# note

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our aim is to eliminate the Lagrangian multipliers of the Lagrangian function ,and the get the quadratic form.

For example, in Herry 's thesis, cost function that node $i \to j$ is $\frac{1}{2}a_{ij}(y_{ij} - \tilde{y}_{ij})^2$, combined the constrains of each node, $\Lambda_i + \sum_k y_{ik} = 0$. The Lagrangian function is :

$$\mathcal{L} = \frac{1}{2}y_{ij}^2 + \sum_k a_{kj}(y_{kj} - \tilde{y}_{kj})^2 + \mu(\Lambda_j + y_{ij} + \sum_{k \in \partial i/j} y_{kj})$$

then,

$$\frac{\partial \mathcal{L}}{\partial y_{kj}} = a_{kj}(y_{kj} - \tilde{y}_{kj}) + \mu = 0$$

we eliminate $\mu$, the cost function is

$$E_{ij} = \frac{1}{2}y_{ij}^2 + \frac{1}{2}(\sum_k a_{kj}^{-1})^{-1}(y_{ij} + \Lambda_j + \sum_k \tilde{y}_{kj})^2$$

compared with

$$E_{ij} = \frac{1}{2}a_{ij}(y_{ij} - \tilde{y}_{ij})^2$$

we can get

$$\tilde{a}_{ij} = 1 + (\sum_k a_{kj}^{-1})^{-1}$$

$$\tilde{y}_{ij} = \frac{\Lambda_j + \sum_k \tilde{y}_{kj}}{1 + \sum_k a_{jk}^{-1}}$$

as for the transport network, we use $Z = C\vec{y}_1 - D\vec{y}_2$,

$$F_{12}(y_1, y_2) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - \frac{a_1}{2}(y_1 - \tilde{y}_1)^2 - \frac{a_2}{2}(y_2 - \tilde{y}_2)^2 - \frac{1}{2}(C\vec{y_1} - D\vec{y_2})^T A^{-1}(C\vec{y_1} - D\vec{y_2})$$

The quadratic form is

$$F_{12}(y_1, y_2) = \frac{a_{121}}{2}(y - \tilde{y}_{121})^2 + \frac{a_{122}}{2}(y_2 - \tilde{y}_{122})^2 + b_{12}(y_1 - \tilde{y}_{121})(y_2 - \tilde{y}_{122})$$

so
$$\begin{cases} a_{121} = 1 - C^T A^{-1} C - a_1 \\ a_{122} = 1 - D^T A^{-1} D - a_2 \\ b_{12} = D^T A^{-1} D \end{cases}$$
as for the $\tilde{y}_{121}, \tilde{y}_{122}$,

$$\begin{pmatrix} a_{121} & b_{12} \\ b_{12} & a_{122} \end{pmatrix} \begin{pmatrix} \tilde{y}_{121} \\ \tilde{y}_{122} \end{pmatrix} = - \begin{pmatrix} a_1 \tilde{y}_1 \\ a_2 \tilde{y}_2 \end{pmatrix}$$

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