CSC3001: Discrete Mathematics

Final Exam (Fall 2019)

Instructions:

1. This exam is 120 minute long.
2. The total mark in this exam paper is 110, and the maximum you can get is 100. **You should try to attempt as many questions as possible.**
3. This exam has 14 pages, consisting of 7 questions. **Write down your full working in this exam paper**.
4. You should check the number of questions in the first 30 minutes. **Instructor is not responsible for any missing questions when exam ends.**
5. Calculators are allowed.
6. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.
7. Please note that all the graphs in this exam paper are **SIMPLE GRAPHS**.

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1. (*19 points*)

(a) Let $A = \{x, y\}, B = \{z\}$. Write down the elements of the sets $pow(A) \times pow(B)$ and $pow(A \times B)$ and conclude whether they have equal size. [**6 marks**]

(b) Let $A, B, C \subseteq \mathbb{R}$. Which of the following identities are true? If the identity is true, prove it; otherwise, give a counterexample.

(i) $(A \cap B) \times C = (A \cap C) \times (B \cap C)$ (ii) $(A - B) \times C = (A \times C) - (B \times C)$

(**Note:** $A - B = \{x \in A \mid x \notin B\}$) [**13 marks**]

(a) $pow(A) = \{\{x\}, \{y\}, \{x, y\}, \phi\}$ $pow(B) = \{\{z\}, \phi\}$

$A \times B = \{(x, z), (y, z)\}$

$pow(A) \times pow(B) = \{(\{x\}, \{z\}), (\{x\}, \phi), (\{y\}, \{z\}), (\{y\}, \phi), (\{x, y\}, \{z\}), (\{x, y\}, \phi), (\phi, \{z\}), (\phi, \phi)\}$ size $= 8$

$pow(A \times B) = \{\{(x, z)\}, \{(y, z)\}, \{(x, z), (y, z)\}, \phi\}$ size $= 4$

They don't have equal size.

(b) (i) False. counterexample: $A=\{1,2\}$, $B=\{1\}$. $C=\{3\}$

$(A\cap B)\times C=\{1\}\times\{3\}=\{(1,3)\}$ $(A\cap C)\times(B\cap C)=\emptyset\times\emptyset=\{(\emptyset,\emptyset)\}$

(ii) $(A-B)\times C=(A\times C)-(B\times C)$ (**Note:** $A-B=\{x\in A \mid x\notin B\}$)

(ii) True. (proof by definition)
(Equality $\Rightarrow$ subset of each other)

① LHS $\subseteq$ RHS:

Assume $(x,y)\in(A-B)\times C \Rightarrow x\in(A-B) \Rightarrow x\in A, x\notin B$

$y\in C \Rightarrow (x,y)\in A\times C$ $(x,y)\notin B\times C$

So $(x,y)\in(A\times C)-(B\times C)$

② RHS $\subseteq$ LHS

Assume $(x,y)\in(A\times C)-(B\times C) \Rightarrow (x,y)\in(A\times C), (x,y)\notin(B\times C)$

$\Rightarrow x\in A, x\notin B \Rightarrow x\in(A-B)$. $y\in C$

So $(x,y)\in(A-B)\times C$

LHS = RHS

2. (*10 points*) Let $S \subseteq \{1, 2, \ldots, 100\}$ be of size 25. Prove that there exist mutually distinct $a, b, c, d \in S$ such that $a + b = c + d$.

① Possible numbers of sum of two elements in S ($a+b$/$c+d$)

min: $1+2=3$

max: $99+100=199$

3-199: 197 numbers

② Possible combinations of two numbers in S:

$$\binom{25}{2} = \frac{25 \times 24}{2} = 300$$

By pigeon-hole principle:

- 300 values are distributed into 197 positions, so there must be two equal sums ($a+b$) and ($c+d$)
- $a \neq b$ $c \neq d$ by selection (select 2 different elements from 25 numbers)
- $a \neq b \neq c \neq d$ or they will not have the same sum.

Q.E.D.

For the remaining questions, all graphs are assumed to be **SIMPLE**.

3. (*11 points*) Suppose our library keeps several copies of the book "*The Grant's Solutions*" in the stock. The following are its borrow/return records in this semester.

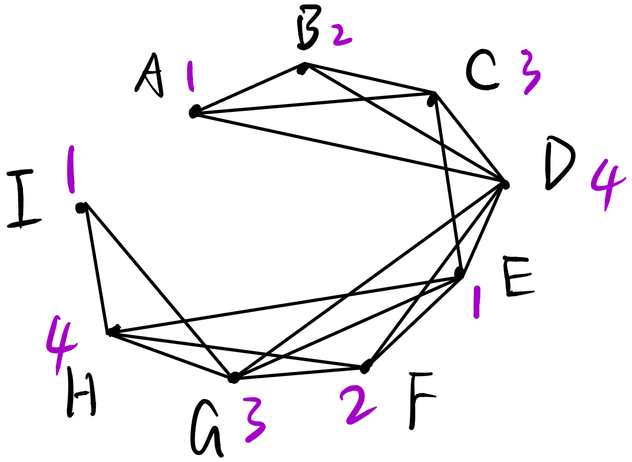
Date \ Status	Borrow	Return
2/9	Li Lei, Ken	
8/9	Han Meimei	
10/9	Anne	
20/9		Li Lei
28/9		Ken
8/10	Sue	
9/10		Han Meimei
17/10	Li Lei	
27/10	Han Meimei	
2/11		Anne
11/11	Peter	
19/11		Li Lei
27/11		Sue
3/12	Li Lei	
10/12		Peter
13/12		Li Lei, Han Meimei

Model this problem as a graph problem and determine the minimum number of copies that are kept in our library.

Represent each (borrow+return) record as a vertex

Join an edge between two vertices if their borrowing period overlaps. (9 vertices, arranged by borrow time)

(A=2/9, Li Lei, B=2/9, Ken
--- I=13/12, HMm)



The chromatic number is 4 (we can 4-color the graph & ABCD form a complete graph K_4)

so the minimum number of copies is 4.

4. (*25 points*) Determine whether a graph exists for the following degree sequences:

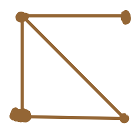
1) $1, 2, 2, 3$ 2) $4, 4, 5, 5, 5, 5$ 3) $2, 2, 3, 3, 3$ 4) $1, 2, 3, 4, 4$ 5) $1, 1, 1, 2, 2, 3$

If it exists, draw all the possibilities up to isomorphism and determine whether these graphs are planar; otherwise, explain why it does not exist.

Use n to represent # of vertices, m : # of edges

$m = \frac{\sum deg(v)}{2}$ $v \in V$; planar graph : $m \leq 3n-6$

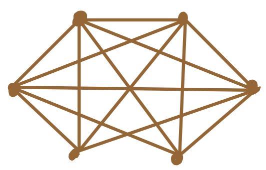
1) Exist. $n=4$ $m=4$



planar

$(4 \leq 3\times4-6)$

2) Exist. $n=6$ $m=14$



not planar

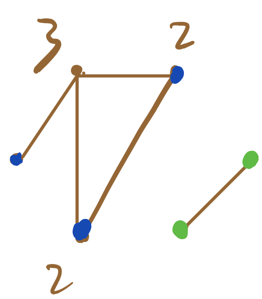
$(14 > 3\times6-6=12)$

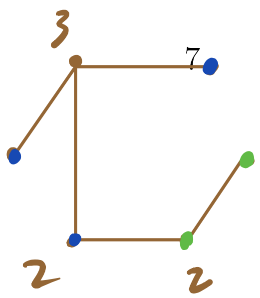
3) Not exist. $n=5$ $m = \frac{\sum deg(v)}{2} = \frac{13}{2} = 6.5$ (hand-shaking theorem)

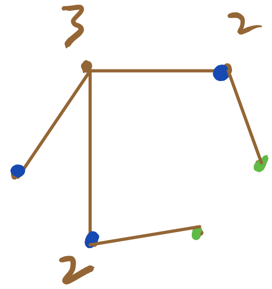
impossible

4) Not exist. $n=5$ $m=7$. For two vertices of degree 4, they connect to all other vertices in the graph, making the least degree 2, but there is a 1-degree. Impossible!

5) Exist. $n=6$ $m=5$.







planar

$(5 < 3\times6-6=12)$

5. (*10 points*) Suppose the preferences between 4 boys and 4 girls are as follows.

Boys' preference

boy	preference order
1	C,B,A,D
2	A,C,B,D
3	C,D,A,B
4	D,C,B,A

Girls' preference

girl	preference order
A	4,3,1,2
B	2,3,4,1
C	4,3,1,2
D	3,4,2,1

Determine whether the following matching is stable or not.

$$\{(1,A),(2,D),(3,B),(4,C)\}$$

If yes, prove it; otherwise list all the unstable pairs.

Not stable matching.

((1,D),(1,B),(1,C)(2,A)···)

Unstable pairs: (check all possibilities 3x4=12)

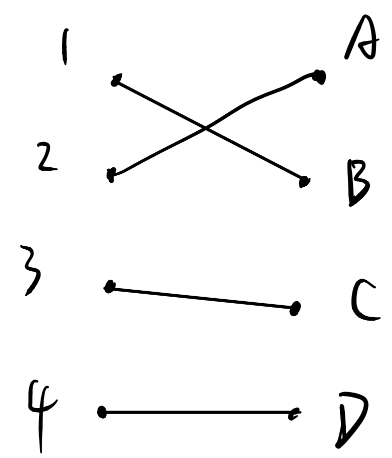
(2,B)

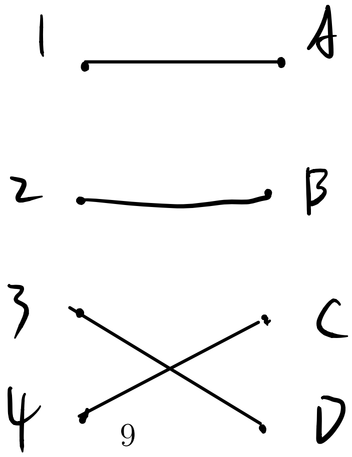
(3,A) (3,D) ← 渣男!

(4,D)

4 pairs

(can verify that)





boy propose

girl propose

6. (*15 points*) Let $n \geq 2$ be an integer and let $G = (V, E)$ be a bipartite graph with $V = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ and $E = \{a_i b_j | i \neq j\}$.

(a) Show that G has a perfect matching. **[5 marks]**

(b) Count the total number of different perfect matchings in G. **[10 marks]**

(a) G is a bipartite graph. $G=(A,B;E)$

Consider a subgraph of G. a subset S of A, $S \subseteq A$ $1 \leq |S| \leq n$

- If $|S|=1$, each vertex a_i connects to $(n-1)$ vertices except b_i. $|N(S)|=(n-1)$
- If $|S|>1$, the subset will connect to all vertelces in B.
  $|N(S)|=n$

$n \geq 2$, $|S| \leq n$, so $|N(S)| \geq |S|$ for every subset S of A.
(For B. It is similiar)

By Hall's Theorem, G has a perfect matching.

(Or say G is $(n-1)$-regular graph $\Rightarrow$ perfect matching)

(b) Assume: For the bipartite graph G:

A_1: perfect matching with edge a_1b_1

A_2: perfect matching with edge a_2b_2

...

A_n: perfect matching with edge a_nb_n

# of perfect matchings with $E=\{a_ib_j \mid i\neq j\}$

(inclusion-exclusion)

$$= |\bar{A_1}\cap\bar{A_2}\cap\cdots\cap\bar{A_n}| = |\overline{A_1\cup A_2\cup\cdots\cup A_n}| = N-|A_1\cup A_2\cup\cdots\cup A_n|$$

$$= N-\left(\sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n+1}|A_1\cap A_2\cap\cdots\cap A_n|\right)$$

$$= n! - \left(\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots + (-1)^{n+1}\binom{n}{n}(n-n)!\right)$$

no constraint
a_1 has n choices
a_2 has (n-1) choices
⋮
a_n has 1 choice

select an A_i

only 1 edge identified

2 edges identified

select two $A_i A_j$ in $A_1 \sim A_n$

$$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n\binom{n}{n}0!$$

$$= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)!$$

7. (*20 points*) Let G be a k-regular graph with n vertices such that

- every pair of adjacent vertices has λ common neighbors; and
- every pair of non-adjacent vertices has μ common neighbors.

for some $k, n \in \mathbb{Z}^+$ and $\lambda, \mu \in \mathbb{N}$.

(i) Prove that $(n-k-1)\mu = k(k-\lambda-1)$. **[10 marks]**

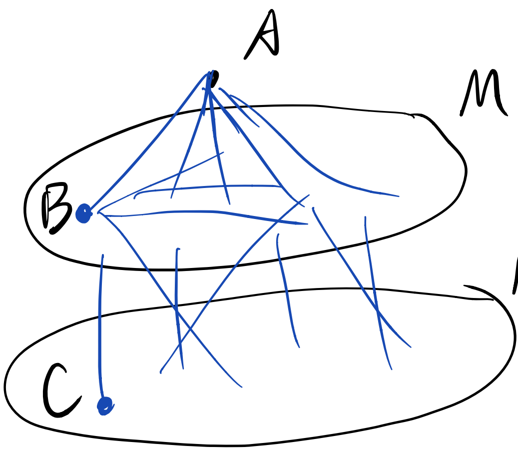
(ii) Suppose $k \geq 2$. When will G be disconnected? What is the relation between λ and n when G is disconnected? **[10 marks]**

(1) pick a vertex A. divide the other vertices into 2 categories.

- M: all the neighboring vertices of A (adjacent)
- N: all the non-adjacent vertices of A

k-regular

n vertices



⇐ edges between M and N

① pick a vertex B in M:

A、B are adjacent vertices, they have λ common neighbors, and they must all lie in M (neighbors of A)

so B connects to λ vertices in M

B is k-degree ⇒ B connects to $(k-\lambda-1)$ vertices in N

② pick a vertex C in N:

A.C are non-adjacent pairs, they have μ common neighbors, and they must all lie in M. (neighbors of A)

so C connects to μ vertices in M

③ A is k-degree. There are k vertices in M, n in total
so there are $(n-k-1)$ vertices in N

Consider # of edges between M and N

LHS: $(n-k-1)\mu$ (# of vertices in N × # of edges connecting M for each vertex)

RHS: $k(k-\lambda-1)$ (# of vertices in M × # of edges connecting N for each v)

∗ LHS = RHS

(2) G is disconnected when $\mu=0$ and $(n>k+1)$
(# of edges between M and N is zero)

From (1): $k(k-\lambda-1)=0$ $k\neq 0 \Rightarrow k-\lambda-1=0$

• consider A, A is connected to k vertices in M and no vertex in N. They form a $(k+1)$ vertex cluster. Here A can be any vertex in the Graph. For any vertex in the cluster, it connects to all other vertices since the graph is k-regular. They form a complete graph $K_{(k+1)}$.

There are many $(k+1)$ complete graph in the whole graph

As a result: $(k+1) \mid n \Rightarrow (\lambda+2) \mid n$