Tut 10

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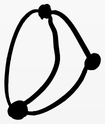
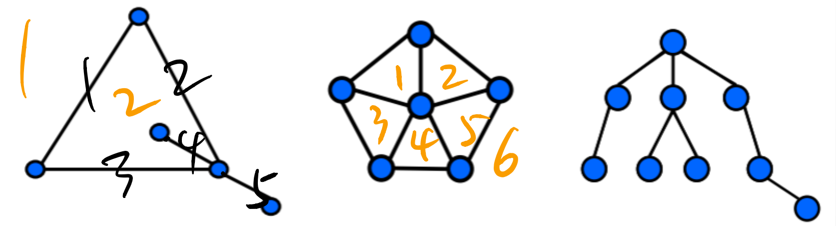
(not necessarily simple)

* If a connected planar graph has n vertices, m edges, and f faces, then

$$n - m + f = 2$$



$n=3$ $m=4$
$f=3$



n=5, m=5, f=2 n=6, m=10, f=6 n=9, m=8, f=1

Tree:
$n=m+1$

* **Claim.** If G is a simple planar graph with at least 3 vertices, then

$$m \le 3n-6 \qquad (n \geq 3)$$

1 **Let G be a simple planar graph with at least 3 vertices. Every vertex of G has degree at least five, and at least one vertex of G has degree eight. Show that G has at least fifteen vertices.**

$n - m + f = 2$ ($n \to$ vertices, $m \to$ edges, $f \to$ faces)

$m \le 3n - 6$

$$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

handshaking

$2m = \sum \deg(v)$

$n \geq 3$

$\deg(v) \geq 5$, $v \in V$

$\deg(v_i) \geq 8$

$\Rightarrow n \geq 15$

$5n + (8-5) = 5n + 3$

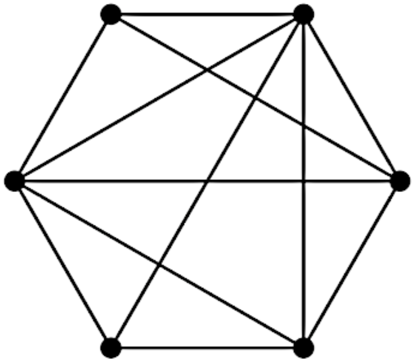
$\sum \deg(v) = 2m \geq 5n + 3$

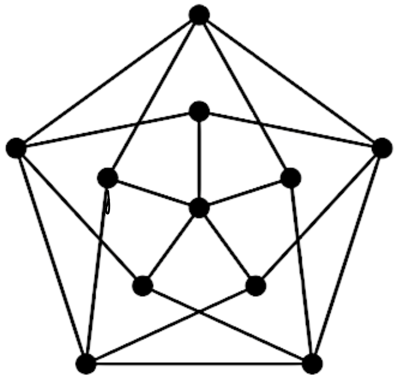
$m \le 3n - 6$

$6n - 12 \geq 2m \geq 5n + 3 \Rightarrow n \geq 15$

2

Determine if the following graphs are planar or not.





isomorphism

$n-m+f=2$

$m \le 3n-6$

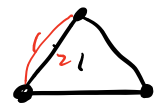
A graph is ***planar*** if there is a way to draw it in a plane without edges crossing.

证明：

$m \le 3n-6$ $\qquad (n-m+f=2)$

prove: Face length ;

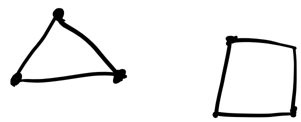
$$2m = \sum_{i=1}^{f} F_i \ge 3f$$



$F_1 = 3$

$F_2 = 3$

$m = 3$



smallest ⇒ Δ

最小面长为3

$2m \ge 3f$

$$n-m+f=2 \Rightarrow m = n+f-2 \le n+\frac{2}{3}m-2$$

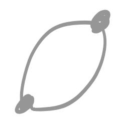
$$\Rightarrow \frac{1}{3}m \le n-2$$

$$m \le 3n-6$$

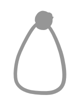
simple graph

(if not simple, face length can be smaller than 3)

$F \ge 3f$

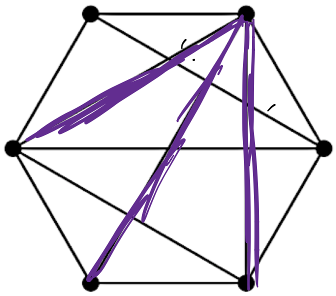


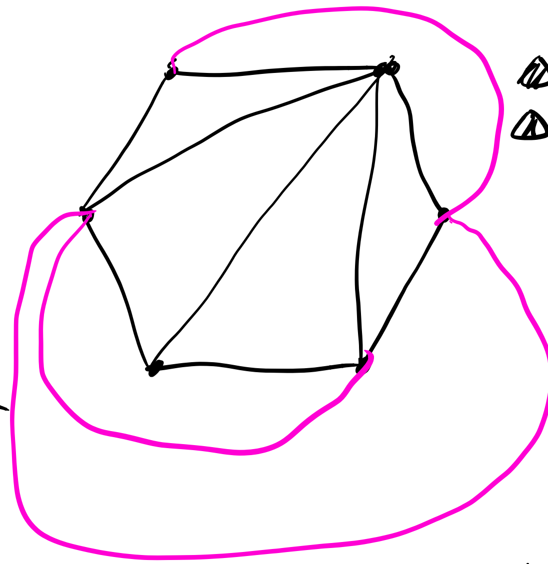
$F=2$



$F=1$

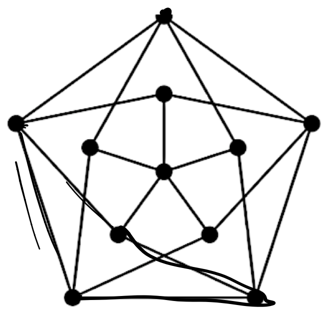
(课件上也有证明)





$n=6$
$m=12$
$\boxed{f=8}$

$$\begin{cases} n-m+f=2 \rightarrow \text{planar} \\ m \leq 3n-6 \end{cases}$$



$n=11$
$m=20$
$\Downarrow$
$f=11$

(if it is a planar graph, then it should have 11 faces)

Face length:
$2m=\sum_{i=1}^{11} F_i = 40$

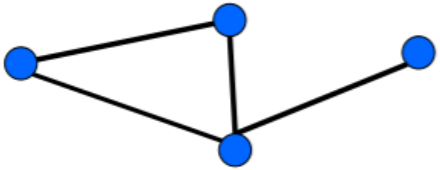
Face number: 11

no triangle
$\sum_{i=1}^{11} F_i \geq 4f = 44$

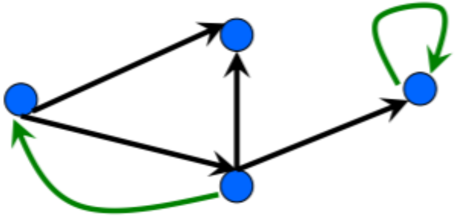
contradiction
$44 > 40$

Types of Graphs

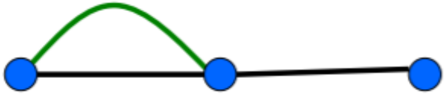
Simple Graph
(no multiedges, no loops, no directed edges)



Directed Graph (digraph)



Multi-Graph



Eulerian path problem

Unless otherwise specified, all graphs in this course are simple.

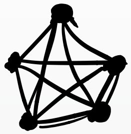
3 **Determine whether a graph exists for each of the following degree sequences:**

1) 1,2,3,4,5 $n=5$ $2m=15$ X

2) 3,3,3,2,2 $n=5$ $2m=13$ X

3) 4,4,4,4,4 $n=5$ $2m=20 \Rightarrow m=10$

K_5



4) 4,4,3,2,1

5) 3,3,2,2,2

If it exists, draw all the possibilities up to isomorphism and determine whether these graphs are planar.

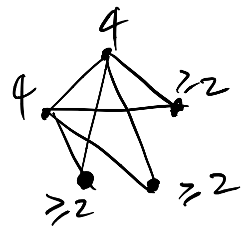
$$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

① Calculate $n \cdot m$

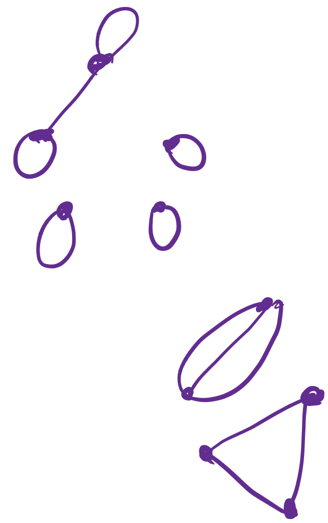
② see the vertex with largest degree

4) 4,4,3,2,1
5) 3,3,2,2,2

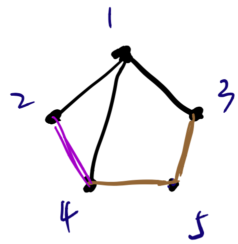
4) $n=5 \quad 2m=14 \quad m=7$



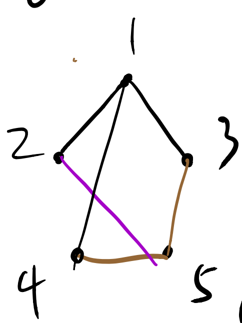
contradiction



5) $n=5 \quad 2m=12 \quad m=6$



(2)



(3)

4

Let G be a **simple connected** graph whose vertex degrees are all $\leq k$, and there exists a vertex whose degree is strictly smaller than k. Prove by induction on number of vertices that G is k-colorable. $k \in \mathbb{Z}^+$

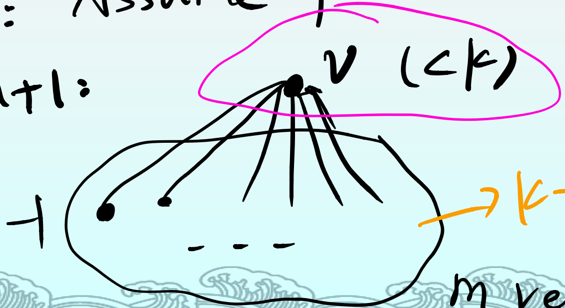
n

$P(n)$: G is k-colorable. $\deg(v) \leq k$, $v \in V$ $\deg(v_i) < k$ $(k \in \mathbb{Z}^+)$

Base case: $n=1$ •   1-colorable ✓

Inductive step: Assume $P(m)$ is true for $n=m$, $m \in \mathbb{Z}^+$

For $n=m+1$:



→ k-colorable

m vertices $(\leq k)$

→ k-colorable

Remove v $\underline{deg(v) \le k}$, $v \in V$ $deg(v_i) < k$ ✓

($k \in \mathbb{Z}^+$)

m vertices $\to$ k-colorable (by $p(m)$)

color v with one of k colors

$p(m+1)$ is true ✓