Tut 12 & Review

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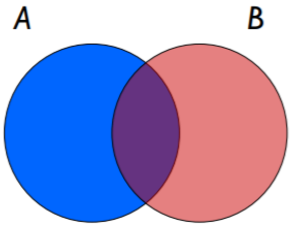
Inclusion-Exclusion (2 sets)

For two arbitrary sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Inclusion-Exclusion (n sets)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

+ - + -

Q1

How many shuffles are there of a deck of cards, such that $A\heartsuit$ is not directly on top of $K\heartsuit$, and $A\spadesuit$ is not directly on top of $K\spadesuit$?

52 cards

13 $\heartsuit$ 13 $\diamondsuit$ 13 $\spadesuit$ 13 $\clubsuit$

A: $A\heartsuit$ on top of $K\heartsuit$

B: $A\spadesuit$ on top of $K\spadesuit$

$\boxed{\begin{matrix}A\\K\end{matrix}} \to$ 2 to 1

$\boxed{\begin{matrix}A\\K\end{matrix}} \to 2 \to$ to

$$|\bar{A}\cap\bar{B}| = |\overline{A\cup B}| = N - |A\cup B| = N - (|A|+|B|-|A\cap B|)$$

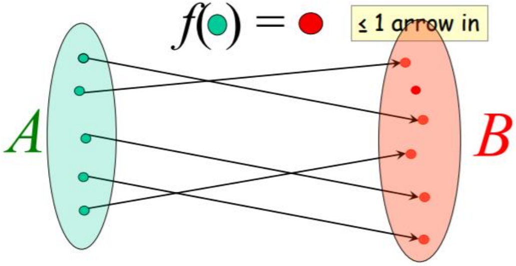
$$= 52! - (51! + 51! - 50!)$$

$$= 52! - 2\times 51! + 50!$$

Injections

$f: A \to B$ is injective if no two inputs have the same output.

单射

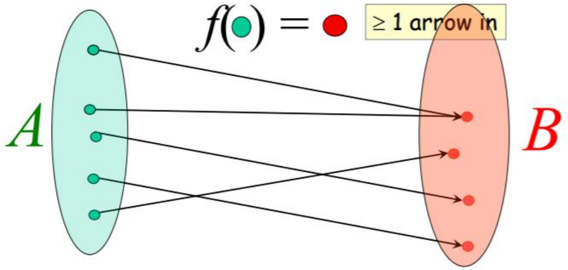


$$\boxed{|A| \leq |B|}$$

Surjections

$f: A \to B$ is surjective if every output is possible.

满射

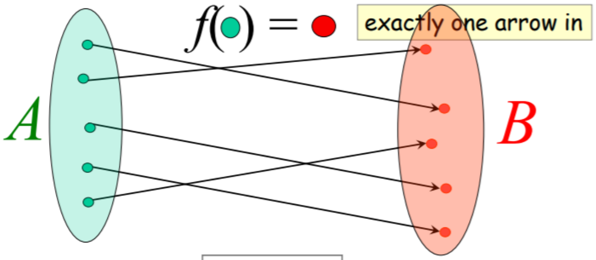


$$\boxed{|A| \geq |B|}$$

Bijections

$f: A \to B$ is bijective if it is both surjective and injective.

全射



$$\boxed{|A| = |B|}$$

Q2

How many flags can we make with 7 stripes, if we have 2 white, 2 red and 3 green stripes?

$$\binom{7}{2,2,3} = \frac{7!}{2!2!3!}$$

aa. bb. ccc

abccb ac

$$\frac{7!}{2!2!3!}$$

eliminate order

Q3

We go to a pizza party, and there are 5 types of pizza. We have starved for days, so we can eat 13 slices, but we want to sample each type at least once. In how many ways can we do this? Order does not matter.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 13 \\ x_1, x_2, x_3, x_4, x_5 \geq 1 \end{cases}$$

12 positions

$$00|000|00|000|000$$

2 3 2 3 3

type A B C D E

$$\binom{12}{4}$$

0: pizzas

1: divider

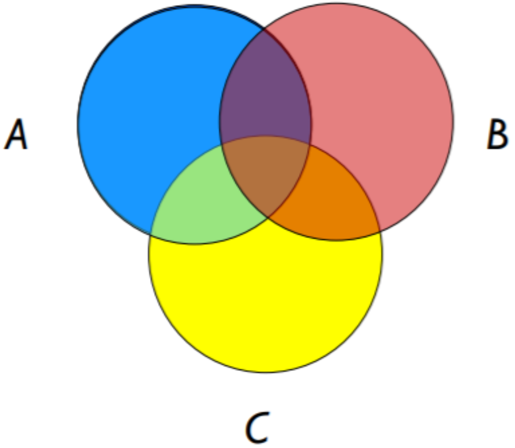
$$y_1 + y_2 + y_3 + y_4 + y_5 = 8$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

$$\binom{8+5-1}{5-1} = \binom{12}{4}$$

Inclusion-Exclusion (3 sets)

$$
\begin{aligned}
|A \cup B \cup C| = |A| + |B| + |C| \\
- |A \cap B| - |A \cap C| - |B \cap C| \\
+ |A \cap B \cap C|
\end{aligned}
$$



Q4

There are five people of different height. In how many ways can they stand in a line, so there is no 3 consecutive people with increasing height?

1 2 3 4 5 (height)

12345 X

21345 X

3 1245 X

54321 ✓

45231 ✓

Distinguish height & position

method 1:

$\boxed{1\,2\,3}\,0\,0 \quad 0\,\boxed{1\,2\,3}\,0 \quad 0\,0\,\boxed{1\,2\,3}$

$\boxed{1\,2\,3}\,4\,5$

$1\,\boxed{2\,3\,4}\,5$

3 people

4 people $\boxed{1\,2\,3\,4}\,0 \quad 0\,\boxed{1\,2\,3\,4}$

5 people

$$\binom{5}{3}\cdot 2!\cdot 3-\binom{5}{4}\cdot 1!\cdot 2+\binom{5}{5}$$

take inclusion-exclusion

A
B
C

1 2 3 4 5

$3-2=1$

(ignore this!)

---

1 2 3 4 5 (height)

watch for positions

method 2:

$\overline{P_1}\ \overline{P_2}\ \overline{P_3}\ \overline{P_4}\ \overline{P_5}$

A: P_1, P_2, P_3 increasing orders

B: P_2, P_3, P_4

C: P_3, P_4, P_5

Should define set A, B, C well

$$(\bar{A}\cap\bar{B}\cap\bar{C})=|\overline{A\cup B\cup C}|=N-|A\cup B\cup C|$$

$$=5!-\left(3\cdot\binom{5}{3}\cdot 2!-\left(\binom{5}{4}-\binom{5}{4}-1\right)+1\right)$$

$(|A|+|B|+|C|)$ $\quad$ $(|A\cap B|+|A\cap C|+|B\cap C|)$ $\quad$ $|A\cap B\cap C|$

$$=120-(60-5-5)=70$$

- Mid-term (updated)
  average: 74.82
  median: 78
  max: 98
  (forget it! Final is the most important!)

- Final exam: (harder than mid-term)
  7 Qs, 100/110

- Suggestions:
  Do more practices (FQ, tut, asg, lec)
  Summarize the techniques to solve each type of questions.

  Search related questions on Google.

What we have learned…

- 1. Before midterm …
- 2. Graph Theory
- 3. Graph matching
- 4. Graph coloring
- 5. Combinatorial proof
- 6. Pigeon - hole theorem
- 7. Counting

1. Before midterm …

Logic: circuit, valid check

Set: pow(S) {subset}

GCD: pouring water problem.

$$\begin{array}{ll} & p \wedge q \to \neg r \\ \text{(i)} & p \to \neg q \\ & \neg q \to p \\ \hline \therefore & \neg r \end{array}$$

→ valid

T → F ✗

Theorem. Given water jugs of capacity a and b with a ≤ b, it is possible to have exactly k (≤ b) gallons in one jug if and only if k is a multiple of gcd(a,b).

wop

Mathematical induction

Recursion modular

2. Graph Theory

4. (*25 points*) Determine whether a graph exists for the following degree sequences:

1) 1, 2, 2, 3 2) 4, 4, 5, 5, 5, 5 3) 2, 2, 3, 3, 3 4) 1, 2, 3, 4, 4 5) 1, 1, 1, 2, 2, 3

If it exists, draw all the possibilities up to isomorphism and determine whether these graphs are planar; otherwise, explain why it does not exist.

$n=6 \quad m=14$

① calculate n, m
② start from large

$$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

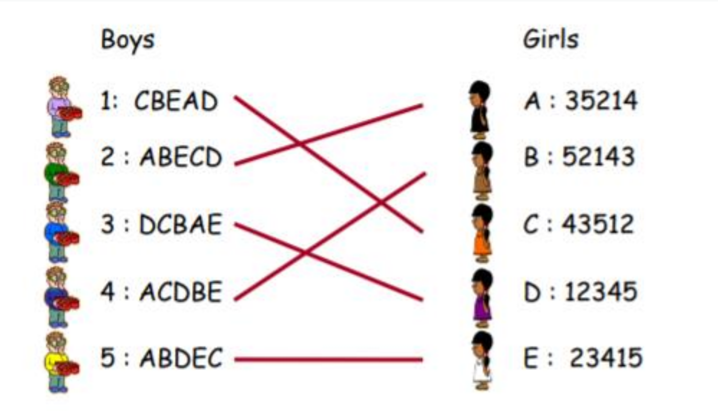
If a connected planar graph has n vertices, m edges, and f faces, then

$$n - m + f = 2$$

Claim. If G is a simple planar graph with at least 3 vertices, then

$$m \le 3n-6$$

3. Graph matching



Boys' preference

boy	preference order
1	C,B,A,D
2	A,C,B,D
3	C,D,A,B
4	D,C,B,A

Girls' preference

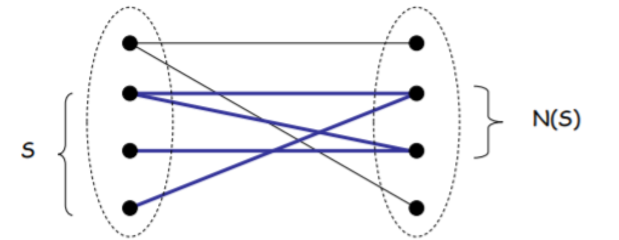
girl	preference order
A	4,3,1,2
B	2,3,4,1
C	4,3,1,2
D	3,4,2,1

Should be very familiar with the procedure (boy propose first)

Boy optimal.

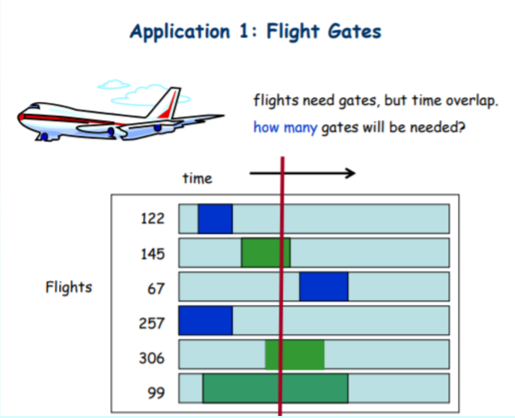
3. Graph matching

Hall's Theorem. A bipartite graph G=(V,W;E) has a perfect matching **<u>if and only if</u>** |N(S)| ≥ |S| for each subset S of V and for each subset S of W.



4. Graph coloring

- **Graph Formulation:**
- Register allocation, borrow books, airline, exam…



$$
\begin{array}{lrcl}
 & \text{Inputs:} & & a, b \\
\text{Step 1.} & c & = & a + b \\
2. & d & = & a * c \\
3. & e & = & c + 3 \\
4. & f & = & c - e \\
5. & g & = & a + f \\
6. & h & = & f + 1 \\
 & \text{Outputs:} & & d, g, h
\end{array}
$$

5. Combinatorial proof

7. (*20 points*) Let G be a k-regular graph with n vertices such that

- every pair of adjacent vertices has λ common neighbors; and
- every pair of non-adjacent vertices has μ common neighbors.

for some $k, n \in \mathbb{Z}^+$ and $\lambda, \mu \in \mathbb{N}$.

(i) Prove that $(n-k-1)\mu = k(k-\lambda-1)$. **[10 marks]**

(ii) Suppose $k \geq 2$. When will G be disconnected? What is the relation between λ and n when G is disconnected? **[10 marks]**

(ii) For $n \geq 2$, establish the following by combinatorial argument

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1)$$

Choose (a, b, S) $S \subseteq \{1, 2, \cdots, n\}$

$a, b \in S$

a, b are elements

S is set

Solution: (referenced)

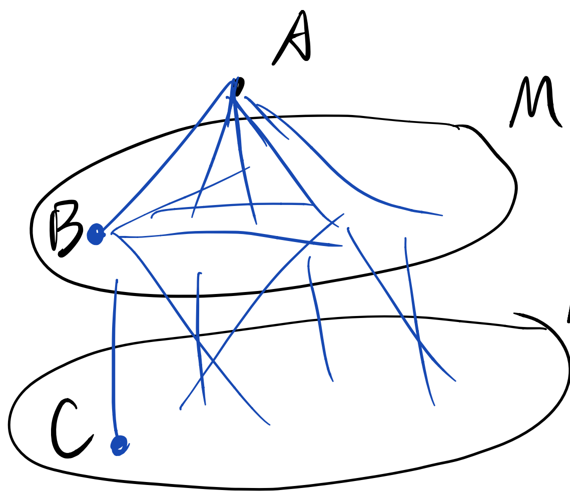
(1) Pick a vertex A, divide the other vertices into 2 categories:

M: all the neighboring vertices of A (adjacent)

N: all the non-adjacent vertices of A

k-regular

n vertices



⇐ edges between M and N

① pick a vertex B in M:

A, B are adjacent vertices, they have λ common neighbors, and they must all lie in M (neighbors of A)

so B connects to λ vertices in M

B is k-degree ⇒ B connects to $(k-\lambda-1)$ vertices in N

② pick a vertex C in N:

A, C are non-adjacent pairs, they have μ common neighbors, and they must all lie in M. (neighbors of A) so C connects to μ vertices in M

③ A is k-degree. There are k vertices in M, n in total

so there are $(n-k-1)$ vertices in N

Consider # of edges between M and N

LHS: $(n-k-1)\mu$ (# of vertices in N $\times$ # of edges connecting M for each vertex)

RHS: $k(k-\lambda-1)$ (# of vertices in M $\times$ # of edges connecting N for each v)

$*$ LHS = RHS

(2) G is disconnected when $\mu=0$ and $(n>k+1)$
(# of edges between M and N is zero)

From (1): $k(k-\lambda-1)=0$ $k\neq 0 \Rightarrow k-\lambda-1=0$

- consider A, A is connected to k vertices in M and no vertex in N. They form a $(k+1)$ vertex cluster. Here A can be any vertex in the Graph. For any vertex in the cluster, it connects to all other vertices since the graph is k-regular. K_{k+1}

There are many $(k+1)$ complete graph in the whole graph

As a result: $(k+1) \mid n \Rightarrow (\lambda+2) \mid n$

6. Pigeon - hole theorem

2. (*10 points*) Let $S \subseteq \{1, 2, \ldots, 100\}$ be of size 25. Prove that there exist mutually distinct $a, b, c, d \in S$ such that $a + b = c + d$.

* recommended exercise: FQ 11-8, 9, 10

Solution: (referenced)

① Possible numbers of sum of two elements in S ($a+b$/$c+d$)

min: $1+2=3$

max: $99+100=199$ $\qquad$ $3 \sim 199$: 197 numbers

② Possible combinations of two numbers in S:

$$\binom{25}{2} = \frac{25 \times 24}{2} = 300$$

By pigeon-hole principle:

- 300 values are distributed into 197 positions, so there must be two equal sums $(a+b)$ and $(c+d)$.
- $a \neq b$ $c \neq d$ by selection (select 2 different elements from 25 numbers)
- $a \neq b \neq c \neq d$ or they will not have the same sum.

Q.E.D.

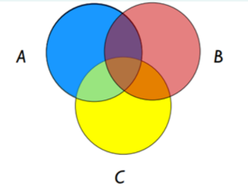
7. Counting

solutions

3. (*19 points*) Count the number of integer solutions for the following inequality:

$$a + b + c < 28$$

with $0 \leq a, b, c \leq 10$.



$$\boxed{\begin{aligned} |A \cup B \cup C| = |A| + |B| + |C| \\ - |A \cap B| - |A \cap C| - |B \cap C| \\ + |A \cap B \cap C| \end{aligned}}$$

USTF evaluation: