CSC3001 Discrete Mathematics

Tutorial 1

Logical operators

$\neg$ ::= NOT

P	$\neg P$
T	F
F	T

$\wedge$ ::= AND

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

$\vee$ ::= OR

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive-Or

Is there a more systematic way to construct such a formula?

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

Idea 1: Look at each true row

Find a formula so that it is only true when having exactly the same input: the second row gives a T value when p is true but q is false, i.e. $(p \wedge \neg q)$; likewise do this for all rows with T value and OR them together

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$

This sub-formula is true only when the input is the second row

And the formula is true exactly when the input is the second row or the third row.

Analogous to ordinary numbers, $\wedge$ may be viewed as product, and $\vee$ may be viewed as sum. The above approach is often called **sum-of-products**.

Idea 2: Look at the false rows

Find a formula so that it is only false when having exactly the same input. The second row gives a F value when both p and q are true, i.e. . $\neg(p \wedge q)$; likewise, do this for all rows with F value and AND them together, giving:

$$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q).$$

By DeMorgan's Law, this becomes

$$(\neg p \vee \neg q) \wedge (p \vee q),$$

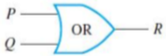
which is a product-of-sums

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Question 1

$2\neg$ $1\wedge$ $1\vee$

1. (*14 points*) Suppose that you are given two "**NOT**"s, one "**AND**", and one "**OR**" of the following electronic components:

<table>
<tr><th>Type of Gate</th><th>Symbolic Representation</th><th>Action</th></tr>
<tr><td>NOT</td><td>
</td><td>
<table>
<tr><th>Input</th><th>Output</th></tr>
<tr><th>P</th><th>R</th></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>0</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>AND</td><td>
</td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table>
</td></tr>
<tr><td>OR</td><td>
</td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table>
</td></tr>
</table>

Design a circuit so that it has the following input/output table.

P	Q	R	output
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	1

(and: using ·(multiply) to represent) (or: using + to represent)

(If you don't like this representation, ignore it, if you have EIE2050, use it)

Idea 1: true row

$$pq\bar r+\bar p q r+\bar p q\bar r+\bar p\bar q r+\bar p\bar q\bar r$$

$$= pq\bar r+\bar p r\underbrace{(q+\bar q)}_{1}+\bar p\bar r\underbrace{(q+\bar q)}_{1}$$

$$= pq\bar r+\bar p\underbrace{(r+\bar r)}_{1} = pq\bar r+\bar p$$ [$(p\wedge q\wedge\bar r)\vee\bar p=(p\vee\bar p)\wedge(q\vee\bar p)\wedge(\bar r\vee\bar p)$]

$$= (p+\bar p)(q+\bar p)(\bar r+\bar p)$$ [$(\bar p+q)(\bar p+\bar r)=\bar p(1+q+\bar r)+q\bar r$]

$$= \bar p+q\bar r$$ [$(A+B)(A+C)=A\cdot A+A\cdot B+A\cdot C+B\cdot C$ $=A(1+B+C)+B\cdot C=A+BC$]

alternative: $pq\bar r+\bar p=\overline{\overline{pq\bar r+\bar p}}=\overline{(\bar p+\bar q+r)\cdot p}=\overline{(\bar q+r)\cdot p}=q\cdot\bar r+\bar p$

Idea 2: false row

$$(\bar p+\bar q+\bar r)(\bar p+q+\bar r)(\bar p+q+r)$$ [$(A+B)(A+C)=A+BC$]

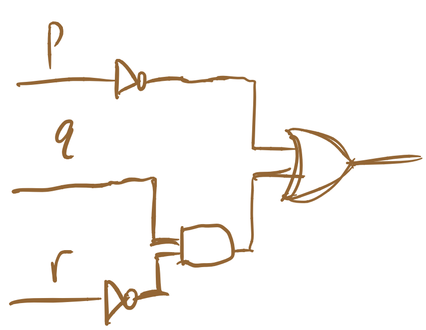
$$=((\bar p+\bar r)+\underbrace{q\bar q}_{0})(\bar p+q+r)$$

$$=(\bar p+\bar r)(\bar p+q+r)$$

$$=\bar p+\bar r(q+r)$$

$$=\bar p+q\bar r$$

$\neg p\vee(q\wedge\neg r)$



Question 2

SQ In a remote village, there are three types of people: Knights – those who always tell the truth; Knaves – those who always lie; Spies – those who can lie or tell the truth.

(a) You have encountered three villagers, and among them there is exactly one spy, one knight and one knave. They address you as follow:

A: C is a knave.
B: A is a knight.
C: I am the spy.

What are A, B and C?

(b) You then encouter another three villagers, and among them there is exactly one spy, one knight and one knave. They address you as follow:

D: E is the spy!
E: No, F is the spy!
F : No, E is definitely the spy.

What are D, E and F ?

(2)

	A	B	C
knight	✓		
knave			✓
spy		✓	

	D	E	F
knight		✓	
knave	✓		
spy			✓

(3)

p	q	$p\to q$	$q\to p$	$p\vee q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	F

✗

invalid

p	q	$p\to q$	$q\to p$	$(p\to q)\vee(q\to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

tautology ✓

$\rightarrow$ ::= IMPLIES

If p then q

$$p \longrightarrow q$$

p implies q

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

p is called the hypothesis; q is called the conclusion

An argument is valid if:

whenever all the assumptions are true, then the conclusion is true.

A **tautology** is a statement that is always true.

A **contradiction** is a statement that is always false.

Question 3

(a) Determine whether the following argument is valid by truth tables.

$$\begin{array}{r} p \to q \\ q \to p \\ \therefore p \vee q \end{array}$$

(b) Decide the following statement is true or not.

$(p \to q) \vee (q \to p)$ is a tautology.

Extra Question 1

4. **SQ** Construct and then simplify logical formulas for $f(p, q, r)$ using only the $\neg$, $\vee$, $\wedge$ operators.

(a)

p	q	r	$f(p, q, r)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

(b)

p	q	r	$f(p, q, r)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

(a) $pqr + \bar{p}\bar{q}r + \bar{p}\bar{q}\bar{r}$

$= pq\bar{r} + \bar{p}\bar{q}(r + \bar{r})$

$= pq\bar{r} + \bar{p}\bar{q}$

(b) $(\bar{p}+\bar{q}+r)(p+\bar{q}+r)(p+q+r)$

$= ((\bar{q}+r)+p\bar{p})(p+q+r)$

$= r + (p+q)\cdot\bar{q}$

$= r + p\bar{q}$

Extra Question 2

$2\neg$ $2\wedge$ $2\vee$

1. (*18 points*) Suppose that you are given two "**NOT**"s, two "**AND**"s, and two "**OR**"s of the following electronic components:

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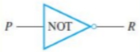
<tr><td>NOT</td><td>
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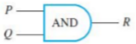
<tr><td>AND</td><td>
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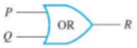
<tr><td>OR</td><td>
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Design a circuit so that it has the following input/output table.

P	Q	R	output
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

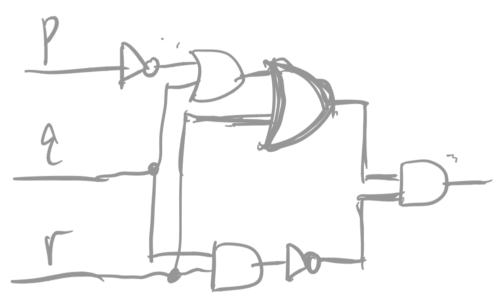
Look at the false row:

$$(\bar p+\bar q+\bar r)(\bar p+q+r)(p+\bar q+\bar r) \qquad (A+B)(A+C)=A+BC$$

$$=(\bar p+q+r)(\bar q+\bar r+\underbrace{\bar p p}_{0})$$

$$=(\bar p+q+r)\cdot\overline{qr}$$

$$(\neg p\lor q\lor r)\land\neg(q\land r)$$



A method used in EIE2050

r \ pq	00	01	11	10
0	1	1	1	0
1	1	0	0	1

$0 \rightarrow (\bar p+q+r)$

$0\ 0 \downarrow (\bar q+\bar r)$

$$(\bar p+q+r)(\bar q+\bar r)$$
$$=(\bar p+q+r)\cdot\overline{q\cdot r}$$