CSC3001: Discrete Mathematics

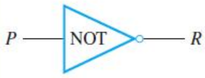
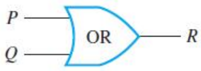
Final Exam (Fall 2018)

Instructions:

1. This exam is 120 minute long.
2. The total mark in this exam paper is 120, and the maximum you can get is 100. **You should try to attempt as many questions as possible.**
3. This exam has 16 pages, consisting of 8 questions. **Write down your full working in this exam paper**.
4. You should check the number of questions in the first 30 minutes. **Instructor is not responsible for any missing questions when exam ends.**
5. Calculators are allowed.
6. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.
7. Please note that all the graphs in this exam paper are **SIMPLE GRAPHS**.

Student Number: ____________________ **Name:** _______________

1. (*18 points*) Suppose that you are given two "**NOT**"s, two "**AND**"s, and two "**OR**"s of the following electronic components:

<table>
<tr><th>Type of Gate</th><th>Symbolic Representation</th><th>Action</th></tr>
<tr><td>NOT</td><td>
</td><td>
<table>
<tr><th>Input</th><th>Output</th></tr>
<tr><th>P</th><th>R</th></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>0</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>AND</td><td>
</td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table>
</td></tr>
<tr><td>OR</td><td>
</td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table>
</td></tr>
</table>

Design a circuit so that it has the following input/output table.

P	Q	R	output
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

2. (*15 points*) Determine the minimum number of registers needed so that the following program executes most efficiently. Model it as a grpah problem and deduce your answer.

Input:	a, c, e
Step 1.	$b = a - 2$
Step 2.	$f = b \cdot e$
Step 3.	$d = c + 1$
Step 4.	$e = 3^a$
Step 5.	$c = b \cdot \frac{1}{4}$
Step 6.	$g = e^2 - f$
Step 7.	$h = b - d$
Output:	f, g, h

3. (*19 points*) Count the number of integer solutions for the following inequality:

$$a + b + c < 28$$

with $0 \leq a, b, c \leq 10$.

4. (*25 points*) Determine whether a graph exists for the following degree sequences:

1) $1, 1, 2, 3$ 2) $5, 5, 5, 5, 5, 5$ 3) $2, 2, 2, 3, 3$ 4) $1, 2, 3, 4, 4$ 5) $1, 1, 2, 3, 3, 4$

If it exists, draw all the possibilities up to isomorphism and determine whether these graphs are planar; otherwise, explain why it does not exist.

5. (*8 points*) The preferences between 4 boys and 4 girls are partially specified as follows.

Boys' preference

boy	preference order
1	$*, *$,B,A
2	$*, *$,A,B
3	C,D,$*, *$
4	D,C,$*, *$

Girls' preference

girl	preference order
A	$*, *$,1,2
B	$*, *$,2,1
C	4,3,$*, *$
D	3,4,$*, *$

Prove that

$$\{(1, A), (2, B), (3, C), (4, D)\}$$

is a stable matching no matter what unspecified preferences are.

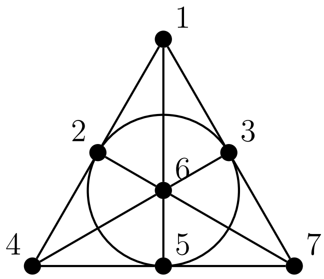
6. (*10 points*) Let G be a bipartite graph with bipartition (A, B). Let $k, d \in \mathbb{Z}^+$. Suppose the vertices in A are all of degree k, and the vertices in B are all of degree d. Give a self-contained proof to show that G has a perfect matching if and only if $k = d$. (**Note:** you cannot cite any result from the lecture notes except Hall's Theorem.)

7. (*17 points*) Let $b, v \in \mathbb{Z}^+$. Let P be a set of *points*, $\mathcal{B} \subseteq \mathsf{Pow}(P)$ a set of *blocks* such that $|P| = v, |\mathcal{B}| = b$. A (b, v, r, k, λ)-*design* is a $(P, \mathcal{B})$-pair such that

- each block contains exactly k points; and
- each point lies in exactly r blocks; and
- each pair of points occurs together in exactly λ blocks.

for some $k, r, \lambda \in \mathbb{Z}^+$.

(i) By defining $P, \mathcal{B}$, show that the following graph is a (b, v, r, k, λ)-design, and hence determine the values of r, k, λ. **[5 marks]**



(ii) Given a (b, v, r, k, λ)-design, prove the following identities:

$$bk = vr, \qquad\qquad r(k-1) = \lambda(v-1)$$

[12 marks]

8. (*8 points*) Let $n \in \mathbb{Z}^+$. Find a subset $S \subseteq A = \{1, 2, \ldots, 2n\}$ of the greatest size such that for any distinct $a, b \in S$ we have $a \nmid b$. Prove your claim.