Distinct-Roots Theorem

Suppose a sequence $(a_0, a_1, a_2, a_3, \ldots)$ satisfies a recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2}$$

If $t^2 - At - B = 0$ has two *distinct* roots r and s,

then $a_n = Cr^n + Ds^n$ for some C and D.

$$a_{n+1} = pa_n + qa_{n-1} \qquad (x^2 = px + q)$$

$$(a_{n+1} - sa_n) = t \cdot (a_n - sa_{n-1}) \quad \triangle$$

$$(a_{n+1} - ta_n) = s \cdot (a_n - sa_{n-1})$$

$$a_{n+1} = (t+s)a_n - tsa_{n-1}$$

$$\begin{cases} p = t+s \\ q = ts \end{cases}$$

WD $\Rightarrow$ t, s are the solutions of

$$1 \cdot x^2 - px - q = 0$$

$$ax^2 + bx + c = 0$$

$$\begin{cases} x_1 + x_2 = -\frac{b}{a} \\ x_1 x_2 = \frac{c}{a} \end{cases}$$

$\{a_n - sa_{n-1}\}$

$$a_n - sa_{n-1} = (a_1 - sa_0) \cdot t^{n-1} \quad ①$$

$$a_n - ta_{n-1} = (a_1 - ta_0) \cdot s^{n-1} \quad ②$$

t, s are symmetric

$$① - ②: \quad \cancel{(t-s)} a_{n-1} = \boxed{(a_1 - sa_0)} \cdot t^{n-1} - \boxed{(a_1 - ta_0)} \cdot s^{n-1}$$

$$\frac{A}{t-s} \qquad -\frac{B}{t-s}$$

$$a_n = C \cdot t^n + D \cdot s^n$$

(1) $t = s$: $a_n = (A + Bn) \cdot s^n$

(2) no solutions: $\{a_n\}$ is periodic

Question 1

Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recursion relation, $a_k = 3a_{k-1} - 2a_{k-2}$ for all integers $k \geq 2$.

(a) Show that the explicit formula $a_n = C\,2^n + D$, where C and D are real numbers.

(b) For $a_2 = 1$, $a_3 = -15$, determine the value of C and D

① Distinct-root theorem or induction

$p(t)$: $a_n = C\cdot 2^n + D$, $C, D \in R$. $\forall n \in N$, $n \leq t$

base case: $n=0$ $a_0 = C\cdot 2^0 + D = C+D$

$n=1$ $a_1 = C\cdot 2^1 + D = 2C+D$

$n=2$ $a_2 = C\cdot 2^2 + D = 4C+D$

inductive step:

Assume $p(k)$ is true for some $k \in N$, $k \geq 3$

$p(k+1)$: $a_{k+1} = 3a_k - 2a_{k-1} = 3\cdot(C\cdot 2^k + D) - 2\cdot(C\cdot 2^{k-1} + D)$

$= C 2^{k+1} + D$

so $p(k+1)$ is true

so $p(t)$ is true for $\forall t \in N$

(2) $x^2 = 3x - 2$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x_1 = 1$ $x_2 = 2$

$a_n = C\cdot 2^n + D\cdot 1^n$

$$\begin{cases} 4C+D = 1 \\ 8C+D = -15 \end{cases} \Rightarrow \begin{cases} C = -4 \\ D = 17 \end{cases}$$

- $a_k - a_{k-1} = 2(a_{k-1} - a_{k-2})$

① $a_k - a_{k-1} = 2^{k-1}\cdot(\underbrace{a_1 - a_0}_{A})$

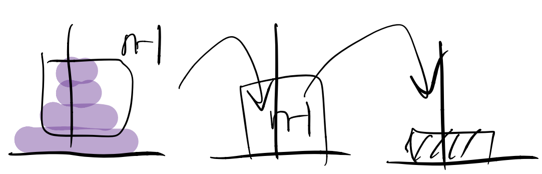
- $a_k - 2a_{k-1} = a_{k-1} - 2a_{k-2}$

② $a_k - 2a_{k-1} = 1^{k-1}\cdot(\underbrace{a_1 - 2a_0}_{B})$

$2\times① - ②$:

$$a_k = 2A\cdot 2^{k-1} - B\cdot 1^{k-1}$$

$$a_n = C\cdot 2^n + D$$



$h_n = 2h_{n-1} + 1$

Tower of Hanoi

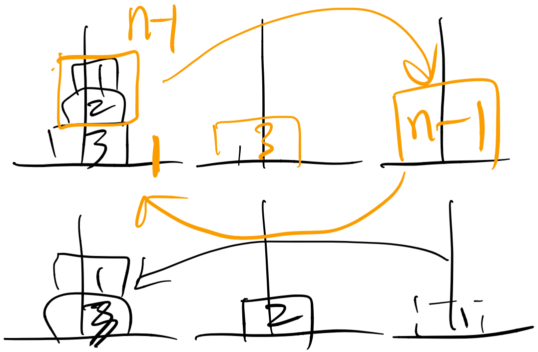
The goal is to move all the disks to post 3.

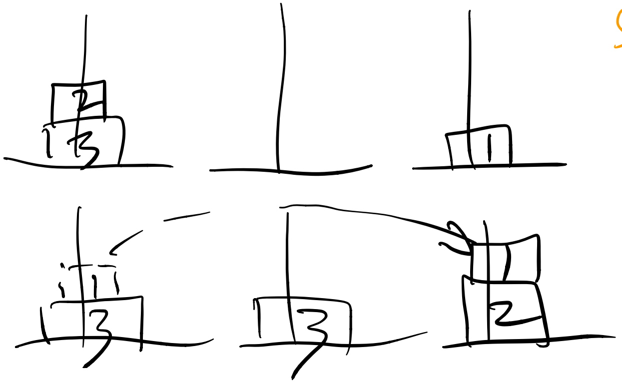
The rule is that a bigger disk cannot be placed on top of a smaller disk.

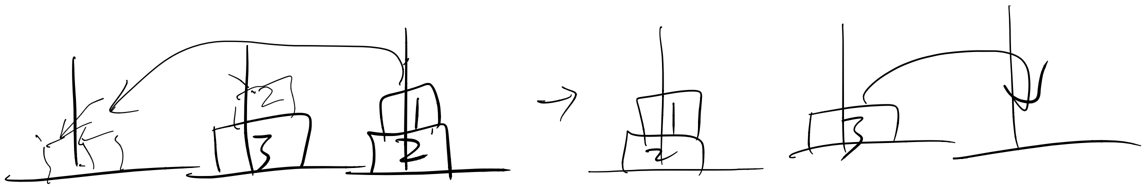
Question 2

For the classical game of Hanoi Tower, if we add the constraint that no disk is allowed to be moved across between pole 1 and pole 3 in one step(you need to move it to pole 2 first, and then it can be moved to pole 3). Find the recurrence relation of the minimum number of steps needed to move n disks from pole 1 to pole 3. (~~No~~ need to solve it)

H_n : n disks $1 \mapsto 3$ (move n disks from 1 to 3, not one step)







$$H_n = H_{n-1} + 1 + H_{n-1} + 1 + H_{n-1}$$
$$(1\to3)\quad(1\to2)\quad(3\to1)\quad(2\to3)\quad(1\to3)$$

$$H_n = 3H_{n-1} + 2$$

$$x = 3x + 2$$

$$x = -1$$

$$(H_n - (-1)) = 3(H_{n-1} - (-1))$$

$$H_n + 1 = 3(H_{n-1} + 1)$$

$$\{H_n + 1\}$$

$$H_n + 1 = \overset{2}{(H_1 + 1)} \cdot 3^{n-1}$$

$$H_n = 3^n - 1$$

$h_n = 2^n - 1$ (original Hanoi)

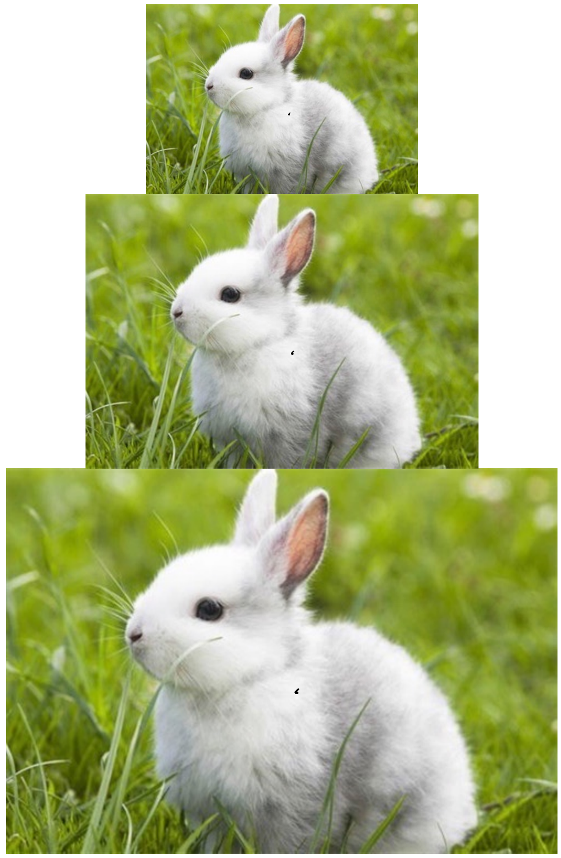
```
def Hanoi(start, middle, end, n)
    Hanoi(start, middle, end, n-1)
        start → middle
    Hanoi(end, middle, start, n-1)
        middle → end

    Hanoi(start, middle, end, n-1)
```

Question 3

- We have a single pair of rabbits (male and female) initially. Assume that:
- The rabbit pairs are not fertile during their first two months of life, but thereafter give birth to three new male/female pairs at the end of every month;
- The rabbits will never die.

Find the recurrence relation for the number of rabbit pairs in n month, denoted by .



⇓

① ② ③ ④ ⑤ ⑥ ⑦

001 010 100 103 133 433 7312 ……

亿元

$\{a_n\}_{小}$ $\{b_n\}_{中}$ $\{c_n\}_{大}$

$r_n = a_n + b_n + c_n$ [new, old]

$$\begin{cases} b_n = a_{n-1} & ① \\ c_n = (\text{new}+\text{old}) = b_{n-1} + c_{n-1} & ② \\ a_n = 3c_{n-1} & ③ \end{cases}$$

$$r_n = 3c_{n-1} + \boxed{a_{n-1} + (b_{n-1} + c_{n-1})}\ r_{n-1}$$

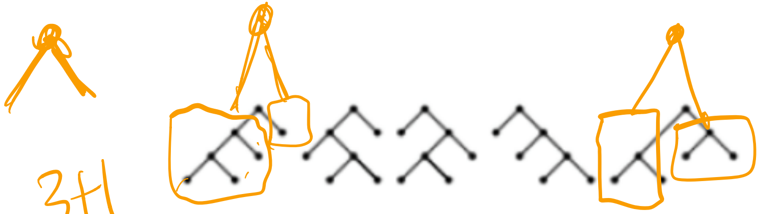
$$= 3(b_{n-2} + c_{n-2})$$

$$= 3\boxed{(a_{n-3} + (b_{n-3} + c_{n-3}))}\ r_{n-3}$$

$$r_n = r_{n-1} + 3r_{n-3}$$

(old) (new)

Question 5. A full binary tree is a rooted binary tree where every vertex has either two children or no children. Let B_n be the number of full binary trees with n leaves, e.g. $B_4 = 5$. Find a recurrence relation of B_n. Explain your answer.



3+1

2+2

n leaves

i leaves (n-i) leaves

n leaves

$$B_n = B_1B_{n-1} + B_2B_{n-2} + \cdots + B_{n-1}B_1 = \sum_{i=1}^{n-1} B_iB_{n-i}, \quad B_1 = 1$$

Parenthesis

How many valid ways to add n pairs of parentheses?

E.g. There are 5 valid ways to add 3 pairs of parentheses.

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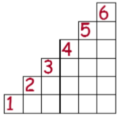
Stairs

Let r_n be the number of ways to fill the n-stair by n rectangles.

How do we compute it using $r_1, r_2, \ldots, r_{n-1}$?



Catalan Number

How many valid ways to add n pairs of parentheses?

$$r_n = \sum_{k=1}^{n} r_{k-1} r_{n-k}$$

So the recursion for the stair problem is the same as the recursion for the parentheses problem. It can be shown that

$$r_n = \frac{1}{n+1}\binom{2n}{n}$$

This is well known as the n-th Catalan number.

A useful formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$