CSC3001: Discrete Mathematics

Final Exam (Fall 2019)

Instructions:

1. This exam is 120 minute long.
2. The total mark in this exam paper is 110, and the maximum you can get is 100. **You should try to attempt as many questions as possible.**
3. This exam has 14 pages, consisting of 7 questions. **Write down your full working in this exam paper**.
4. You should check the number of questions in the first 30 minutes. **Instructor is not responsible for any missing questions when exam ends.**
5. Calculators are allowed.
6. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.
7. Please note that all the graphs in this exam paper are **SIMPLE GRAPHS**.

Student Number: ____________________ **Name:** _______________

1. (*19 points*)

(a) Let $A = \{x, y\}, B = \{z\}$. Write down the elements of the sets $pow(A) \times pow(B)$ and $pow(A \times B)$ and conclude whether they have equal size. [**6 marks**]

(b) Let $A, B, C \subseteq \mathbb{R}$. Which of the following identities are true? If the identity is true, prove it; otherwise, give a counterexample.

(i) $(A \cap B) \times C = (A \cap C) \times (B \cap C)$ (ii) $(A - B) \times C = (A \times C) - (B \times C)$

(**Note:** $A - B = \{x \in A \mid x \notin B\}$) [**13 marks**]

2. (*10 points*) Let $S \subseteq \{1, 2, \ldots, 100\}$ be of size 25. Prove that there exist mutually distinct $a, b, c, d \in S$ such that $a + b = c + d$.

3. (*11 points*) Suppose our library keeps several copies of the book "*The Grant's Solutions*" in the stock. The following are its borrow/return records in this semester.

Date \ Status	Borrow	Return
2/9	Li Lei, Ken	
8/9	Han Meimei	
10/9	Anne	
20/9		Li Lei
28/9		Ken
8/10	Sue	
9/10		Han Meimei
17/10	Li Lei	
27/10	Han Meimei	
2/11		Anne
11/11	Peter	
19/11		Li Lei
27/11		Sue
3/12	Li Lei	
10/12		Peter
13/12		Li Lei, Han Meimei

Model this problem as a graph problem and determine the minimum number of copies that are kept in our library.

For the remaining questions, all graphs are assumed to be **SIMPLE**.

4. (*25 points*) Determine whether a graph exists for the following degree sequences:

1) $1, 2, 2, 3$ 2) $4, 4, 5, 5, 5, 5$ 3) $2, 2, 3, 3, 3$ 4) $1, 2, 3, 4, 4$ 5) $1, 1, 1, 2, 2, 3$

If it exists, draw all the possibilities up to isomorphism and determine whether these graphs are planar; otherwise, explain why it does not exist.

For the remaining questions, all graphs are assumed to be **SIMPLE**.

5. (*10 points*) Suppose the preferences between 4 boys and 4 girls are as follows.

Boys' preference

boy	preference order
1	C,B,A,D
2	A,C,B,D
3	C,D,A,B
4	D,C,B,A

Girls' preference

girl	preference order
A	4,3,1,2
B	2,3,4,1
C	4,3,1,2
D	3,4,2,1

Determine whether the following matching is stable or not.

$$\{(1, A), (2, D), (3, B), (4, C)\}$$

If yes, prove it; otherwise list all the unstable pairs.

For the remaining questions, all graphs are assumed to be **SIMPLE**.

6. (*15 points*) Let $n \geq 2$ be an integer and let $G = (V, E)$ be a bipartite graph with $V = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ and $E = \{a_i b_j | i \neq j\}$.

(a) Show that G has a perfect matching. [**5 marks**]

(b) Count the total number of different perfect matchings in G. [**10 marks**]

For the remaining questions, all graphs are assumed to be **SIMPLE**.

7. (*20 points*) Let G be a k-regular graph with n vertices such that

- every pair of adjacent vertices has λ common neighbors; and
- every pair of non-adjacent vertices has μ common neighbors.

for some $k, n \in \mathbb{Z}^+$ and $\lambda, \mu \in \mathbb{N}$.

(i) Prove that $(n-k-1)\mu = k(k-\lambda-1)$. [**10 marks**]

(ii) Suppose $k \geq 2$. When will G be disconnected? What is the relation between λ and n when G is disconnected? [**10 marks**]

For the remaining questions, all graphs are assumed to be **SIMPLE**.