1. Back Substitution and Forward Substitution

Background

The *back-substitution algorithm*, which is useful for solving a linear system of equations that has an upper-triangular coefficient matrix.

Definition (Upper-Triangular Matrix). An $n \times n$ matrix $\mathbf{A} = [a_{i,j}]$ is called ***upper-triangular*** provided that the elements satisfy $a_{i,j} = 0$ whenever $i > j$.

If **A** is an upper-triangular matrix, then $\mathbf{AX} = \mathbf{B}$ is said to be an upper-triangular system of linear equations.

$$
\begin{array}{rcl}
a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1,n}x_n & = & b_1 \\
a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n-1}x_{n-1} + a_{2,n}x_n & = & b_2 \\
a_{3,3}x_3 + \cdots + a_{3,n-1}x_{n-1} + a_{3,n}x_n & = & b_3 \\
\vdots \qquad\qquad \vdots & & \vdots \\
a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n & = & b_{n-1} \\
a_{n,n}x_n & = & b_n
\end{array}
\tag{1}
$$

Theorem (Back Substitution). Suppose that $\mathbf{AX} = \mathbf{B}$ is an upper-triangular system with the form given above in (1). If $a_{i,i} \neq 0$ for $i = 1, 2, \ldots, n$ then there exists a unique solution.

The back substitution algorithm. To solve the upper-triangular system $\mathbf{AX} = \mathbf{B}$ by the method of back-substitution. Proceed with the method only if all the diagonal elements are nonzero. First compute

$$x_n = \frac{b_n}{a_{n,n}}$$

and then use the rule

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j}x_j}{a_{i,i}} \quad \text{for } i = n-1, n-2, \ldots, 1$$

Or, use the "generalized rule"

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j}x_j}{a_{i,i}} \quad \text{for } i = n, n-1, \ldots, 1$$

where the "understood convention" is that $\sum_{j=n+1}^{n} a_{n+1,j}x_j$ is an "empty summation" because the lower index of summation is greater than the upper index of summation.

Example 1. Use the back-substitution method to solve the upper-triangular linear system

$$\begin{pmatrix} 4 & -1 & 2 & 3 \\ 0 & -2 & 7 & -4 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 20 \\ -7 \\ 4 \\ 6 \end{pmatrix}.$$

Solution 1.

The *forward-substitution algorithm*, which is useful for solving a linear system of equations that has a lower-triangular coefficient matrix.

Definition (Lower-Triangular Matrix). An $n \times n$ matrix $\mathbf{A} = [a_{i,j}]$ is called ***lower-triangular*** provided that the elements satisfy $a_{i,j} = 0$ whenever $i < j$.

If **A** is an lower-triangular matrix, then $\mathbf{AX} = \mathbf{B}$ is said to be a lower-triangular system of linear equations.

$$(2) \quad \begin{array}{lllllll} a_{1,1}x_1 & & & & & & = b_1 \\ a_{2,1}x_1 + & a_{2,2}x_2 & & & & & = b_2 \\ a_{3,1}x_1 + & a_{3,2}x_2 + & a_{3,3}x_3 & & & & = b_3 \\ & & & \vdots & & \vdots & \vdots \\ a_{n-1,1}x_1 + & a_{n-1,2}x_2 + & a_{n-1,3}x_3 + \cdots + & a_{n-1,n-1}x_{n-1} & & & = b_{n-1} \\ a_{n,1}x_1 + & a_{n,2}x_2 + & a_{n,3}x_3 + \cdots + & a_{n,n-1}x_{n-1} + & a_{n,n}x_n & & = b_n \end{array}$$

Theorem (Forward Substitution). Suppose that $\mathbf{AX} = \mathbf{B}$ is an lower-triangular system with the form given above in (2). If $a_{i,i} \neq 0$ for $i = 1, 2, \ldots, n$ then there exists a unique solution.

The forward substitution algorithm. To solve the lower-triangular system $\mathbf{AX} = \mathbf{B}$ by the method of forward-substitution. Proceed with the method only if all the diagonal elements are nonzero. First compute

$$x_1 = \frac{b_1}{a_{1,1}}$$

and then use the rule

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{i,j}x_j}{a_{i,i}} \quad \text{for } i = 2, 3, \ldots, n.$$

Remark. The loop control structure will permit us to use one formula

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{i,j}\, x_j}{a_{i,i}} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Example 2. Use the forward-substitution method to solve the lower-triangular linear system

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -2 & -1 & 0 \\ 1 & -2 & 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 4 \\ 2 \end{pmatrix}.$$

Solution 2. Perform forward-substitution.

Example 1. Use the back-substitution method to solve the upper-triangular linear system

$$\begin{pmatrix} 4 & -1 & 2 & 3 \\ 0 & -2 & 7 & -4 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 20 \\ -7 \\ 4 \\ 6 \end{pmatrix}.$$

Solution 1. Perform back-substitution.

$$3 x_4 == 6$$
$$x_4 = 2$$

$$6 x_3 + 5 x_4 == 4$$
$$10 + 6 x_3 == 4$$
$$6 x_3 == -6$$
$$x_3 = -1$$

$$-2 x_2 + 7 x_3 - 4 x_4 == -7$$
$$-15 - 2 x_2 == -7$$
$$-2 x_2 == 8$$
$$x_2 = -4$$

$$4 x_1 - x_2 + 2 x_3 + 3 x_4 == 20$$
$$8 + 4 x_1 == 20$$
$$4 x_1 == 12$$
$$x_1 = 3$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -1 \\ 2 \end{pmatrix}$$

Example 2. Use the forward-substitution method to solve the lower-triangular linear system

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -2 & -1 & 0 \\ 1 & -2 & 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 4 \\ 2 \end{pmatrix}.$$

Solution 2. Perform forward-substitution.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{23}{3} \\ -\frac{43}{3} \\ \frac{305}{6} \end{pmatrix}$$