Strong Induction

Strong induction

Prove $P(0)$.

Then prove $P(n+1)$ assuming *all* of

$P(0), P(1), \ldots, P(n)$ (instead of just $P(n)$).

Conclude $\forall n.P(n)$

$\forall n_0 \le n$

equivalent

Ordinary induction

$0 \rightarrow 1,\ 1 \rightarrow 2,\ 2 \rightarrow 3,\ \ldots,\ n\text{-}1 \rightarrow n.$

So by the time we get to n+1, already know *all* of

$P(0), P(1), \ldots, P(n)$

The point is: assuming P(0), P(1), up to P(n), it is often easier to prove P(n+1).

Question 1

The formula for the nth term a_n of the Fibonacci sequence,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

is given by,

$$a_n = \begin{cases} 1 \text{ for } n = 1 \text{ and } 2 \\ a_{n-2} + a_{n-1} \text{ for } n > 2 \end{cases}$$

Prove by mathematical induction that,

$$a_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{\sqrt{5}\, 2^n}$$

$P(n)$: for Fibonacci sequence: $\forall n \in \mathbb{Z}^+$

strong induction $a_n = \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{\sqrt{5} \cdot 2^n}$

(1) base case:

when $n=1$ $\quad a_1 = \dfrac{(1+\sqrt{5}) - (1-\sqrt{5})}{\sqrt{5} \cdot 2^1} = 1$

when $n=2$ $\quad a_2 = \dfrac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{\sqrt{5} \cdot 2^2} = 1$

(2) inductive step:

Assume $P(t)$ is true for $\forall 1 \le t_0 \le t$, $t_0, t \in \mathbb{Z}^+$

$$a_t = \frac{(1+\sqrt{5})^t - (1-\sqrt{5})^t}{\sqrt{5} \cdot 2^t} \qquad a_{t-1} = \frac{(1+\sqrt{5})^{t-1} - (1-\sqrt{5})^{t-1}}{\sqrt{5} \cdot 2^{t-1}}$$

TF

For $n = t+1$:

$$a_{t+1} = a_t + a_{t-1}$$

$$= \frac{(1+\sqrt{5})^t - (1-\sqrt{5})^t + 2(1+\sqrt{5})^{t-1} - 2(1-\sqrt{5})^{t-1}}{\sqrt{5} \cdot 2^t}$$

$6+2\sqrt{5} = (\sqrt{5}+1)^2 \Leftarrow 2(3+\sqrt{5}) \qquad 2(3-\sqrt{5}) \rightarrow (\sqrt{5}-1)^2$

$$= \frac{(1+\sqrt{5}+2)(1+\sqrt{5})^{t-1} - (1-\sqrt{5}+2)(1-\sqrt{5})^{t-1}}{\sqrt{5} \cdot 2^t \cdot 2}$$

$$= \frac{(1+\sqrt{5})^{t+1} - (1-\sqrt{5})^{t+1}}{\sqrt{5} \cdot 2^{t+1}}$$

$\Rightarrow P(n)$ is true for $\forall n \in \mathbb{Z}^+$

Question 2

Use strong induction to prove $n \leq 3^{\frac{n}{3}}$, for any natural number n.

$p(n): \forall n \in \mathbb{N},\ n \leq 3^{\frac{n}{3}}$

(1) base case: $n=0:\ 0 \leq 3^{\frac{0}{3}} = 1$

$n=1:\ 1 \leq 3^{\frac{1}{3}}\ (\sqrt[3]{3}) \Leftarrow 1^3 \leq (3^{\frac{1}{3}})^3 = 3$

$n=2:\ 2 \leq 3^{\frac{2}{3}} \Leftarrow 8 = 2^3 \leq (3^{\frac{2}{3}})^3 = 9$

(2) inductive step: Assume $p(n)$ is true for $\forall\, 0 \leq t_0 \leq t$

$t \geq 3,\ t_0, t \in \mathbb{N}$

script

$t \leq 3^{\frac{t}{3}}$;

$\left(t+1 \leq 3^{\frac{t+1}{3}}\right)$

$\downarrow$

$3^{\frac{t}{3}+\frac{1}{3}}$

$t-1 \leq 3^{\frac{t-1}{3}}$

$2 \leq 3^{\frac{2}{3}}$

$t+1 \leq 2(t-1) \leq 3^{\frac{t+1}{3}}$

$\big(2(t-1)-(t+1)$

$= 2t-2-t-1$

$= t-3 \geq 0 \big)$

$\Rightarrow 3^{\frac{t+1}{3}} \geq t+1$

process:

$3^{\frac{t+1}{3}} = 3^{\frac{t-1}{3}} \cdot 3^{\frac{2}{3}}$

$\geq (t-1) \times 2$

$= 2t-2$

$= t+(t-2)$

$\geq t+1$

$(t \geq 3)$

$p(t+1)$ is true

$\Rightarrow p(n)$ is true for $\forall n \in \mathbb{N}$

Question 3

Write the numbers $1, 2, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k, write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. **Prove that this integer is odd.**

n is odd parity 奇偶性

Let S: sum of all the numbers on the blackboard.

At first: $S = \sum_{i=1}^{2n} i = \frac{(1+2n)\cdot 2n}{2} = n(2n+1)$

$\Downarrow$

odd

For each operation:

$$\Delta S = (j+k) - |j-k|$$

$$= \begin{cases} j+k-(j-k) = 2k \\ (j+k)-(k-j) = 2j \end{cases}$$

ΔS is even.

So after certain operations, the integer is still odd.

Well Ordering Principle

$\mathbb{N}$

Axiom Every nonempty set of *natural numbers* has a *least element.*

$S = \{x^2 \mid x \in \mathbb{N}\}$

This axiom is in fact a consequence of mathematical induction.

Question 4

Using **Well-ordering Principle** to show that the equation $a^2 + b^2 = 3(s^2 + t^2)$ has no non-zero integer solution.

Let $S=\{a^2 \mid a^2+b^2=3(s^2+t^2), a,b,s,t \in \mathbb{Z}^+\}$

Assume the equation has non-zero integer solution. $S \neq \phi$

$\exists a_0^2 \in S$ where a_0^2 is the smallest among all elements.

$$a_0^2+b_0^2 = 3(s_0^2+t_0^2)$$

$(a_0^2+b_0^2)$ should be a multiple of 3

$a_0=3k \quad (3k)^2=9k^2 = 3k_1 \rightarrow 0$

$a_0=3k+1 \quad (3k+1)^2=9k^2+6k+1 = 3k_2+1 \rightarrow 1$

$a_0=3k+2 \quad (3k+2)^2=9k^2+12k+4 = 3k_3+1 \rightarrow 1$

so a_0, b_0 are also a multiple of 3.

Let $a_0=3a_1 \quad b_0=3b_1$

$$9a_1^2+9b_1^2 = 3(s_0^2+t_0)^2$$

3^2 0

1

2 1

$$\Rightarrow \quad 3(a_1^2+b_1^2) = s_0^2+t_0^2$$

but applying the same argument

s_0, t_0 are a multiple of 3

let: $s_0 = 3s_1$, $t_0 = 3t_1$

$$3(a_1^2+b_1^2) = 9(s_1^2+t_1^2)$$

$$a_1^2+b_1^2 = 3(s_1^2+t_1^2)$$

$$(a_1^2, b_1^2, s_1^2, t_1^2) \in S$$

$$\left(\frac{a_0}{3}\right)^2 \left(\frac{b_0}{3}\right)^2 \left(\frac{s_1}{3}\right)^2 \left(\frac{t_1}{3}\right)^2$$

$$a_1 = \frac{a_0}{3} < a_0$$

contradiction!

so $S = \emptyset$

no solution.

3. (*20 points*)

(a) Translate the following statement into logical formula without predicates.

> For each $a, b \in \mathbb{Z}^+$ with $a \leq b$, we have
>
> $$\frac{a}{b} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m}$$
>
> for some mutually distinct $d_1, \ldots, d_m \in \mathbb{Z}^+$.

(b) Use mathematical induction to prove the statement in (a).
(Full mark will be given **ONLY** if you use mathematical induction.)

⑤ (may be wrong)

$\forall a \leq b \leq n$, $a, b, n \in \mathbb{Z}^+$.

$$\frac{a}{b} = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_m}$$

for mutually distinct $d_1, \cdots, d_m \in \mathbb{Z}^+$

$P(a)$: $\forall b \in \mathbb{Z}^+$, $a \leq b$, $\frac{a}{b}$ can be decomposed

(1) base case:

$a = 1$: $\frac{1}{b}$ ✓

(2) inductive case:

Assume $P(t)$ is true for some $t \in \mathbb{Z}^+$

$\frac{t}{b}$ can be decomposed

For. $\frac{t+1}{b} = \frac{t}{b} + \frac{1}{b}$

repeated:

$$\frac{1}{b} = \underline{\frac{1}{b+1}} + \underline{\frac{1}{b(b+1)}}$$

...