CSC3001: Discrete Mathematics

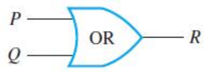
Midterm Exam (Spring 2018)

Instructions:

1. This exam is 110 minute long, and worth 100 points.
2. This exam has 12 pages, consisting of 6 questions, all to be attempted. **Write down your full working in this exam paper**.
3. No calculator is allowed.
4. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.

Student Number: ____________________ **Name:** _______________

1. (*14 points*) Suppose that you are given two "**NOT**"s, one "**AND**", and one "**OR**" of the following electronic components:

<table>
<tr><th>Type of Gate</th><th>Symbolic Representation</th><th>Action</th></tr>
<tr><td>NOT</td><td>
</td><td><table><tr><th>Input</th><th>Output</th></tr><tr><th>P</th><th>R</th></tr><tr><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td></tr></table></td></tr>
<tr><td>AND</td><td>
</td><td><table><tr><th colspan="2">Input</th><th>Output</th></tr><tr><th>P</th><th>Q</th><th>R</th></tr><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table></td></tr>
<tr><td>OR</td><td>
</td><td><table><tr><th colspan="2">Input</th><th>Output</th></tr><tr><th>P</th><th>Q</th><th>R</th></tr><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table></td></tr>
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Design a circuit so that it has the following input/output table.

P	Q	R	output
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

2. (*23 points*) Solve the following system of modular equations:

$$\begin{cases} 95x \equiv 5 \pmod{40} \\ 21x \equiv -9 \pmod{60} \\ x \equiv 16 \pmod{15} \end{cases}$$

3. (*14 points*) Let $a, b, c \in \mathbb{Z}^+$ be mutually coprime. Suppose $n \in \mathbb{N}$ is divisible by a, b, c. Prove that $abc|n$.

4. (*21 points*) Consider the set of all strings of a's, b's and c's. Let r_n be the number of strings of a's, b's and c's of length n that do not contain the pattern aa. ($n \in \mathbb{Z}^+$)

(a) List all the strings of length one, two and three that do not contain the pattern aa, and hence give the values of r_1, r_2, r_3. [**6 marks**]

(b) Find the recurrence relation for $\{r_n\}$. [**5 marks**]

(c) Find the closed form for r_n. [**10 marks**]

5. (*18 points*)

(i) Use **only** $\forall, \exists, \neg, \wedge, \vee, =, \neq$ to translate the following statement into a first-order logical formula. (You are **NOT** allowed to use any other symbols like $\rightarrow, >, <$, etc.)

$S(n) =$ "The number n cannot be written as the sum of three or more consecutive positive integers." [**5 marks**]

(ii) Prove that an odd number n greater than 1 is a prime if and only if it cannot be written as the sum of three or more consecutive positive integers. [**13 marks**]

6. (*10 points*) Use **Well Ordering Principle** to prove that

$$n^3 \leq 3^n \qquad \forall n \in \mathbb{N}$$

(**Note:** you may use the fact that $\sqrt[3]{3} \approx 1.44$.)