CSC3001: Discrete Mathematics

Midterm Exam (Fall 2019)

Instructions:

1. This exam is 120 minute long, and worth 100 points.
2. This exam has 12 pages, consisting of 6 questions, all to be attempted. **Write down your full working in this exam paper**.
3. Calculator is allowed.
4. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.

Student Number: ____________________ **Name:** GGTF

1. (*11 points*) Find $\mathsf{gcd}(2019! + 1, 2020! + 1)$.

$\gcd(2020!+1, 2019!+1)$

$= \gcd(2019!+1, 2019!-2018)$

$= \gcd(2019!-2018, 2019)$

$= \gcd(2019, 1)$

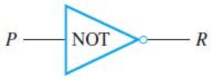
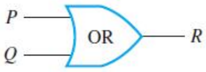
$= 1$

$2020!+1 = (2019!+1)\cdot 2019 + 2019!-2018$

$2019!+1 = (2019!-2018)\cdot 1 + 2019$

$2019!-2018 = (2019)(2018!-1)+1$

2. (*18 points*) Suppose that you are given two “**NOT**”s, two “**AND**”s, and two “**OR**”s of the following electronic components:

Type of Gate	Symbolic Representation	Action
NOT		<table><tr><th>Input</th><th>Output</th></tr><tr><th>P</th><th>R</th></tr><tr><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td></tr></table>
AND		<table><tr><th colspan="2">Input</th><th>Output</th></tr><tr><th>P</th><th>Q</th><th>R</th></tr><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table>
OR		<table><tr><th colspan="2">Input</th><th>Output</th></tr><tr><th>P</th><th>Q</th><th>R</th></tr><tr><td>1</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table>

Design a circuit so that it has the following input/output table.

P	Q	R	output
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

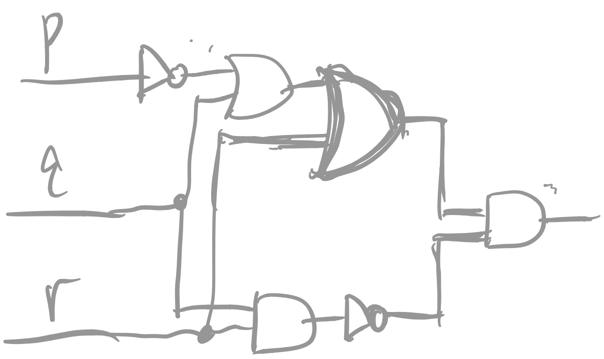
Look at the false row:

$$(\bar{p}+\bar{q}+\bar{r})(\bar{p}+q+r)(p+\bar{q}+\bar{r}) \qquad (A+B)(A+C)=A+BC$$

$$= (\bar{p}+q+r)(\bar{q}+\bar{r}+\underbrace{\bar{p}p}_{0})$$

$$= (\bar{p}+q+r)\cdot\overline{q\cdot r}$$

$$(\neg p\vee q\vee r)\wedge\neg(q\wedge r)$$



A method used in EIE2050

r \ pq	00	01	11	10
0	1	1	1	0
1	1	0	0	1

$0 \rightarrow (\bar{p}+q+r)$

$0\ 0 \downarrow (\bar{q}+\bar{r})$

$$(\bar{p}+q+r)(\bar{q}+\bar{r})$$
$$= (\bar{p}+q+r)\cdot\overline{q\cdot r}$$

3. (*19 points*) Consider the set of all strings of a's, b's and c's. Let r_n be the number of strings of a's, b's and c's of length n that do not contain the patterns aa and ab. ($n \in \mathbb{Z}^+$)

(a) Find the values of r_1, r_2, r_3 by enumerating the strings. **[6 marks]**

(b) Find the recurrence relation for $\{r_n\}$. **[5 marks]**

(c) Find the closed form for r_n. **[8 marks]**

(a) $n=1$: a b c

$n=2$: ac ba bb bc ca cb cc

$n=3$: aca acb acc bac bba bbb bbc

bca bcb bcc cac cba cbb cbc

cca ccb ccc

$r_1 = 3$ $r_2 = 7$ $r_3 = 17$

(b) cases:

$\boxed{n \text{ strings}}$

$a + c + \boxed{(n-2) \text{ strings}}$

$b + \boxed{(n-1) \text{ strings}}$

$c + \boxed{(n-1) \text{ strings}}$

$r_n = 2r_{n-1} + {}^{5}r_{n-2}$

(c) $r_n = 2r_{n-1} + r_{n-2} \quad (n \in \mathbb{Z}^+)$

Use distinct-root theorem

$x^2 = 2x + 1$

$x^2 - 2x - 1 = 0$

$x_1 = 1 + \sqrt{2} \quad x_2 = 1 - \sqrt{2}$

$r_n = C \cdot (1+\sqrt{2})^n + D(1-\sqrt{2})^n$

$n=1$: $\quad C(1+\sqrt{2}) + D(1-\sqrt{2}) = 3$

$n=2$ $\quad C(1+\sqrt{2})^2 + D(1-\sqrt{2})^2 = 7$

$$\begin{cases} C = \frac{1+\sqrt{2}}{2} \\ D = \frac{1-\sqrt{2}}{2} \end{cases}$$

$r_n = \frac{1}{2}(1+\sqrt{2})^{n+1} + \frac{1}{2}(1-\sqrt{2})^{n+1} \quad (n \in \mathbb{Z}^+)$

4. (*24 points*) Find the smallest positive integer x satisfying the following:

$$\begin{cases} 95x \equiv 5 \pmod{40} & ① \\ 21x \equiv -9 \pmod{60} & ② \\ 2x \equiv 152 \pmod{75} & ③ \end{cases}$$

①: $95x = 5 + 40n \Rightarrow 19x = 1 + 8n$

$\Rightarrow 19x \equiv 1 \pmod 8$

$\Rightarrow 3x \equiv 1 \pmod 8$ $\qquad 3 \cdot 3x \equiv 3 \cdot 1 \pmod 8$

$\Rightarrow x \equiv 3 \pmod 8$

②: $21x = -9 + 60n \Rightarrow 7x = -3 + 20n$

$7x \equiv -3 \pmod{20}$

$\Rightarrow 7x \equiv 17 \pmod{20}$ $\qquad 3 \cdot 7x \equiv 3 \cdot 17 \pmod{20}$

$x \equiv 51 \pmod{20}$

$x \equiv 11 \pmod{20}$

③: $2x \equiv 152 \pmod{75}$

$2x \equiv 2 \pmod{75}$ $\qquad$ (or $2 \cdot 38x \equiv 2 \cdot 38 \pmod{75}$)

$x \equiv 1 \pmod{75}$

$$\begin{cases} x \equiv 3 \pmod{8} \\ x \equiv 11 \pmod{20} \\ x \equiv 1 \pmod{25} \end{cases}$$

Let $x = 3 + 8a \quad (a \in \mathbb{Z})$

$3 + 8a \equiv 11 \pmod{20} \Rightarrow 8a \equiv 8 \pmod{20}$

$2a \equiv 2 \pmod{5}$

$a \equiv 1 \pmod{5}$

$a = 1 + 5b \quad (b \in \mathbb{Z})$

$x = 3 + 8(1+5b) = 40b + 11$

$40b + 11 \equiv 1 \pmod{75}$

$40b \equiv -10 \pmod{75}$

$8b \equiv -2 \pmod{15}$

$2 \cdot 8b \equiv 2 \cdot (-2) \pmod{15}$

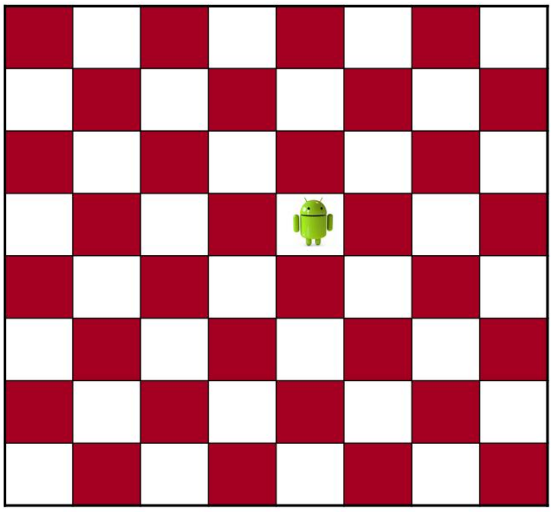
$b \equiv 11 \pmod{15}$

$b = 11 + 15c \quad (c \in \mathbb{Z})$

$x = 40(11 + 15c) + 11 = 600c + 451 \quad (c \in \mathbb{Z})$

Smallest x: $600 \times 0 + 451 = 451$

5. (*10 points*) A robot is cleaning floor on an $n \times n$ square grid. Each step of its movement is to move from a square to its adjacent square (up/down/left/right). Discuss

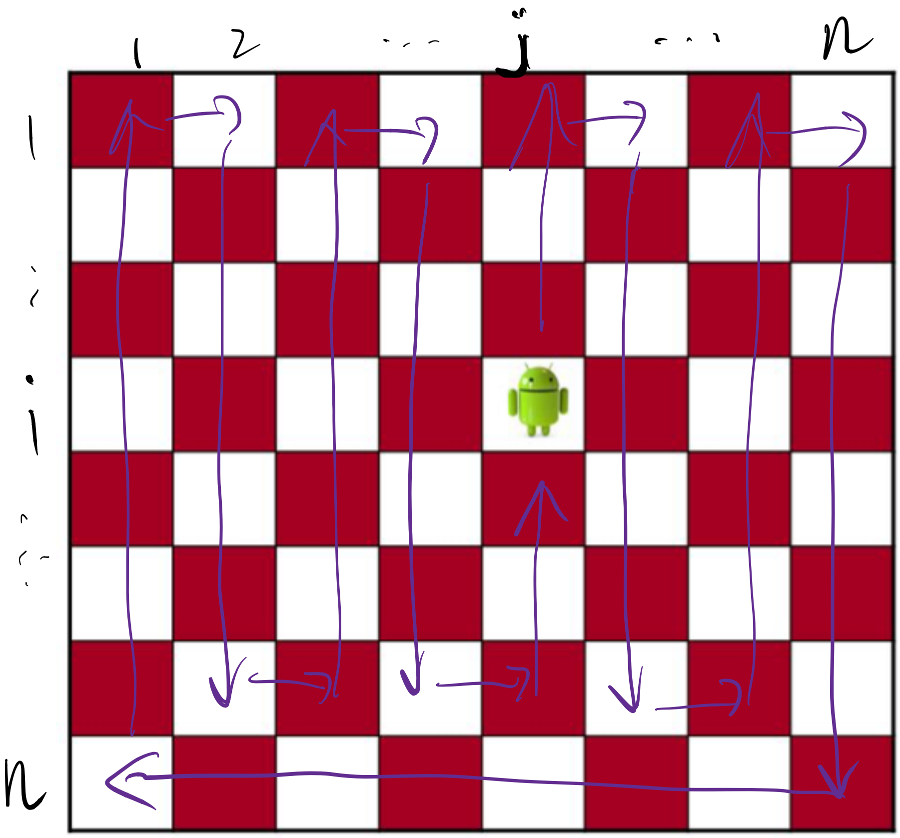


for which n it can find an n^2-step path so that it can clean each square and return to the initial squre in the last step. When it is possible, present such a path; otherwise prove the infeasibility.

① n is odd: impossible

Each step the robot moves from a red square to a white square. If it wants to return to the initial, it should go through the same number of white and red squares (white → white), which has even steps. n^2 is odd, impossible.

② n is even: possible



start (i,j)

$(i,j) \to (1,j)$ (if j is odd, turn right, or turn left)

odd j: $(1, j+1) \to (n-1, j+1) \to \cdots (i,j)$

even j: $(1, j-1) \to (n-1, j-1) \to \cdots (i,j)$

You can design your own path.

6. (*18 points*) A confectionery company is designing an assorted pack of confectionery consisting of chocolate (15g/bag), marshmallow (6g/bag) and toffee (10g/bag). Show that for any pack with an integer weight at least 61g (i.e., 61g, 62g, 63g, etc), there is always a way to mix these three kinds of confectionery so that the pack contains some (≥ 1 bag) of each confectionery.

① $P(n)$: Any integer weight $w \in \{61, 62, 63, \cdots n\}$ can be mixed by three kinds of confectionery.

② Base case:

If $n=61$, then $w = 61 = 15\times1 + 6\times1 + 10\times4$, $P(61)$ is true.

③ Inductive step:

Assume $P(t)$ is true for some $t \geq 61$, for $n = t+1 \geq 62$, consider:

1) If $n=62$, then $m = 61.62$, only consider $m=62$

$62 = 15\times2 + 6\times2 + 10\times2$, So $P(t+1)$ is true.

2) If $n=63$, $63 = 15\times1 + 6\times3 + 10\times3$, $P(t+1)$ true

3) If $n=64$, $64 = 15\times2 + 6\times4 + 10\times1$, $P(t+1)$ true

4) If $n=65$, $65 = 15\times1 + 6\times5 + 10\times2$, $P(t+1)$ true

5) If $n=66$, $66 = 15\times2 + 6\times1 + 10\times3$, $P(t+1)$ true

6) If $n \geq 67$, then $s = n-6 \in \{61, 62, \cdots t\}$, which can be mixed by three kinds of confectionery.

$$s = 15a + 6b + 10c \quad , a、b、c \in Z^+$$

Then:

$$n = s + 6 = 15a + 6(b+1) + 10c$$

So $p(t+1)$ is true

Therefore, $p(n)$ is true for any $n \geq 61$.