9 March 2017

**Generalized Belief Propagation in Ising Model on a Square Lattice**

1. **Conventional Approach**

In the conventional approach, the compatibility constraints of the probabilities are replaced by the form obtained from Möbius inversion, resulting in the appearance of many messages in the recursion relations.

Specifically, the constraint  is replaced by



⇒ 

The constraint  is replaced by





Introduce the energy terms

, , 

Then the average energy is rewritten as



The free energy is given by











Interchanging the order of summation in the boxed term in the second last line,



Interchanging the order of summation in the first boxed term in the last line,



Interchanging the order of summation in the second boxed term in the last line,



Considering both nodes *i* and *j* connected by the factor node *b*,



Summarizing the changes, the free energy is given by

 where











Gradient with respect to :



 where 

Gradient with respect to :





Gradient with respect to :





To derive the cluster variation equations, we let







Compatibility between the probabilities at node *i* and cluster *a*:







Next, we consider the compatibility between the probabilities of cluster *a* and region *σ*:







To see which terms are canceled from both sides, we break the terms on both sides.











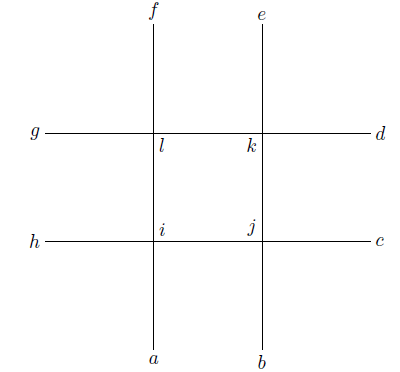
1. **Application to 2D Ising Model**

Using Pelizzola’s notation,







For *h* = 0,  and 

Factor to node message:





Dividing,



Near critical point,





Region to factor message:





For (*si*, *sj*) = (+, +),





For (*si*, *sj*) = (−, −),





Dividing and expanding to 1st order near critical point,









For (*si*, *sj*) = (+, +) and (−, −) added,



For (*si*, *sj*) = (+, −),



Dividing and for *m*1 = *m*2 = 0 in the paramagnetic phase,





Now consider when the generalized belief propagation converges to the ferromagnetic solution.

Let and 





 ⇒  ⇒ 

⇒ 

Hence the critical temperature estimated by the algorithm is 2.426*J*.

Compared with the belief propagation result of 2.885*J*, this result is closer to the exact result of 2.885*J*.

1. **Alternative Approach**

**In this approach, we apply the compatibility constraints of the messages directly, instead of decomposing them into separate constraints of the regions, clusters and spins.**

***h***

***j***

**4**

***f***

***d***

***l***

***b***

**1**

**3**

***γ***

***β***

***α***

***δ***

**2**

**5**

***e***

***c***

***g***

***i***

***a***

***k***

A network of square plaquettes. The circles represent the Ising spins as the variable nodes located on vertices, and the squares represent the factor nodes located at edges. A cluster is centered at a factor node, and a region consists of 4 clusters each centered at an edge of a square. Each variable node is a member of 4 clusters and 4 regions. For example, node 1 is a member of the clusters *a*, *d*, *g* and *l*, and is also a member of the regions *α*, *β*, *γ* and *δ*. Each factor node is a member of 2 regions. For example, factor *a* is a member of the regions *α* and *δ*.

To calculate the total entropy in the cluster variation method for the network of square plaquettes, we first sum over the entropy of the regions. Since each factor node is a member of 2 regions, we have to subtract from it the entropy of the clusters to give the correct counting of the factor nodes. Then in this entropy sum, since each variable node is a member of 4 clusters and 4 regions, it is counted 4 – 4 = 0 times. Hence we have to add back the entropy of the nodes. In summary, the total entropy is the entropy of the regions, minus the entropy of the clusters, plus the entropy of the nodes.

Introduce the energy terms

, , 

Then the average energy is rewritten as



The free energy becomes









The free energy can be decomposed into

 where











Gradient with respect to :





Gradient with respect to :





Gradient with respect to :





To derive the cluster variation equations, we let







Compatibility between the probabilities at node *i* and cluster *a*:







Collecting those terms independent of  into an integral ,

 where



Similar expressions can be found for the other 3 clusters to which *i* belongs. Multiplying,



Next we can proposed two form of the iterative equation. The choice of the proposals will depend on their performance.

Proposal 1: Keeping the product intact.



This enables us to eliminate the sum of the cluster messages in the original equation, yielding



Note that the calculation of  processes messages collected from the neighbors *j* connected through factor nodes *a*. In this step, we observe the outward information flow from the neighbor nodes to the factor nodes.

The next step is the calculation of  which processes messages  collected from the neighboring factor nodes *a*. In this step, we observe the inward information flow from the factor nodes to the variable nodes.

The spin distribution is given by



Proposal 2: Factoring the product but respecting the order of iteration.



Next, we consider the compatibility between the probabilities of cluster *a* and region *σ*:





Collecting those terms independent of  into a trace ,

 where



A similar expression can be found for the other region to which *a* belongs. Multiplying,





This enables us to eliminate the sum of the region messages in the original equation, yielding



Note that  since cluster *a* belongs to only 2 regions.

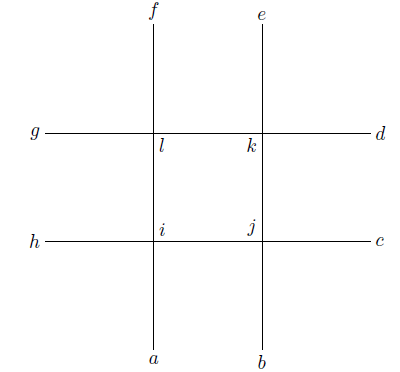
Note that the calculation of  processes messages collected from the neighboring factor nodes *b* connected through regions *σ*. In this step, we observe the outward information flow from the factor nodes to the regions.

The next step is the calculation of  which processes messages  collected from the neighboring region *τ*. In this step, we observe the inward information flow from the regions to the factor nodes.

1. **Application to 2D Ising Model**

Using Pelizzola’s notation,

















Let  and 

, 

, 



Near critical point,



⇒ 





For (*si*, *sj*) = (+, +),









Near the critical point,















Now consider when the generalized belief propagation converges to the ferromagnetic solution.

Let  and 



Note that  ⇒ 











Same result