CSC3001: Discrete Mathematics

Midterm Exam (Fall 2018)

Instructions:

1. This exam is 120 minute long, and worth 100 points.
2. This exam has 12 pages, consisting of 6 questions, all to be attempted. **Write down your full working in this exam paper**.
3. Calculator is allowed.
4. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.

Student Number: ____________________ **Name:** _______________

1. (*18 points*) Given statements p, q, r, s, which of the following arguments are valid? (**Note:** you need to give the reason in order to obtain full mark.)

(i)
$$\begin{array}{ll} & (p \vee q) \to \neg r \\ & p \to \neg q \\ & \neg q \to p \\ \hline \therefore & \neg r \end{array}$$

(ii)
$$\begin{array}{ll} & p \to q \\ & q \to \neg p \\ \hline \therefore & p \to \neg p \end{array}$$

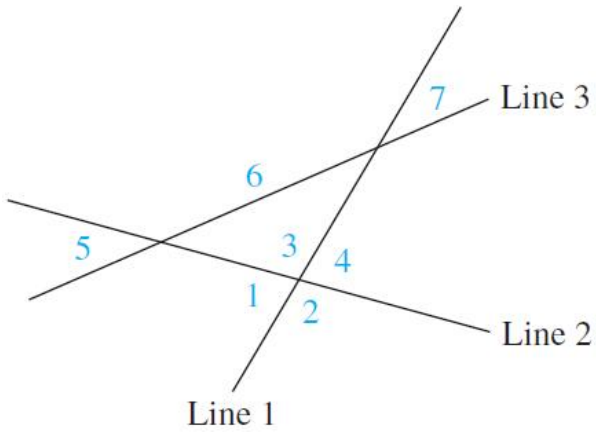
(iii)
$$\begin{array}{ll} & (q \wedge r) \to p \\ & (p \vee q) \to r \\ \hline \therefore & s \leftrightarrow s \end{array}$$

2. (*12 points*) For all integers $n \geq 4$, use **Well Ordering Principle** to prove that

$$n^2 \leq 2^n$$

3. (*20 points*) Find the smallest positive four consecutive integers such that they are divisible by $4, 9, 25, 49$ respectively.

4. (*20 points*) A single line divides a plane into 2 regions; two lines (by crossing) divide a plane into 4 regions; three lines divde a plane into 7 regions (as below). Let r_n denote the maximum number of regions that n lines can divide into.



(a) Find the recurrence relation for $r_n,\ n \in \mathbb{N}$. [**4 marks**]

(b) Apply **generating function** to find the closed form for r_n. [**16 marks**]
$\left(\textbf{Note:} \text{ you may use the formula } (1 + x + \ldots + x^n + \ldots)^3 = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} x^k\right)$

5. (*20 points*) Find an example of irrational numbers a, b such that a^b is also irrational. (You need to make your choice of a, b, and show that a, b, a^b are all irrational. You may use the fact that any rational number can be written as $\frac{m}{n}$ with $\mathsf{gcd}(m, n) = 1$.)

6. (*10 points*) A triangular board has been cut into 100 small triangular cells by the lines parallel to its sides. Two cells that share a side are called neighbors. In each cell there is a grasshoper. All at once, the grasshoppers hop from their cells to neighbor cells. Prove that at least 10 cells will be empty after 99 times of hops.