Efficient generation of the cartesian coordinates of truncated icosahedron and related polyhedra

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Efficient algorithms for deriving the analytical expressions of the rectangular coordinates of the vertices of regular polyhedra and truncated icosahedron inscribed in a cube is described and the results are exposed. Various characteristic quantities of the geometrical structure of truncated icosahedron are obtained. Kaleidoscopes for projecting the truncated icosahedron are discussed. © 2001 by Elsevier Science Inc.

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INTRODUCTION

From the time of Plato and Archimedes, polyhedra, especially of high symmetry, such as regular and semi-regular ones, have been drawing the attention of not only mathematicians and scientists but also artists, architects, and beauty-appreciating people of all times and places,[1–9] not to mention their geometrical analysis. There has also been a huge accumulation of know-how on constructing paper models of polyhedra.[1,5–7]

Recently, however, owing to the rapid progress in computer software, it has become quite easy to draw complicated molecular structures, such as fullerenes and the related polyhedra, even without detailed knowledge of their geometries and cartesian coordinates of the vertices. Those numerical values can in principle be derived from the data of the regular polyhedra, through available formula-transforming software, such as Mathematica.[10] The number of digits carried by this software is large enough for constructing more complicated polyhedra. However, these coordinates are not suitable for further study because symmetry is not carefully taken into account and analytical expression is lacking there. On the other hand, although various characteristic quantities for the regular and semi-regular polyhedra seem to have been well documented,[1,6] many of those data are found to be inaccurate or sometimes incorrect (for example, almost all the characteristic quantities of TI given in Williams[6] are in error beyond the fifth digit of significant figures).

The purpose of this article is to expose analytical and symmetric cartesian coordinates and characteristic quantities of the truncated icosahedron and the related polyhedra together with the algorithm for deriving these quantities. Kaleidoscopes to project an image of a soccerball are also discussed.[11]

Generation of Regular Polyhedra

Consider a $2 \times 2 \times 2$ cube located in the 3-D rectangular coordinate system so that the eight vertices occupy the positions at $(\pm1, \pm1, \pm1)$, called set C. In this note the combined use of $\pm$ signs is meant to take all the possible combinations. The rectangular coordinates of the four other regular polyhedra inscribed in this cube can quite easily be derived as follows.

A regular tetrahedron can be generated by taking a set of four such vertices from C that are not connected with each other (See Figure 1a). The set F of six face-centers of the cube, i.e., $(\pm1, 0, 0)$, $(0, \pm1, 0)$, and $(0, 0, \pm1)$, forms a regular octahedron.

For generating dodecahedron and icosahedron the three-fold rotational symmetry of the four diagonals of the cube is used (See Figure 1b, 1c). Namely, one of the corners, say (1, 1, 1), is surrounded by a triplet of the vertices, $(1, c, 0)$, $(0, 1, c)$, and $(c, 0, 1)$, forming a regular triangle (see Figure 1b). These vertices belong to the set A, $(\pm1, \pm c, 0)$, $(0, \pm1, \pm c)$, and $(\pm c, 0, \pm1)$ with $0 < c < 1$. Similarly to (1, 1, 1) each corner can find a nearest-neighbor triangle determined by three vertices belonging to A. Draw six segments by joining a pair of vertices on each square, and one gets the inscribed icosahedron, E, which becomes regular if c is the reciprocal of the golden ratio, or $c = (\sqrt{5} - 1)/2$.

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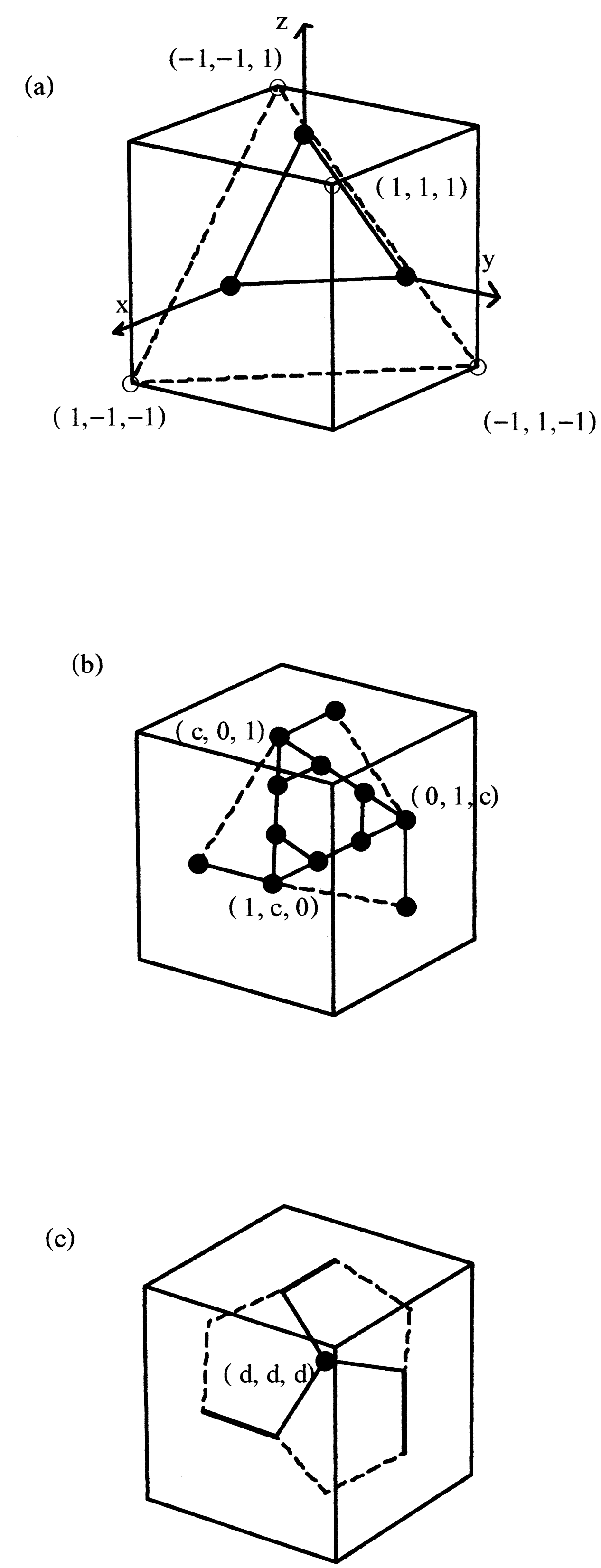

Figure 1. Illustrations for showing the relation between the $2 \times 2 \times 2$ cube and inscribed polyhedra. Refer to Tables 1–3. Only key vertices and edges are drawn. (a) Regular tetrahedron and regular octahedron. (b) Regular icosahedron and truncated icosahedron. (c) Regular dodecahedron.

Choose eight vertices D defined by $(\pm d, \pm d, \pm d)$ with $d = (\sqrt{5} - 1)/2$ instead of C, combine them with A but with $c = d^2 = (3 - \sqrt{5})/2$, and one gets the inscribed regular dodecahedron with twenty vertices (Figure 1c). Thus, one could obtain all the coordinates of the vertices of the four kinds of the regular polyhedra inscribed in a $2 \times 2 \times 2$ cube. These coordinates are summarized in Table 1.

Generation of Truncated Icosahedron

Trisect all the edges of an icosahedron, draw twenty hexagons by using these new vertices, and one gets a so-called soccerball-shaped truncated icosahedron, TI (See Figure 1b). The coordinates of the 60 vertices are given in Table 2. The eight triangles facing toward the corners, C, of the parent cube generate 48 vertices, while 12 vertices come from the six squares. Figure 2a shows a perspective view of the relation between the cube and the inscribed truncated icosahedron in the first octant. Note that the expressions for all the vertices of a TI are symmetrical with respect to the three planes determining the rectangular coordinate system. This means that the generated TI has exactly the same geometrical structure in all the eight octants. If three identical square mirrors are assembled to form a corner of a cube as in Figure 2 and a regular hexagon with three legs (as shown with bold segments and filled circles) is attached to it, a special kaleidoscope can be constructed to project up a TI.

Characteristic Quantities of TI

One can calculate various characteristic quantities representing the geometrical features of TI from the coordinates of the vertices, centers of the pentagons and hexagons, and the midpoint of an edge. The rectangular coordinates of the center of gravity P of the pentagon which vertically cuts the xy-plane as seen in Figure 2 can be calculated to be $((10 + \sqrt{5})/15, (9\sqrt{5} - 5)/30, 0)$. The coordinates of the center M of the central hexagon in Figure 2 are obtained to be (f, f, f) with $f = (1 + \sqrt{5})/6$, while those of the center H of vertical hexagon are $(1/3, (3 + \sqrt{5})/6, 0)$.

The TI can be decomposed into twenty hexagonal pyramids and twelve pentagonal pyramids by drawing segments from all the vertices of TI toward the origin. The height of the latter is the radius of the inscribed sphere, while the distance from the origin to a vertex is the circumradius. The analytical expressions and numerical values of various characteristic quantities for the two TIs (i) with a unit edge and (ii) inscribed in a $2 \times 2 \times 2$ cube, are given in Table 3. As already mentioned in the Introduction, many of the hitherto available data are inaccurate or incorrect.

Kaleidoscope of TI

The above-mentioned kaleidoscope with three-fold rotational symmetry is rather awkward both from manufacturing and aesthetic points of view. In our view, Betsumiya[11,12] is the first person who has designed and constructed the elegant kaleidoscope of TI with fivefold symmetry, or the soccerball kaleido-

Table 1. Rectangular coordinates of the four kinds of regular polyhedra inscribed in a 2 × 2 × 2 cube

Regular tetrahedron	(1, 1, 1)	(1, −1, −1)	(−1, 1, −1)	(−1, −1, 1)
Regular octahedron	(±1, 0, 0)	(0, ±1, 0)	(0, 0, ±1)	
Regular dodecahedron	(±1, ±c, 0)	(0, ±1, ±c)	(±c, 0, ±1)	(±d, ±d, ±d)
Regular icosahedron	(±1, ±d, 0)	(0, ±1, ±d)	(±d, 0, ±1)	

$c = (3-\sqrt{5})/2 = 0.38196601$, $d = (\sqrt{5}-1)/2 = 0.61803399$.

Table 2. Rectangular coordinates of the truncated icosahedron inscribed in a 2 × 2 × 2 cube

(±1/3, ±a, ±b)	(±b, ±1/3, ±a)	(±a, ±b, ±1/3)
(±2/3, ±c, ±b/2)	(±b/2, ±2/3, ±c)	(±c, ±b/2, ±2/3)
(±1, ±b/2, 0)	(0, ±1, ±b/2)	(±b/2, 0, ±1)

$a = (3 + \sqrt{5})/6 = 0.87267800$, $b = (\sqrt{5}-1)/3 = 0.41202266$, $c = \sqrt{5}/3 = 0.74535599$.

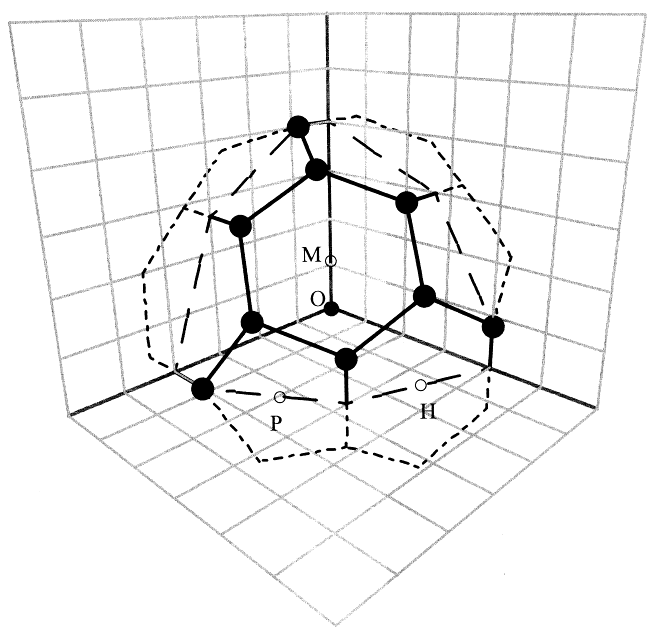

Figure 2. Perspective view of the truncated icosahedron in the first octant of the rectangular coordinate system. P is the center of gravity of a pentagon and M and H are the centers of hexagons.

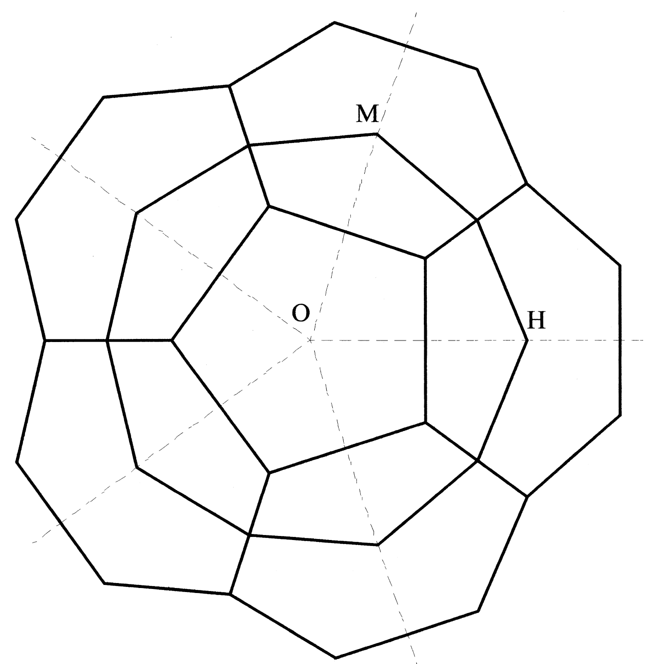

Figure 3. Projected view of a truncated icosahedron reflected in the kaleidoscope of five-fold rotational symmetry.

scope, the top view of which is sketched as in Figure 3. A regular pentagon surrounded by five trisected regular hexagons is embedded in a pentagonal pyramid composed of five identical triangular mirrors. The vertical angle, θ, of the isosceles triangular mirror is supposed to obey $\sin \theta = 2/3$ or $\cos \theta = \sqrt{5}/3$, which is determined by the triangle OHM formed by the centers of the two adjacent hexagons and the origin (see Figures 2 and 3).

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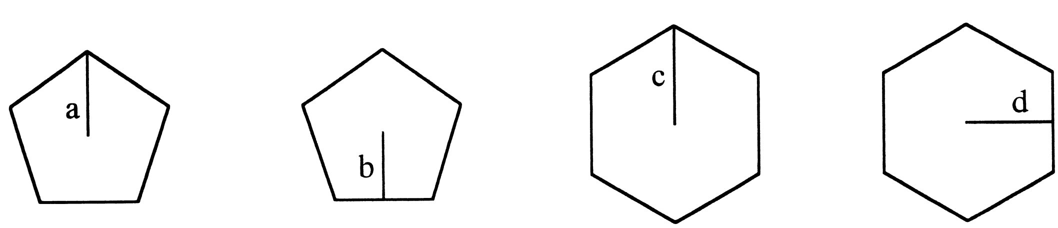
Table 3. Characteristic quantities of the geometrical structure of truncated icosahedron

Characteristic quantity		Analytical expression and numerical value: edge = 1	Analytical expression and numerical value: inscribed in $2 \times 2 \times 2$ cube, edge = $(\sqrt{5} - 1)/3$
Pentagon P			
Center of P to vertex	a	$\sqrt{(5 + \sqrt{5})/10}$ 0.85065081	$\sqrt{2(5 - \sqrt{5})/45}$ 0.35048741
Center of P to mid-edge	b	$\sqrt{(5 + 2\sqrt{5})/20}$ 0.68819096	$\sqrt{(5 + \sqrt{5})/90}$ 0.28355027
Height of P	$a + b$	$\sqrt{5 + 2\sqrt{5}}/2$ 1.53884177	$\sqrt{(5 + \sqrt{5})/18}$ 0.63403768
Area of P	S_5	$\sqrt{5(5 + 2\sqrt{5})}/4$ 1.72047740	$\sqrt{10(5 - \sqrt{5})}/18$ 0.29207284
Pentagonal pyramid			
Height	H_5	$\sqrt{(125 + 41\sqrt{5})/40}$ 2.32743844	$\sqrt{(85 - \sqrt{5})/90}$ 0.95895737
Volume	V_5	$(35 + 13\sqrt{5})/48$ 1.33476841	$(9\sqrt{5} - 5)/162$ 0.093361801
Hexagon H			
Center of H to vertex	c	1 1.00000000	$(\sqrt{5} - 1)/3$ 0.41202266
Center of H to mid-edge	d	$\sqrt{3}/2$ 0.86602540	$(\sqrt{5} - 1)/2\sqrt{3}$ 0.35682209
Area of H	S_6	$3\sqrt{3}/2$ 2.59807621	$(3 - \sqrt{5})/\sqrt{3}$ 0.44105636
Hexagonal pyramid			
Height (Inscribed radius)	$H_6 = r$	$\sqrt{3}(3 + \sqrt{5})/4$ 2.26728394	$\sqrt{3}(\sqrt{5} + 1)/6$ 0.93417236
Volume	V_6	$3(3 + \sqrt{5})/8$ 1.96352549	$(\sqrt{5} - 1)/9$ 0.13734089
Truncated icosahedron (TI)			
Surface	$12\ S_5 + 20\ S_6$	$30\sqrt{3} + 3\sqrt{5(5 + 2\sqrt{5})}$ 72.60725376	$20\sqrt{3}(3-\sqrt{5})/3$ $+2\sqrt{10(5 - \sqrt{5})}/3$ 12.32600126
Volume	$12\ V_5 + 20\ V_6$	$(125 + 43\sqrt{5})/4$ 55.28773078	$(78\sqrt{5} - 70)/27$ 3.86715934
Circumradius (Center of TI to vertex)	R	$\sqrt{29 + 9\sqrt{5}}/8$ 2.47801866	$\sqrt{(21 - \sqrt{5})/18}$ 1.04244067



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