Convolution Sum

Definition

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k] = x(n) \cdot h(n)$$

⊛ **Example:**

Consider a discrete LTI sytem:

$$y(n) = x(n+1) - 2\,x(n)$$

Determine $y(n)$ given $x(n) = [2, \underset{\uparrow}{1}, 3, 2]$

<u>Solution</u>:

$$y(n) = x(n+1) - 2 \cdot x(n) = [1, \underset{\uparrow}{-2}]$$

$$x(n) = [2, \underset{\uparrow}{1}, 3, 2]$$

+ step 1: fix a signal, in this case, fix $x(n)$
+ step 2: reverse the other signal

$$y[n] = [1, \underset{\uparrow}{-2}] \xrightarrow{\text{reverse}} [\underset{\uparrow}{-2}, 1]$$

+ step 3: calculate

$$y[0]: \left.\begin{matrix} [2, \underset{\uparrow}{1}\ 3, 2] \\ [\underset{\uparrow}{-2}, 1] \end{matrix}\right\} \rightarrow y[0] = -2 \cdot 1 + 1 \cdot 3 = 1$$

$$y[-1]: \left.\begin{matrix} [2, \underset{\uparrow}{1}, 3, 2] \\ [-2, \underset{\uparrow}{1}] \end{matrix}\right\} \rightarrow y[-1] = -2 \cdot 2 + 1 \cdot 1 = -3$$

$y[1]$: $\left.\begin{array}{l}[2, \underset{\uparrow}{1}, \underline{3}, 2] \\ [\underset{\uparrow}{0}, -2, 1]\end{array}\right\} \rightarrow y_1 = 0.1 - 2.3 + 1.2 = -4$

$y[2]$: $\left.\begin{array}{l}[2, \underset{\uparrow}{1}, 3, \underline{2}] \\ [\underset{\uparrow}{0}, 0, -2, 1]\end{array}\right\} \rightarrow y[2] = 0 + 0 - 4 = -4$

$y[-2]$: $\left.\begin{array}{l}[2, \underset{\uparrow}{1}, 3, 2] \\ [-2, 1, 0]\end{array}\right\} \rightarrow y[-2] = 2$

$$\Rightarrow \boxed{y[n] = [2, -3, \underset{\uparrow}{1}, -4, -4]}$$

Execise 1 : (signal and system p. 159 / answer p. 942)

$$x[n] = \delta[n] + 2\,\delta[n-1] - \delta[n-3]$$

$$h[n] = 2\,\delta[n+1] + 2\,\delta[n-1]$$

Compute and plot each of the following convolutions:

a) $y_1[n] = x[n] * h[n]$
b) $y_2[n] = x[n+2] * h[n]$
c) $y_3[n] = x[n] * h[n+2]$

$$x[n] = [\underset{\uparrow}{1}, 2, 0, -1] \qquad h[n] = [2\ \underset{\uparrow}{0}\ 2]$$

a) $x[n] * h[n]$

$$y[0]: \left.\begin{matrix} [\underset{\uparrow}{1}, 2, 0, -1] \\ [2, \underset{\uparrow}{0}, 2] \end{matrix}\right\} \rightarrow y[0] = 0.1 + 2.2 = 4$$

$$y[-1]: \left.\begin{matrix} [1\ 2\ 0\ -1] \\ [2\ 0\ 2] \end{matrix}\right\} \rightarrow y[-1] = 2$$

$$y[1]: \left.\begin{matrix} [1\ 2\ 0\ -1] \\ [2\ 0\ 2] \end{matrix}\right\} \rightarrow y[1] = 2.1 + 0 + 0 = 2$$

$$y[2]: \left.\begin{matrix} [1\ 2\ 0\ -1] \\ [0\ 2\ 0\ 2] \end{matrix}\right\} \rightarrow y[2] = 0 + 2.2 + 0 - 2.1 = 2$$

$$y[3]: \left.\begin{matrix} [1\ 2\ 0\ -1] \\ [0\ 0\ 2\ 0\ 2] \end{matrix}\right\} \rightarrow y[3] = 0 + 0 + 0 - 0 = 0$$

$$y[4]: \left.\begin{matrix} [1\ 2\ 0\ -1] \\ [0\ 0\ 0\ 2\ 0\ 2] \end{matrix}\right\} \rightarrow y[4] = 0 + 0 + 0 - 2.1 = -2$$

$$\rightarrow \boxed{y[n] = [2, \underset{\uparrow}{4}, 2, 2, 0, -2]}$$

b) $y_2[n] = x[n+2] * h[n]$

$x[n+2] = [1\ \ 2\ \ \underset{\uparrow}{0}\ \ -1]$ $\qquad h[n] = [2,\underset{\uparrow}{0},2]$

$y_2[0] = 2.2 + 0 - 2.1 = 2$

$y_2[1] = 0$, $y_2[2] = -2$

$y_2[-1] = 1.2 + 2.0 + 2.0 = 2$

$y_2[-2] = 1.0 + 2.2 = 4$

$y_2[-3] = 1.2 + 0 = 2$

$\rightarrow \ y_2[n] = [2,4,2,\underset{\uparrow}{2},0,-2] = y[n+2]$

c) $y_3[n] = x[n] * h[n+2]$

$x[n] = [\underset{\uparrow}{1},2,0,-1]$ and $h[n+2] = [2\ \ 0\ \ 2\ \ \underset{\uparrow}{0}]$

$\rightarrow [\underset{\uparrow}{0},2,0,2]$

$$\left.\begin{aligned} y_3[0] &= 2 \\ y_3[1] &= 0 \\ y_3[2] &= -2 \\ y_3[-1] &= 2 \\ y_3[-2] &= 4 \\ y_3[-3] &= 2 \end{aligned}\right\} \Rightarrow y_3[n] = [2,4,2,\underset{\uparrow}{2},0,-2] = y_2[n]$$

Exercise 2: (signal and system p.160 / ans p. 942)

Consider an input $x[n]$ and a unit impulse response $h[n]$ given by:

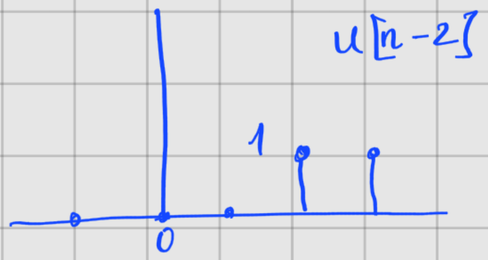
$$x[n] = \left(\frac{1}{2}\right)^{n-2} \cdot u[n-2]$$

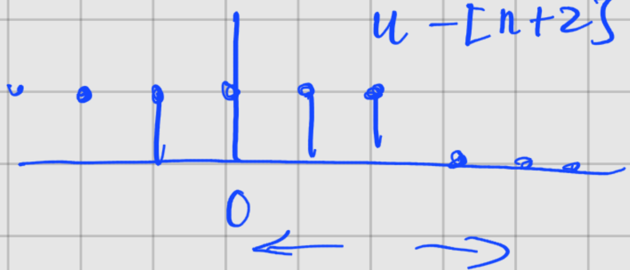
$$h[n] = u[n+2]$$

Determine and plot the output $y[n] = x[n] * h[n]$

<u>Solution:</u>

$$u[n-2] = \begin{cases} 1 & n \geq 2 \\ 0 & n < 2 \end{cases}$$





$x[0] = x[1] = 0$

(reverse and move $h[n]$ to the right)

$y[n] = 0$ if $n < 0$

$y[0] = 1$

$y[1] = 1 + \frac{1}{2}$

$$1 + q + q^2 + \dots + q^k = \frac{1 - q^{k+1}}{1 - q}$$

$$y[n] = \boxed{\frac{1 - q^{n+1}}{1 - q}}$$

$$y_2 = 1 + \left(\frac{1}{2}\right)^{2-2}$$

$$y_0 = \frac{1}{2}^{-2} =$$