Tut 9

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The Marrying Procedure

Morning: boy propose to their favourite girl
Afternoon: girl rejects all but favourite (i.e. top suitor)
Evening: rejected boy writes off the girl

FQ9-10

BOY	1	2	3	4	5
Adam	~~Beth~~	Amy	Diane	Ellen	Cara
Bill	~~Diane~~	~~Beth~~	~~Amy~~	Cara	Ellen
Carl	Beth	Ellen	Cara	Diane	Amy
Dan	~~Amy~~	~~Diane~~	~~Cara~~	~~Beth~~	Ellen
Eric	~~Beth~~	Diane	Amy	Ellen	Cara

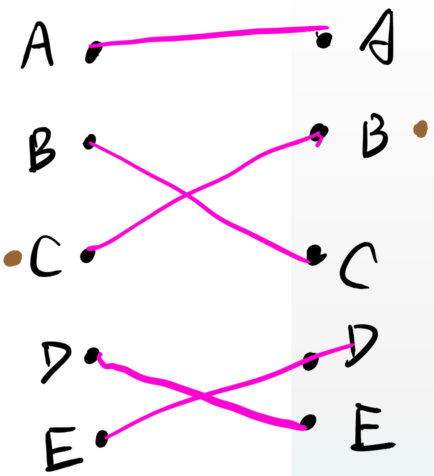
Boys' Preferences

GIRL	1	2	3	4	5
Amy	Eric	Adam	Bill	Dan	Carl
Beth	Carl	Bill	Dan	Adam	Eric
Cara	Bill	Carl	Dan	Eric	Adam
Diane	Adam	Eric	Dan	Carl	Bill
Ellen	Dan	Bill	Eric	Carl	Adam

Girls' Preferences

Figure 8: Question 5, (c)

Boy:



Girl: match in the first round

Find a stable matching using the Gale-Shapley algorithm with boys making proposals. Find a stable matching using the Gale-Shapley algorithm with girls making proposals.

Boy optimal proof (10.1)

Claim. *The marrying procedure is boy optimal.*

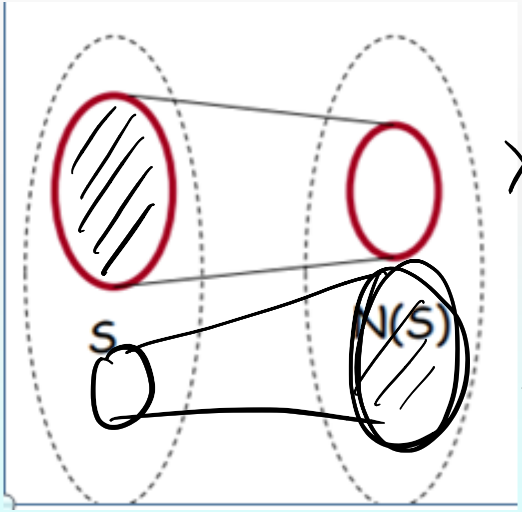
Proof. Suppose that the marrying procedure is not boy optimal. Then there exists some boy not matched with his best valid girl.
Let x be the first such boy during the procedure. Then x must be rejected by some valid girl, say the first one is A. Then A is matched with another boy y.
Since x is the first boy being rejected by his valid girls, it follows that y is not yet rejected by any other of his valid girls. So y prefers A rather than any other of his valid girls. In the meantime, A rejects x, so A prefers y rather than x.
Consider that A is valid for x, so (x, A) should be a couple in another stable matching. However, both y, A have incentive to leave their partners in this stable matching, so (y, A) is an unstable pair in this matching, which is a contradiction. □

A拒绝了x，故A更喜欢y

① boy $x \longrightarrow$ girl A (first rejected) (A, y) * $A: y > x$

② (x, A) (y, B) (y, A) × y先追求A * $y: A >$ others

Hall's Theorem. A bipartite graph $G=(V,W;E)$ with $|V|=|W|$ has a perfect matching if and only if $|N(S)| \geq |S|$ for every subset S of V.



X

✓

Exercise 1-4. Consider the problem of perfectly tiling a subset of a checkerboard (i.e. a collection of unit squares, see example below) with dominoes (a domino being 2 adjacent squares).

1. Show that this problem can be formulated as the problem of deciding whether a bipartite graph has a perfect matching.

2. Can the following figure be tiled by dominoes? Give a tiling or a short proof that no tiling exists.

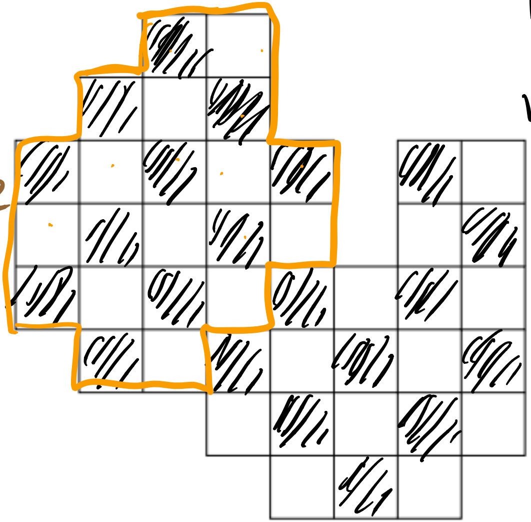
invariant

square → vertex

adjacent → add an edge

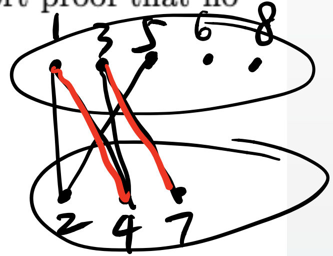
bipartite: color black & white

11 black 10 white



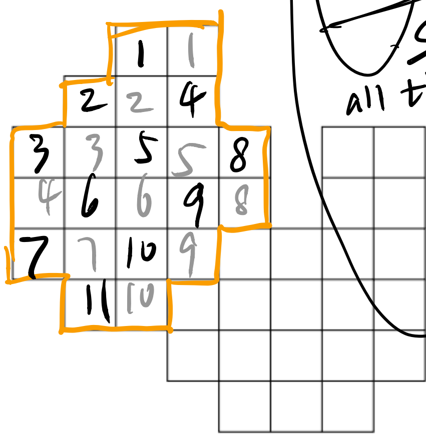
black

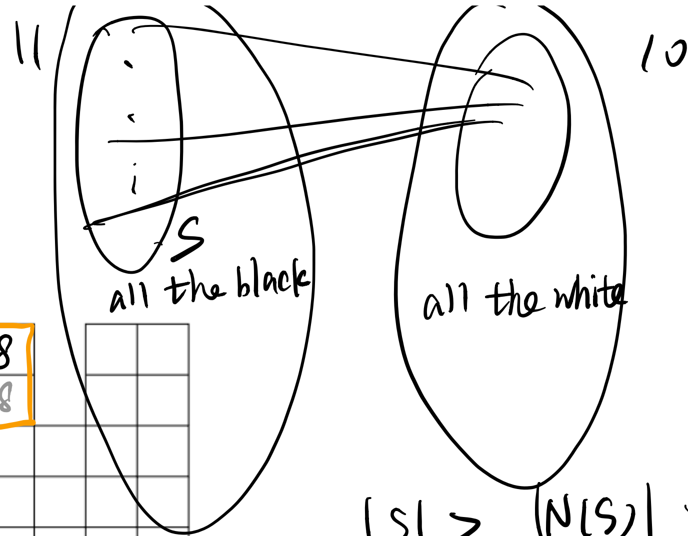
white



Hall's theorem

$|S| \geq |N(S)|$



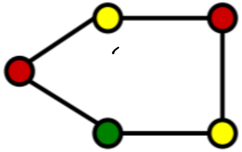


$|S| > |N(S)|$ ×

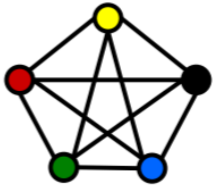
Contradiction:

找組黑，找所有相鄰的白，|黑| > |白|，矛盾！

Definition. min #colors for G is chromatic number, $\chi(G)$



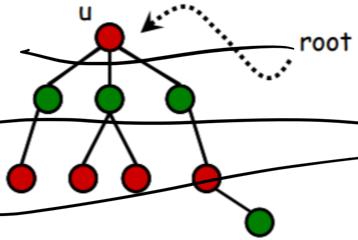
$$\chi(C_{odd}) = 3$$



$$\chi(K_n) = n$$

K_5

Trees



2-colorable: tree,

$$\chi(G) \geq \omega(G)$$

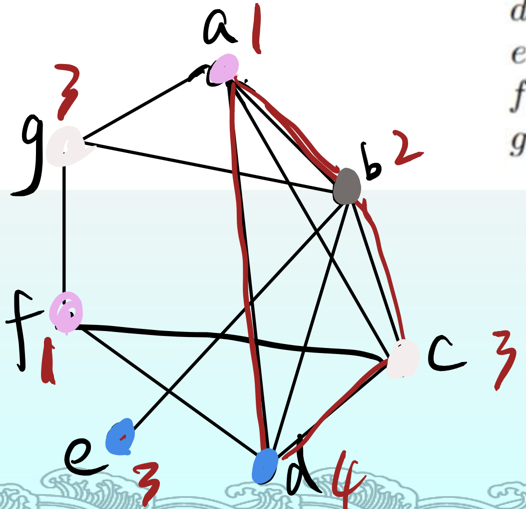
A schedule for finals is to be drawn up for a group of 7 classes, a through g. Two classes may not be scheduled at the same time if there exists a student in both classes. The table below shows the classes which may not be scheduled at the same time (marked with a •). What is the minimum number of time slots are needed to schedule all 7 classes?

color: time slots

	a	b	c	d	e	f	g
a		•	•	•			•
b	•		•	•	•		•
c	•	•		•		•	
d	•	•	•			•	
e		•					
f			•	•			•
g	•	•				•	

complete graph K_n

n



找完整图→最少色数

abcd→K_4 ⇐ 4 colors

$\chi(G) \geq 4$

4

A basic example of a simple graph with chromatic number n is the complete graph on n vertices, that is $\chi(K_n) = n$. This implies that any graph with K_n as a subgraph must have chromatic number at least n. It's a common misconception to think that, conversely, graphs with high chromatic number must contain a large complete subgraph. In this problem we exhibit a simple example countering this misconception, namely a graph with chromatic number four that contains no *triangle*—length three cycle—and hence no subgraph isomorphic to K_n for $n \geq 3$. Namely, let G be the 11-vertex graph of Figure 1. The reader can verify that G is triangle-free.

$K_n \to n$ colors

n colors $\to K_n$ ✗

K_3

But now your task is to prove the chromatic number of G is 4.

$\chi(G) = 4$

① 4-colorable

$\chi(G) \leq 4$

② not 3-colorable

$\chi(G) \geq 3$

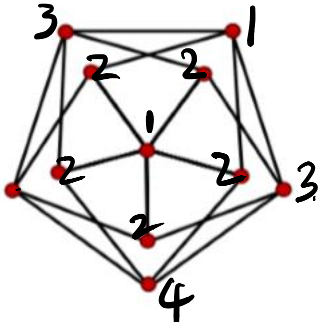
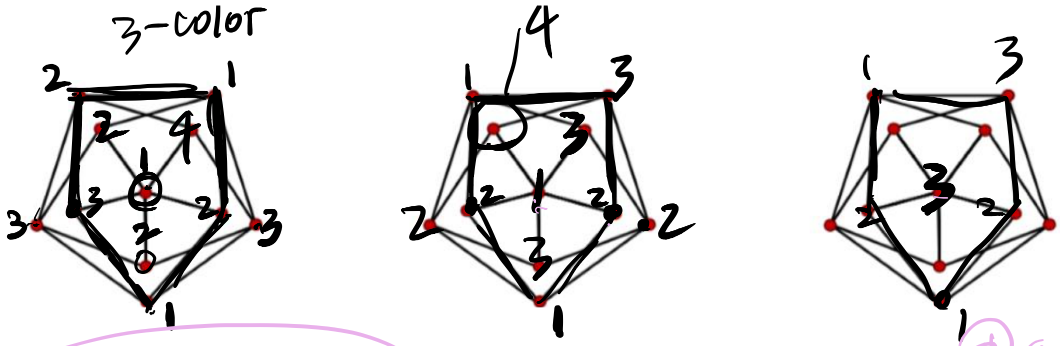


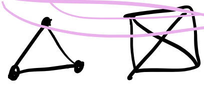
Figure 1 Graph G with no triangles and $\chi(G) = 4$.

K_n

3-color



case analysis



pentagon (3)

① 4 colors ② ③

分类讨论（可以自己分类）