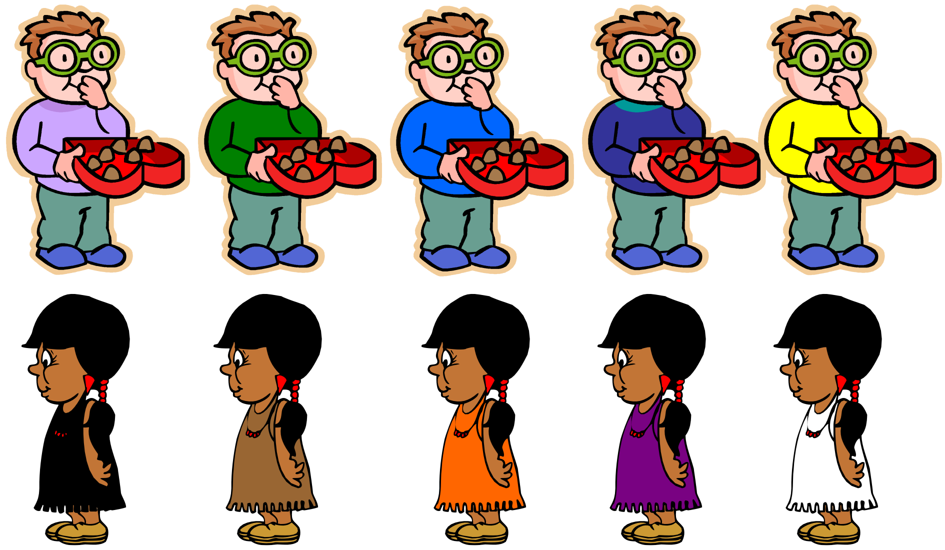
Graph Matching



This Lecture

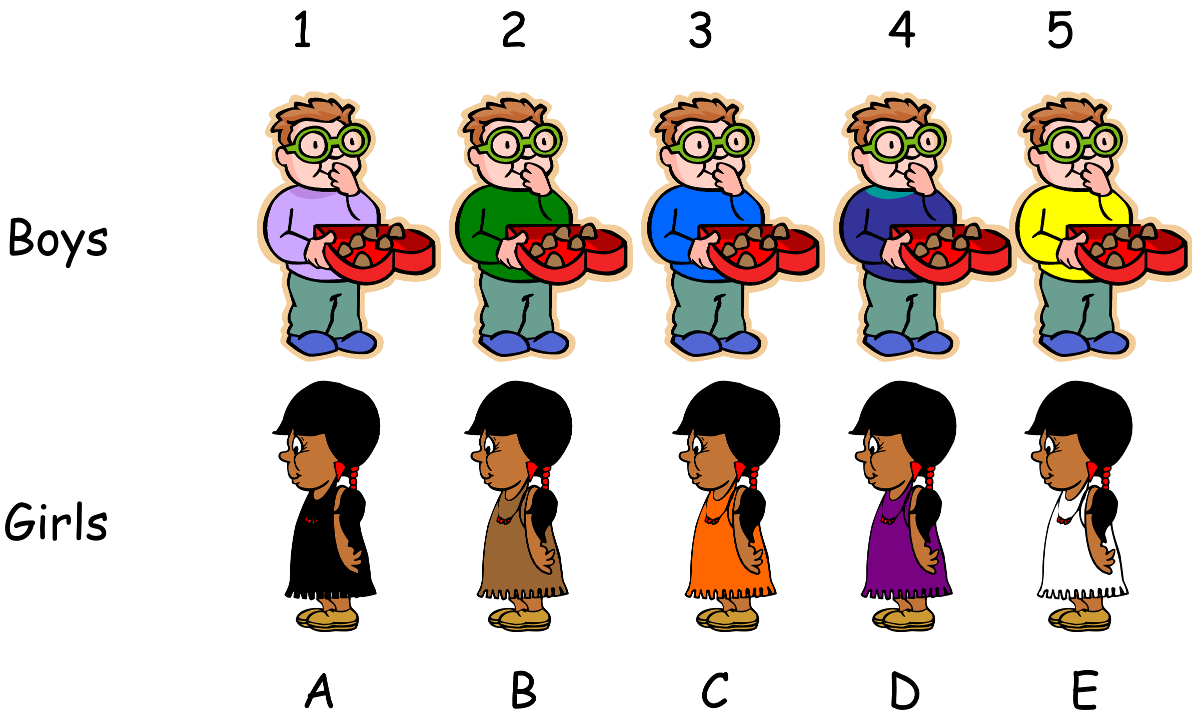
Last week we learnt some basic concepts of graphs.

Today, we study an important problem in graph theory
-- graph matching, which has many applications and
is the basis of more advanced problems.

This lecture's plan:

- Stable matching
- Bipartite matching

Matching



Today's goal: to "match" the boys and the girls in a "good" way.

Matching

Today's goal: to "match" the boys and the girls in a "good" way.

What is a matching?

- Each boy is matched with at most one girl.
- Each girl is matched with at most one boy.

What is a good matching?

Depends on the information we have.

- ***A* stable matching:** They have no incentive to break up…
- ***A* maximum matching:** To maximize the number of matched pairs…

Stable Matching

The Stable Matching Problem:

- There are n boys and n girls.
- For each boy, there is a preference list of the girls.
- For each girl, there is a preference list of the boys.

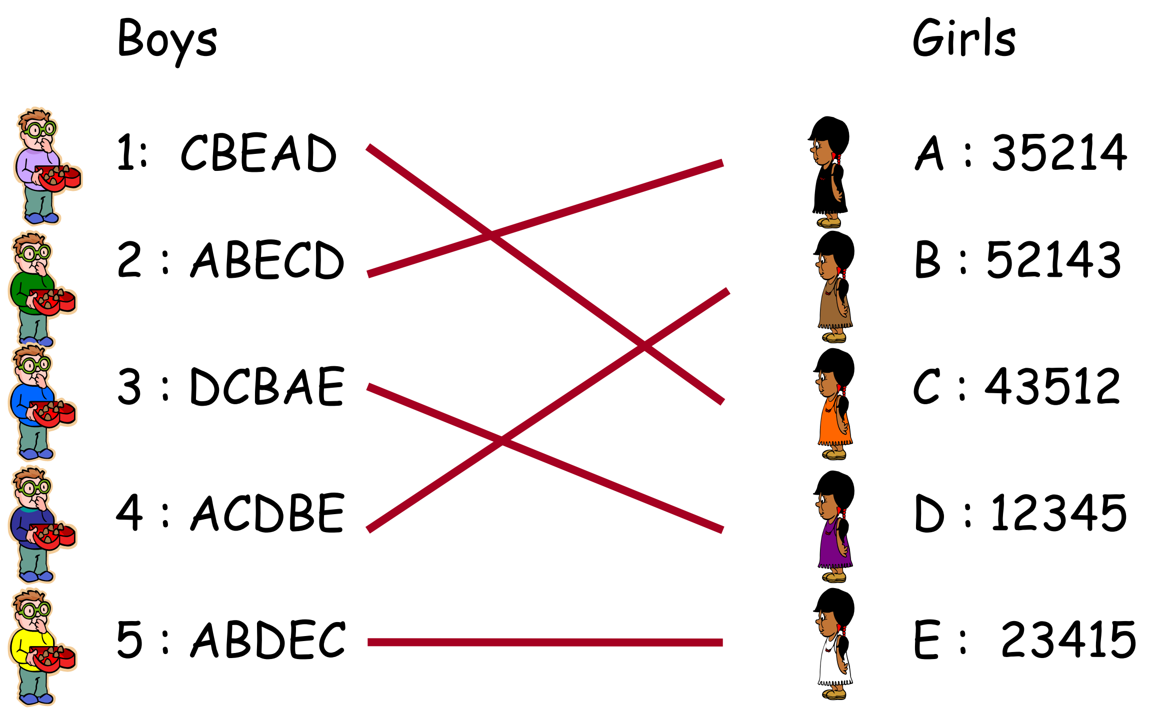
	Boys		Girls
	1: CBEAD		A : 35214
	2 : ABECD		B : 52143
	3 : DCBAE		C : 43512
	4 : ACDBE		D : 12345
	5 : ABDEC		E : 23415

Stable Matching

What is a ***stable*** matching?

Consider the following matching.

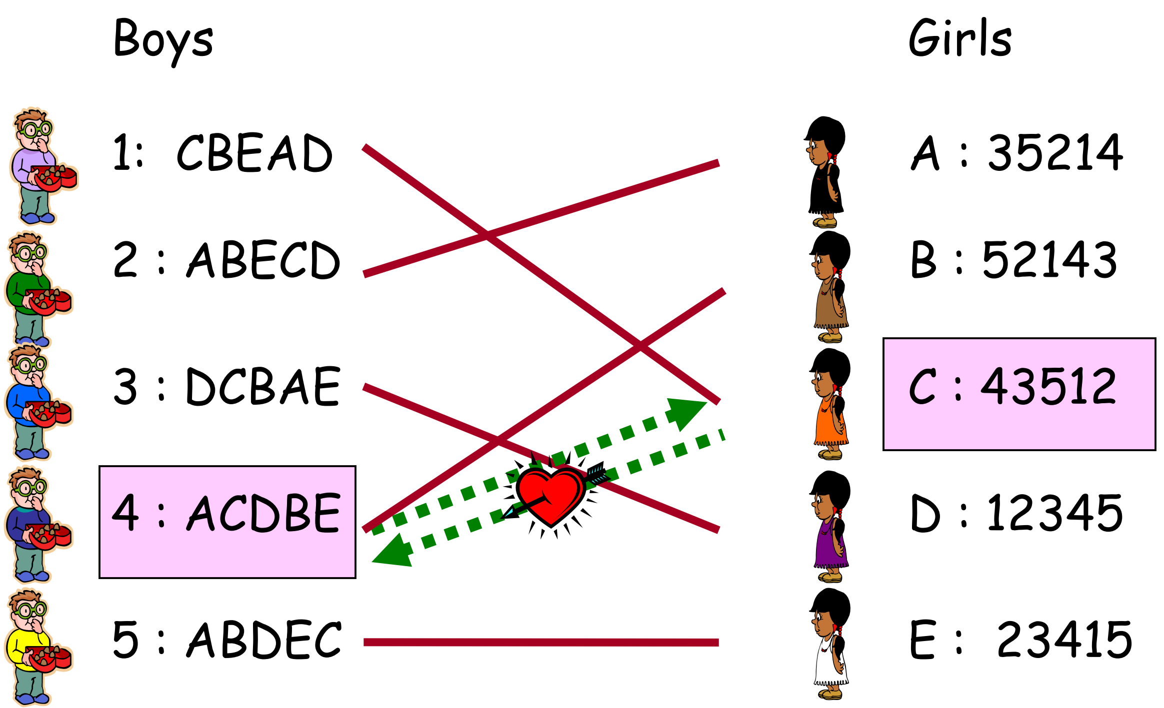
It is **unstable**, why?



Stable Matching

- Boy 4 prefers girl C more than girl B (his current partner).
- Girl C prefers boy 4 more than boy 1 (her current partner).

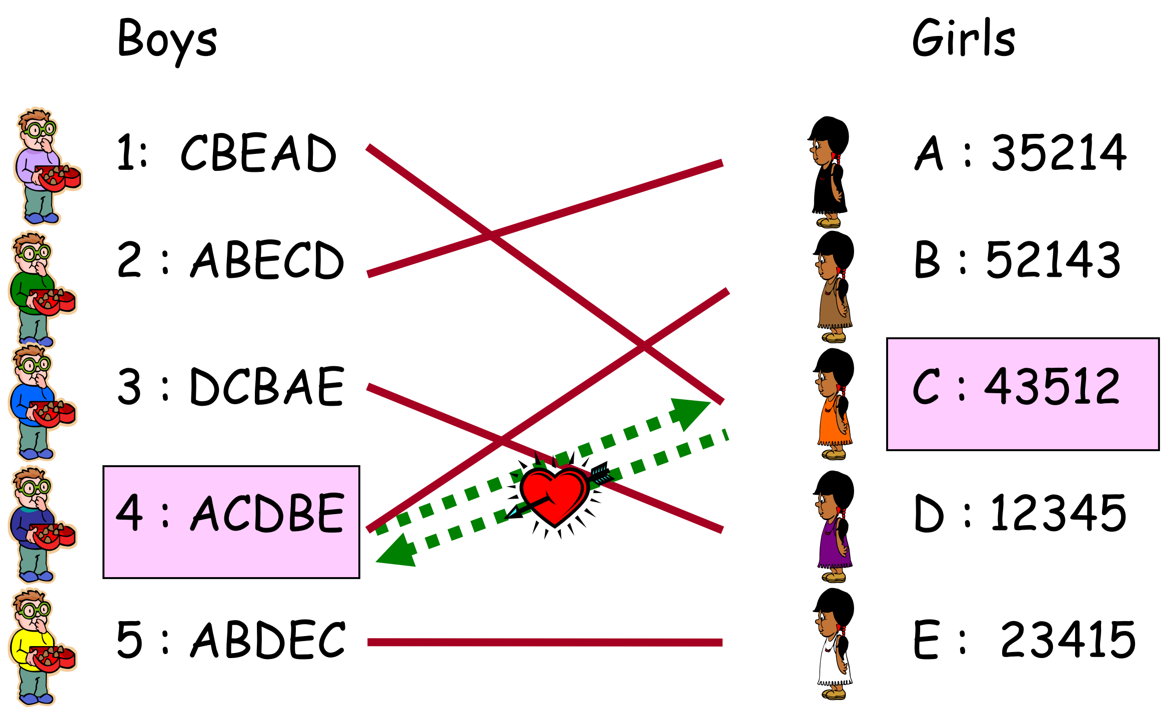
So they have the incentive to leave their current partners, and switch to each other, we call (4, C) an **unstable pair**.



Stable Matching

More formally, given a matching M for the vertex set V, the pair (v,w) is ***unstable*** if

1. (v, w′) and (v′, w) are matched pairs in M, where v ≠ v′, w ≠ w′.
2. v prefers w rather than w′.
3. w prefers v rather than v′.

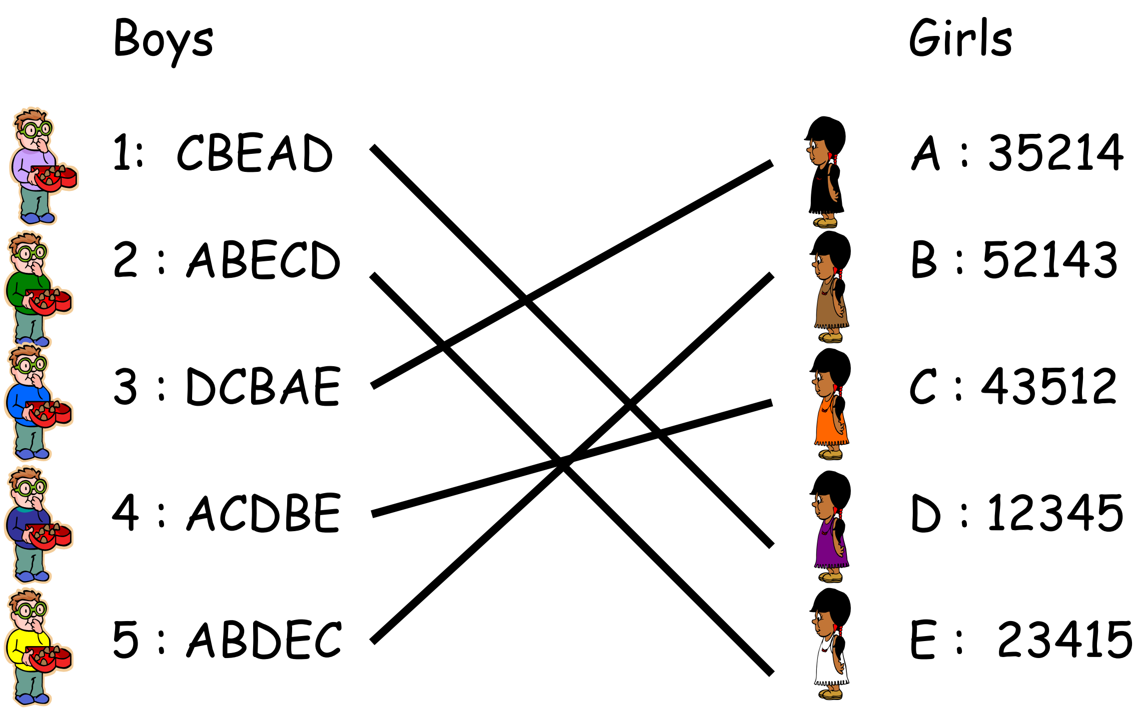


Stable Matching

What is a ***stable*** matching?

A stable matching is a matching with no unstable pair, and everyone is married.

Can you find a stable matching in this case?



Stable Matching

Does a stable matching always exist? Not clear…

Can you find a stable matching in this case?

Boys	Girls
1: CBEAD	A : 35214
2 : ABECD	B : 52143
3 : DCBAE	C : 43512
4 : ACDBE	D : 12345
5 : ABDEC	E : 23415

Stable Roommate

The Stable Roommate Problem:

- There are 2n people.
- There are n rooms, each can accommodate 2 people.
- Each person has a preference list of 2n-1 people.
- Find a stable matching (match everyone and no unstable pairs).

Does a stable matching always exist?

Not clear…

When is it difficult to find a stable matching?

Idea: **triangle relationship**!

Stable Roommate

Idea: **triangle relationship**!

	1	2	3
a	b	c	d
b	c	a	d
c	a	b	d
d	a	b	c

- a prefers b more than c
- b prefers c more than a
- c prefers a more than b
- no one likes d

Any problem with the matching {(a,b),(c,d)} ? (b,c)
Any problem with the matching {(a,c),(b,d)} ? (a,b)
Any problem with the matching {(a,d),(b,c)} ? (a,c)

No stable matching exists!

Stable Matching

Can we find a stable matching in a stable matching problem?

Not clear…

Gale,Shapley [1962]:

There is always a stable matching in the stable matching problem.

Wait, what makes this different from **the Stable Roommate Problem?**

No partition of the set.

This is more than a solution to a puzzle:

- College Admissions (original Gale & Shapley paper, 1962)
- Matching Hospitals & Residents.

Shapley received Nobel Prize 2012 in Economics for it!

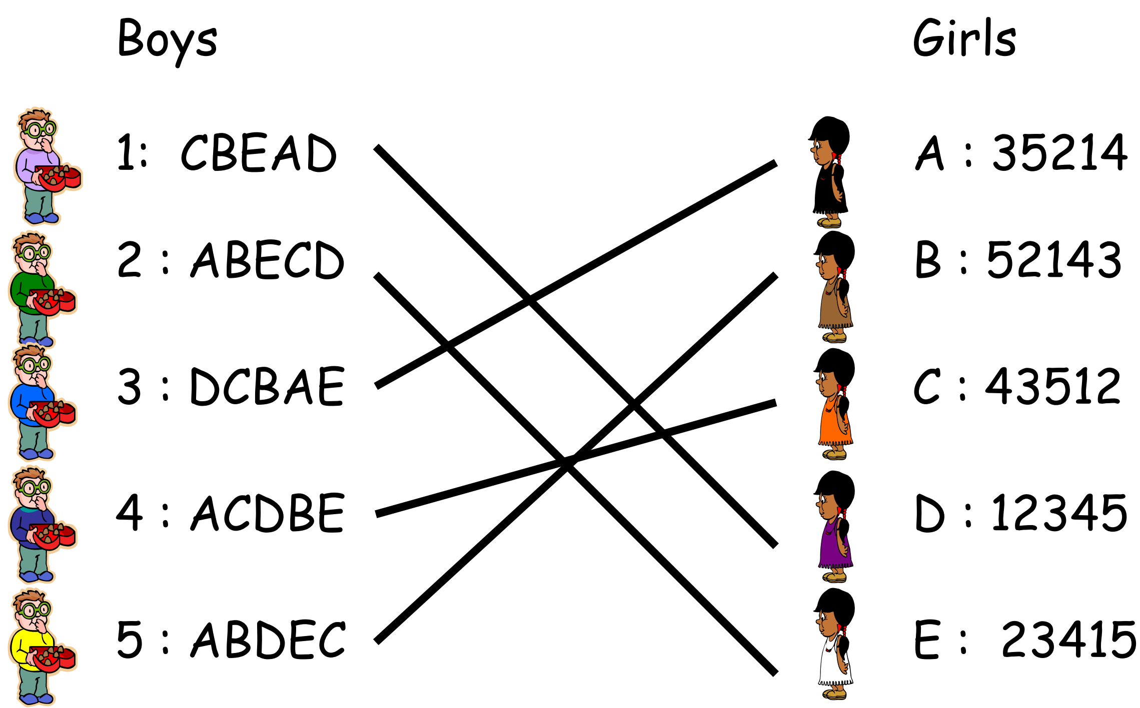
The proof is based on a marriage procedure…

Stable Matching

Why stable matching is easier than stable roommate?

Intuition: It is enough if we only satisfy one side!

This intuition leads us to a very natural approach.



The Marrying Procedure

Morning: boy propose to their favourite girl

Billy Bob

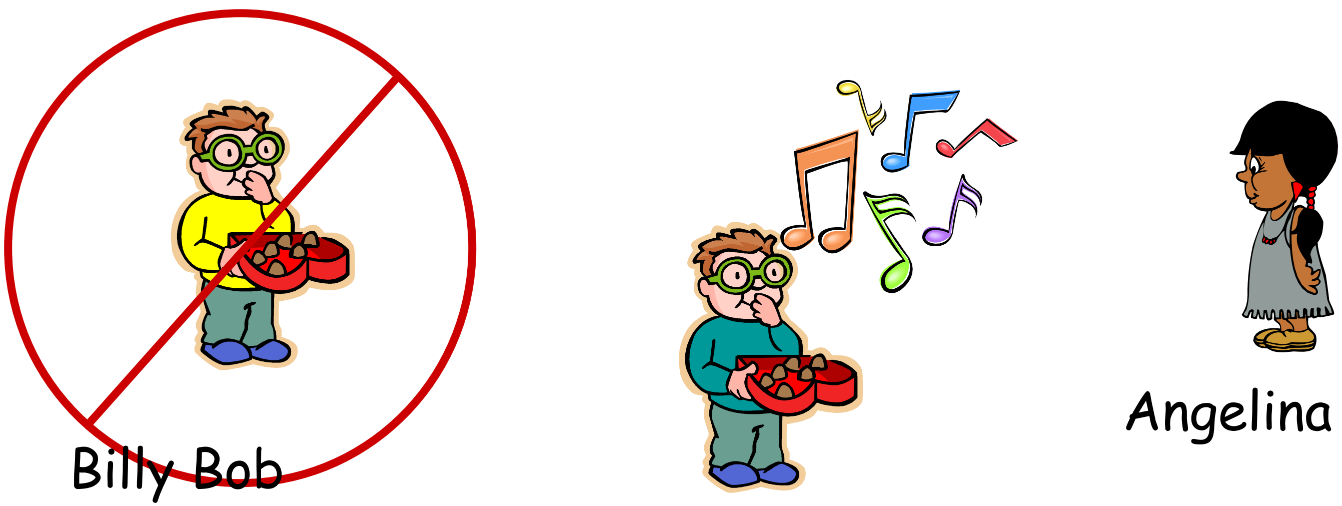
Brad

Angelina

The Marrying Procedure

Morning: boy propose to their favourite girl

Afternoon: girl rejects all but favourite (i.e. top suitor)



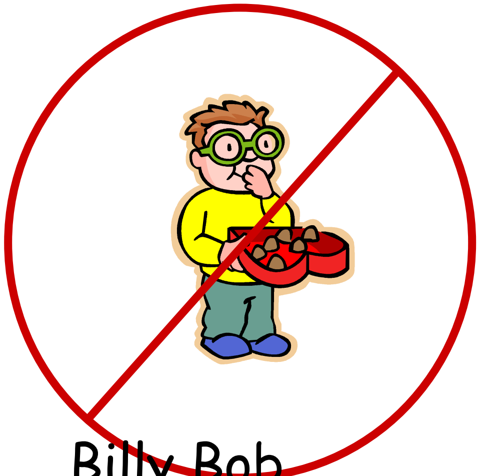
The Marrying Procedure

Morning: boy propose to their favourite girl

Afternoon: girl rejects all but favourite (i.e. top suitor)

Evening: rejected boy writes off the girl

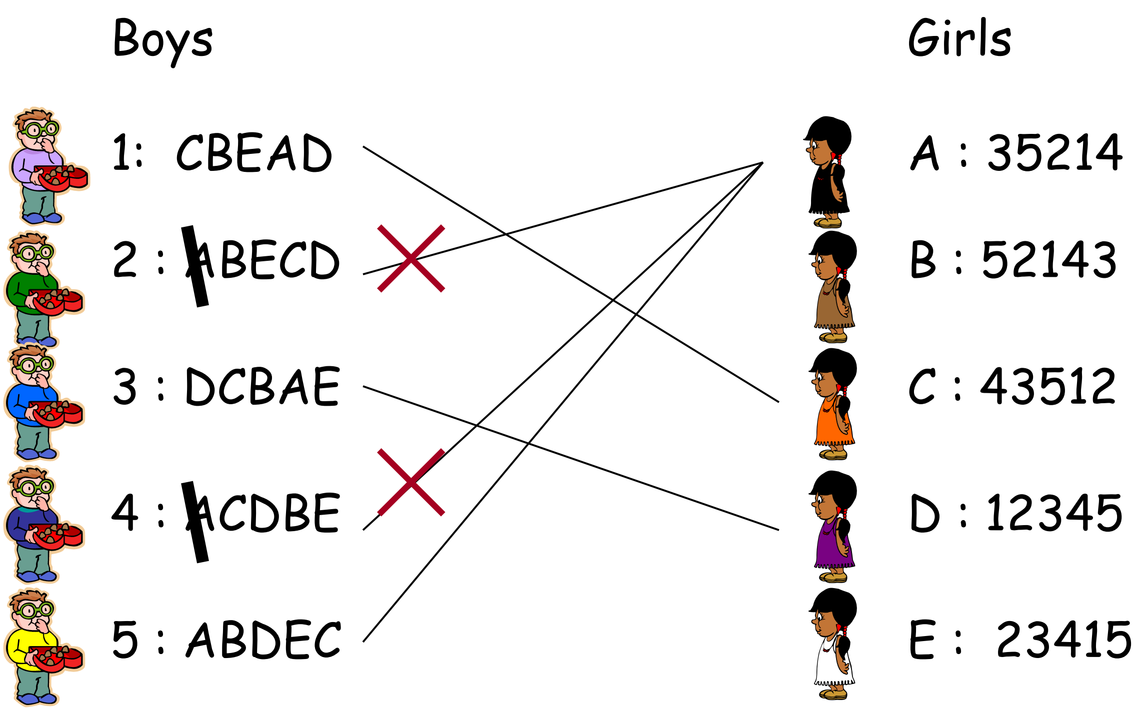
This procedure is then repeated until all boys propose to a different girl.



Billy Bob

Day 1

Morning: boy propose to their favourite girl
Afternoon: girl rejects all but favourite
Evening: rejected boy writes off the girl

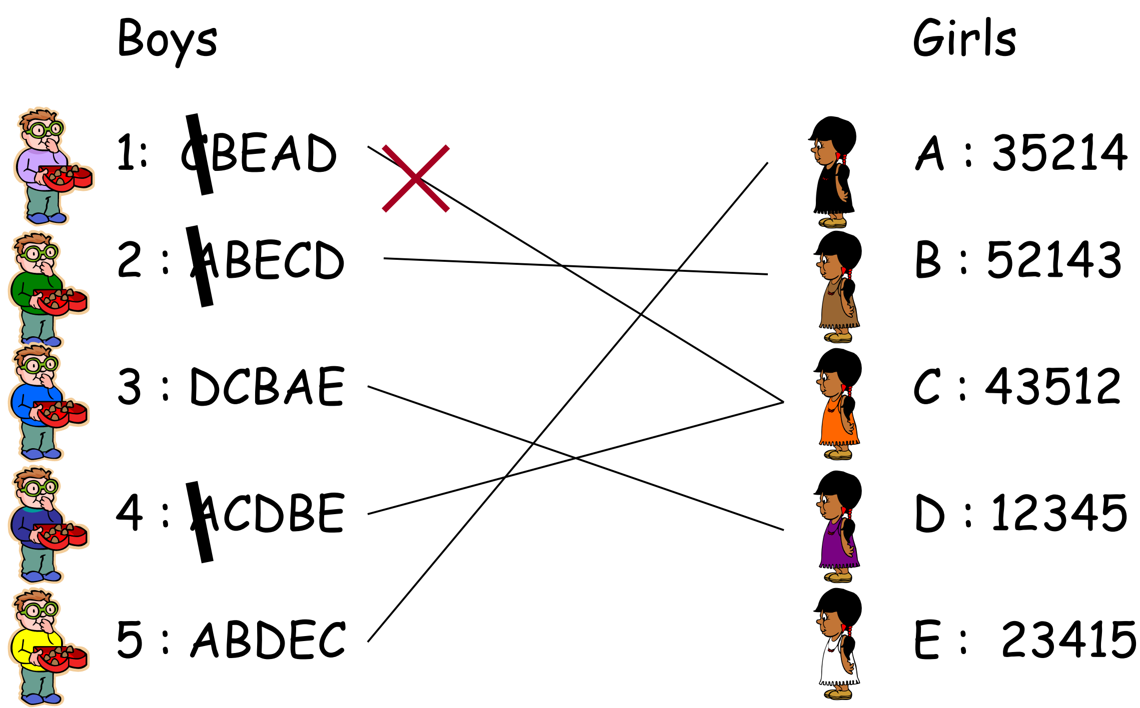


Day 2

Morning: boy propose to their favourite girl

Afternoon: girl rejects all but favourite

Evening: rejected boy writes off the girl

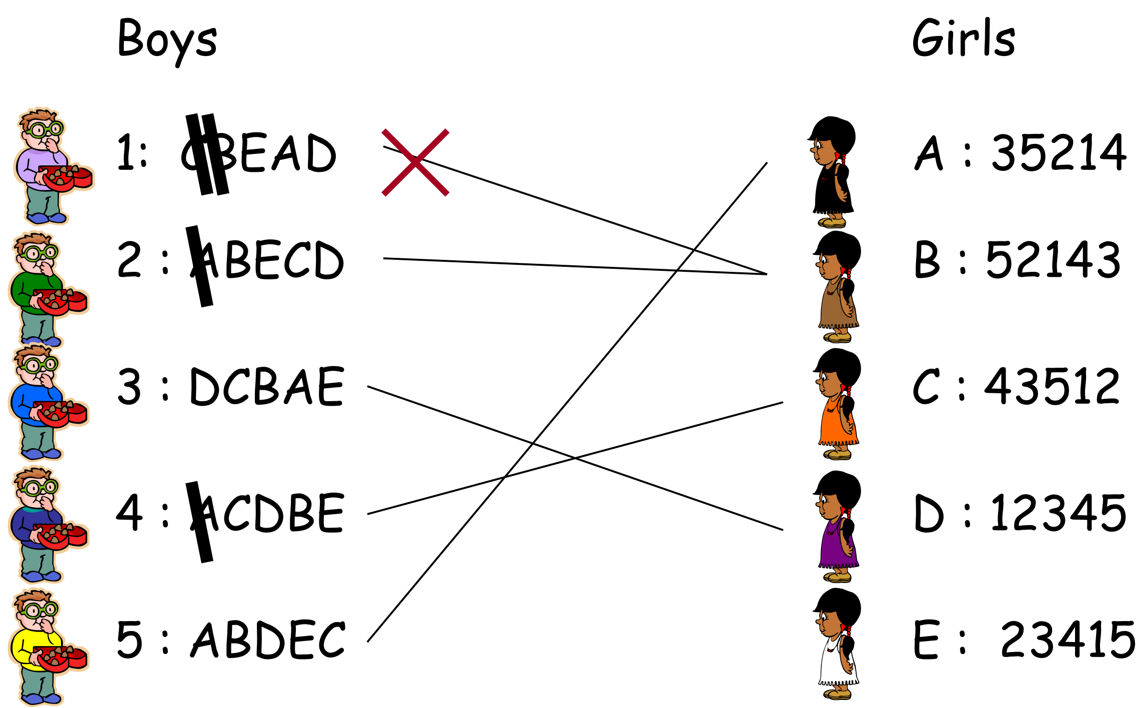


Day 3

Morning: boy propose to their favourite girl

Afternoon: girl rejects all but favourite

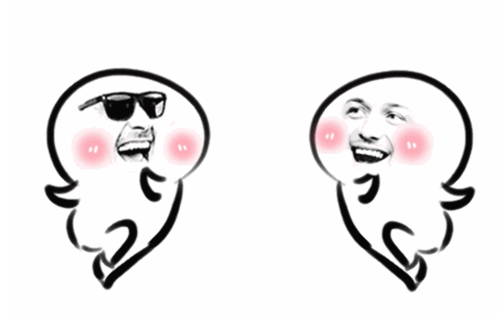
Evening: rejected boy writes off the girl



Day 4

Morning: boy propose to their favourite girl
Afternoon: girl rejects all but favourite
Evening: rejected boy writes off the girl

OKAY, marriage day!



Boys

1: ~~C~~~~B~~EAD
2 : ~~A~~BECD
3 : DCBAE
4 : ~~A~~CDBE
5 : ABDEC

Girls

A : 35214
B : 52143
C : 43512
D : 12345
E : 23415

Gale-Shapley Algorithm
(Sometimes called Deferred Acceptance Algorithm)

INITIALIZE M to empty matching

WHILE some man m is unmatched

  $w \leftarrow m$'s most preferred woman to whom m has not yet proposed

  IF w is unmatched

    Add (m, w) to matching M

  ELSE IF w is matched but prefers m to her current partner m'

    Replace (m', w) by (m, w) in matching M

  ELSE (i.e. w doesn't prefer m over her current partner m')

    w rejects m

RETURN stable matching M

(Note: the algorithm terminates after at most n^2 iterations of the WHILE loop)

Proof of Gale-Shapley Theorem

Gale,Shapley [1962]:

This procedure always finds a stable matching in the stable marriage problem.

What do we need to check?

1. The procedure will terminate.
2. Everyone is married.
3. No unstable pairs.

Step 1 of the Proof

Claim 1. The procedure will terminate in at most n^2 days.

1. If every girl is proposed by at most one boy,
   then all girls are married (since #boys = #girls),
   so the procedure will terminate.

2. Otherwise, there must be a girl receiving more
   than one proposal.

3. She will reject at least one boy in this case.
   These boys will write off that girl from their lists,
   and propose to their next favourite girls.

4. There are n boys whose list has n girls,
   and at least one name will be written off each day.
   So the procedure will last for at most n^2 days.

Step 1 of the Proof

May be tightened to n(n-1) + 1

- If no rejection is received, then the algorithm terminates, and as long as there are rejections, the algorithm will continue.
- In the worst case scenario, every man will need to propose n times, with exactly one man being rejected every day until the algorithm terminates

  ⇒ there are n men with at most (n–1) rejections each

  ⇒ there is at maximum n(n–1)days where a man gets rejected by a woman
- Now supposing that on the n(n–1)th day, the particular man x who gets rejected for the (n–1)th time, he still needs one more day to propose to the nth woman on his list
- This last proposal will not result in a rejection, since if that were the case x won't get matched

  ⇒ the last day will have no rejection which thus terminates the algorithm
- Therefore, the worst case scenario for the stable marriage algorithm is: the sum of the worst case number of days where a man gets rejected and the one day where no man gets rejected, or n(n–1)+1.

Step 2 of the Proof

Claim 2. Everyone is married when the procedure stops.

Proof (by contradiction).

1. Suppose B is not married and his list is empty.
2. Then B was rejected by all girls.
3. A girl rejects a boy only if she already has a more preferable partner.
4. So every girl has her own partner.
5. That is, all girls are married, but some boys are not.
6. This implies boys are more than girls, a contradiction.

Step 3 of the Proof

Claim 3. There is no unstable pair.

Consider any matched pair (B,G) at the end of the procedure.

Case 1: G is the first choice for B.

- Then B married to his most favourite.
- So B has no incentive to leave.

Case 2: G is not B's first choice.

- If (B,G') is unstable, then B prefers G' than G.
- So G' rejected B before.
- That is, G' prefers her current partner than B.
- So G' has no incentive to leave.

Stability: Detailed Proof

Assume (m, w) forms an unstable pair

Possibility 1: m never proposed to w

Since men propose in order of preferences
$\Rightarrow$ m prefers current partner w' over w
$\Rightarrow$ m and w are not unstable

Possibility 2: m proposed to w

since m and w are not matched to each other
$\Rightarrow$ w rejected m at some point, either right away or later
(since women only reject for better partners)
$\Rightarrow$ w prefers current partner m' over m
$\Rightarrow$ m and w are not unstable

Proof of Gale-Shapley Theorem

Gale,Shapley [1962]:

There is always a stable matching in the stable marriage problem.

Claim 1. The procedure will terminate in at most n^2 days.

Claim 2. Everyone is married when the procedure stops.

Claim 3. There is no unstable pair.

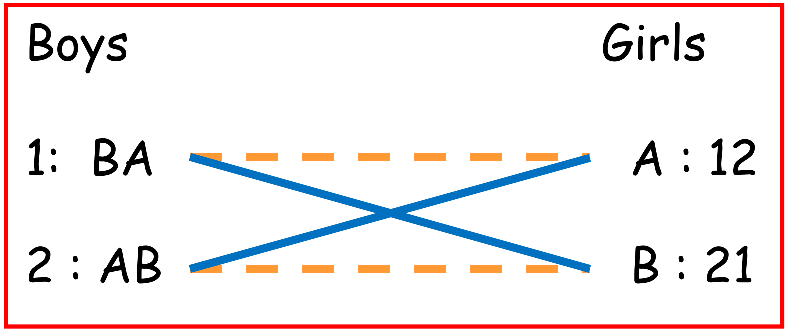
So the theorem follows.

More Questions

Gale,Shapley [1962]:

There is always a stable matching in the stable marriage problem.

Is the stable matching always unique when exists? NO!



- {(1,B), (2,A)} is a stable matching;
- {(1,A), (2,B)} is also a stable matching.

More Questions

Intuition: It is enough if we only satisfy one side!

Is this marrying procedure better for boys or for girls??

- All boys get the best partners!
- All girls get the worst partners!

That is, among all possible stable matchings, boys get the best possible partners, girls get the worst possible.

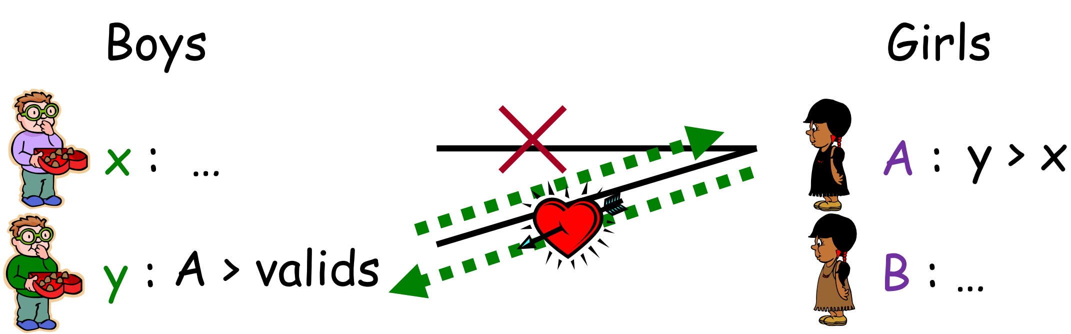
Why?

Boy Optimality

Claim 1. The marrying procedure is boy optimal.

Def. A girl G (or boy B) is valid for boy B (or girl G) if (B,G) is a couple in some stable matching.

- Suppose not ⇒ first boy x rejected by first valid girl A, while A married to y.
- What if (x,A) is a couple in another stable matching? (y,A) is unstable!



Boy Optimality – Detailed Proof

Claim 1. The marrying procedure is boy optimal.

- Men propose in order ⇒ at least one man was rejected by a valid partner
- Let m and w be the first such reject in S
- This happens because w chose some $m' > m$ (i.e. we have the prevailing pairing (m', w) in S)

  ⇒ *w prefers m' over m*
- Let S' be a stable matching with m, w paired (S' exists by definition of valid)
- Let w' be partner of m' in S'
- m' was not rejected by valid woman in S before m was rejected by w (by assumption), and due to the prevailing pairing of (m', w) in S, m' cannot have proposed to w' before proposing to w - otherwise we would have the prevailing pairing (m', w') in S instead at the time m was rejected by w

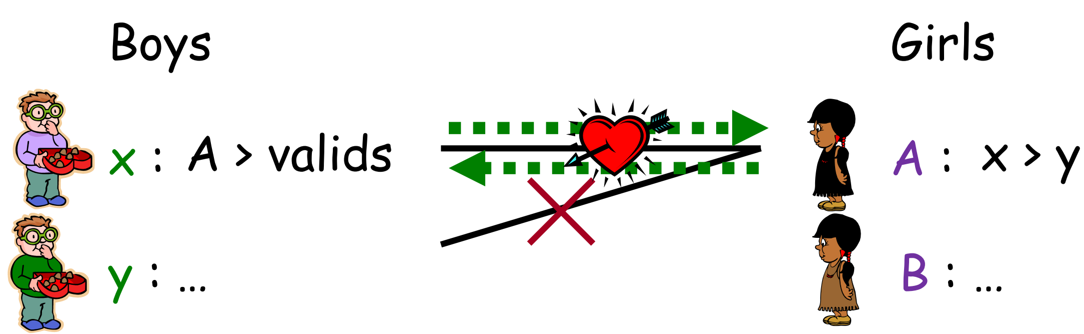
  ⇒ *m' prefers w to w'*
- Thus, m' and w form an unstable pair in S', a contradiction
- Thus, no man is rejected by a valid partner

Girl Pessimality

Claim 2. The marrying procedure is girl pessimal.

Def. A girl G (or boy B) is valid for boy B (or girl G) if (B,G) is a couple in some stable matching.

- Suppose not ⇒ boy x married to A, while A has a worse valid partner y.
- What if (y,A) is a pair in another stable matching? (x,A) is unstable!

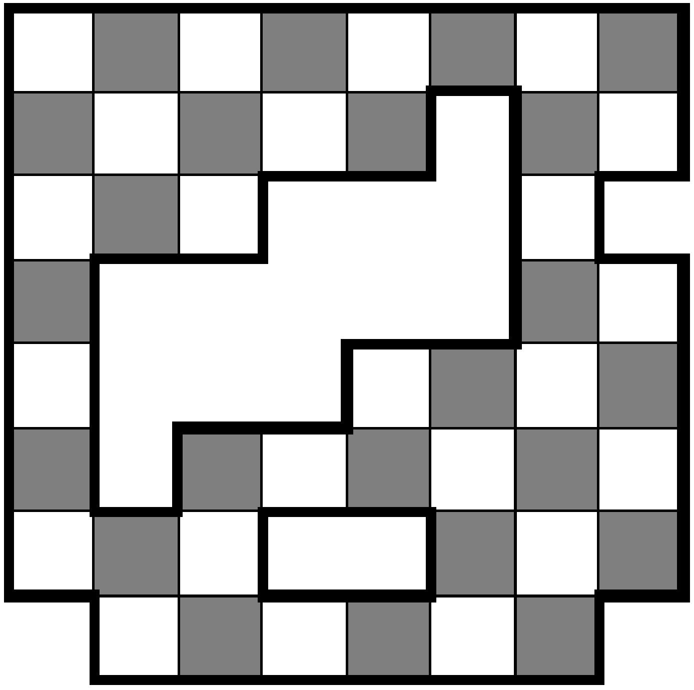


Girl Pessimality – Detailed Proof

Claim 2. The marrying procedure is girl pessimal.

- Let m and w matched in S, and m is not the worst valid partner for w
  $\Rightarrow$ exists stable S' with w paired to m' whom w prefers less than m, i.e.
  w prefers m over m'
- Let w' be partner of m in S'
- By Boy Optimality, w is the best valid partner for m
  $\Rightarrow$ *m prefers w over w'*
- Thus, m and w form an unstable pair in S', a contradiction that S' is stable
- Thus, there is no worse valid partner for woman

Bipartite Matching



This Lecture

In the last lecture we consider the stable matching problem.

Here we will study the bipartite matching problem.

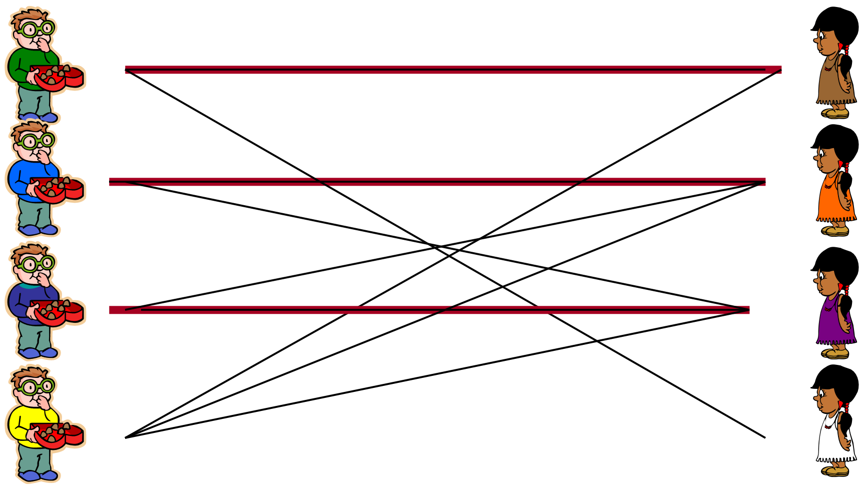
- Problem and Hall's theorem
- Reductions and Applications
- Proof of Hall's theorem (optional)

Bipartite Matching

The Bipartite Marriage Problem:

- There are n boys and n girls.
- Each boy/girl can only marry to some girls/boys.

Goal: to maximize the number of matched pairs.

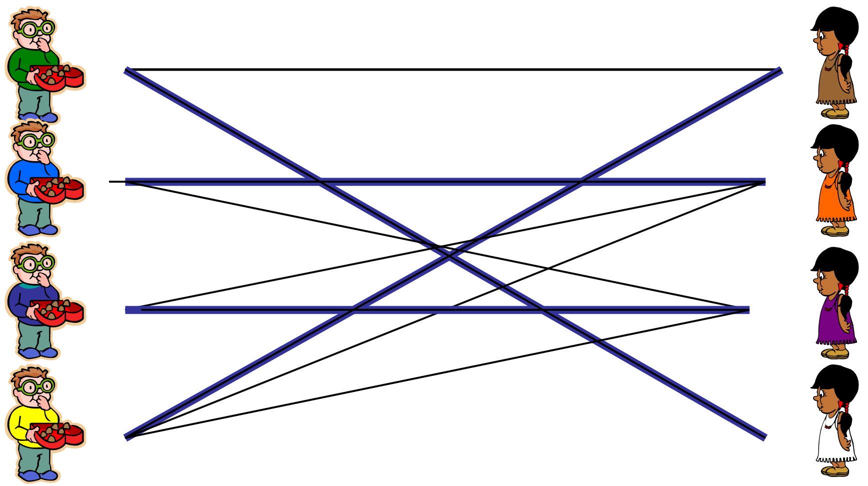


Bipartite Matching

The Bipartite Marriage Problem:

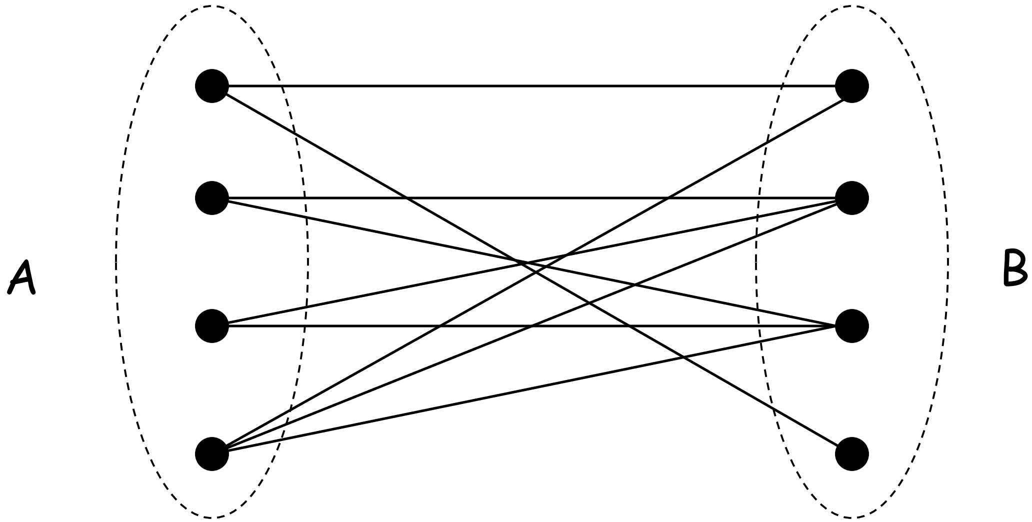
- There are n boys and n girls.
- Each boy/girl can only marry to some girls/boys.

Goal: to maximize the number of matched pairs.



Graph Problem

A graph is **bipartite** if its vertex set can be partitioned into two subsets A and B so that each edge has one endpoint in A and the other endpoint in B.

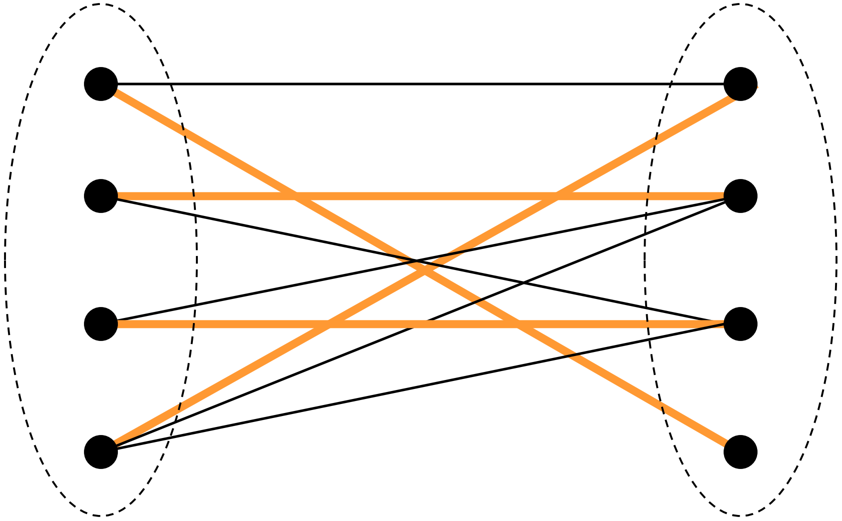


A **matching** is a subset of edges so that every vertex has degree at most one.

Maximum Matching

The bipartite matching problem:

Find a matching with the maximum number of edges.



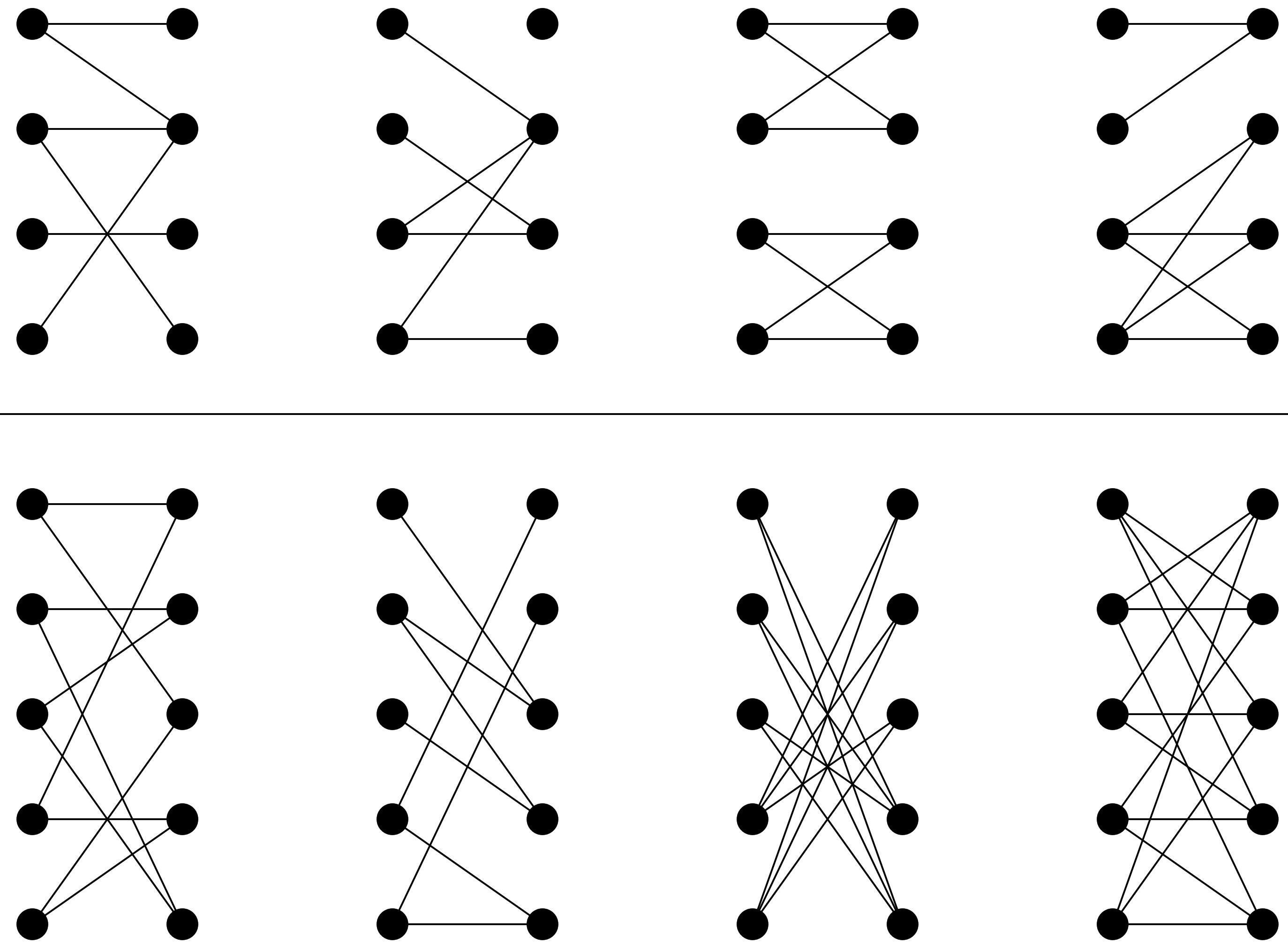
A **perfect matching** is a matching that every vertex is matched (i.e. of degree 1).

The perfect matching problem: Is there a perfect matching?

Once you know how to solve perfect matching, you can also do maximum matching.

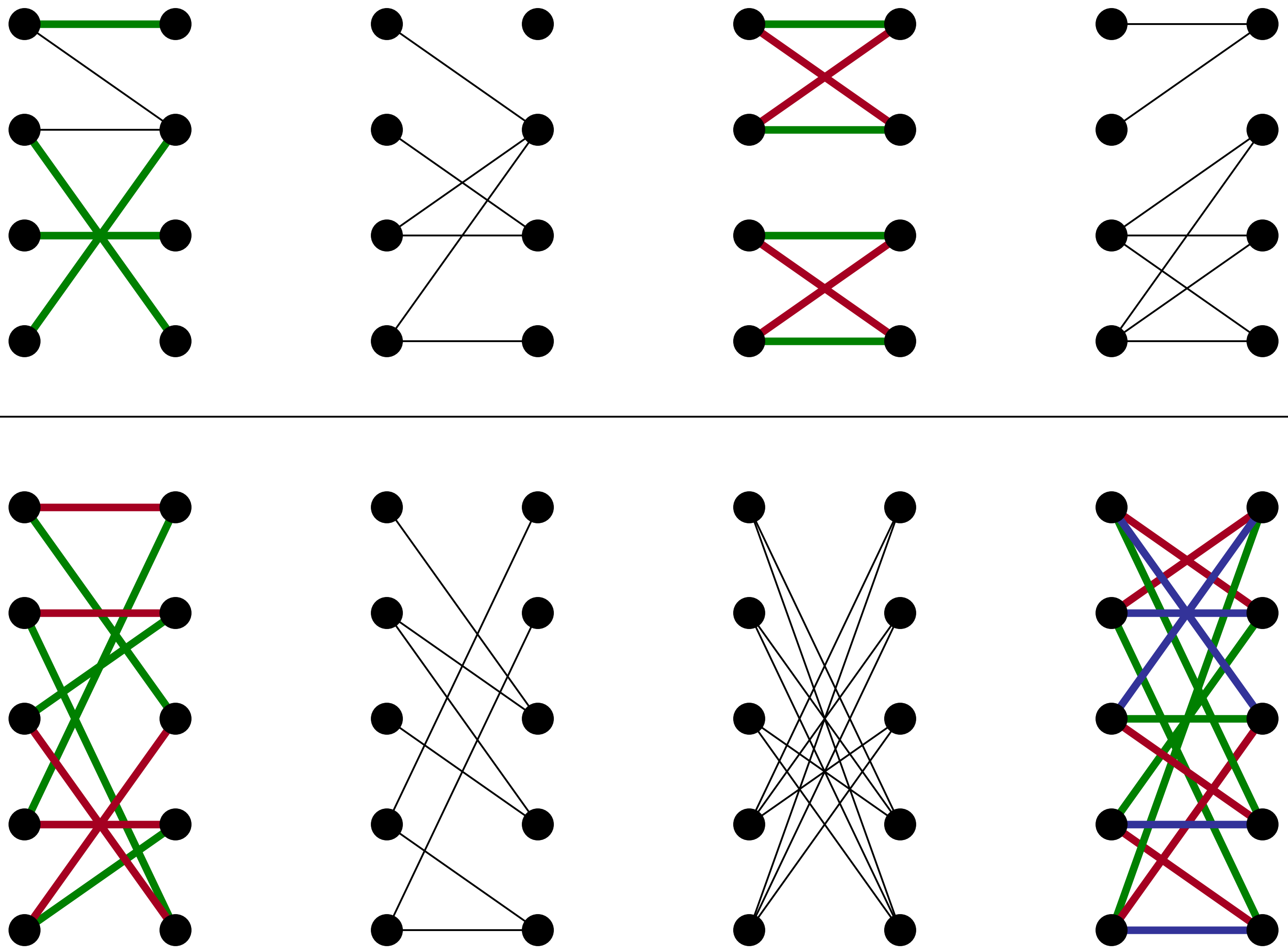
Examples

Which bipartite graph has a perfect matching?



Examples

Which bipartite graph has a perfect matching?



Working for the King

Suppose you work for the King, and your job is to find a perfect matching between 200 men and 200 women.

If there is a perfect matching, then you can show it to the King.

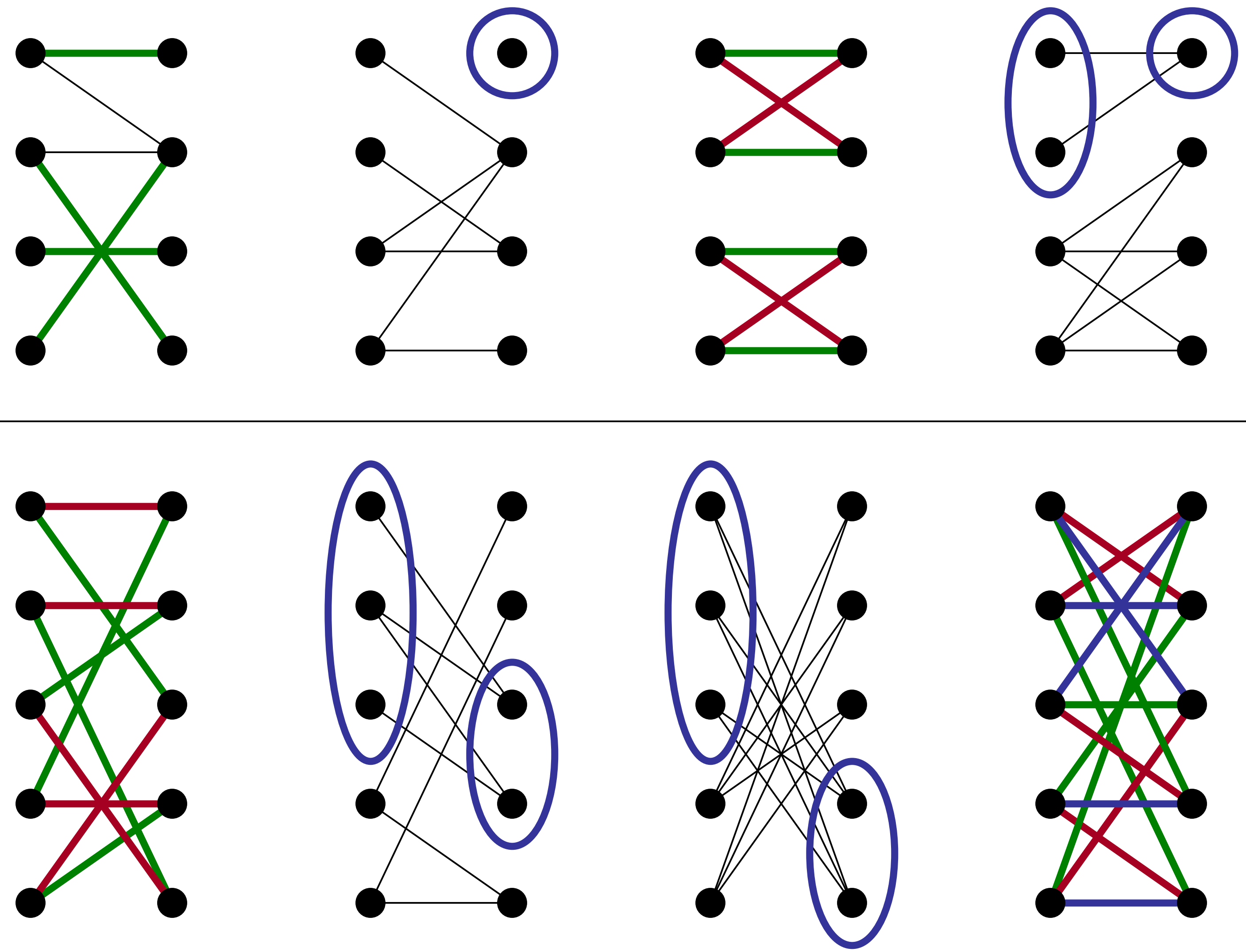
But suppose there is no perfect matching, how can you convince the King this fact (i.e. there is really no perfect matching, not because you are incompetent)?

One attempt is to try all the possibilities and show that none works, but you can imagine the King won't have the time and patience for that.

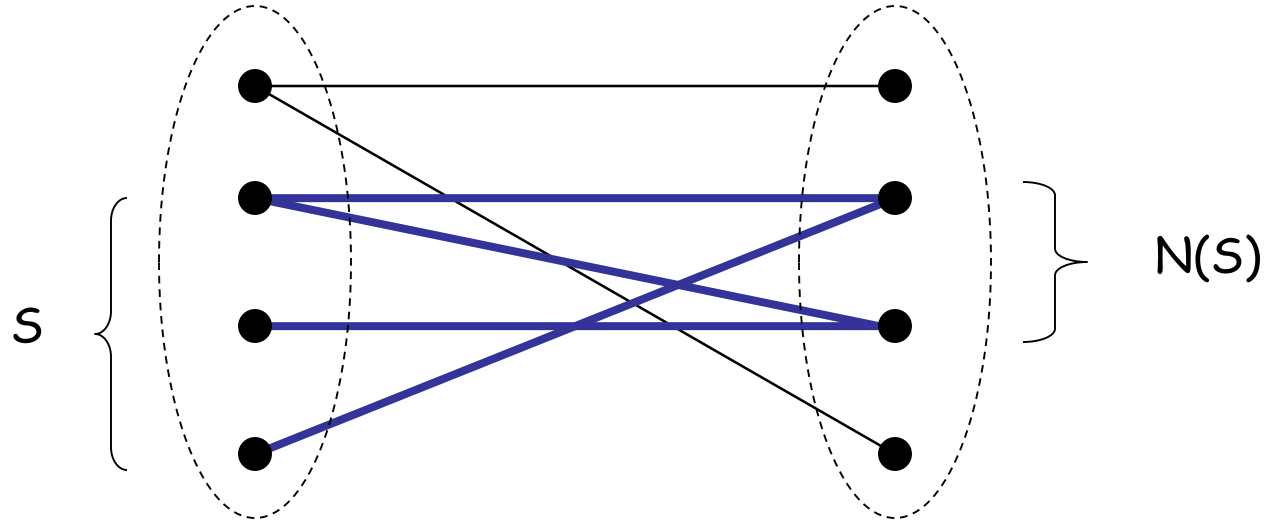
Is there a smarter way? It is difficult to argue that no solution exists.

Examples

Which bipartite graph has a perfect matching?



Some Notations



Let S be a subset of vertices of a graph. (e.g. a bipartite graph)

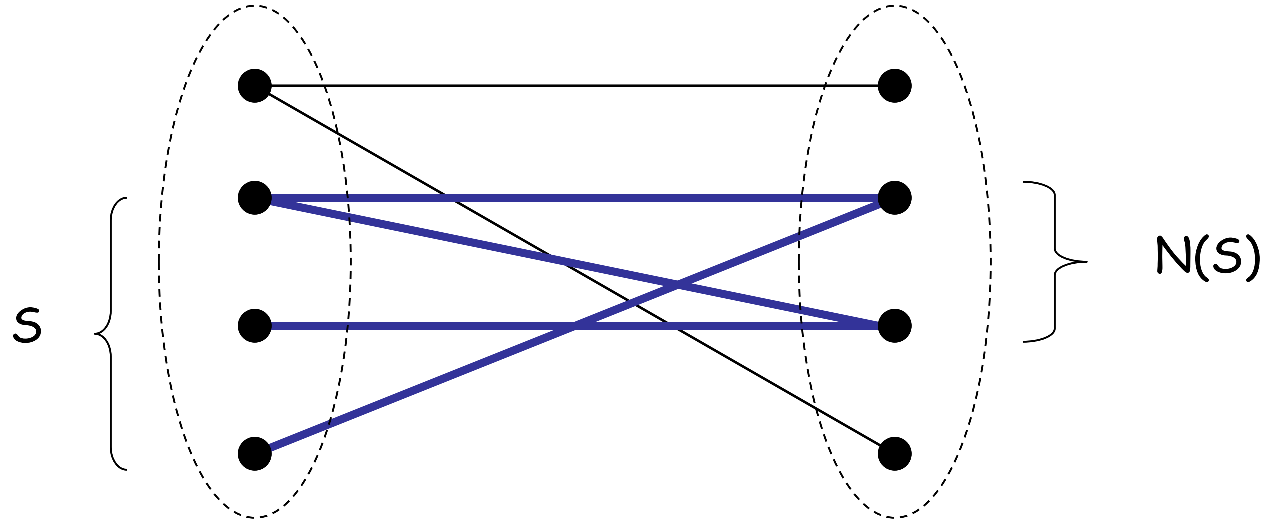
We denote the ***neighbor set*** of S by

N(S) = { v | v is a neighbor of some vertex in S}.

So |S| is the number of vertices in S,

and |N(S)| is the number of neighbors of S.

A Necessary Condition



If $|N(S)| < |S|$ for some S, then it is impossible to have a perfect matching.

In other words, in order to have a perfect matching,
a necessary condition is that for **each** subset S on either side,
we must have $|N(S)| \geq |S|$.

Hall's Theorem

Is this the only case that a bipartite graph does not have a perfect matching?

Hall said yes in 1935.

Hall's Theorem. A bipartite graph $G=(V,W;E)$ has a perfect matching **if and only if** $|N(S)| \geq |S|$ for each subset S of V and for each subset S of W.

This is a deep theorem.
It characterizes when exactly a bipartite graph does not have a perfect matching.
(Now you can convince the king.)

This Lecture

- Problem and Hall's theorem
- Reductions and Applications
- Proof of Hall's theorem (optional)

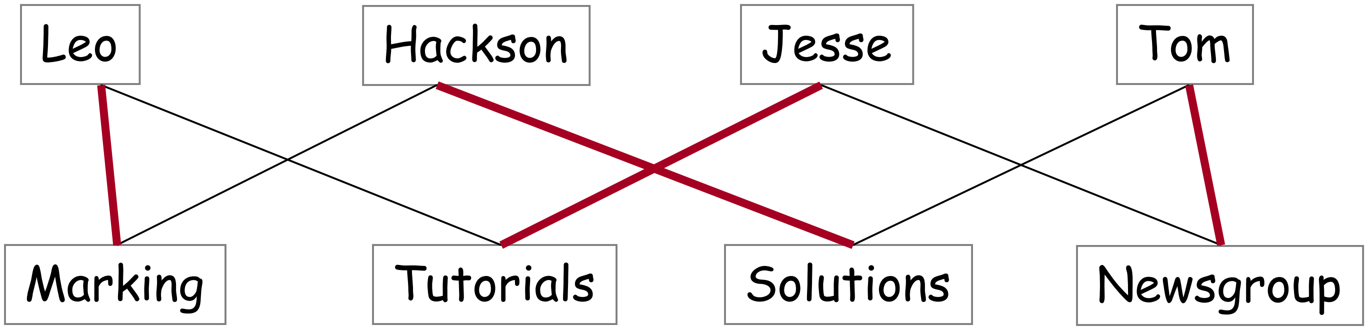
Application 1: Job Assignment

Job Assignment Problem:

There are n persons and n jobs.

Each person is willing to do a subset of jobs.

Can you find an assignment so that all jobs are taken care of while everyone is responsible for at most one job?



A perfect assignment corresponds to a perfect matching.

Application 1: Job Assignment

Job Assignment Problem:

There are n persons and n jobs.

Each person is willing to do a subset of jobs.

Can you find an assignment so that all jobs are taken care of while everyone is responsible for at most one job?

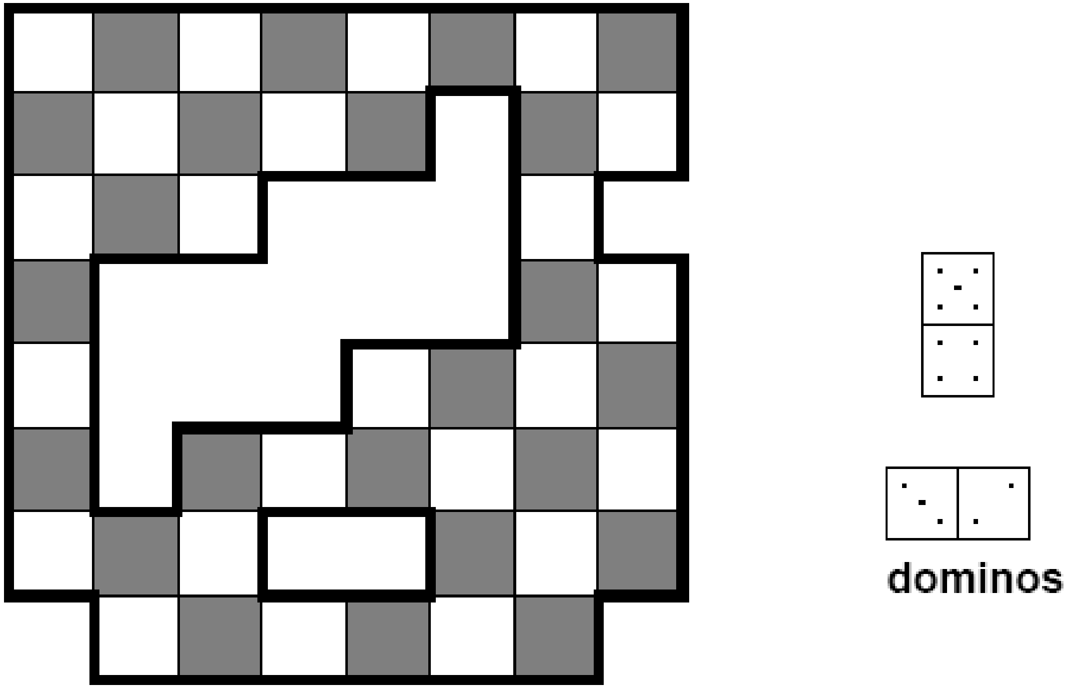
We can model the job assignment problem as a bipartite matching problem.

We create a vertex for each person, and we create a vertex for each job.

If a person is willing to do the job, then we add an edge between them.

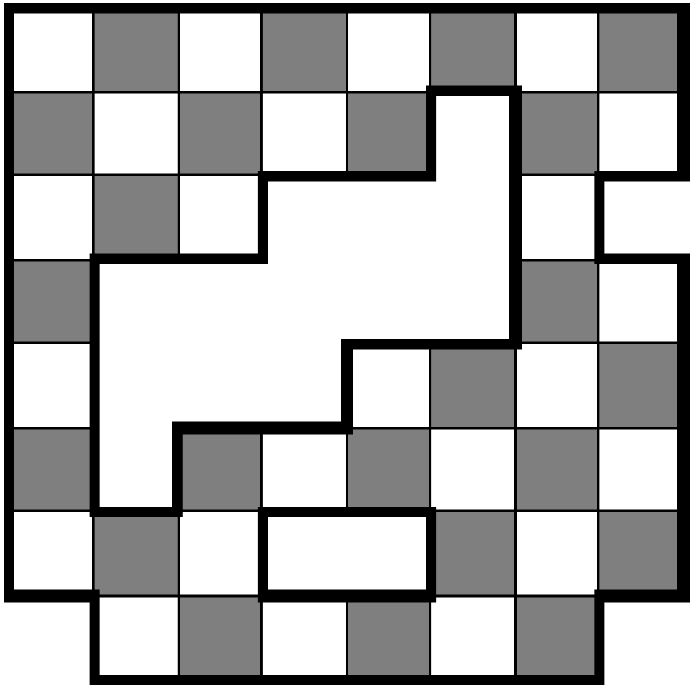
Then a perfect matching corresponds to a perfect "assignment".

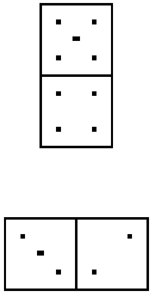
Application 2: Domino Puzzle



Can you fill an (irregular) chessboard perfectly with dominos?

Application 2: Domino Puzzle





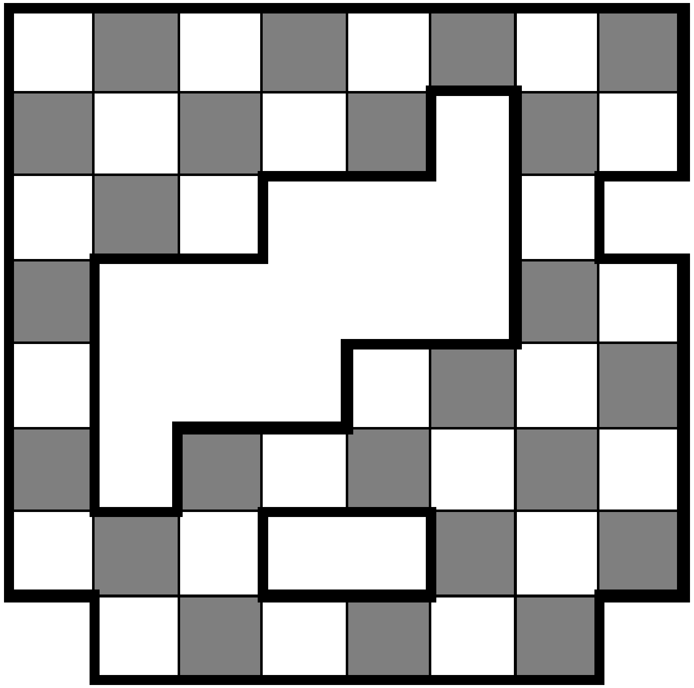
Create a vertex for each square in the board.

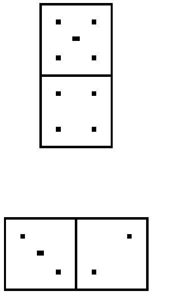
Add an edge between two squares if they are adjacent.

This is a bipartite graph with the black and white vertices.

A perfect matching in this graph corresponds to a placement plan of dominos.

Application 2: Domino Puzzle





This is another example where we can model a problem as a graph problem.

Application 3: Partial Latin Square

Latin Square: an nxn square, the goal is to fill the square with numbers from 1 to n so that:

- Each row contains every number from 1 to n.
- Each column contains every number from 1 to n.

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

So, all numbers in each row/column must be distinct.

Application 3: Partial Latin Square

Suppose you are given a partial Latin Square which some rows have been filled.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

Can you always extend it to a full Latin Square?

Yes, we can prove this using bipartite matching.

The proof consists of two steps:

(1) Interpret the problem as a bipartite matching problem.

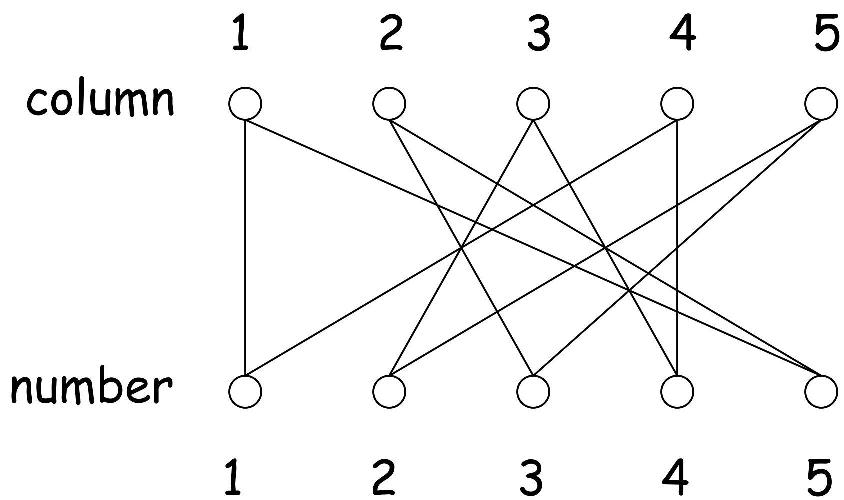
(2) Prove that a solution always exists by Hall's theorem.

Step 1: Interpreting as Bipartite Matching

First, we interpret the problem as a bipartite matching problem.

Given a partial Latin square, we construct a bipartite matching to <u>fill the next row</u>.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4



We want to "match" the numbers with the columns.

Create one vertex for each column and each number.

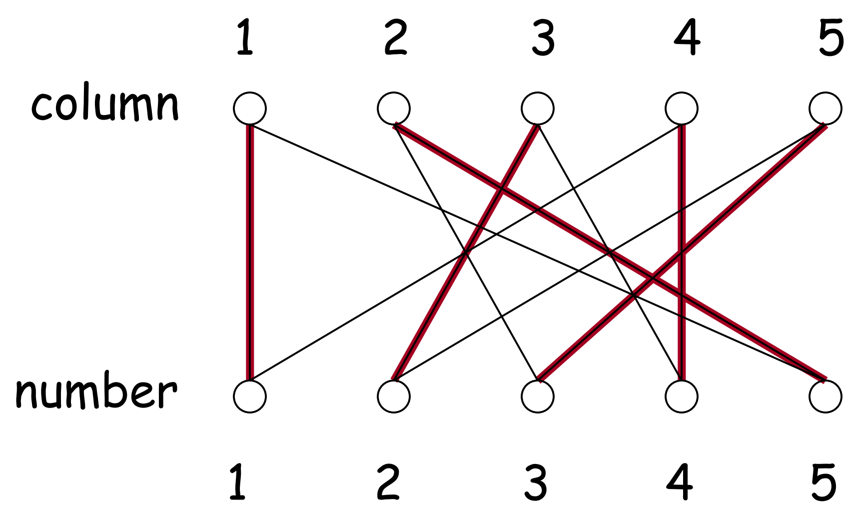
Add an edge between column i and number j if number j can be put in column i.

Step 1: Interpreting as Bipartite Matching

First, we interpret the problem as a bipartite matching problem.

Given a partial Latin square, we construct a bipartite matching to fill the next row.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4
1	5	2	4	3



Each perfect matching corresponds to a valid assignment of the next row.

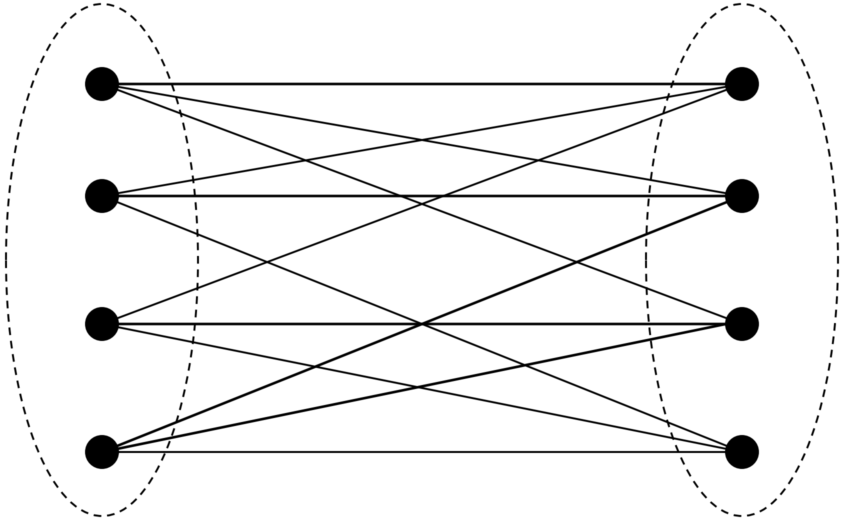
If we can always complete the next row, then we are done by induction.

The key is to prove that these bipartite graphs always have perfect matchings.

Step 2: Using Hall's Theorem

Hall's Theorem. A bipartite graph G=(V,W;E) has a perfect matching if and only if $|N(S)| \geq |S|$ for every subset S of V and W.

A graph (not necessarily bipartite) is **k-regular** if every vertex is of degree k.

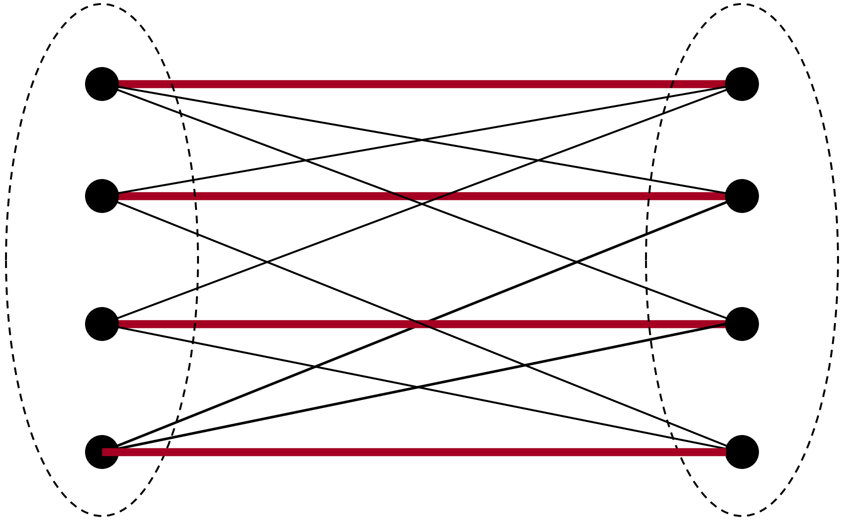


A 3-regular bipartite graph

Step 2: Using Hall's Theorem

Hall's Theorem. A bipartite graph G=(V,W;E) has a perfect matching if and only if $|N(S)| \geq |S|$ for every subset S of V and W.

Claim. Every k-regular bipartite graph has a perfect matching.



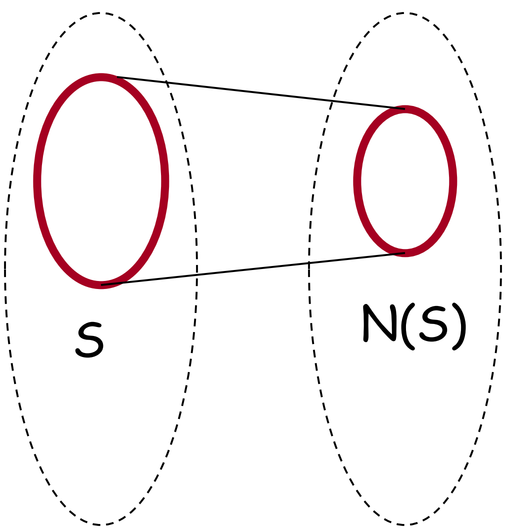
A 3-regular bipartite graph

Step 2: Using Hall's Theorem

Hall's Theorem. A bipartite graph G=(V,W;E) has a perfect matching if and only if $|N(S)| \geq |S|$ for every subset S of V and W.

Claim. Every k-regular bipartite graph has a perfect matching.

To prove this claim by Hall's theorem, we need to verify $|N(S)| \geq |S|$ for each subset S.



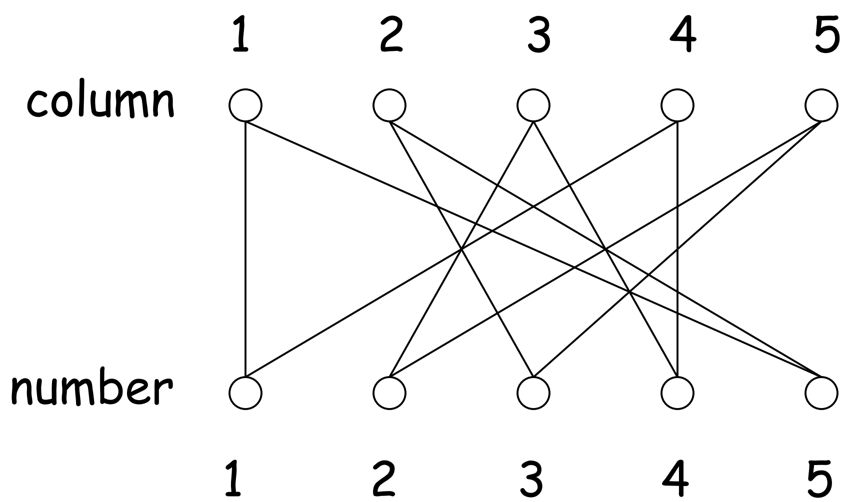
Proof by contradiction:

1. Suppose there is a subset S with $|S| > |N(S)|$.
2. Each edge connects S also connects N(S).
3. There are totally $k|S|$ edges connecting S.
4. But #edges connecting N(S) = $k|N(S)| < k|S|$.
5. A contradiction.

Completing Latin Square

Claim. Every k-regular bipartite graph has a perfect matching.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4



The bipartite graphs arising from Latin square are always regular because:

Suppose there are k unfilled rows.

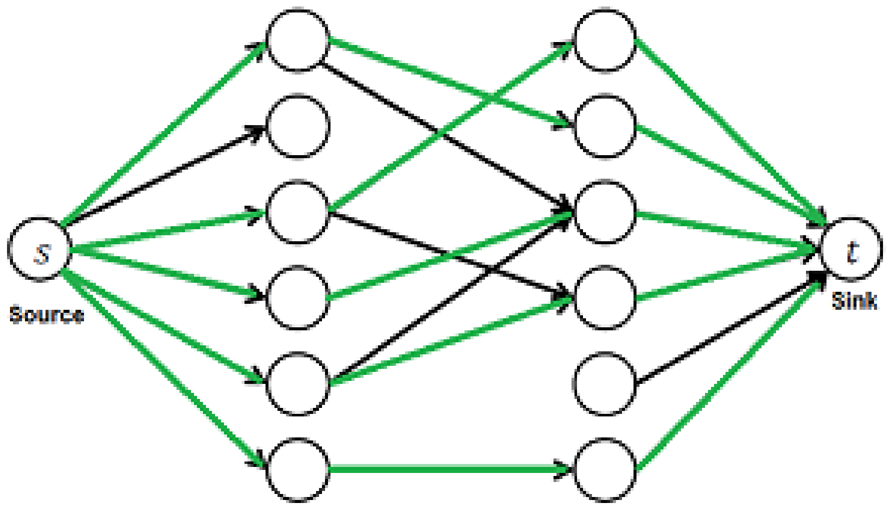
Then each column needs to fill k more numbers, so connects to k numbers.

Each number needs to be filled in k columns below, so connects to k columns.

So, the bipartite graph is k-regular, and thus always has a perfect matching.

More Applications (Optional)

One important application of the bipartite matching problem is the "maximum flow problem".



The maximum flow from source to sink is five units. Therefore, maximum five people can get jobs.

We need to find the maximum flow from source to sink.

Assume each edge has capacity 1 unit.

Then a maximum flow corresponds to a bipartite matching between applicants and jobs.

This Lecture

- Problem and Hall's theorem
- Reductions and Applications
- Proof of Hall's theorem (optional)

Hall's Theorem

Hall's Theorem. A bipartite graph $G=(V,W;E)$ has a perfect matching if and only if $|N(S)| \geq |S|$ for every subset S of V and W.

- If $S = V$, then $|V| \leq |N(V)| \leq |W|$.
- If $S = W$, then $|W| \leq |N(W)| \leq |V|$.

$\Rightarrow$ $|V| = |W|$

So the theorem can be restated as:

Hall's Theorem. A bipartite graph $G=(V,W;E)$ with $|V|=|W|$ has a perfect matching if and only if $|N(S)| \geq |S|$ for every subset S of V.

Proof of Hall's Theorem (easy direction)

Hall's Theorem. A bipartite graph G=(V,W;E) with |V|=|W| has a perfect matching if and only if |N(S)| ≥ |S| for every subset S of V.

The forward direction is easy: if there is a perfect matching, then |N(S)| ≥ |S| for every subset S of V.

Just consider the neighbors of S in a perfect matching.

Proof of Hall's Theorem (difficult direction)

Hall's Theorem. A bipartite graph $G=(V,W;E)$ with $|V|=|W|$ has a perfect matching if and only if $|N(S)| \geq |S|$ for every subset S of V.

The other direction is more interesting…

Hall's Claim. If $|N(S)| \geq |S|$ for every subset S of V, then there is a perfect matching.

How to prove such a statement?

Proof by strong induction on the number of edges m.

P(m): Hall's claim holds for all graphs $G=(V,W;E)$ with $|E| \leq m$.

Proof of Hall's Claim

Hall's Claim. If $|N(S)| \geq |S|$ for every subset S of V, then there is a perfect matching.

Base case: $m = 0$.

- There is some subset S of V such that $0 = |N(S)| < |S|$.
  (of course also for all S, but we don't need it)
- This also holds when the graph has no perfect matching.
- So no perfect matching implies $|N(S)| < |S|$ for some subset S.
- Then its contrapositive also holds.

Proof of Hall's Claim

Hall's Claim. If $|N(S)| \geq |S|$ for every subset S of V, then there is a perfect matching.

Inductive step: Assume $P(m)$ true. Consider $P(m+1)$.

Case 1: Every proper subset S has $|N(S)| > |S|$. (Easy case)

① Just delete an edge, and $|N(S)|$ will decrease by at most 1.

② Since $|N(S)| > |S|$ before,

③ we still have $|N(S)| \geq |S|$ after deleting an edge.

④ This is a smaller graph (with m edges).

⑤ By $P(m)$ there is a perfect matching in this smaller graph.

⑥ Hence there is a perfect matching in the original graph.

Proof of Hall's Claim

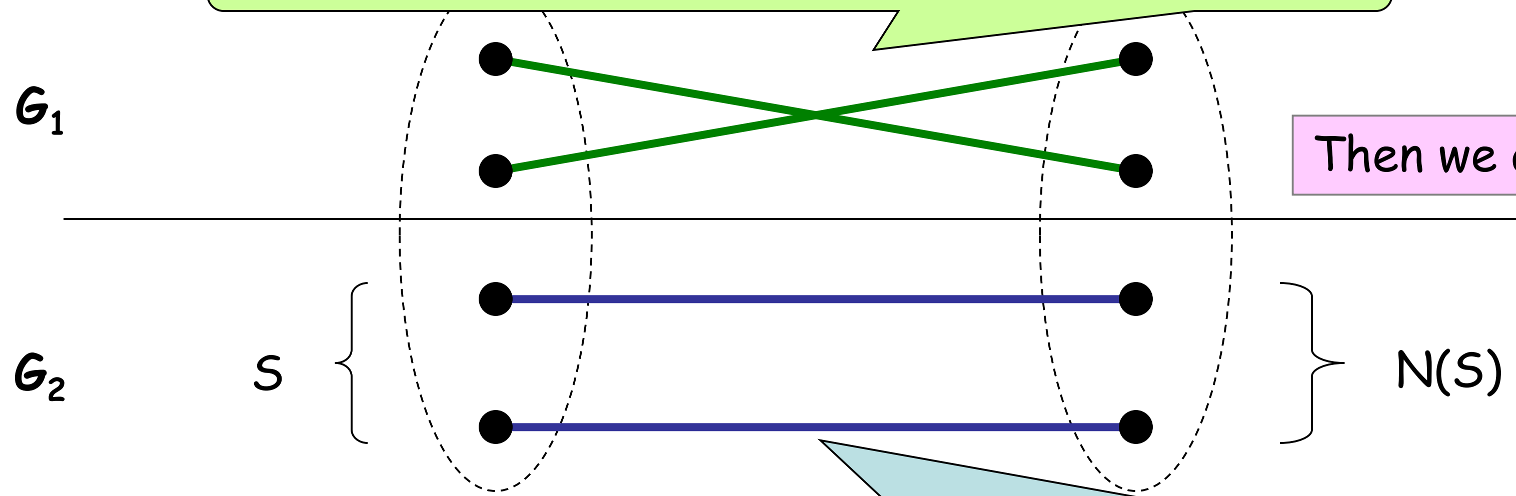
Hall's Claim. If $|N(S)| \geq |S|$ for every subset S of V, then there is a perfect matching.

Inductive steps:

Case 2: Suppose there is a proper subset S with $|N(S)| = |S|$.

Divide the graph into two smaller graphs G_1 and G_2 (so we can apply induction)

Find a perfect matching in G_1 by induction.



Then we are done.

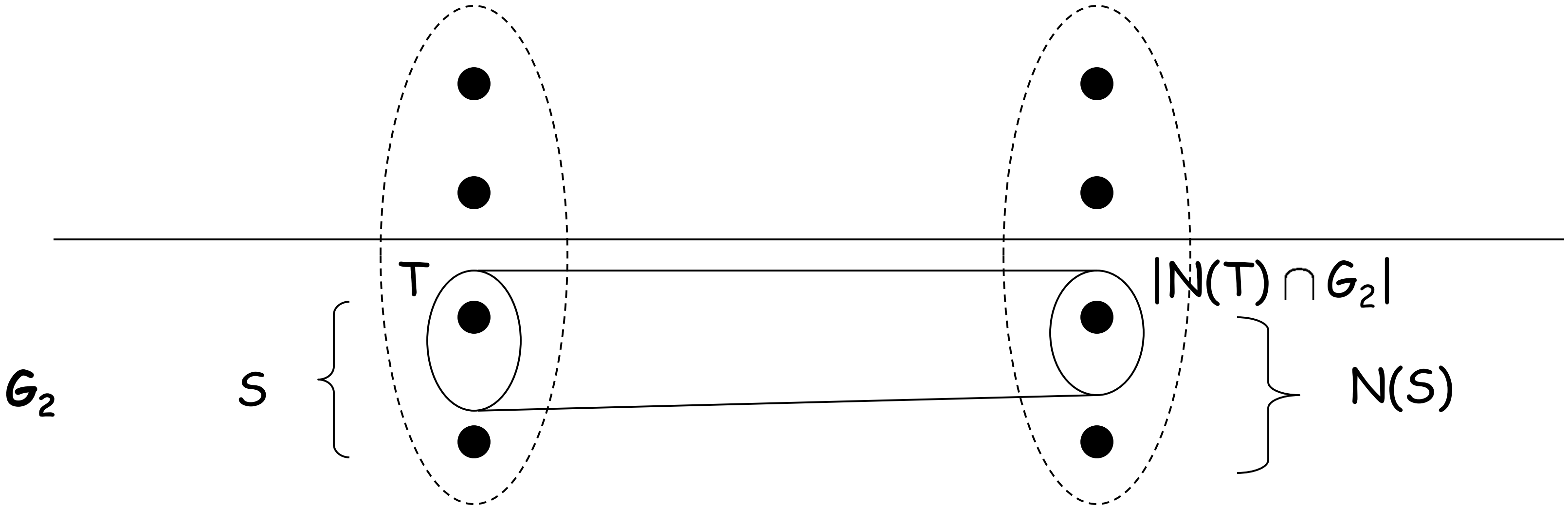
Find a perfect matching in G_2 by induction.

Proof of Hall's Claim

Why there is a perfect matching in G_2?

Hall's Claim. If $|N(S)| \geq |S|$ for every subset S of V, then there is a perfect matching.

For applying $P(m)$, we need to show $|N(T) \cap G_2| \geq |T|$ for any $T \subset S$.



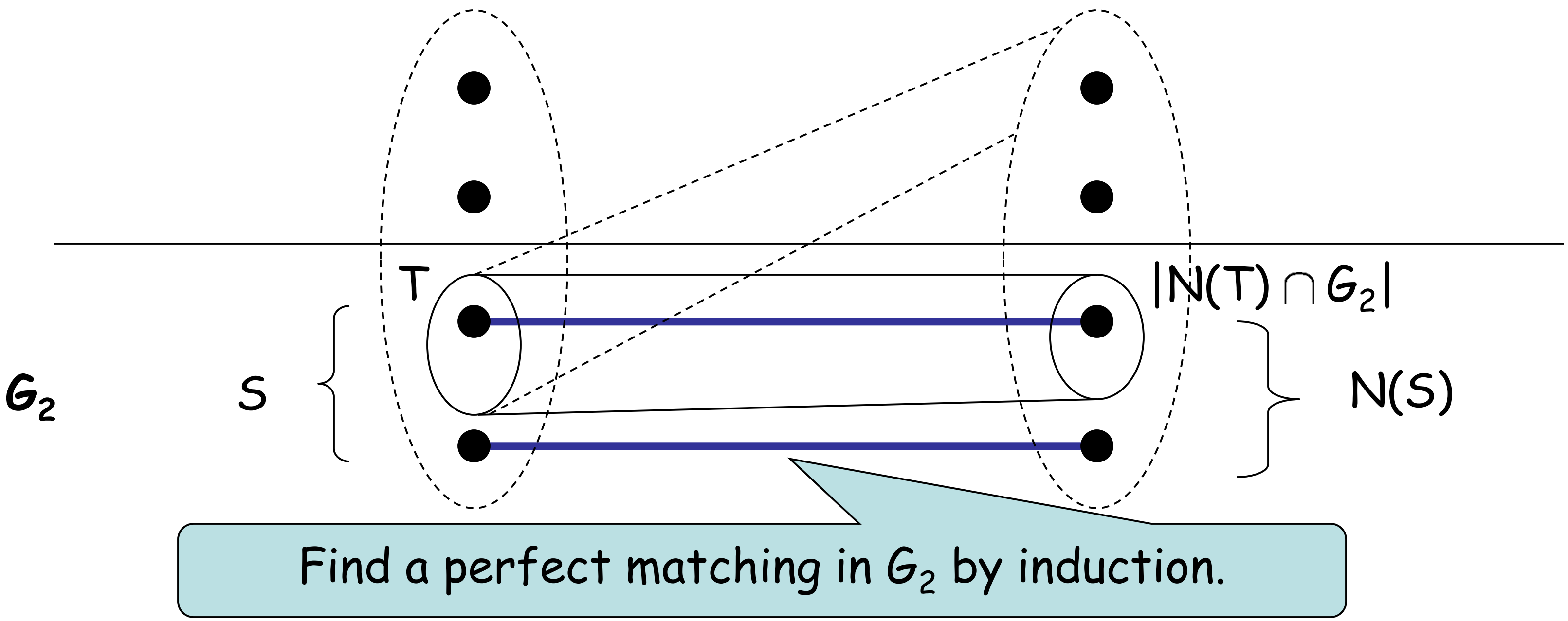
Proof of Hall's Claim

Why there is a perfect matching in G_2?

For any proper subset $T \subset S$, $N(T) \subseteq N(S)$ is in G_2.

Hence, $|N(T) \cap G_2| = |N(T)| \geq |T|$.

Therefore, by induction, there is a perfect matching in G_2.



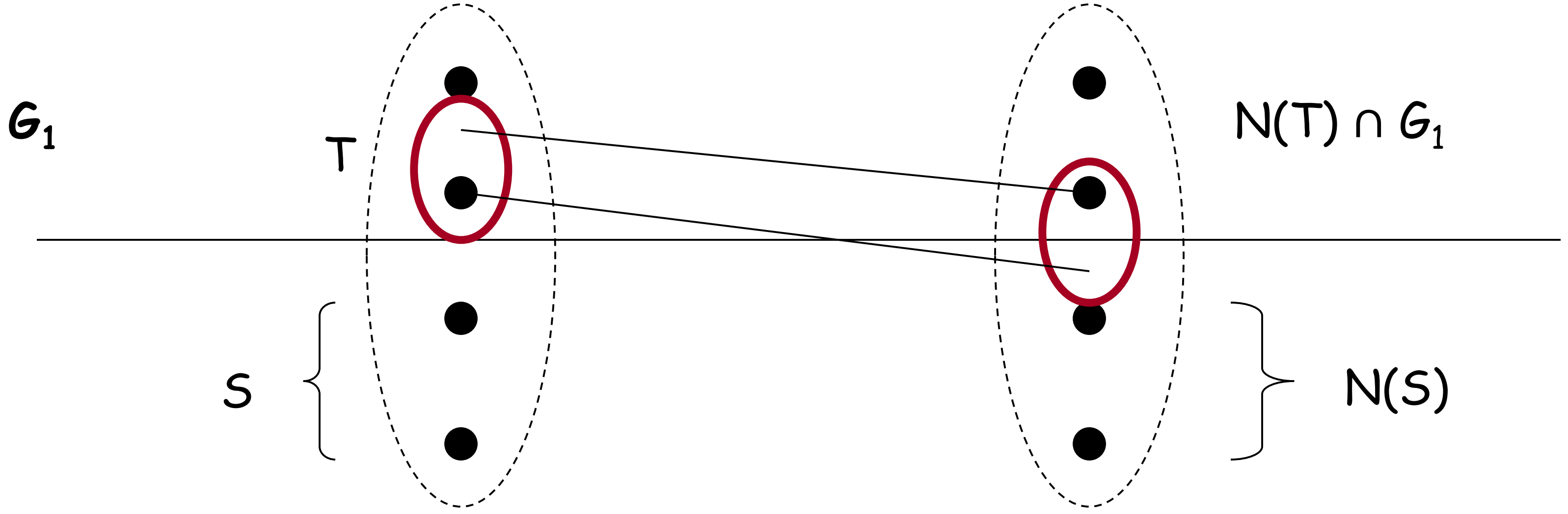
Proof of Hall's Claim

Why there is a perfect matching in G_1?

For any subset T in G_1, we hope $|N(T) \cap G_1| \geq |T|$ so that we can apply induction.

- Consider T, by assumption, $|N(T)| \geq |T|$.
- Can we conclude that $|N(T) \cap G_1| \geq |T|$?
- No, because N(T) might intersect N(S)!

Now what?

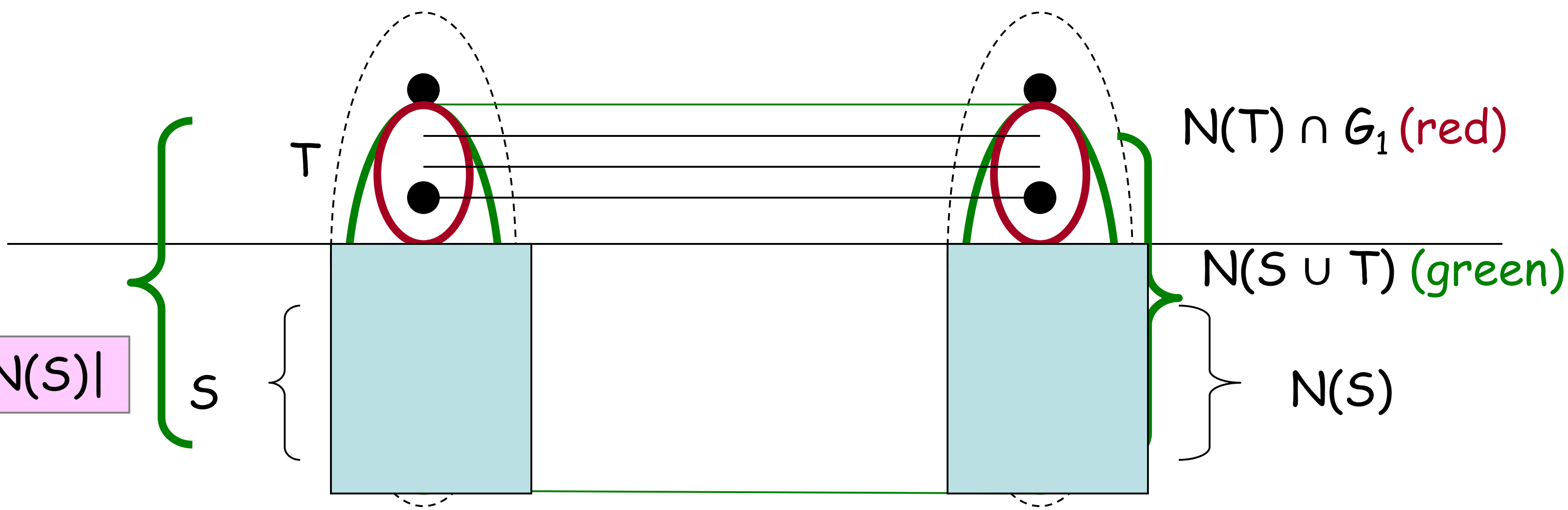


Proof of Hall's Claim

Why there is a perfect matching in G_1?

For any subset T in G_1, we hope $|N(T) \cap G_1| \geq |T|$ so that we can apply induction.

1. Consider $S \cup T$, by assumption, $|N(S \cup T)| \geq |S \cup T|$ (green areas).
2. Since $|S| = |N(S)|$, $|N(S \cup T) - N(S)| \geq |S \cup T - S|$ (red areas).
3. So $|N(T) \cap G_1| = |N(S \cup T) - N(S)| \geq |S \cup T - S| = |T|$, as desired.



|S|=|N(S)|

Proof of Hall's Claim

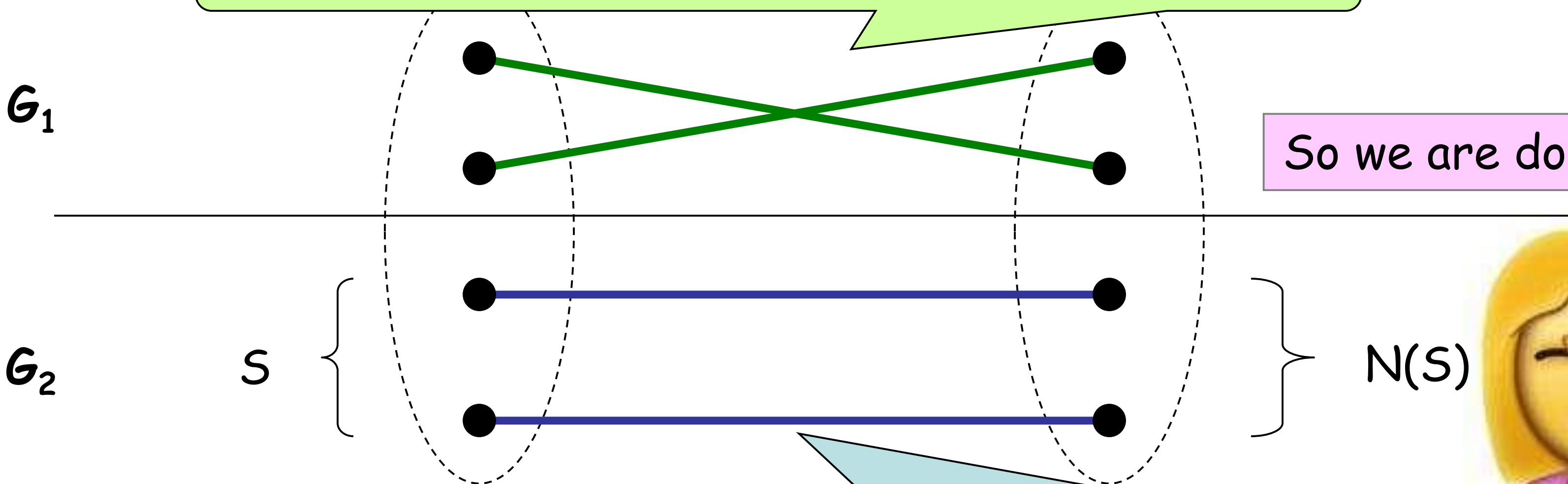
Hall's Claim. If $|N(S)| \geq |S|$ for every subset S of V, then there is a perfect matching.

Inductive steps:

Case 2: Suppose there is a proper subset S with $|N(S)| = |S|$.

Divide the graph into two smaller graphs G_1 and G_2 (so we can apply induction)

Find a perfect matching in G_1 by induction.



So we are done.

Find a perfect matching in G_2 by induction.

Bipartite Matching and Hall's Theorem

Hall's theorem is a fundamental theorem in graph theory.

In this course, it is important to learn

1. how to use bipartite matching to solve problems, and
2. how to apply Hall's theorem.

The proof of Hall's theorem is optional.