CSC3001 Tutorial 3

- Direct proof intuition
- Contrapositive If P then Q $\rightleftharpoons$ If $\neg Q$ then $\neg P$
- Proof by contradiction Assume ... true/false $\Rightarrow$ common
- Proof by cases

Q1. Prove

If a, b are integers, then $a^2-4b \neq 2$

Q2. Prove

$\log_2 3$ is irrational.

①

proof by contradiction

Assume $\exists a.b \in \mathbb{Z}$ $a^2 - 4b = 2$

$\Rightarrow$ $a^2 = 4b+2 = 2(2b+1)$ (even)

a^2 is even $\Rightarrow a$ is even : $a = 2k$, $k \in \mathbb{Z}$

$(2k)^2 = 2(2b+1) \Rightarrow 4k^2 = 2(2b+1) \Rightarrow 2k^2 = 2b+1$

$\Rightarrow$ $\underbrace{2(k^2-b)}_{\text{even}} = \underbrace{1}_{\text{odd}}$ contradiction!

$\Rightarrow a^2 - 4b \neq 2$

②

proof by contradiction

Assume $\log_2 3$ is rational

$\log_2 3 = \frac{a}{b}$ ($a.b \in \mathbb{Z}^+$, a,b are coprime)

$\gcd(a.b) = 1$

$\Downarrow$

$2^{\frac{a}{b}} = 3$

$2^a = 3^b$ $\Rightarrow$ LHS is even

RHS is odd

contradiction! $\Rightarrow$ $\log_2 3$ is irrational

Q3. Prove

If k is a positive integer and $\sqrt{k}$ is not a integer, then $\sqrt{k}$ is irrational.

proof by contrapositive

If $k \in \mathbb{Z}^+$ and $\sqrt{k} \notin \mathbb{Z}$, then $\sqrt{k}$ is irrational

$(k \in \mathbb{Z}^+) \wedge (\sqrt{k} \notin \mathbb{Z}) \longrightarrow (\sqrt{k}$ is irrational$)$

$\sqrt{k}$ is rational $\longrightarrow (k \notin \mathbb{Z}^+) \vee (\sqrt{k} \in \mathbb{Z})$

Assume $\sqrt{k}$ is rational $\quad \sqrt{k} = \frac{a}{b} \; (a, b \in \mathbb{Z}^+, \gcd(a.b)=1)$ (coprime)

$\Rightarrow k = \frac{a^2}{b^2} \Rightarrow kb^2 = a^2$

① $k \notin \mathbb{Z}^+$ ✓

② $k \in \mathbb{Z}^+$, $\gcd(a.b)=1 \Rightarrow \gcd(a^2, b^2)=1 \Rightarrow \gcd(kb^2, b^2)=1$

$\gcd(kb^2, b^2) = b^2 \qquad b^2 = 1 \Rightarrow b = 1$

$\sqrt{k} = a \in \mathbb{Z}^+ \Rightarrow \sqrt{k} \in \mathbb{Z}$ ✓

Q.E.D.

The Induction Rule

0 and (from n to $n+1$),

proves 0, 1, 2, 3,....

Very easy to prove

Much easier to prove with P(n) as an assumption.

induction rule (an axiom)

$$\frac{P(0),\ \forall n \in Z\ \ P(n) \rightarrow P(n+1)}{\forall m \in Z\ \ P(m)}$$

The point is to use the knowledge on smaller problems to solve bigger problems (i.e. can assume P(n) to prove P(n+1)).

Compare it with the universal generalization rule.

Q4.

Prove or disprove that $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^3 + 3n^2 + 2n = 3k$.

Q4.modified

Use mathematical induction to Prove or disprove that $\forall n \in \mathbb{Z}^+, \exists k \in \mathbb{Z}^+, n^3 + 3n^2 + 2n = 3k.$

④ (a) $n^3+3n^2+2n = n(n^2+3n+2) = n(n+1)(n+2)$

$n \in Z$ three consecutive integers

There must be a multiple of 3

has a factor 3

$3k$

(b)

step:

① write down $P(n)$

② base case:

when $n = n_0$ (0, 1, ...)

$P(n_0)$ is true

③ inductive step:

Assume $P(t)$ is true for some t,

For $n = t+1 \Rightarrow P(t+1)$ is true

④ $P(n)$ is true for $\forall n \in N$

$P(n)$: $\exists k \in Z^+$: $n^3+3n^2+2n = 3k$

(1) base case:

when $n=1$ $n^3+3n^2+2n = 6 = 3 \times \underset{k}{2}$

$P(1)$ is true

(2) inductive step:

Assume $P(t)$ is true for some $t \in Z^+$

$\exists k_0 \in Z^+$ $t^3+3t^2+2t = 3k_0$

For $n = t+1$:

$(t+1)^3 + 3(t+1)^2 + 2(t+1)$

$= (t^3+3t^2+3t+1) + 3(t^2+2t+1) + 2t+2$

$= (t^3+3t^2+2t) + 3(t^2+3t+2)$

$= 3(k_0+t^2+3t+2) \Rightarrow P(t+1)$ is true

$P(n)$ is true for $\forall n \in Z^+$