CSC3001: Discrete Mathematics

Final Exam (Fall 2020)

Instructions:

1. This exam is 120 minute long.
2. The total mark in this exam paper is 110, and the maximum you can get is 100. **You should try to attempt as many questions as possible.**
3. This exam has 14 pages, consisting of 7 questions. **Write down your full working in this exam paper**.
4. You should check the number of questions in the first 30 minutes. **Instructor is not responsible for any missing questions when exam ends.**
5. Calculators are **NOT** allowed.
6. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.
7. Please note that all the graphs in this exam paper are **SIMPLE GRAPHS**.

Student Number: ____________________ **Name:** _______________

1. (*10 points*) Let $n \in \mathbb{Z}^+$, $A = \{1, 2, \ldots, n\}$ and $S \subseteq A$ with $|S| \geq \frac{n}{2}$. Prove that there always exist distinct $x, y \in S$ such that $x + y = n$.

2. (*10 points*) Given the following program, what is the minimum number of registers needed to store values of the variables throughout the computation? Deduce your claim.

Input:	a
Step 1.	$b = a - 2$
Step 2.	$c = b \cdot \frac{1}{4}$
Step 3.	$d = c + 1$
Step 4.	$e = 3^a$
Step 5.	$f = b \cdot e$
Step 6.	$g = e^2 - f$
Step 7.	$h = b - d$
Output:	f, g, h

3. (*15 points*) For the stable matching problem with equal number of boys and girls, prove that there is always a girl who does not receive any proposals until the last day of the marrying procedure.

4. (*20 points*) Let $n \in \mathbb{Z}^+$ and set

$$S = \{1, 2, \ldots, n\}, \qquad T = \{a, b, c\}$$

Determine the number of surjective functions from S to T.

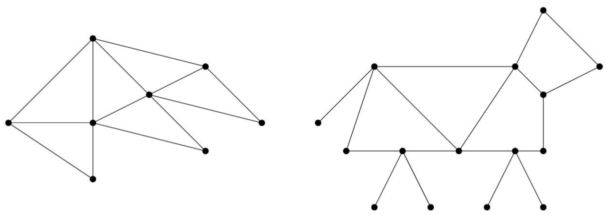
5. (*20 points*) Given $m \in \mathbb{Z}^+$, prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0, & \text{if } n = 2m-1; \\ (-1)^m \binom{2m}{m}, & \text{if } n = 2m. \end{cases}$$

6. (*15 points*) Let G be a bipartite graph with a bipartition (A, B) so that $|A| = |B| = n$. Suppose that each vertex in A has a mutually distinct positive degree. Prove that G has a perfect matching.

For the remaining questions, all graphs are assumed to be **SIMPLE**.

7. (*20 points*) Let G be a planar graph drawn on a plane with no edge crossing. If all the vertices of G lie on the unbounded face in this case, then G is *outerplanar*. The following are two examples of outerplanar graphs.



(a) Give an example of a planar graph that is not outerplanar. [**2 marks**]

(b) Prove that any outerplanar graph is 3-colorable. [**8 marks**]

(c) Suppose that G has $n \geq 2$ vertices and m edges. Determine the maximum value of m and prove your claim. [**10 marks**]

For the remaining questions, all graphs are assumed to be **SIMPLE**.