeed,

Color Plates for this article are on page 42.

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Table 1. Symbol table

α	Projection factor, also used for depth cueing
E	Vector pointing from the principal point of the eye to the origin of the paper coordinates
M	Magnification applied to the object
O	Vector pointing from the origin of paper coordinates to the centroid of the object
Q	Projection onto the paper of **R**
R	Point within the object, measured from its centroid, O
S	Scaled sight line to the object origin; $\mathbf{S} = (\mathbf{E} + \mathbf{O})/M$
X, Y	Paper coordinates of **Q**, so that $\mathbf{Q} = \mathbf{X}X + \mathbf{Y}Y$
X, **Y**	Vectors defining the plane of the paper and (implicitly) the paper coordinate system
$\overline{\mathbf{E}}^{\mathrm{T}}$, $\overline{\mathbf{X}}^{\mathrm{T}}$, $\overline{\mathbf{Y}}^{\mathrm{T}}$	Constant vectors (strictly, covectors) for a given view

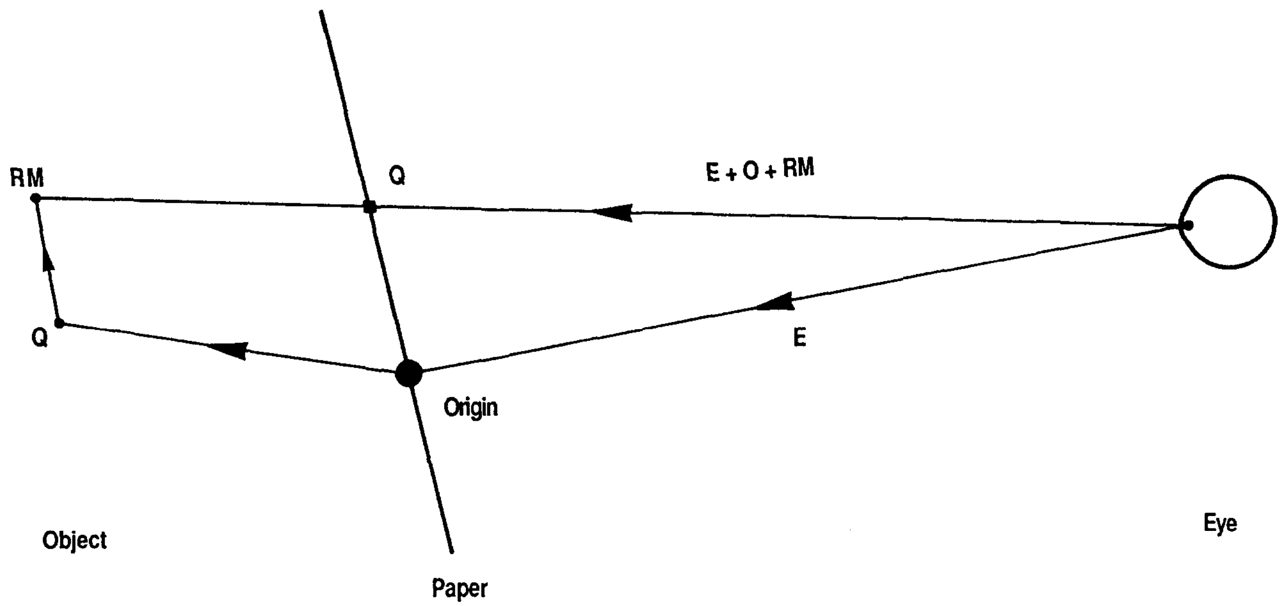

Figure 1. Calculation of perspective for viewing directly.

already sufficient for direct viewing of filter-separation (e.g., red/green) stereo.

CORRECT PERSPECTIVE AND DIRECT-VIEWING STEREO

It is necessary to know how to print perspective correctly before any questions of convergence and focus cueing in stereoception can be addressed. Perspective is properly calculated by establishing where a ray from a point within the object to be viewed intersects the plane of the paper on its path to the pupil of the eye. The geometry of this is shown in Figure 1 in a form that will be extended later. The surface on which the image is printed or displayed is assumed to be flat.

A typical point within an object centered on O is called R; this is printed or displayed at the point Q, which can be described by two scalar coordinates X and Y. The plane of the paper or screen can be described by two vectors, **X** and **Y** so that $\mathbf{Q} = \mathbf{X}X + \mathbf{Y}Y$, which is

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \begin{pmatrix} X_x \\ X_y \\ X_z \end{pmatrix} X + \begin{pmatrix} Y_x \\ Y_y \\ Y_z \end{pmatrix} Y \qquad (1)$$

in full component notation, with x, y, and z being any convenient Cartesian axes. Normally, **X** and **Y** would be mutually perpendicular vectors, say 1 mm long, but this restriction is neither required nor indeed necessarily useful. On graphics equipment, for example, they could be of different lengths, and be used to span the pixel grid directly; then X and Y would be just the conventional pixel coordinates.

It is often useful to be able to magnify the object being viewed, so the position of a general point within it can be written as $(\mathbf{O} + \mathbf{R}M)$, where M is the desired magnification, and **R** is measured from **O**. The position of the required mark on the paper must be the position of the eye, $-\mathbf{E}$ (whose sign is chosen to reduce the number of minus signs in the final results) plus some as yet unknown multiple of the vector from the eye to the object point being plotted, which is $\mathbf{E} + \mathbf{O} + \mathbf{R}M$. This multiple is specified so that the sum of the two vectors lies in the plane of the paper. Writing it as α/M (with the benefit of hindsight) gives the determining equation for $\mathbf{Q}$, the position of the printed image of R:

$$[\mathbf{X} \quad \mathbf{Y}]\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{Q} = (\mathbf{E} + \mathbf{O} + \mathbf{R}M)\frac{\alpha}{M} - \mathbf{E}$$
$$= (\mathbf{S} + \mathbf{R})\alpha - \mathbf{E} \tag{2}$$

The frequently occurring constant term

$$\mathbf{S} = \left(\frac{\mathbf{E} + \mathbf{O}}{M}\right) \tag{3}$$

is used to simplify the final results. The factor α will turn out to be related to depth cueing, and for this reason was first divided by M, forcing it to behave correctly in this connection without an extra subsidiary correction for the chosen magnification.

A minor rearrangement of Equation 2:

$$[\mathbf{E} \quad \mathbf{X} \quad \mathbf{Y}]\begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = (\mathbf{S} + \mathbf{R})\alpha \tag{4}$$

can be premultiplied by the inverse of the matrix $[\mathbf{E} \quad \mathbf{X} \quad \mathbf{Y}]$,

$$[\mathbf{E} \quad \mathbf{X} \quad \mathbf{Y}]^{-1} = \begin{bmatrix} \overline{\mathbf{E}}^{\mathrm{T}} \\ \overline{\mathbf{X}}^{\mathrm{T}} \\ \overline{\mathbf{Y}}^{\mathrm{T}} \end{bmatrix} \tag{5}$$

to give:

$$\begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{E}}^{\mathrm{T}} \\ \overline{\mathbf{X}}^{\mathrm{T}} \\ \overline{\mathbf{Y}}^{\mathrm{T}} \end{bmatrix}(\mathbf{S} + \mathbf{R})\alpha \tag{6}$$

which is written in such a way that exploits the fact that $\overline{\mathbf{E}}^{\mathrm{T}}$, $\overline{\mathbf{X}}^{\mathrm{T}}$, $\overline{\mathbf{Y}}^{\mathrm{T}}$, and $\mathbf{S}$ are all constant for a given eye position. It is easy to see from the top row of Equation 6 that the unknown α is just:

$$\alpha = \frac{1}{\overline{\mathbf{E}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})} \tag{7}$$

which gives, equally directly:

$$X = \frac{\overline{\mathbf{X}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})}{\overline{\mathbf{E}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})} = \alpha\,\overline{\mathbf{X}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R}) \tag{8}$$

and

$$Y = \frac{\overline{\mathbf{Y}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})}{\overline{\mathbf{E}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})} = \alpha\overline{\mathbf{Y}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R}) \tag{9}$$

This means that for each image point $\mathbf{Q}$, it is necessary only to sum a pair of vectors $(\mathbf{S} + \mathbf{R})$, take three scalar (i.e., dot) products (with $\overline{\mathbf{E}}^{\mathrm{T}}$, $\overline{\mathbf{X}}^{\mathrm{T}}$, $\overline{\mathbf{Y}}^{\mathrm{T}}$), and two scalar divides by $1/\alpha$, to print an exact perspective view of $\mathbf{R}$. The equations here are more general than those commonly used, because no right angles have been used anywhere in the construction, and neither the paper nor the viewer have been tied to a specific reference frame.

The correctly depth-cued width of a short line segment on the paper is its width in the object multiplied by α, as can be verified from the gradient of the paper coordinates with respect to small changes in the object position:

$$\nabla_{\mathbf{R}}X \equiv \frac{\partial X}{\partial \mathbf{R}} = \frac{\overline{\mathbf{E}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})\overline{\mathbf{X}}^{\mathrm{T}} - \overline{\mathbf{X}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})\overline{\mathbf{E}}^{\mathrm{T}}}{[\overline{\mathbf{E}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})]^2}$$
$$= \alpha(\overline{\mathbf{X}}^{\mathrm{T}} - X\overline{\mathbf{E}}^{\mathrm{T}}) \tag{10}$$

Thus

$$\delta X = \nabla_{\mathbf{R}}X\,\delta\mathbf{R} \equiv \frac{\partial X}{\partial \mathbf{R}}\delta\mathbf{R} = \alpha(\overline{\mathbf{X}}^{\mathrm{T}} - X\overline{\mathbf{E}}^{\mathrm{T}})\,\delta\mathbf{R} \tag{11}$$

The latter term describes the change of image scale as the object point R moves towards or away from the viewer, while more pertinently, the first term indicates that the lateral width of a feature (i.e., in the direction of $\mathbf{X}$) scales directly as α (remembering that $\overline{\mathbf{X}}^{\mathrm{T}}$ is constant, as is $\overline{\mathbf{E}}^{\mathrm{T}}$). Similar equations hold for Y by direct substitution of Y for X and of $\overline{\mathbf{Y}}^{\mathrm{T}}$ for $\overline{\mathbf{X}}^{\mathrm{T}}$.

All of the equations necessary to print perspective correctly have now been established. Stereo follows easily by specifying two different positions for the eye. Typical values of the defining vectors for this method are (assuming for the sake of example the use of the modern graphics language 'PostScript' with the default paper orientation[9]):

$$\mathbf{X} = \begin{pmatrix} 0\,\text{mm} \\ 0\,\text{mm} \\ -1\,\text{mm} \end{pmatrix} \tag{12}$$

for the paper X-axis,

$$\mathbf{Y} = \begin{pmatrix} 0\,\text{mm} \\ 1\,\text{mm} \\ 0\,\text{mm} \end{pmatrix} \tag{13}$$

for the paper Y-axis. Then, for a viewing distance of 500 mm and a typical interocular separation of 66 mm, the two eye positions are

$$-\mathbf{E}_{\mathrm{L}} = \begin{pmatrix} 500\,\text{mm} \\ -33\,\text{mm} \\ 0\,\text{mm} \end{pmatrix} \tag{14}$$

for the left eye, and

$$-\mathbf{E}_{\mathrm{R}} = \begin{pmatrix} 500\,\text{mm} \\ +33\,\text{mm} \\ 0\,\text{mm} \end{pmatrix} \tag{15}$$

for the right eye. These vectors are all expressed on a conventional right-handed coordinate system familiar to crystallographers, which has the x-axis pointing toward the viewer, the y-axis pointing to the right, and the z-axis pointing upwards. Graphics calculations have often been referred to another frame, which suffers from the marked disadvantage of confusing the Cartesian three-dimensional reference axes x, y, and z with the vectors $\mathbf{X}$ and $\mathbf{Y}$ defining the plane of the paper or screen. A frequently defined Z-axis often compounds this confusion by being arranged so that $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ form an otherwise rarely used left-handed set (e.g., Angell[10]).

It is necessary to be able to separate the two images, and an old method now little used is to print one in red and one in

Table 2. Further symbols

$\mathbf{A}$, $\mathbf{A}'$	Entrance and exit principal axes of the viewing optic
$\mathcal{J}$	Transformation matrix describing a viewing optic
$\mathbf{P}$	Vector pointing from the entrance principal point of viewing optic to the paper origin
$\mathbf{P}'$	Vector pointing from the exit principal point of viewing optic to the paper origin
$\mathbf{U}$, $\mathbf{U}'$	First principal lateral axis of entrance and exit principo-nodal planes of the viewing optic
$\mathbf{V}$, $\mathbf{V}'$	Second principal lateral axis of entrance and exit principo-nodal planes of the viewing optic

green. Filters in front of each eye force one of the two images to disappear, and have a negligible effect on the viewing geometry, so the equations are not affected. To maximize the success of what will inevitably be a rather artificial-looking view with differently colored filters, the object should be centered on the plane of the paper, which optimally matches the eyes' focusing cues with the convergence cues. This means that the vector $\mathbf{O}$ should be in the plane of the paper, easily achieved by setting it to be the null vector $\mathbf{O} = \mathbf{0}$. When using PostScript, however, the origin of paper coordinates will not be in the center of the sheet, and it may then be necessary to translate the image accordingly. For A4 paper, the appropriate translations would be $+105.1$ mm along $\mathbf{X}$ and $+148.7$ mm along $\mathbf{Y}$. It is also possible to achieve the same results by repositioning the origin of the paper coordinates directly, swinging both of the vectors $\mathbf{O}$ and $\mathbf{E}$.

The techniques of this section are also appropriate when the image separation is achieved with polarizing filters. This is quite a common method of displaying stereo transparencies, but the only printed example known to the author is an obviously expensive stereoception test card used by optometrists, which does, however, induce an extremely good impression of solidity.

SOME METHODS OF GEOMETRICAL OPTICS

A stereoscope is normally inserted between the eye and the paper or screen; such devices are designed to help the viewer see in stereo, but as was noted earlier they achieve this with widely varying degrees of success. There are many types in use, consisting variously of electrooptical shutters, polarizers, Ortony mirrors, or strongly converging lenses. The latter choice is the most popular for viewing stereo pairs printed in journals, where space is at a premium, and is usually the least successful because of inappropriate optical design. Despite the apparently bewildering variety of optics, it is possible to use a single analytical description regardless of the optic chosen by making recourse to the general geometrical theory of paraxial optics. Fortunately, very little of this theory is needed: it suffices to know that any complete imaging system can be described by principal entrance and exit axes and by special planes known as principal and nodal planes.[11] Principal planes are characterized by the property that a ray penetrating the input principal plane a certain distance from the input axis will leave the output principal plane at the same distance from the output axis, but at a generally different relative angle. The intersection of a principal axis and the corresponding principal plane is called a principal point. Nodal points are defined similarly as the intersection of a nodal plane with the appropriate axis. Nodal planes are characterized by the different property that a ray entering the input nodal point at a certain angle from the axis will exit from the output nodal point at the same angle relative to the output axis. When, as is certainly the case with a stereoscope, the device is an ordinary optical one surrounded by a single medium like air, the principal and nodal planes coincide helpfully in what can be called a principo-nodal plane. The position of the entrance principal point relative to the origin of the paper coordinates can be called $-\mathbf{P}$, from which radiate the principal entrance axis $\mathbf{A}$, and the two defining axes of the entrance principo-nodal plane, $\mathbf{U}$ and $\mathbf{V}$. It is necessarily possible to define $\mathbf{A}$, $\mathbf{U}$, and $\mathbf{V}$ as a righthanded set of mutually perpendicular unit vectors for any normal optical device, but this restriction is also neither required nor necessarily useful in the present discussion. The usual notation when working with elementary optical devices is to prime (i.e., mark with ′) the exit quantities. The exit properties of the viewing device are thus represented similarly by $\mathbf{A}'$, $\mathbf{U}'$, and $\mathbf{V}'$, all tied to $-\mathbf{P}'$.

Functionally, the eight vectors describing the viewing device fall naturally into two families: $\mathbf{P}$ and $\mathbf{P}'$ describe translations of the image, while $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{U}$, $\mathbf{U}'$, $\mathbf{V}$, and $\mathbf{V}'$ describe any rotational (or other) transformation. These latter six are most conveniently combined for the present purposes into a matrix such as:

$$\mathcal{J} = [\mathbf{A} \quad \mathbf{U} \quad \mathbf{V}] \begin{bmatrix} \overline{\mathbf{A}'^{\mathrm{T}}} \\ \overline{\mathbf{U}'^{\mathrm{T}}} \\ \overline{\mathbf{V}'^{\mathrm{T}}} \end{bmatrix}, \tag{16}$$

the righthand part of which is the inverse of $[\mathbf{A}' \quad \mathbf{U}' \quad \mathbf{V}']$. The matrix $\mathcal{J}$ as defined above represents the change in direction of a ray passing through the viewing device in the reverse direction. The utility of this will be seen shortly. Explicitly, the right half of $\mathcal{J}$ resolves an exit ray into its three components on the $\mathbf{A}'\,\mathbf{U}'\,\mathbf{V}'$ reference frame, whereupon the left half creates the entrance ray with those same components.

PERSPECTIVE WITH GENERAL OPTICS

This section reworks the calculations of the second section with the inclusion of an optical device characterized by the $\mathcal{J}$, $\mathbf{P}$, and $\mathbf{P}'$ as defined above. It is illustrated by Figure 2, whose main purpose is to show that if the eye is focused correctly on $\mathbf{R}M$, it is possible to use the stereoscope exit principal point, $\mathbf{P}'$, instead of the input principal point of the eye, $\mathbf{E}$, to simplify the calculations. This construction is sufficient to handle an ordinary mirror stereoscope where no further complications arise.

The eye is shown with its pupil centered on the appropriate principal output axis, $\mathbf{A}'$, of the stereoscope, which is thus

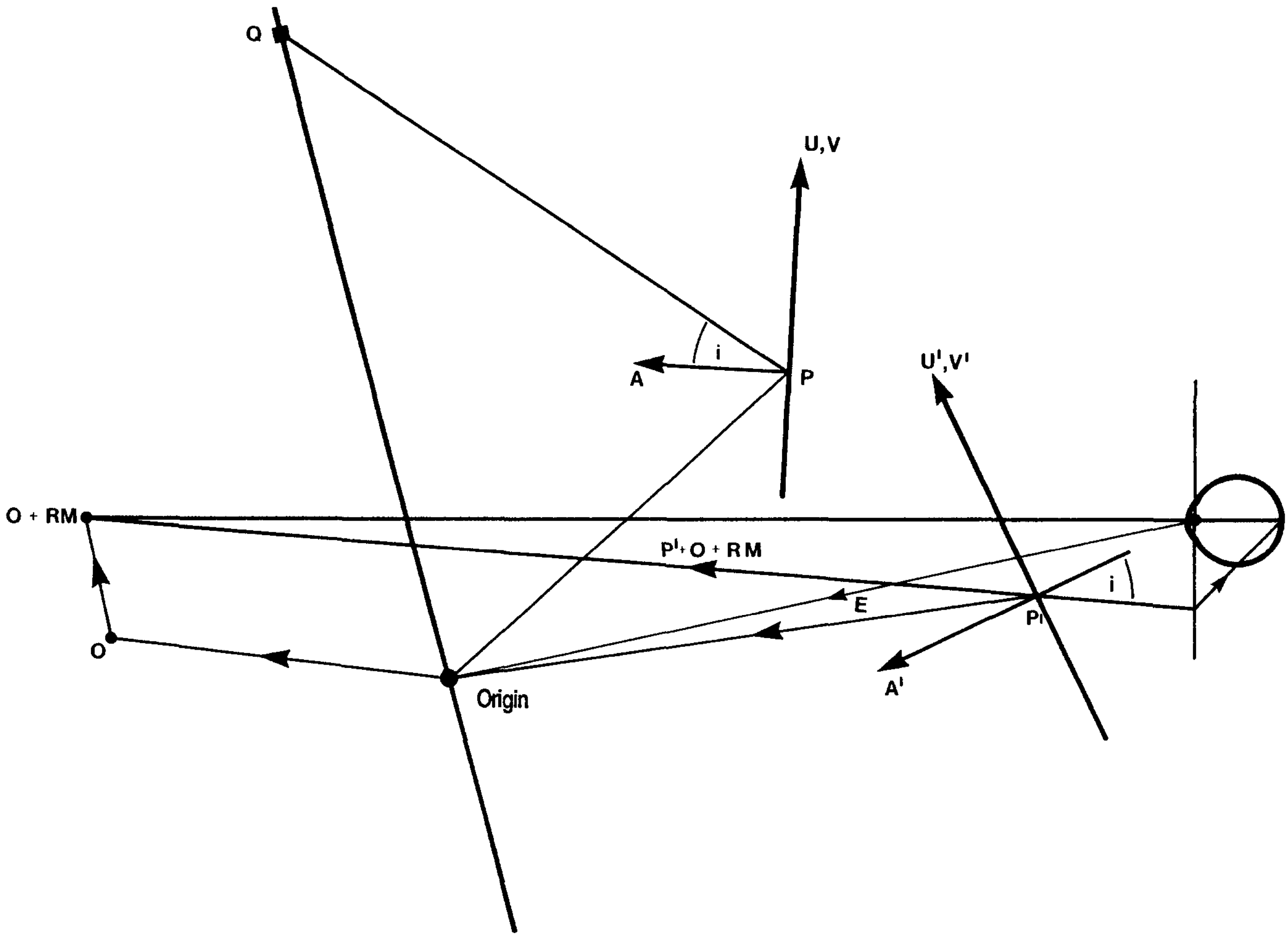

Figure 2. Perspective through a general viewing device.

assumed to be properly adjusted. The eye is taken to point toward and be focused on the point being visualized. This introduces an unavoidable approximation unless the point being viewed is in the correct focal plane, but it can be argued that as long as the entire visualized image lies within the conventionally defined depth of field, no problems should arise. This rejoinder is, however, rather glib, and perhaps a bit on the pessimistic side. The reasons are that brighter illumination could be used, contracting the pupils and increasing the depth of field; practice makes the viewer more tolerant; and few people have "perfect" (i.e., diffraction limited) vision, though an effect of the implied residual aberrations can be to increase the perceived depth of field. (Indeed, camera lens designers are often more lenient on spherical aberration than might be supposed because it is particularly useful in this regard.) Assuming a pupil of 3-mm diameter (which is typical in normal office lighting), a viewing distance of 720 mm and light of 5000-Å wavelength, gives a depth of field of ± 57.6 mm, using Formula 14–18 in Longhurst,[11] corresponding to a maximum object depth of *ca.* 120 mm. The author's experience is that this estimate is accurate and can be exceeded only slightly for unpracticed viewing. People with practice or with less than perfect vision will find it over-stringent. The imaging resolution within the object under the conditions specified above is *ca.* 0.12 mm.

With the condition of correct focus, the imagined ray from $\mathbf{R}M$ to the effective center of the pupil of the eye, $\mathbf{E}$, can be replaced by a generally noncentral ray radiating from $\mathbf{R}M$ that will nonetheless still land at the same point on the retina. (The small angle between these two rays is exaggerated for clarity of illustration.) This new ray can be chosen without any loss of generality to intersect the exit principal point of the stereoscope, $\mathbf{P}'$. Exploiting the fact that this point is also a nodal point, the ray from $\mathbf{Q}$ to $\mathbf{P}$ that would produce the required ray from $\mathbf{R}M$ to $\mathbf{P}'$ is determined by setting equal the two angles, i, from the relevant principal axes.

Reexpressing this mathematically, the vector used earlier, $\mathbf{E} + \mathbf{O} + \mathbf{R}M$, is now replaced by $\mathbf{P}' + \mathbf{O} + \mathbf{R}M$, which is the sight line from the exit principal point of the viewer to the general point within the object being visualized. This is referred to the entrance of the viewing optic to establish where the required input ray would intersect the paper. Only the direction of the input ray is required, however, which is obtained by premultiplying by $\mathcal{I}$, since the absolute position of the ray is defined by being tied to the input principal point, $\mathbf{P}$, of the viewing optic. Thus Equation (2) becomes instead:

$$[\mathbf{X} \quad \mathbf{Y}]\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{Q} = \mathcal{I}(\mathbf{P}' + \mathbf{O} + \mathbf{R}M)\frac{\alpha}{M} - \mathbf{P}$$
$$= \mathcal{I}(\mathbf{S} + \mathbf{R})\alpha - \mathbf{P} \tag{17}$$

where

$$\mathbf{S} = \frac{(\mathbf{P}' + \mathbf{O})}{M} \tag{18}$$

replacing the earlier definition.

Rearranging Equation 17 as before and premultiplying by the inverse of the matrix [**P** **X** **Y**], gives

$$\begin{bmatrix} 1 \\ X \\ Y \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{P}}^{\mathrm{T}} \\ \overline{\mathbf{X}}^{\mathrm{T}} \\ \overline{\mathbf{Y}}^{\mathrm{T}} \end{bmatrix} \mathcal{I}(\mathbf{S} + \mathbf{R})\alpha = \begin{bmatrix} \overline{\mathcal{P}}^{\mathrm{T}} \\ \overline{\mathcal{X}}^{\mathrm{T}} \\ \overline{\mathcal{Y}}^{\mathrm{T}} \end{bmatrix} (\mathbf{S} + \mathbf{R})\alpha \tag{19}$$

which is also written in such a way that exploits the fact that $\overline{\mathcal{P}}^{\mathrm{T}} = \overline{\mathbf{P}}^{\mathrm{T}}\mathcal{I}$, $\overline{\mathcal{X}}^{\mathrm{T}} = \overline{\mathbf{X}}^{\mathrm{T}}\mathcal{I}$, $\overline{\mathcal{Y}}^{\mathrm{T}} = \overline{\mathbf{Y}}^{T}\mathcal{I}$, and **S** are all constant for a given viewing position, regardless of the vagaries of the viewing device. Then α becomes

$$\alpha = \frac{1}{\overline{\mathcal{P}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})} \tag{20}$$

so that

$$X = \frac{\overline{\mathcal{X}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})}{\overline{\mathcal{P}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})} = \alpha\overline{\mathcal{X}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R}) \tag{21}$$

and

$$Y = \frac{\overline{\mathcal{Y}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})}{\overline{\mathcal{P}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R})} = \alpha\overline{\mathcal{Y}}^{\mathrm{T}}(\mathbf{S} + \mathbf{R}) \tag{22}$$

just like before. These results are always true, regardless of the complexity of the viewing optics, providing only that they do not distort. This is still true if the device rotates the image, and is true even if the optical magnification is anisotropic, because this can be accommodated by varying the vectors **U**, **U**′, **V**, and **V**′. The correctly depth-cued width of a short line segment on the paper is also still given by its width in the object multiplied by α.

A MIRROR STEREOSCOPE

With the advent of easily affordable high quality A3 and A4 laser printers, it now seems most convenient for everyday use to print rather large stereo pairs filling the page, and to view them with an inexpensive mirror stereoscope. Optically, mirror stereoscopes offer the advantage of complete absence of chromatic aberration, and so long as the mirrors are flat, of not perturbing the focus. It can therefore be argued that they offer an almost ideal combination of optimal optical performance at minimal cost. Present-day laser printers already meet the resolution requirement implied by the imaging resolution given in the last section, if they are used with a mirror stereoscope as suggested here.

Figure 3 shows one half of a mirror stereoscope in parallel alignment. It is not possible to define either principal axes or principal points uniquely for plane mirrors, which allows a choice to be made purely for analytical convenience. The choice of axis is easy: it is taken simply as an extension of the principal axis of the eye, which is well-defined. The principal points are then chosen to be the reflections in the two mirrors of the point which bisects the principal ray on its passage between the two mirrors. The difference in position between the two principal points is then given rather conveniently as twice the normal separation of the mirrors. The justification for this choice is that the principal points so defined move only slightly when the mirrors are not exactly parallel (which is not true of points further along the sight lines), and can then be used as a good approximation to the true positions even when the mirrors are not quite parallel, as is usually the case. The rather striking offset between the input and output principal points shows immediately the utility of the equations developed in the last section.

For illustration, suitable values for all of the defining vectors are tabulated for printing on the whole area of both A3 and A4 papers. These tables are calculated as follows (by a FORTRAN routine): **O**, **X**, and **Y** take the values previously

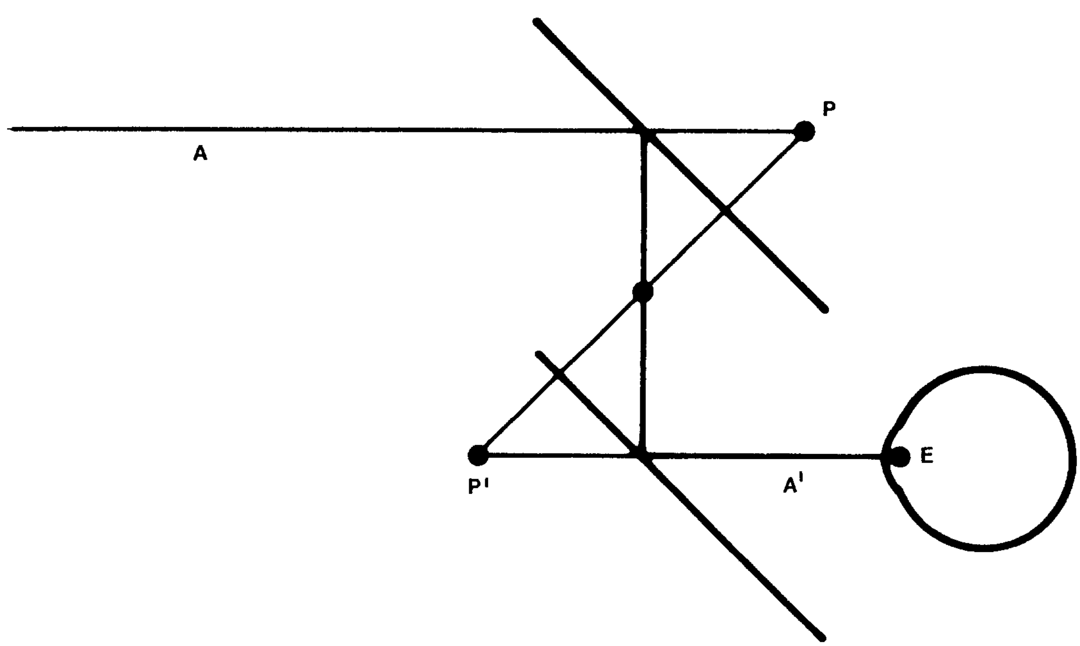

Figure 3. Principal axes of a mirror stereoscope.

specified as suitable for use with a 'PostScript' printer, **O** being the null vector to minimize discrepancies between focus and convergence cues; **P**′ is set to the nominal viewing distance of 720 mm less half the longitudinal distance between the two principal points, and then further offset sideways by ±33 mm for the two eyes; **A**′ is a unit vector parallel to **P**′ that defines the axis along which each eye can be expected to look (these axes converge on the object origin); **U**′ and **V**′ are set to be unit vectors mutually perpendicular to **A**′, with **V**′ being perpendicular to **A**′ and to the line between the two eyes; **P** is shifted from **P**′ backwards and outwards (sideways), each by 33.5 mm, describing the effect of an inexpensive and popular commercially available mirror stereoscope; **A** is a unit vector pointing from **P** to a position one-quarter of the paper's length away from the center of the paper in the direction of the intended eye, to maximize the usable area; finally, **U** and **V** follow the same calculations as their primed counterparts.

Table 3 shows values appropriate for printing stereo on A3 paper, the upper three rows being the x, y, and z components of the vectors for the left image, while the lower three rows are the same for the right image. Table 4 shows the same vectors for A4 paper. Note that only **A** and **U** have changed. An example of stereo printed with these values is displayed in Color Plate 1.

LENS STEREOSCOPE

Lens stereoscopes display the new feature of changing the focus. If the focal length of the lenses is f and the distance from the lens to the paper is ℓ, then the printed image will appear to be in focus a distance p behind the true plane of the paper, given by:

$$p = \frac{\ell^2}{f - \ell} \tag{23}$$

which is approximately the same as $|\mathbf{A}|^2/(f - |\mathbf{A}|)$ or $|\mathbf{A}'|^2/(f - |\mathbf{A}'|)$, assuming normal geometries. To be practically useful, this means that the stereoscope lenses should be between 1 diopter and 1.4 diopters weaker than Rule's prescription, giving viewing distances of 1 m to *ca.* 700 mm, respectively. Thin symmetrical lenses have their two principal points rather close to each other, and it is a commonly accepted approximation to assume that they are in fact the same. This assumption is certainly adequate for the type of lens stereoscope commonly used for viewing printed stereo pairs of protein structures, and a set of vectors calculated this way is shown as Table 5. It is not possible to bring the principal point of the eye, about 1.5 mm behind the front of the cornea, into juxtaposition with the output principal point of a simple stereoscope lens; this must mean that some

Table 3. Stereo parameter values for A3 paper

		O	**X**	**Y**	**P**	**A**	**U**	**V**	**P′**	**A′**	**U′**	**V′**
	x	0.0	0.0	0.0	−736.75	−0.9985	−0.0553	0.0	−703.25	−0.9990	0.0437	0.0
L	y	0.0	0.0	1.0	64.28	−0.0553	0.9985	0.0	30.78	0.0437	0.9990	0.0
	z	0.0	−1.0	0.0	0.00	0.0000	0.0000	−1.0	0.00	0.0000	0.0000	−1.0
	x	0.0	0.0	0.0	−736.75	−0.9985	0.0553	0.0	−703.25	−0.9990	−0.0437	0.0
R	y	0.0	0.0	1.0	−64.28	0.0553	0.9985	0.0	−30.78	−0.0437	0.9990	0.0
	z	0.0	−1.0	0.0	0.00	0.0000	0.0000	−1.0	0.00	0.0000	0.0000	−1.0

Table 4. Stereo parameter values for A4 paper

		O	**X**	**Y**	**P**	**A**	**U**	**V**	**P′**	**A′**	**U′**	**V′**
	x	0.0	0.0	0.0	−736.75	−0.9999	−0.0136	0.0	−703.25	−0.9990	0.0437	0.0
L	y	0.0	0.0	1.0	64.28	−0.0136	0.9999	0.0	30.78	0.0437	0.9990	0.0
	z	0.0	−1.0	0.0	0.00	0.0000	0.0000	−1.0	0.00	0.0000	0.0000	−1.0
	x	0.0	0.0	0.0	−736.75	−0.9999	0.0136	0.0	−703.25	−0.9990	−0.0437	0.0
R	y	0.0	0.0	1.0	−64.28	0.0136	0.9999	0.0	−30.78	−0.0437	0.9990	0.0
	z	0.0	−1.0	0.0	0.00	0.0000	0.0000	−1.0	0.00	0.0000	0.0000	−1.0

Table 5. Stereo parameter values for thin lenses

		O	**X**	**Y**	**P**	**A**	**U**	**V**	**P′**	**A′**	**U′**	**V′**
	x	−720.0	0.0	0.0	−120.0	−0.9993	0.0379	0.0	−120.0	−0.9993	0.0379	0.0
L	y	0.0	0.0	1.0	31.9	0.0379	0.9993	0.0	31.9	0.0379	0.9993	0.0
	z	0.0	−1.0	0.0	0.0	0.0000	0.0000	−1.0	0.0	0.0000	0.0000	−1.0
	x	−720.0	0.0	0.0	−120.0	−0.9993	−0.0379	0.0	−120.0	−0.9993	−0.0379	0.0
R	y	0.0	0.0	1.0	−31.9	−0.0379	0.9993	0.0	−31.9	−0.0379	0.9993	0.0
	z	0.0	−1.0	0.0	0.0	0.0000	0.0000	−1.0	0.0	0.0000	0.0000	−1.0

enlargement of the printed image occurs, which can be modeled in accord with the accompanying effect of the plane of the paper appearing to rise towards the viewer. The author has found no advantage in making the appropriate corrections, however, which seem to have little effect other than shrinking each half of the printed image.

CONCLUSION

The analysis given above shows that stereo can actually be printed more accurately and reliably with simpler equations than the rotations and orthogonal projection in common use. The equations here are not tied undesirably to any particular reference frame, and being fully vectorial in form do not impose restrictive right angles unnecessarily. The equations automatically determine the relative positions of the left and right images on the paper or screen, and being free from rotations, do not allow the possibility of the all too frequent inversions of hand. Depth cueing is also naturally calculable within the same framework.

By making recourse to some elementary methods of geometrical optics, it is possible to minimize the disagreement between focus and convergence cues for any type of stereoscope. This makes viewing much more relaxed and natural, and experience has shown it to be acceptable to many people who normally find stereo images difficult or impossible to use.

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