CSC3001 Discrete Mathematics Tutorial 2

Example – prime（p）

A **prime number** (or a **prime**) is a natural number greater than 1 that has no positive divisors other than 1 and itself.

$$(p > 1) \wedge (p \in \mathbb{Z}) \wedge (\forall a, b \in \mathbb{Z}^{+}, (p \neq a \cdot b) \vee (a = 1) \vee (a = p))$$

Example – FQ2-1

- 1. Express the following sentence using first order logic.
- Define：
- P(s,t) = student s goes to tutorial t
- Q(t,c) = tutorial t is in course c
- S = {Students} T = {Tutorials} C = {Courses}
- Some student goes to at least one tutorial of each course.

① ②

$\exists s \in S, \forall c \in C, \exists t \in T$

$P(s,t) \wedge Q(t,c)$

$\begin{cases} \exists s \in S, \exists t \in T, P(s,t) \\ \quad \forall c \in C, \exists t \in T, Q(t,c) \end{cases}$

Q1. Define A = { the students taking this course }

B = { the lectures of this course }

C = { the questions in this exam }

$P(x, y)$ = "student x attends lecture y"

$Q(x, z)$ = "student x finishes question z"

Use ONLY $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $=$ and above defined sets, predicates to translate the following question:

*Nobody attends some lectures or every student finishes question 1 and question 2 in this exam.

at least

*Modify: Nobody attends some lectures or every student finishes **two questions** in this exam.

① (a)

$$(\exists b\in B, \forall a\in A, \neg P(a,b)) \vee (\forall a\in A, Q(a,1)\wedge Q(a,2))$$

(b)

$$(\forall a\in A, \exists c_1.c_2\in C, Q(a,c_1)\wedge Q(a,c_2)\wedge(c_1\neq c_2)$$

② condition: $A\oplus C = B\oplus C$

$x\in A$ or $x\in C$ but $x\notin A\cap C$ $\quad$ $x\in B$ or $x\in C$ but $x\notin B\cap C$

prove: $A=B$

proof of set equivalence: two sets belong to each other. Assume $x\in A$, conclude $x\in B$. Assume $x\in B \Rightarrow x\in A$. (相等⇌互为子集)

① $x\in A$

(1) $x\in C$

$x\in A\cap C \Rightarrow x\notin A\oplus C \Rightarrow x\notin B\oplus C$

$x\in B$

(2) $x\notin C \Rightarrow x\in A\oplus C \Rightarrow x\in B\oplus C \Rightarrow x\in B \Rightarrow A\subseteq B$

② $x\in B$

$\Rightarrow x\in A$ (symmetric)

......

$B\subseteq A$

so $A=B$

Q2. The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

$A = B$ (A is equal to B) $\Longleftrightarrow$ $A \subseteq B$ and $B \subseteq A$.

Q3. Proof:

1. If A is a set, then $A \times \emptyset = \emptyset$ and $\emptyset \times A = \emptyset$
2. If A and B are sets. $A \times B = B \times A$ if and only if $A = B$, or $A = \emptyset$, or $B = \emptyset$.

Definition: Given two sets A and B, the **Cartesian product** A x B is the set of all **ordered** pairs (a,b), where a is in A and b is in B. That is,

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

③ (1) Assume $A\times\emptyset\neq\emptyset$, then $\exists a, b$, $(a.b)\in A\times\emptyset$

$\Rightarrow a\in A$, $b\in\emptyset$ → contradiction! (no element in empty set)

$\emptyset\times A=\emptyset$ (similiar)

(2) if and only if: two directions

$\Rightarrow A\times B=B\times A$ $\begin{cases} A\times B=\emptyset \Rightarrow A=\emptyset \text{ or } B=\emptyset \\ A\times B\neq\emptyset, \text{ let } a\in A, b\in B, (a,b)\in A\times B \\ \Rightarrow (a,b)\in B\times A \Rightarrow a\in B, b\in A \end{cases}$

$A\subseteq B$ $B\subseteq A \Rightarrow A=B$

$\Leftarrow$ $A=B$ or $A=\emptyset$ or $B=\emptyset$ prove: $A\times B=B\times A$

$\to A\times B=\emptyset\times B=\emptyset=B\times A$

$\downarrow$

$A\times B=A\times A=B\times A$

Q4. Suppose A, B are sets.

Prove that

$$pow(A) \cap pow(B) = pow(A \cap B)$$

$$pow(A) \cup pow(B) \subseteq pow(A \cup B)$$

pow({a,b}) = {∅, {a}, {b}, {a,b}} (power set of {a,b})

set

pay attention to the difference between $\in$ (element) and $\subseteq$ (subset)

(1)

$x \in pow(A) \cap pow(B) \Leftrightarrow$

$x \in pow(A)$ and $x \in pow(B)$

$x \subseteq A$ and $x \subseteq B \Leftrightarrow x \subseteq A \cap B \Leftrightarrow x \in pow(A \cap B)$

$\Rightarrow pow(A) \cap pow(B) \subseteq pow(A \cap B)$

(反推也成立)

$pow(A \cap B) \subseteq pow(A) \cap pow(B)$

so

$pow(A \cap B) = pow(A) \cap pow(B)$

(2)

$x \in pow(A) \cup pow(B)$

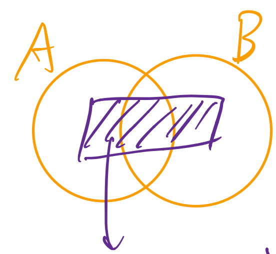
$\Rightarrow x \in pow(A)$ or $x \in pow(B)$

$\Rightarrow x \subseteq A$ or $x \subseteq B$

$\cancel{\Uparrow}$

$\Rightarrow x \subseteq A \cup B \Rightarrow x \in pow(A \cup B)$

(这一步反推不成立)



not a subset of A or a subset of B but a subset of $A \cup B$

(counter-example:

$A = \{1,2\}$ $B = \{2,4\}$ $A \cup B = \{1,2,4\}$

$pow(A) = \{\{1\},\{2\},\{1,2\},\emptyset\}$

$pow(B) = \{\{2\},\{4\},\{2,4\},\emptyset\}$

$pow(A) \cup pow(B) = \{\{1\},\{2\},\{4\},\{1,2\},\{2,4\},\emptyset\}$

$pow(A \cup B) = \{\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{1,2,4\},\emptyset\}$