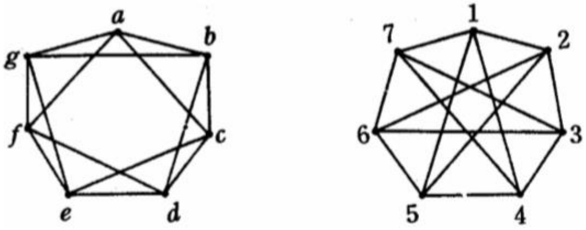
Tut 8

Jingyu Li

1. (*20 points*) Determine whether the following two graphs are isomorphic or not.



If yes, give an isomorphism; if not, give a reason.

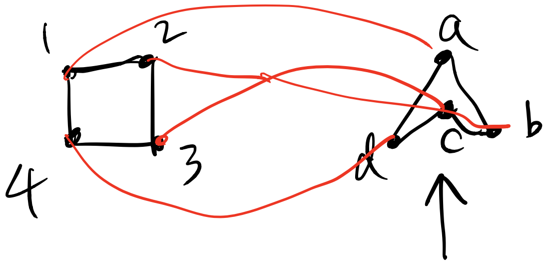
G_1 *isomorphic* to G_2 means there is a one-to-one mapping (***isomorphism***) of the vertices that is edge-preserving.

$\exists$ *one-to-one mapping* $f: V_1 \to V_2$

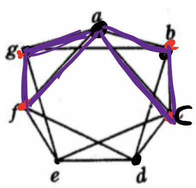
$u - v$ in E_1 iff $f(u) - f(v)$ in E_2

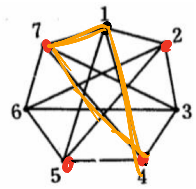
uv is an edge in G_1

f(u)f(v) is an edge in G_2



1-a 3-c
2-b 4-d





找三角形

① check degrees
② try map using triangle

a-1 b-2 g-7 c-4 f-5

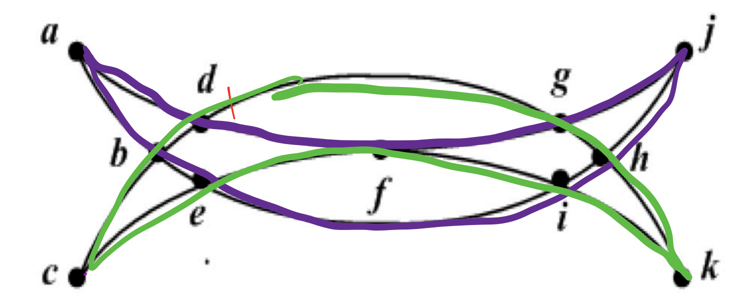
3

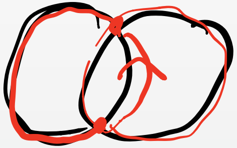
a-1 g-7 f-4 b-2 c-5 d-3
e-6

2. Is the following "***Khanjar***" an Eulerian Graph? If YES, find an Euler cycle. If NOT, explain.

(a) degrees

(b) cycles





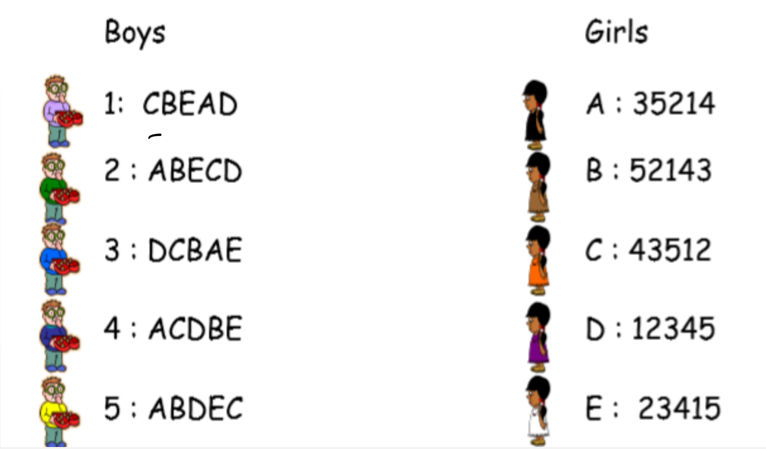
purple

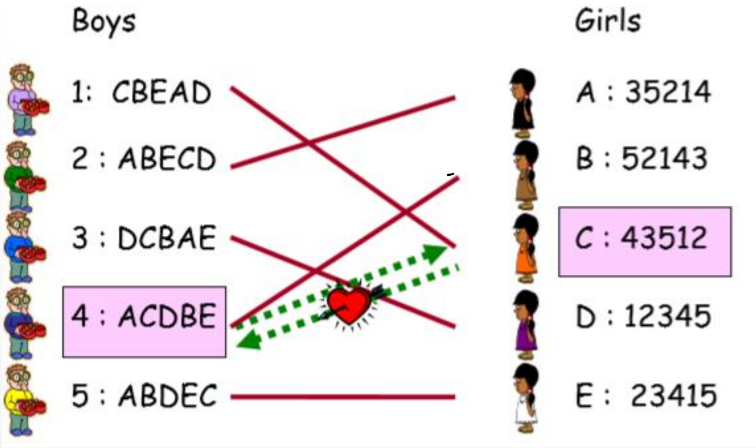
a→e → e→a

从某a出发，从任意点进绕，绕一圈出来，

再绕完整

Euler's theorem. A connected graph has an Eulerian cycle if and only if every vertex is of even degree.





(4,B) (1,C)

(4,C)

More formally, given a matching M for the vertex set V, the pair (v,w) is ***unstable*** if

1. (v, w′) and (v′, w) are matched pairs in M, where v ≠ v′, w ≠ w′.
2. v prefers w rather than w′.
3. w prefers v rather than v′.

boy-optimal boy-pessimal

Use boy-proposing algorithm and girl-proposing algorithm to find a stable matching for the following stable marrying problem.

Boys		Girls	
1	B A C D	A	1 2 3 4
2	B A C D	B	2 1 3 4
3	A C D B	C	1 4 2 3
4	A C B D	D	3 2 1 4

Boys		Girls	
1	B A C D	A	1 2 3 4
2	B A C D	B	2 1 3 4
3	A C D B	C	1 4 2 3
4	A C B D	D	3 2 1 4

What can you conclude from the two solutions.

The Marrying Procedure

Morning: boy propose to their favourite girl
Afternoon: girl rejects all but favourite (i.e. top suitor)
Evening: rejected boy writes off the girl

- Since the boy-proposing process is boy optimal while the girl-proposing process is boy pessimal. So the partner of every boy in the first matching is the best partner while the one in the second matching is the worst partner. And since they are the same, we can conclude that there is only one valid girl for every boy. Similarly, there is also only one valid boy for every girl. So in this problem, the stable marrying matching is unique.

3. (*20 points*) The preferences between 4 boys and 4 girls are partially specified as follows.

Boys' preference

boy	preference order
1	A,B,∗, ∗
2	B,A,∗, ∗
3	∗, ∗,D,C
4	∗, ∗,C,D

Girls' preference

girl	preference order
A	2,1,∗, ∗
B	1,2,∗, ∗
C	∗, ∗,3,4
D	∗, ∗,4,3

(i) Prove that

$$\{(1, A), (2, B), (3, C), (4, D)\}$$ ✓

is a stable matching no matter what unspecified preferences are.

(ii) Show that the stable matching in (i) cannot be obtained from marrying procedure.

(i) consider boys 1.2 ✓ (favourite girls)
possible (3.D) (4.C)
C has no incentive to leave

D ..

(2) boy optimal / boy pessimal

$\{(1, A), (2, B), (3, C), (4, D)\}$ $\quad$ $\{(1, B), (2, A), (3, D), (4, C)\}$

↓

not boy optimal

1.2 not worst valid (boy optimal)

3.4 not best valid (boy pessimal)

× marrying procedure

What we have learned…

- 1. Logic and Set
- 2. Proof I (basic, Invariant method, WOP)
- 3. Proof II (Mathematical induction)
- 4. Recursion
- 5. Greatest common divisors
- 6. Modular arithmetic (CRT)

1. (*18 points*) Suppose that you are given one "**NOT**", three "**AND**"s, and one "**OR**" of the following electronic components:

6

Design a circuit so that it has the following input/output table.

$(A+B)(A+C)=A+BC$

P	Q	R	output
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	0

$\wedge \quad \vee$

$(\bar p+\bar q+\bar r)(\bar p+q+r)(p+q+r)$

$=\overline{p\cdot q\cdot r}\,(q+r+\underbrace{p\bar p}_{0})$

$=\overline{p\cdot q\cdot r}\cdot(q+r)$

2. (*20 points*) Solve the congruence equation

$$609x \equiv 1 \pmod{2020}$$

$\downarrow$ $4 \times 5 \times 101$

$$\begin{cases} 609x \equiv 1 \pmod{4} \\ \cdots \pmod{5} \\ \cdots \pmod{101} \end{cases} \rightarrow \begin{cases} x \equiv 1 \pmod{4} \\ x \equiv 4 \pmod{5} \\ x \equiv 34 \pmod{101} \end{cases}$$

faster/CRM

$x \equiv 1549 \pmod{2020}$

3. (*11 points*) Prove that $15n + 4$ and $10n + 3$ are coprime for any $n \in \mathbb{N}$.

$\gcd(15n+4, 10n+3) = 1$ Euclid

4. (*11 points*) Suppose there are $m + n$ coins along a line where m of them with head facing up and the other n coins with tail facing up. Can we can flip over pairs of adjacent coins so that all the coins have their heads facing up? Discuss for which case we cannot; and for the cases we can, describe a method to flip the coins.

invariant m heads n tails

↓

\# of tails (change)

$H, H \to T, T$

$H, T \to T, H$

$T, T \to H, H$

↓

$0, \pm 2 \to$ even

反例: THT

↓

HTH

↓

HHT → ?

↑

m, n are both even

H T H .. T

5. *(20 points)* Let $\{F_n\}$ be defined by $F_1 = 3, F_2 = 4$ and $F_{n+2} = F_n + F_{n+1}$. Prove that $\gcd(F_n, F_{n+1}) = 1$ for all $n \in \mathbb{Z}^+$.

MI

$P(n)$: $\gcd(F_k, F_{k+1}) = 1$, $\forall k \le n$ (strong)

base case : 1, 2, 3 ...

inductive step: $P(m)$ ✓ $P(m+1) \Rightarrow \gcd(F_{m+1}, F_{m+2}) = 1$ g

$g \mid F_{m+2} - F_{m+1} = F_m$ $g \mid F_{m+1}$

$g \mid F_{m+1} - F_m = F_{m-1}$ $g \mid \gcd(F_{m-1}, F_m) = 1$

$g = 1$

6. (*20 points*) Let $n \in \mathbb{Z}^+$. Denote by r_n the number of n-bit binary strings that do not contain a substring "001". Use **generating function** to find the recurrence relation and the closed form of r_n. (**Note:** Full mark will be given **ONLY** if you use generating function. You may use the notations $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.)

$$r_n = \left(\frac{2}{\sqrt{5}} + 1\right)\alpha^n - \left(\frac{2}{\sqrt{5}} - 1\right)\beta^n - 1$$

$\boxed{0} + \boxed{0} + \boxed{0} + \boxed{0} + \boxed{0} \cdots 1$ $00 \cdots 0$

$\boxed{1}\ \boxed{(n-2)}$

$\boxed{1} + \boxed{(n-1)}$

$r_n = r_{n-1} + r_{n-2} + 1$

What we have tested…

- 1. Logic and Set
- 4. Proof I (basic, Invariant method, WOP)
- 5. Proof II (Mathematical induction)
- 6. Recursion
- 3. Greatest common divisors
- 2. Modular arithmetic (CRT)