CSC3001: Discrete Mathematics

Assignment 4

Instructions:

1. Print out this question paper (**two-sided**) and write down your full working on the blank area.
2. You can have discussions with your classmates. However, make sure all the solutions you submit are your own work. Any plagirism will be given **ZERO** mark.
3. Submission of this assignment should **NOT** be later than **5:00pm on 20th of December**.
4. Before your submission, please **make a softcopy** of your work for further discussion in a tutorial.
5. After making your softcopy, submit your assignment to the dropbox located on the 4th floor in Chengdao Building.

Student Number: ____________________ **Name:** _______________

1. (*20 points*) In order to find an outstanding undergraduate teaching fellow for CSC3001, Grant gave 81 challenging questions to the candidate who eventually scored 90. Assume that the candidate scored integral points and at least 1 point on each question. Prove that the candidate scored exactly 18 points on some consecutive questions.

Proof. Suppose that the candidate scored P_i points on the i-th question. Set $s_n = \sum_{i=1}^{n} P_i$. Then $s_m - s_n$ gives the score from the $(n+1)$-th question to the m-th question.
The candidate scored at least 1 point on each question, so s_n are mutually distinct, and the problem becomes assigning s_n to distinct values ranging between 1 and 90 so that $s_m - s_n \neq 18$ for all m, n. Partition the set $\{1, 2, \ldots, 90\}$ as follows:

$$\{1, 19\}, \{2, 20\}, \ldots, \{18, 36\}, \{37, 55\}, \ldots, \{54, 72\}, \{73\}, \ldots, \{90\}$$

Then at least $81 - 18 = 63$ distinct s_n will be mapped to the following 36 sets:

$$\{1, 19\}, \{2, 20\}, \ldots, \{18, 36\}, \{37, 55\}, \ldots, \{54, 72\}$$

By pigeonhole principle, at least one of the above sets will be mapped by two distinct s_n, whose difference is precisely 18. So the candidate scored exactly 18 points on some consecutive questions. □

2. (*20 points*) Let a, b, c, r, s, t, n be non-negative integers such that

$$n \geq r + s, \qquad n \geq r + t, \qquad n \geq s + t.$$

Count the number of solutions for the following inequality:

$$a + b + c = n$$

where $a < r, b < s, c < t$. (**Note:** You do not need to simplify the expression.)

Solution. Define

$$P_1 = \{a \geq r | a+b+c = n\}, \qquad P_2 = \{b \geq s | a+b+c = n\}, \qquad P_3 = \{c \geq t | a+b+c = n\}$$

Then the number of solutions for the inequality is

$$|\overline{P_1} \cap \overline{P_2} \cap \overline{P_3}| = N - |P_1| - |P_2| - |P_3| + |P_1 \cap P_2| + |P_1 \cap P_3| + |P_2 \cap P_3| - |P_1 \cap P_2 \cap P_3|$$

where

$$N = |\{(a, b, c) | a + b + c = n\}| = \binom{n+2}{2}$$

We also have

$|P_1| = |\{a' \geq 0 | a' + b + c = n - r\}| = \binom{n-r+2}{2}$

$|P_2| = |\{b' \geq 0 | a + b' + c = n - s\}| = \binom{n-s+2}{2}$

$|P_3| = |\{c' \geq 0 | a + b + c' = n - t\}| = \binom{n-t+2}{2}$

$|P_1 \cap P_2| = |\{a', b' \geq 0 | a' + b' + c = n - r - s\}| = \binom{n-r-s+2}{2}$

$|P_1 \cap P_3| = |\{a', c' \geq 0 | a' + b + c' = n - r - t\}| = \binom{n-r-t+2}{2}$

$|P_2 \cap P_3| = |\{b', c' \geq 0 | a + b' + c' = n - s - t\}| = \binom{n-s-t+2}{2}$

$|P_1 \cap P_2 \cap P_3| = |\{a', b', c' \geq 0 | a' + b' + c' = n - r - s - t\}| = \begin{cases} 0, \text{ if } n < r+s+t; \\ \binom{n-r-s-t+2}{2}, \text{ if } n \geq r+s+t \end{cases}$

So, if $n < r + s + t$, then the number of solutions is

$$\binom{n+2}{2} - \binom{n-r+2}{2} - \binom{n-s+2}{2} - \binom{n-t+2}{2} + \binom{n-r-s+2}{2} + \binom{n-r-t+2}{2} + \binom{n-s-t+2}{2}.$$

if $n \geq r + s + t$, then the number of solutions is

$$\binom{n+2}{2} - \binom{n-r+2}{2} - \binom{n-s+2}{2} - \binom{n-t+2}{2} + \binom{n-r-s+2}{2} + \binom{n-r-t+2}{2} + \binom{n-s-t+2}{2} - \binom{n-r-s-t+2}{2}.$$

3. (*20 points*)

(i) Given $r \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$. Define

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

Show that for any $n \in \mathbb{Z}^+$ we have

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

(ii) For $n \geq 2$, establish the following by combinatorial argument

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1)$$

Solution.

(i)

$$\binom{-n}{r} = \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} = (-1)^r \frac{n(n+1)\cdots(n+r-1)}{r!} = (-1)^r \binom{n+r-1}{r}$$

(ii) Consider n people where now we want to count the total number of committees of size k with a chairperson and a secretary. We can select all subsets of size k in $\binom{n}{k}$ ways. Given a subset of size k, there are k choices for the chairperson and k choices for the secretary giving

$$k^2 \binom{n}{k}$$

committees of size k with a chair and a secretary. The total number of these is then given by summing this result or

$$\sum_{k=1}^{n} \binom{n}{k} k^2$$

Now consider first selecting the chair which can be done in n ways. Then selecting the secretary which can either be the chair or one of the $n-1$ other people. If we select the chair and the secretary to be the same person we have $n-1$ people to choose from to represent the committee. All possible subsets from a set of $n-1$ elements is given by 2^{n-1}, giving in total $n2^{n-1}$ possible committees with the chair and the secretary the same person. If we select a different person for the secretary this chair/secretary selection can be done in $n(n-1)$ ways and then we look for all subsets of a set with $n-2$ elements (i.e. 2^{n-2}) so in total we have $n(n-1)2^{n-2}$. Combining these we obtain

$$n2^{n-1} + n(n-1)2^{n-2} = n2^{n-2}(2+n-1) = 2^{n-2}n(n+1)$$

4. (*20 points*) The following data were given in a study of a group of 1000 subscribers to a certain magazine. In reference to job, marital status, and education, there were

312 professionals,
470 married persons,
525 college graduates,
42 professional college graduates,
147 married college graduates,
86 married professionals,
25 married professional college graduates.

(i) Let a person be picked at random. Determine the probability that the person is married or professional or college graduate.

(ii) Explain why the numbers reported in the study must be incorrect.

Solution.

(i) Denote
M the set of people who are married,
W the set of people who are working professionals, and
G the set of people who are college graduates.
From the given data each individual event probability can be estimated as

$$\begin{aligned} P(M) &= \frac{470}{1000} \\ P(G) &= \frac{525}{1000} \\ P(W) &= \frac{312}{1000} \\ P(M \cap G) &= \frac{147}{1000} \\ P(M \cap W) &= \frac{86}{1000} \\ P(W \cap G) &= \frac{42}{1000} \\ P(M \cap W \cap G) &= \frac{25}{1000} \end{aligned}$$

By inclusion-exclusion formula, we find that

$$P(M \cup W \cup G) = 0.47 + 0.525 + 0.312 - 0.147 - 0.086 - 0.042 + 0.025 = 1.057$$

(ii) This probability is greater than one, in contradiction to the rules of probability. Thus the data cannot be correct.

5. (*20 points*) In an election, candidate A receives n votes and candidate B receives m votes, where $n > m$. Assuming that all of the $(n+m)!/n!m!$ orderings of the votes are equally likely, let $P_{n,m}$ denote the probability that A is always ahead in the counting of the votes.

(a) Find $P_{n,1}$, $P_{n,2}$

(b) Based on (a), conjecture the value of $P_{n,m}$.

(c) Derive a recursion for $P_{n,m}$ in terms of $P_{n-1,m}$ and $P_{n,m-1}$ by conditioning on who receives the last vote.

(d) Use part (c) to verify your conjecture in part (b) by an inductive proof on $n + m$.

Solution.

(a)

$$P_{n,1} = P[A \text{ receives the first 2 votes}] = \frac{n(n-1)}{(n+1)n} = \frac{n-1}{n+1}$$
$$P_{n,2} = P[A \text{ receives the first 2 votes and at least 1 of the next 2}]$$
$$= \frac{n}{n+2} \times \frac{n-1}{n+1} \times \left[1 - \frac{2.1}{n(n-1)}\right] = \frac{n-2}{n+2}$$

(b) $P_{n,m} = \dfrac{n-m}{n+m}$

(c)

$$\begin{aligned} P_{n,m} =& P[A \text{ always ahead}] \\ =& P[A \text{ always ahead} \mid A \text{ receives last vote}] \cdot \frac{n}{n+m} \\ &+ P[A \text{ always ahead} \mid B \text{ receives last vote}] \cdot \frac{m}{n+m} \\ =& \frac{n}{n+m} P_{n-1,m} + \frac{m}{n+m} P_{n,m-1} \end{aligned}$$

(d) The conjecture of (b) is true when $n+m=1$ ($n=1, m=0$). Assume this to be true for $n+m=k$. Now suppose that $n+m=k+1$. From (c) and the induction hypothesis, we have

$$P_{n,m} = \frac{n}{n+m} \times \frac{n-1-m}{n-1+m} + \frac{m}{n+m} \times \frac{n-m+1}{n+m-1} = \frac{n-m}{n+m}$$

6. (*10 points*) [**Bonus question**] For $k \in \mathbb{Z}^+$, set $S_k := \{x^2 \pmod k | x \in \mathbb{Z}\}$. Prove that given $m, n \in \mathbb{Z}^+$, if $\mathsf{gcd}(m,n) = 1$, then $|S_m| \cdot |S_n| = |S_{mn}|$.

Proof. Consider the map

$$\begin{array}{cccc} f: & S_{mn} & \to & S_m \times S_n \\ & a & \mapsto & (a,a) \end{array}$$

Note that

$$a' \equiv a \pmod{mn} \quad \Rightarrow \quad a' \equiv a \pmod m \quad \text{and} \quad a' \equiv a \pmod n$$

so f is well defined.
Let $b \in \mathbb{Z}$ be such that $b \equiv a \pmod m$ and $b \equiv a \pmod n$. Since $\mathsf{gcd}(m,n) = 1$, by Chinese Remainder Theorem we have $b \equiv a \pmod{mn}$, and so f is injective.
Pick $x^2 \in S_m, y^2 \in S_n$. By Chinese Remainder Theorem there exists $z \in \mathbb{Z}$ such that

$$z \equiv x \pmod m \text{ and } z \equiv y \pmod n \quad \Rightarrow \quad z^2 \equiv x^2 \pmod m \text{ and } z^2 \equiv y^2 \pmod n$$

So f is surjective.
Thus, f is a bijection, and hence the conclusion follows. □