CSC3001: Discrete Mathematics

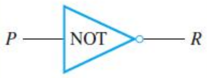
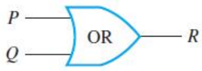
Midterm Exam (Fall 2020)

Instructions:

1. This exam is 120 minute long, and worth 100 points.
2. This exam has 12 pages, consisting of 6 questions, all to be attempted. **Write down your full working in this exam paper**.
3. Calculator is NOT allowed.
4. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.
5. As a bonus of reading this instruction, here is a hint for Question 5: induction.

Student Number: ____________________ **Name:** _______________

1. (*18 points*) Suppose that you are given one “**NOT**”, three “**AND**”s, and one “**OR**” of the following electronic components:

<table>
<tr><th>Type of Gate</th><th>Symbolic Representation</th><th>Action</th></tr>
<tr><td>NOT</td><td>
</td><td>
<table>
<tr><th>Input</th><th>Output</th></tr>
<tr><th>P</th><th>R</th></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>0</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>AND</td><td>
</td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table>
</td></tr>
<tr><td>OR</td><td>
</td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R</th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table>
</td></tr>
</table>

Design a circuit so that it has the following input/output table.

P	Q	R	output
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	0

2. (*20 points*) Solve the congruence equation

$$609x \equiv 1 \pmod{2020}$$

3. (*11 points*) Prove that $15n + 4$ and $10n + 3$ are coprime for any $n \in \mathbb{N}$.

4. (*11 points*) Suppose there are $m + n$ coins along a line where m of them with head facing up and the other n coins with tail facing up. Can we can flip over pairs of adjacent coins so that all the coins have their heads facing up? Discuss for which case we cannot; and for the cases we can, describe a method to flip the coins.

5. (*20 points*) Let $\{F_n\}$ be defined by $F_1 = 3, F_2 = 4$ and $F_{n+2} = F_n + F_{n+1}$. Prove that $\mathsf{gcd}(F_n, F_{n+1}) = 1$ for all $n \in \mathbb{Z}^+$.

6. (*20 points*) Let $n \in \mathbb{Z}^+$. Denote by r_n the number of n-bit binary strings that do not contain a substring "001". Use **generating function** to find the recurrence relation and the closed form of r_n. (**Note:** Full mark will be given **ONLY** if you use generating function. You may use the notations $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.)