RiNNAL-POP: A Low-rank ALM for Polyhedral-SDP and Moment-SOS Relaxations of Polynomial Optimization

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Polynomial Optimization

$$\min_{w} \left\{ f_0(w) \ \middle|\ f_i(w) = 0,\ i \in [m],\ w \in D \right\}, \qquad \text{(POP)}$$

- D: semialgebraic domain, typically $\mathbb{R}^n$ or $\mathbb{R}^n_+$.
- $f_i(w)$: real-valued multivariate polynomials.
- Inequalities: handled via standard or squared slack variables (Ding and Wright, 2025).
- Applications: QCQP, tensor problems, portfolio optimization ...

Moment-SOS hierarchy (J. B. Lasserre, 2001)

- The SDP relaxations converge asymptotically as the relaxation order τ increases.
- But scales extremely poorly:

$$\text{variables} = \Omega(n^{2\tau}), \quad \text{constraints} = \Omega(n^{2\tau}).$$

Convex conic reformulation (Burer, 2009; Kim, Kojima, and Toh, 2020)

- Exact at finite order τ.
- But involves intractable cones.
- Build tractable and tight **polyhedral–SDP relaxations**.

Example: Standard Quadratic Programming

$$\min_{w} \left\{ w^\top Q w \mid e^\top w = 1,\ w \in \mathbb{R}^n_+ \right\}. \tag{StQP}$$

Homogeneous lifting. Choose monomial basis $x = [1;\ w] \geq 0$ and $X = xx^\top$, then

$$\text{(StQP)} \quad \Longleftrightarrow \quad \min_{X} \left\{ \langle Q^0, X \rangle \mid \langle Q^1, X \rangle = 0,\ \langle H^0, X \rangle = 1,\ X \in K \right\},$$

where

$$Q^0 = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}, \quad Q^1 = \begin{bmatrix} -1 \\ e^\top \end{bmatrix} \begin{bmatrix} -1 \\ e^\top \end{bmatrix}^\top, \quad H^0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad K = \left\{ xx^\top \mid x \in \mathbb{R}^{n+1}_+ \right\}.$$

Convex conic reformulation. Replace K by its covnex hull $\mathrm{co}(K)$

$$\min_{X} \left\{ \langle Q^0, X \rangle \mid \langle Q^1,\ X \rangle = 0,\ \langle H^0, X \rangle = 1,\ X \in \mathrm{co}(K) \right\}. \tag{COP}$$

(COP) and (StQP) are **equivalent**.

Convex Conic Reformulation: General Polynomial Case

$$\min_{X} \Big\{ \langle Q^0, X\rangle \mid \langle Q^i, X\rangle = 0,\ i \in [m],\ \langle H^0, X\rangle = 1,\ X \in \mathrm{co}(K) \Big\}. \qquad \text{(COP)}$$

From StQP to general POP.

- (Polynomial degree) arbitrary order $\tau > 0$, handled via higher-order monomial basis

$$u^{\mathcal{A}_\tau}(x) \;=\; \big(x^\alpha\big)_{\alpha \in \mathcal{A}_\tau}, \qquad \mathcal{A}_\tau \;=\; \{\alpha \in \mathbb{N}^{n+1} \mid |\alpha| \le \tau\}.$$

- (Sparse basis) choose $\mathcal{A} \subseteq \mathcal{A}_\tau$, only the monomials appearing in f_i are retained

$$K \;=\; \{\, u^{\mathcal{A}}(x)\big(u^{\mathcal{A}}(x)\big)^\top \mid x \in \overline{D}\,\}, \qquad \overline{D} := \mathbb{R}_+ \times D \subseteq \mathbb{R}^{n+1}.$$

- (Constraint number) arbitrary number of polynomial constraints

$$J \;=\; \{\, X \in \mathrm{co}(K) \mid \langle Q^i, X\rangle = 0,\ i \in [m]\,\}.$$

Theorem 1 (Kim, Kojima, and Toh, 2020)

COP equivalence. *If (1) K is nonempty; (2) $\mathrm{val}\,(\text{COP})$ is finite; (3) $\mathrm{co}(K \cap J) = J$. Then*

$$\mathrm{val}\,(\text{POP}) = \mathrm{val}\,(\text{COP}).$$

Computational Issues

Convex problem can still be difficult:

$$\min_{X} \left\{ \langle Q^0, X \rangle \mid \langle Q^i, X \rangle = 0,\ i \in [m],\ \langle H^0, X \rangle = 1,\ X \in \mathrm{co}(K) \right\}. \tag{COP}$$

Issue. (COP) involves intractable cone $\mathrm{co}(K)$.

- For (POP) with $D = \mathbb{R}^n_+$, $\mathrm{co}(K)$ is the the completely positive cone

$$\mathrm{co}(K) = \mathrm{CPP}^{n+1} = \mathrm{co}\left\{ xx^\top \mid x \in \mathbb{R}^{n+1}_+ \right\}.$$

- Even testing membership in CPP is NP-hard (Dickinson and Gijben, 2014).

Tractable Conic Relaxation

Tractable Relaxation. Replace $\mathrm{co}(K)$ by an outer approximation

$$\mathrm{co}(K) \ \subseteq\ \mathbb{S}_+^{\mathcal{A}} \ \cap\ \mathcal{P}^{\mathcal{A}} \ \cap\ \mathcal{L}^{\mathcal{A}}.$$

The resulting relaxation yields a lower bound on the original problem.

1. (Positive semidefinite cone $\mathbb{S}_+^{\mathcal{A}}$)

$$\mathbb{S}_+^{\mathcal{A}} = \{X \in \mathbb{S}^{\mathcal{A}} \mid X \succeq 0\}. \tag{SDP}$$

2. (Polyhedral cone $\mathcal{P}^{\mathcal{A}}$) problem dependent, for example

$$D = \mathbb{R}_+^n \quad \Longrightarrow \quad \mathcal{P}^{\mathcal{A}} = \{X \in \mathbb{S}^{n+1} \mid X \geq 0\}. \tag{DNN}$$

3. (Consistency cone $\mathcal{L}^{\mathcal{A}}$) from the fact that $x^{\alpha+\beta} = x^{\gamma+\delta}$ if $\alpha + \beta = \gamma + \delta$:

$$\mathcal{L}^{\mathcal{A}} = \left\{X \in \mathbb{S}^{\mathcal{A}} \,\middle|\, X_{\alpha,\beta} = X_{\gamma,\delta} \text{ if } \alpha+\beta = \gamma+\delta\right\}. \tag{Mom-SOS}$$

Identify Hidden Facial Structure

Facial constraints. Many equality constraints in (COP) have the following structure: for $X \succeq 0$,

$$AX = 0 \quad \Longleftrightarrow \quad \langle A^\top A, X \rangle = 0, \tag{RLT}$$

which arise naturally in

- Reformulation-linearization techniques (RLT) (Sherali, 2007)
- Moment-SOS hierarchies (J. B. Lasserre, 2001)
- Example: facial constraints in (StQP)

$$e^\top x - 1 = 0 \quad \Longrightarrow \quad \begin{bmatrix} -1 & e \end{bmatrix} X = 0 \quad \Longleftrightarrow \quad \left\langle \begin{bmatrix} -1 \\ e^\top \end{bmatrix} \begin{bmatrix} -1 \\ e^\top \end{bmatrix}^\top, X \right\rangle = 0.$$

Implication. Enable us to

- Strengthen the relaxation by adding RLT constraints.
- Eliminate a large number of linear constraints.

Polyhedral-SDP Relaxation

$$\min_{X} \left\{ \langle Q^0, X \rangle \;\middle|\; \begin{array}{l} \langle H^0, X \rangle = 1, \;\; AX = 0, \\ \langle Q^i, X \rangle = 0, \;\; \forall i \in [\ell], \\ X \in \mathbb{S}_+^{\mathcal{A}} \cap \mathcal{P}^{\mathcal{A}} \cap \mathcal{L}^{\mathcal{A}} \end{array} \right\}. \tag{Poly-SDP}$$

- Assume that $A \in \mathbb{R}^{m \times n}$ has full row-rank.
- A unified framework that includes SDP/DNN/RLT/Mom-SOS relaxations.

Theorem 2 (Informal)

Tightness. *Poly-SDP relaxation is as tight as moment-SOS relaxation.*

$$\begin{array}{ccc} \textit{Original POP} & \longrightarrow & \textit{Moment-SOS} \\ \Updownarrow & & \Updownarrow \\ \textit{Reformulated POP} & \longrightarrow & \textit{Poly–SDP} \end{array}$$

How to Solve (Poly-SDP)?

$$\min_{X} \left\{ \langle Q^0, X \rangle \;\middle|\; \begin{array}{l} \langle H^0, X \rangle = 1,\; AX = 0, \\ \langle Q^i, X \rangle = 0,\; \forall i \in [\ell], \\ X \in \mathbb{S}_+^{\mathcal{A}} \cap \mathcal{P}^{\mathcal{A}} \cap \mathcal{L}^{\mathcal{A}} \end{array} \right\}. \tag{Poly-SDP}$$

Challenges.

1. $\Omega(mn^\tau + \ell)$ equality constraints;
2. $\Omega(n^{2\tau})$ nonnegativity constraints when $D = \mathbb{R}_+^n$;
3. $\Omega(n^{2\tau})$ consistency constraints for fixed $\tau \geq 2$;
4. Failure of Slater's condition and strong duality whenever $A \neq 0$.

Contribution.

- Design RiNNAL-POP solver for solving (Poly-SDP).
- A two-phase augmented Lagrangian method (ALM).
- Use low-rank factorization approach to solve the subproblems.

Reformulation of (Poly-SDP)

To apply ALM, rewrite (Poly-SDP) into the following splitting form:

$$\min_{X,Y} \left\{ \langle Q^0, X\rangle + \delta_{\mathcal{M}}(X) + \delta_{\mathcal{P}}(Y) \mid \; X - Y = 0,\; \mathcal{Q}(X) = 0 \right\}, \qquad (\overline{\text{Poly-SDP}})$$

where we split the feasible set into three parts:

1. (PSD cone with facial constraints)

$$\mathcal{M} = \left\{ X \in \mathbb{S}_+^{\mathcal{A}} \mid AX = 0 \right\}.$$

2. (Polyhedral cone)

$$\mathcal{P} := \left\{ X \in \mathcal{P}^{\mathcal{A}} \cap \mathcal{L}^{\mathcal{A}} \mid \langle H^0, X\rangle = 1 \right\}.$$

3. (General constraints)

$$\mathcal{Q}(X) = 0 \quad \text{with} \quad \mathcal{Q}(X) := \left[\langle Q^1, X\rangle, \ldots, \langle Q^\ell, X\rangle \right]^\top.$$

We can now apply ALM to ($\overline{\text{Poly-SDP}}$).

Augmented Lagrangian Method for ($\overline{\text{Poly-SDP}}$)

Given $\sigma > 0$, augmented Lagrangian function:

$$\begin{aligned} L_\sigma(X, Y; y, W) &:= \langle Q^0, X\rangle - \langle y, \mathcal{Q}(X)\rangle - \langle W, X - Y\rangle + \frac{\sigma}{2}\|\mathcal{Q}(X)\|^2 + \frac{\sigma}{2}\|X - Y\|^2 \\ &= \langle Q^0, X\rangle + \frac{\sigma}{2}\left\|\mathcal{Q}(X) - \sigma^{-1}y\right\|^2 + \frac{\sigma}{2}\|X - Y - \sigma^{-1}W\|^2 - \frac{1}{2\sigma}\|y\|^2 - \frac{1}{2\sigma}\|W\|^2. \end{aligned}$$

ALM at the $(k+1)$-th iteration:

$$\begin{aligned} (X^{k+1}, Y^{k+1}) &= \arg\min_{X,Y} \left\{ L_{\sigma_k}(X, Y; y^k, W^k) : \ X \in \mathcal{M},\ Y \in \mathcal{P} \right\} \\ y^{k+1} &= y^k - \sigma_k(\mathcal{Q}(X^{k+1}) - b), \\ W^{k+1} &= W^k - \sigma_k(X^{k+1} - Y^{k+1}), \end{aligned}$$

where $0 < \sigma_k \uparrow \sigma_\infty \le +\infty$ are penalty parameters.

Observation.

- The optimal solution Y^{k+1} has the form $Y^{k+1} = \Pi_{\mathcal{P}}(X^{k+1} - \sigma^{-1}W^k)$.
- We can first minimize w.r.t. $Y \in \mathcal{P}$ to get a convex problem only in variable X.

ALM Subproblem

$$\min_X \left\{ \begin{array}{r} \phi(X) := \langle Q^0, X\rangle + \frac{\sigma}{2}\|\sigma^{-1}y - \mathcal{Q}(X)\|^2 \\ + \frac{\sigma}{2}\|X - \sigma^{-1}W - \Pi_{\mathcal{P}}(X - \sigma^{-1}W)\|^2 \end{array} \;\middle|\; AX = 0,\ X \in \mathbb{S}^{\mathcal{A}}_+ \right\}, \tag{CVX}$$

Observation.

1. (Low-rank decomposition) Suppose (CVX) has a rank r solution X^*. then there exists $R \in \mathbb{R}^{\mathcal{A}\times r}$ such that
$$X^* = RR^\top.$$
2. (Feasible region) the feasible region w.r.t. R can be obtained by
$$AX = 0,\ X \in \mathbb{S}^{\mathcal{A}}_+ \quad \Longleftrightarrow \quad AR = 0.$$

Thus, (CVX) is equivalent to the following **low-rank reformulation**:

$$\min_R \left\{ \begin{array}{r} \phi(RR) = \langle Q^0, RR^\top\rangle + \frac{\sigma}{2}\|\sigma^{-1}y - \mathcal{Q}(RR^\top)\|^2 \\ + \frac{\sigma}{2}\|RR^\top - \sigma^{-1}W - \Pi_{\mathcal{P}}(RR^\top - \sigma^{-1}W)\|^2 \end{array} \;\middle|\; AR = 0,\ R \in \mathbb{R}^{\mathcal{A}\times r} \right\}. \tag{LR}$$

A Hybrid Method to Solve ALM Subproblem

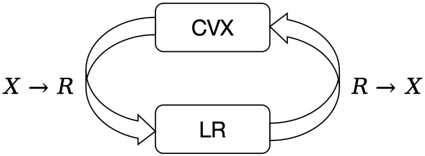

Figure: Two-phase transition

CVX Phase

Projected-gradient step on X

$$\begin{aligned} \min_{X} \quad & \phi(X) \\ \text{s.t.} \quad & AX = 0,\ X \in \mathbb{S}_{+}^{\mathcal{A}}. \end{aligned}$$

- Primal feasibility enforcement.
- Global convergence (Lee et al., 2024).
- Automatic rank selection.

LR Phase

Projected-gradient step on R

$$\begin{aligned} \min_{R} \quad & \phi(RR^{\top}) \\ \text{s.t.} \quad & AR = 0. \end{aligned}$$

- Reduces variable dimension.
- Reduces the number of constraints.
- Reduces the objective value.

Automatic Rank Selection

Observation. During the (CVX) phase, the projected-gradient (PG) iterates:

- Quickly stabilize at the correct rank (Hou, Tang, and Toh, 2025c),
- Long before objective convergence.

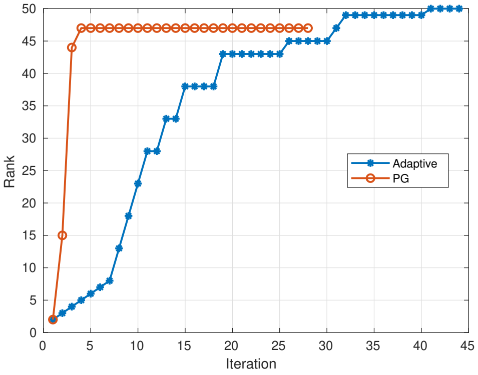

Figure: Initial rank: $r = 2$.

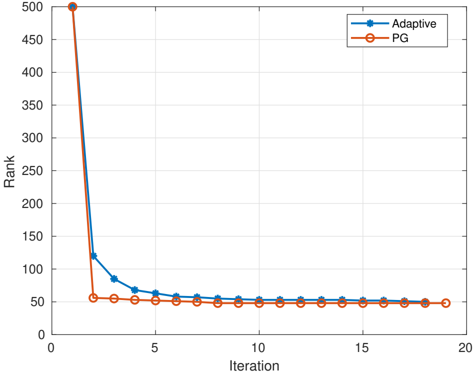

Figure: Initial rank: $r = n$.

Rank Identification

Manifold identification. (Lee et al., 2024)

- The feasible set $\mathcal{M}$ of (CVX) has a fixed-rank manifold structure near the solution.
- The indicator function $\delta_{\mathcal{M}}$ is partly smooth.
- Under a mild nondegeneracy condition, PG automatically identifies this manifold in finite steps.

Theorem 3 (Informal)

Finite-Step Rank Identification. *After finitely many iterations,*

$$\operatorname{rank}(X^k) = \operatorname{rank}(X^\star) \quad \textit{for all sufficiently large } k.$$

Implication. (CVX) phase helps to identify correct rank for (LR) phase automatically.

Efficient Projection onto $\mathcal{M}$ and $\mathcal{P}$

Theorem 4 (Hou, Tang, and Toh, 2025c)

Projection onto $\mathcal{M}$. *The projection onto the intersection of PSD cone and facial constraints*

$$\mathcal{M} = \left\{ X \in \mathbb{S}_+^{\mathcal{A}} \mid AX = 0 \right\}$$

can be computed by

$$\Pi_{\mathcal{M}}(X) = \Pi_{\mathbb{S}_+^n}(JXJ), \quad J := I - A^\top (AA^\top)^{-1} A.$$

One-time computation. J only needs to be precomputed once.

Theorem 5 (Hou, Tang, and Toh, 2025b)

Projection onto $\mathcal{P}$. *The projection onto the intersection of monomial-consistency, componentwise nonnegativity and normalization constraint constraints*

$$\mathcal{P} := \mathcal{N}^{\mathcal{A}} \cap \tilde{\mathcal{L}}^{\mathcal{A}}, \quad \mathcal{N}^{\mathcal{A}} := \left\{ X \in \mathbb{S}^{\mathcal{A}} \mid X \geq 0 \right\}, \quad \tilde{\mathcal{L}}^{\mathcal{A}} := \left\{ X \in \mathcal{L}^{\mathcal{A}} \mid \langle H^0, X \rangle = 1 \right\}$$

can be computed by

$$\Pi_{\mathcal{P}} = \Pi_{\mathcal{N}^{\mathcal{A}}} \circ \Pi_{\tilde{\mathcal{L}}^{\mathcal{A}}}.$$

Linear time. The computational complexity is significantly reduced from $\mathcal{O}(n^{7\tau})$ to $\mathcal{O}(n^{2\tau})$.

Dual Variable Recovery

Assume (Poly-SDP) admits an optimal solution satisfying the KKT conditions

$$AX = 0, \quad \nabla\phi(X) - A^\top U - U^\top A = S,$$
$$\langle X, S\rangle = 0, \quad X \succeq 0, \quad S \succeq 0.$$

Theorem 6 (Recover dual variables)

A matrix $X \in \mathcal{M}$ is a minimizer of (Poly-SDP) *if and only if*

$$\widehat{S} := J(\nabla\phi(X))J \succeq 0$$
$$X\widehat{S} = 0$$

where $J := I - A^\top(AA^\top)^{-1}A$ is the orthogonal projector onto the null space of A.

Implication. This yields a remarkably simple procedure for certifying global optimality.

Extension of RiNNAL-POP for Solving Moment-SOS Relaxation

- Each inequality $h_j(w) \geq 0$ corresponds to a localizing moment matrix:

$$\mathcal{M}_{h_j}(v(w)v(w)^\top) := h_j(w)[w]_{\tau-\lceil h_j \rceil}[w]_{\tau-\lceil h_j \rceil}^\top \succeq 0 \quad \longmapsto \quad \mathcal{M}_{h_j}(X) \succeq 0.$$

- The moment–SOS relaxation introduces additional **localizing positive semidefinite constraints**

$$\mathcal{M}_{h_j}(X) \succeq 0, \quad j \in [p].$$

- These constraints are handled via variable splitting and augmented Lagrangian penalization:

$$\mathcal{M}_{h_j}(X) - Y^{(j)} = 0, \quad Y^{(j)} \succeq 0.$$

- Eliminating $Y, Y^{(j)}$ through projection operators yields a convex reduced subproblem in X.
- All subsequent steps remain identical to the (Poly-SDP) case.

Numerical Experiments

- Machine: MATLAB R2023b on an Intel Xeon E5-2680 v3 workstation with 96 GB RAM.
- Tolerance: relative KKT residue: $= 10^{-6}$; Maxtime: $= 3600s$.
- Baseline solvers:
  1. RiNNAL-POP (ours)
  2. SDPNAL$+$ (Yang, Sun, and Toh, 2015)
  3. GloptiPoly (Henrion, J.-B. Lasserre, and Löfberg, 2009)

Relaxation Comparison

Relaxation	$X \geq 0$	RLT	Localizing PSD	Remarks
Poly-SDP	✓	-	-	Polyhedral SDP relaxation
Poly-SDP-RLT	✓	✓	-	Add RLT constraints
Mom-SOS	-	✓	✓	Moment-SOS relaxation
Poly-Mom-SOS	✓	✓	✓	Add $X \geq 0$

Table: Overview of SDP-based relaxations.

The relaxation quality is evaluated using the relative gap:

$$\%\text{gap} = \frac{v^* - v}{\max\{1, |v^*|\}} \times 100\%,$$

Relaxation Comparison

Order	1st	2nd
Poly–SDP	∞	∞
Poly–SDP–RLT	0	0
Mom–SOS	∞	0
Poly–Mom–SOS	0	0

Table: Relative gaps (%) of the four relaxations for the random instance.

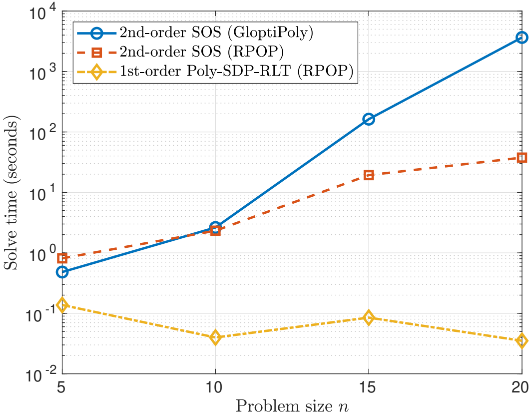

Figure: Computational time comparison between GloptiPoly and RPOP for the random instance.

Relaxation Comparison

Relaxation	1st-order Gap (%)	1st-order Time (s)	2nd-order Gap (%)	2nd-order Time (s)
Poly–SDP	∞	-	∞	-
Poly–SDP–RLT	0.562	0.3	0.280	4.6
Mom–SOS (RPOP)	∞	-	1.049	1875.4
Mom–SOS (GloptiPoly)	∞	-	1.046	6394.0
Poly–Mom–SOS	0.562	0.3	0.000	245.7

Table: Relative gaps and run times for the extended Horn instance ($n = 21$).

Summary:

1. RLT and nonnegativity constraints enhance relaxation strength.
2. Poly–SDP–RLT balances relaxation quality and computational cost.
3. RPOP outperforms GloptiPoly in both efficiency and scalability.

Benchmark Problem Classes

1. Standard quadratic programming:

$$\min\left\{x^\top Qx:\ e^\top x=1,\ x\in\mathbb{R}^n_+\right\}. \tag{StQP}$$

2. Binary quadratic programs:

$$\min\left\{x^\top Qx+c^\top x \mid x\in\{0,1\}^n\right\}. \tag{BIQ}$$

3. Minimum bisection problem:

$$\min\left\{x^\top Lx:\ e^\top x=\frac{n}{2},\ x\in\{0,1\}^n\right\}. \tag{MBP}$$

4. Multiple quadratic knapsack problem:

$$\max\left\{x^\top Q\,x+c^\top x \mid Ax\le b,\ x\in\{0,1\}^n\right\}. \tag{MQKP}$$

Benchmark Problem Classes

5. Ball-constrained quartic minimization:

$$\min\left\{\langle c, [x]_4\rangle \mid \|x\| \leq 1,\ x \in \mathbb{R}^{n-1}\right\}. \tag{BQM}$$

6. Kurtosis-minimization portfolio problem:

$$\min\left\{\mathbb{E}_\xi\left[(x^\top(\xi-\mu))^4\right] \mid \mu^\top x = \mu_0,\ x^\top \Sigma x = \sigma_0^2,\ e^\top x = 1,\ x \geq 0\right\}. \tag{KMP}$$

7. Copositivity of symmetric tensors:

$$\min\left\{\langle \mathcal{B}, \otimes^{\mathrm{t}} x\rangle \mid e^\top x = 1,\ x \geq 0\right\}. \tag{CST}$$

8. Best nonnegative rank-one tensor approximation problem:

$$\min\left\{\|\mathcal{B} - \lambda x^{\otimes\alpha}\|^2 \mid \ \|x^{(i)}\| = 1,\ \lambda \geq 0,\ x^{(i)} \geq 0,\ \forall i \in [p]\right\}. \tag{NTF}$$

Result Summary

Table: Summary of Benchmark Problem Classeses.

Problem	n	Deg	Relax	Eq	Ineq	≥ 0	Speedup
(StQP)	$1000-6000$	2	1	Simplex	-	✓	$34-36$
(StQP)	$20-80$	2	2	Simplex	-	✓	$3-6$
(BIQ)	$20-60$	2	2	Binary	-	✓	$10-25$
(MBP)	$30-60$	2	2	Equipartition	-	✓	$21-30$
(MQKP)	$20-40$	2	2	Binary	Knapsack	✓	$10-30$
(BQM)	$20-80$	4	2	-	Ball	-	$6-16$
(KMP)	$20-60$	4	2	Mean–variance	-	✓	$200-278$
(CST)	$1000-6000$	2	1	Simplex	-	✓	$18-500$
(CST)	$20-60$	4	2	Simplex	-	✓	$13-173$
(NSTF)	$20-80$	$3, 4$	2	Spherical	-	✓	$2-7$
(NTF)	$10-25$	$3, 4$	2	Block spherical	-	✓	$2-10$

Conclusion

Problem	**Relaxation**	**Solver**
MBQP	SDP/DNN	RiNNAL
MBQP	SDP–RLT	RiNNAL+
POP	Poly–SDP	RiNNAL–POP

Ongoing work: global solvers for POPs using RiNNAL-POP.

Di Hou, Tianyun Tang, and Kim-Chuan Toh (2025a). “A low-rank augmented Lagrangian method for doubly nonnegative relaxations of mixed-binary quadratic programs”. In: *Operations Research*

Code: `https://github.com/HouDiOpt/RiNNAL`

Di Hou, Tianyun Tang, and Kim-Chuan Toh (2025c). “RiNNAL+: a Riemannian ALM Solver for SDP-RLT Relaxations of Mixed-Binary Quadratic Programs”. In: *arXiv preprint arXiv:2507.13776*

Code: `https://github.com/HouDiOpt/RiNNALplus`

Di Hou, Tianyun Tang, and Kim-Chuan Toh (2025b). “A Low-rank Augmented Lagrangian Method for Polyhedral–SDP and Moment-SOS Relaxations of Polynomial Optimization”. In: *arXiv preprint arXiv:2512.06359*

References I

Burer, Samuel (2009). "On the copositive representation of binary and continuous nonconvex quadratic programs". In: *Mathematical Programming* 120.2, pp. 479–495.

Dickinson, Peter JC and Luuk Gijben (2014). "On the computational complexity of membership problems for the completely positive cone and its dual". In: *Computational optimization and applications* 57, pp. 403–415.

Ding, Lijun and Stephen J Wright (2025). "On squared-variable formulations". In: *SIAM Journal on Optimization* 35.4, pp. 2265–2293.

Henrion, Didier, Jean-Bernard Lasserre, and Johan Löfberg (2009). "GloptiPoly 3: moments, optimization and semidefinite programming". In: *Optimization Methods & Software* 24.4-5, pp. 761–779.

Hou, Di, Tianyun Tang, and Kim-Chuan Toh (2025a). "A low-rank augmented Lagrangian method for doubly nonnegative relaxations of mixed-binary quadratic programs". In: *Operations Research*.

References II

- Hou, Di, Tianyun Tang, and Kim-Chuan Toh (2025b). "A Low-rank Augmented Lagrangian Method for Polyhedral–SDP and Moment-SOS Relaxations of Polynomial Optimization". In: *arXiv preprint arXiv:2512.06359*.
- — (2025c). "RiNNAL+: a Riemannian ALM Solver for SDP-RLT Relaxations of Mixed-Binary Quadratic Programs". In: *arXiv preprint arXiv:2507.13776*.
- Kim, Sunyoung, Masakazu Kojima, and Kim-Chuan Toh (2020). "A geometrical analysis on convex conic reformulations of quadratic and polynomial optimization problems". In: *SIAM Journal on Optimization* 30.2, pp. 1251–1273.
- Lasserre, Jean B (2001). "Global optimization with polynomials and the problem of moments". In: *SIAM Journal on Optimization* 11.3, pp. 796–817.
- Lee, Ching-pei et al. (2024). "Accelerating nuclear-norm regularized low-rank matrix optimization through Burer-Monteiro decomposition". In: *Journal of Machine Learning Research* 25.379, pp. 1–52.
- Sherali, Hanif D (2007). "RLT: A unified approach for discrete and continuous nonconvex optimization". In: *Annals of Operations Research* 149.1, p. 185.

References III

Yang, Liuqin, Defeng Sun, and Kim-Chuan Toh (2015). "SDPNAL+: a majorized semismooth Newton-CG augmented Lagrangian method for semidefinite programming with nonnegative constraints". In: *Mathematical Programming Computation* 7.3, pp. 331–366.

Violation of $\mathrm{co}(K \cap J) = J$: Reformulation

$$\min\left\{\langle Q^0, X\rangle \mid X \in J,\ \langle H^0, X\rangle = 1\right\}. \tag{COP}$$

Issue 1: The condition $\mathrm{co}(K \cap J) = J$ fails to hold for (StQP).

Reformulation technique:

- A sufficient condition (Kim, Kojima, and Toh, 2020) is that J is a face of $\mathrm{co}(K)$.
- We instead reformulate (StQP) as

$$e^\top w - 1 = 0 \quad \Longleftrightarrow \quad (e^\top w - 1)^2 = 0.$$

- Under this reformulation, the constraint in (COP) becomes

$$\left\langle \begin{bmatrix} -1 & e^\top/2 \\ e/2 & 0 \end{bmatrix}, X \right\rangle = 0 \quad \Longleftrightarrow \quad \left\langle \begin{bmatrix} -1 \\ e^\top \end{bmatrix} \begin{bmatrix} -1 \\ e^\top \end{bmatrix}^\top, X \right\rangle = 0,$$

  which satisfies the above sufficient condition.

Application 1: RLT Constraints

Assume the homogenized equalities satisfy

$$\bar{f}_i(x) = \langle a_i, u^{\mathcal{A}}(x)\rangle, \qquad a_i \in \mathbb{R}^{\mathcal{A}},\ i = 1, \ldots, m,$$

and collect $A := [a_1, \ldots, a_m]^\top \in \mathbb{R}^{m\times\mathcal{A}}$. Then

$$A\, u^{\mathcal{A}}(x) = 0.$$

Multiplying by $u^{\mathcal{A}}(x)^\top$ yields the RLT constraint

$$AX = 0, \qquad X = u^{\mathcal{A}}(x)\, u^{\mathcal{A}}(x)^\top,$$

which is added to the SDP to tighten the relaxation.

$$\begin{bmatrix} \bar{f}_1(x) \\ \vdots \\ \bar{f}_m(x) \end{bmatrix} = 0 \iff \begin{bmatrix} \bar{f}_1(x) \\ \vdots \\ \bar{f}_m(x) \end{bmatrix} \left(u^{\mathcal{A}}(x)\right)^\top = 0$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$A\, u^{\mathcal{A}}(x) = 0 \iff A\, u^{\mathcal{A}}(x)\ \left(u^{\mathcal{A}}(x)\right)^\top = 0 \iff AX = 0.$$

Application 2: Moment–SOS Constraints

Assume the POP includes equalities

$$g_i(w) = 0$$

with $\deg(g_i) \le \tau$. The moment–SOS hierarchy generates the lifted polynomial constraints

$$g_i(w)\,[w]_{2\tau-\deg(g_i)} = 0.$$

Equivalently,

$$\big(g_i(w)[w]_{\tau-\deg(g_i)}\big)[w]_\tau^\top = 0 \iff G_i\,[w]_\tau[w]_\tau^\top = 0,$$

where $G_i \in \mathbb{R}^{d_i \times \bar{n}_\tau}$ is the coefficient matrix and

$$d_i = \binom{n+\tau-\deg(g_i)}{\tau-\deg(g_i)}.$$

Lifting $[w]_\tau[w]_\tau^\top$ to $X \in \mathbb{S}_+^{\bar{n}_\tau}$ yields

$$G_i X = 0, \quad i = 1, \ldots, \ell.$$

Collecting all constraints,

$$A = \begin{bmatrix} G_1 \\ \vdots \\ G_\ell \end{bmatrix} \implies AX = 0.$$