AMIRH Document

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The AMIRH project aims to reproduce the complex atmospheric phenomena in the atmosphere of terrestrial planets. It considers the radiation from the universe and the properties of the planet itself. This model does not take a specific planet as the hypothetical object. I look forward to exploring the relationship between the atmosphere and the planet. With the powerful computing power provided by the GPU, the model can support a very detailed resolution.

1 Where I start

The original idea for this project came from the barotropic atmospheric model proposed by Chung-Ying Hu in 1975. His model is based on,

- ⋆ Equation of motion
- ⋆ Equation of continuity
- ⋆ Equation of gas state

And these assumptions,

- ⋆ Non-divergent
- ⋆ Incompressible
- ⋆ No surface friction
- ⋆ Baroclinic effect

The master equation is,

$$\frac{\partial}{\partial t}\nabla^2 z = J(f+\zeta, z) \tag{1}$$

z is the air pressure at the isobaric surface, f is Coriolis parameter, $J(A,B) = \frac{\partial A}{x}\frac{\partial B}{y} - \frac{\partial A}{y}\frac{\partial B}{x}$. By applying the finite difference and the relaxation method:

$$(\frac{\partial z}{\partial t})^{n+1}_{i,j} = -\frac{1}{4}[(\frac{\partial z}{\partial t})_{i+1,j} + (\frac{\partial z}{\partial t})_{i-1,j} + (\frac{\partial z}{\partial t})_{i,j+1} + (\frac{\partial z}{\partial t})_{i,j-1} - \frac{d^2}{m^2}J(f+\zeta,z)_{i,j}]^n \tag{2}$$

Repeatedly calculate (2) until,

$$\mathrm{max}|(\frac{\partial z}{\partial t})^{n+10} - (\frac{\partial z}{\partial t})^n| \leq \varepsilon \tag{3}$$

d is the real distance corresponding to the grid points, ε is the lowest value for convergence, which Hu sets to 1.0×10^{-6} m/sec. If $t = 0$, by applying the Time-forward method, we could obtain the next z, when $t > 0$ by applying the Center-difference method:

$$\begin{cases} z^{t+\Delta t}_{i,j} = z^t_{i,j} + \Delta t(\frac{\partial z}{\partial t})^t_{i,j} & t = 0 \\ z^{t+\Delta t}_{i,j} = z^{t-\Delta t}_{i,j} + 2\Delta t(\frac{\partial z}{\partial t})^t_{i,j} & t > 0 \end{cases} \tag{4}$$

The boundary condition is Free slip boundary condition, which assumes that the atmosphere can flow freely at the boundary without being affected by frictional resistance,

$$\begin{cases} \frac{\partial v}{\partial x} = 0 \\ \frac{\partial u}{\partial y} = 0 \end{cases} \tag{5}$$

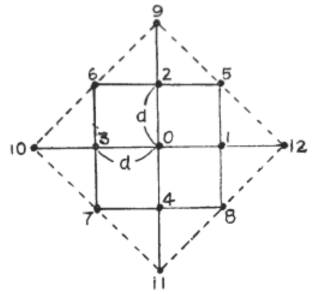

Figure 1: computation domain and grid system

Whihc could be expressed as,

$$z_{outsidetheboundary} - boundary = 2z_{boundary} - z_{insidetheboundary} - boundary \tag{6}$$

The initial condition is by applying several time slots of observations. The J above is solved by 12-points approximation, Fig.1 is the corresponding computation domain and grid system.

The law of conservation of energy is verified.

$$J(A,B) = 2J_1(A,B) - J2(A,B) \tag{7}$$

$$J_1(A,B) = \frac{1}{3}[J^{++}(A,B) + J^{+\times}(A,B) + J^{\times+}(A,B)] \tag{8}$$

$$J^{++}(A,B) = \frac{1}{4d^2}[(A_1 - A_3)(B_3 - B_4) - (A_2 - A_4)(B_1 - B_3)] \tag{9}$$

$$J^{+\times}(A,B) = \frac{1}{4d^2}[A_5(B_2 - B_1) - A_7(B_3 - B_4) + A_6(B_3 - B_2) - A_8(B_4 - B_1)] \tag{10}$$

$$J^{\times+}(A,B) = \frac{1}{4d^2}[A_1(B_5 - B_8) - A_3(B_6 - B_7) - A_2(B_5 - B_6) + A_4(B_8 - B_7)] \tag{11}$$

$$J_2(A,B) = \frac{1}{3}[J^{\times\times}(A,B) + J^{\times+}(A,B) + J^{+\times}(A,B)] \tag{12}$$

$$J^{\times\times}(A,B) = \frac{1}{8d^2}[(A_5 - A_7)(B_6 - B_8) - (A_6 - A_8)(B_5 - B_7)] \tag{13}$$

$$J^{+\times}(A,B) = \frac{1}{8d^2}[A_5(B_9 - B_{12}) - A_7(B_{10} - B_4) - A_6(B_9 - B_{10}) + A_8(B_{12} - B_{11})] \tag{14}$$

$$J^{\times+}(A,B) = \frac{1}{8d^2}[A_9(B_6 - B_5) - A_{11}(B_7 - B_8) + A_{10}(B_7 - B_6) - A_{12}(B_8 - B_5)] \tag{15}$$

Here $A_i, B_i (i = 1, ...12)$ is the corresponding value of A, B at i-th points.

2 3D model

As for the 3D model, the motion equations are,

$$\begin{cases} \frac{du}{dt} = \frac{uv\tan\phi}{r} - \frac{uw}{r} - \frac{1}{\rho}\frac{\partial p}{r\cos\phi\partial\lambda} + fv - \hat{f}\omega + F_\lambda\frac{d}{dt} + \frac{u^2}{\tan\phi}r \\ \frac{dv}{dt} = -\frac{u^2\tan\phi}{r} - \frac{vw}{r} - \frac{1}{\rho}\frac{\partial p}{r\partial\phi} - fu + F_\phi \\ \frac{d\omega}{dt} = \frac{u^2+v^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} - g + \hat{f}u + F_r \end{cases} \tag{16}$$

2.1 sigma vertial coordinate

A vertical coordinate for atmospheric models defined as pressure normalized by its surface value, or as the difference in pressure and its value at the top of the model atmosphere normalized by the surface value of this difference.
Thus,

$$\sigma = \frac{p - p_T}{p_S - p_T} \tag{17}$$

The difference could be seen in figure 2.

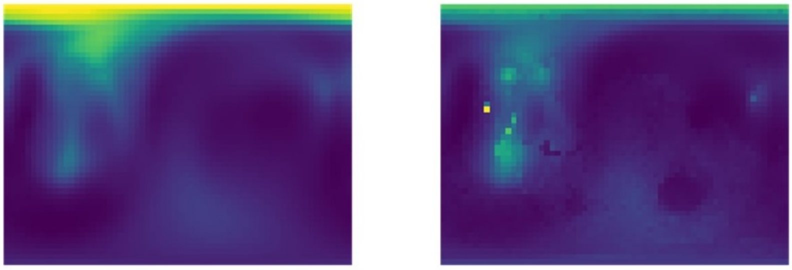

Figure 2: Before sigma and sigma

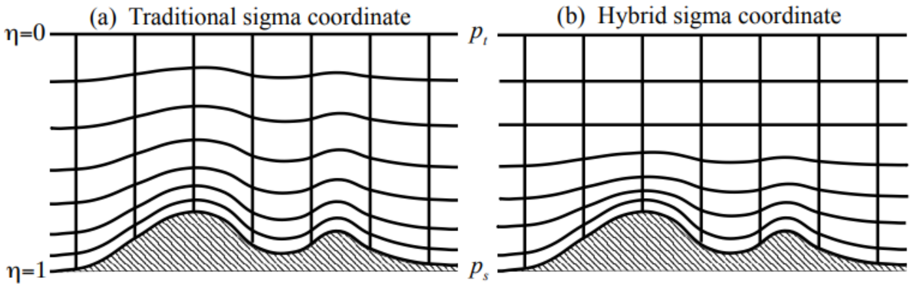

Figure 3: sigma and hybrid sigma

2.2 Hybrid sigma

The hybrid σ coordinate remodified the high pressure level.

$$p_d = B(\eta)(p_s - p_t) + [\eta - B(\eta)](p_0 - p_t) + p_t \tag{18}$$

where p_0 is a reference sea-level pressure. Here, $B(\eta)$ defines the relative weighting between the terrain-coordinate for $B(\eta) = \eta$ and reverts to a hydrostatic pressure coordinate for $B(\eta) = 0$. To smoothly transition from a sigma near the surface to a pressure coordinate at upper levels, $B(\eta)$ is defined by a third order polynomial

$$B(\eta) = c_1 + c_2\eta + c_3\eta^2 + c_4\eta^3 \tag{19}$$

By applying the boundary conditions, there should be,

$$\begin{cases} B(1) = 1 \\ B_\eta = 1 \\ B(\eta_c) = 0 \\ B_\eta(\eta_c) = 0 \end{cases} \tag{20}$$

such that,

$$\begin{cases} c_1 = \frac{2\eta_c^2}{(1-\eta_c)^3} \\ c_2 = \frac{-\eta_c(4+\eta_c+\eta_c^2)}{(1-\eta_c)^3} \\ c_3 = \frac{2\eta_c(1+\eta_c+\eta_c^2)}{(1-\eta_c)^3} \\ c_4 = \frac{-(1+\eta_c)}{(1-\eta_c)^3} \end{cases} \tag{21}$$

where η_c is a specified value of η at which it becomes a pure pressure coordinate. The difference could be seen in figure 3.
What should the η_c be for Mars?
The vertical coordinate metric is defined as

$$\mu_d = \frac{\partial p_d}{\partial \mu} = B_\mu(\mu)(p_s - p_t) + [1 - B_\mu(\mu)](p_0 - p_t) \tag{22}$$

Since $\mu_d\delta\mu = \delta p_d = -g\rho\delta z$ is proportional to the mass per unit area weightin a gird cell, the appripirate flux forms for the prognostic variables are defined as

$$\begin{cases} \mathbf{V} = \mu_d \mathbf{v} = (U, V, W) \\ \Omega = \mu_d \omega \\ \Theta_m = \mu_d \theta_m \\ Q_m = \mu_d q_m \end{cases} \tag{23}$$

Here, $\mathbf{v} = (u, v, \omega)$ are the covariant velocities in the horizontal and vertical directions, while $\omega = \dot{\mu}$ is the contravariant 'vertical' velocity. $\theta_m = \theta(1 + (\frac{R_v}{R_d})q_v) \approx \theta(1 + 1.61q_v)$ is the moist potential temperature and $Q_m = \mu d q_m$, where $q_m = q_v, q_c, q_r \cdots$ represnets the mixing ratio of moisture variables (water vapor, cloud water, rain water,...). Although the geopotential $\phi = gz$ is also a prognostic variable in the governing equations of the model, it is not written in flux-form as $\mu_d\phi$ is not a conserved quantity.

2.3 Coupled Algorithm

The pressure based solver allows us to solve the problem in either a segregated or coupled manner. Using the coupled approach offers some advantages over the non-coupled or segregated approach. The coupled scheme obtains a robust and efficient single phase implementation for steady-state flows, with superior performance compared to the segregated solution schemes. This pressure-based coupled algorithm offers an alternative to the density-based and pressure-based segregated algorithm with SIMPLE-type pressure-velocity coupling. For transient flows, using the coupled algorithm is necessary when the quality of the mesh is poor, or if large time steps are used.

The pressure-based segregated algorithm solves the momentum equation and pressure correction equations separately. This semi-implicit solution method results in slow convergence.

2.4 Flux-Form Euler Equations Under Sphere Coordinate

Using the variables defined above, the flux-form Euler equations can be written as

$$\begin{aligned} \partial_t U + (\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V u) + \frac{1}{r\sin\theta}\frac{\partial W u}{\partial\psi}) + \\ \mu_d\alpha(\sin\theta\cos\psi\frac{\partial p}{\partial r} + \frac{\cos\theta\cos\psi}{r}\frac{\partial p}{\partial\theta} - \frac{\sin\psi}{r\sin\theta}\frac{\partial p}{\partial\psi}) + \\ (\frac{\alpha}{\alpha_d}\partial_\mu p(\sin\theta\cos\psi\frac{\partial\phi}{\partial r} + \frac{\cos\theta\cos\psi}{r}\frac{\partial\phi}{\partial\theta} - \frac{\sin\psi}{r\sin\theta}\frac{\partial\phi}{\partial\psi})) \\ = F_U \end{aligned} \tag{24}$$

$$\begin{aligned} \partial_t V + (\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U v) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V v) + \frac{1}{r\sin\theta}\frac{\partial W v}{\partial\psi}) + \\ \mu_d\alpha(\sin\theta\sin\psi\frac{\partial p}{\partial r} + \frac{\cos\theta\sin\psi}{r}\frac{\partial p}{\partial\theta} + \frac{\cos\psi}{r\sin\theta}\frac{\partial p}{\partial\psi}) + \\ (\frac{\alpha}{\alpha_d}\partial_\mu p(\sin\theta\cos\psi\frac{\partial\phi}{\partial r} + \frac{\cos\theta\cos\psi}{r}\frac{\partial\phi}{\partial\theta} - \frac{\sin\psi}{r\sin\theta}\frac{\partial\phi}{\partial\psi})) \\ = F_V \end{aligned} \tag{25}$$

$$\begin{aligned} \partial_t W + (\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U \omega) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V \omega) + \frac{1}{r\sin\theta}\frac{\partial W \omega}{\partial\psi}) + \\ -g[\frac{\alpha}{\alpha_d}\partial_\eta p - \mu_d] = F_W \end{aligned} \tag{26}$$

$$\partial_t\Theta_m + (\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U\theta_m) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V\theta_m) + \frac{1}{r\sin\theta}\frac{\partial W\theta_m}{\partial\psi}) = F_{\Theta_m} \tag{27}$$

$$\partial_t\mu_d + (\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V) + \frac{1}{r\sin\theta}\frac{\partial W}{\partial\psi}) = 0 \tag{28}$$

$$\partial_t\phi + \mu_d^{-1}[U\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U\phi) + V\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V\phi) + W\frac{1}{r\sin\theta}\frac{\partial W\phi}{\partial\psi} - gW] = 0 \tag{29}$$

$$\partial_t Q_m + (\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U q_m) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V q_m) + \frac{1}{r\sin\theta}\frac{\partial W q_m}{\partial\psi}) = F_{Q_m} \quad (30)$$

with the diagnostic equation for dry hydrostatic pressure

$$\partial_\eta \phi = -\alpha_d \mu_d \quad (31)$$

and the diagnostic relation for the full pressure (dry air plus water vapor)

$$p = p_0(\frac{R_d \theta_m}{p_0 \alpha_d})^\gamma \quad (32)$$

In these equations, α_d is the inverse density of the dry air $\frac{1}{\rho_d}$ and α is the inverse density taking into account the full parcel density $\alpha = \alpha_d(1 + q_v + q_c + q_r + q_i + \ldots)^{-1}$. $\gamma = \frac{c_p}{c_v} = 1.4$ is the ratio of the heat capacities for dry air, R_d is the gas constant for dry air, and p_0 is a refernce surface pressure (On earth typically 10^5 Pascals). The right-hand-side (RHS) terms F_U, F_V, F_W, and F_Θ represent forcing terms arising from model physics, turbulent mixing, spherical projections, and the planet's rotation.

2.5 Map Projections, Coriolis and Curvature Terms

Map scale factors m_x and m_y are defined as the ratio of the distance in computational space to the corresponding distance on the planet's surface:

$$(m_x, m_y) = \frac{(\delta x)\delta y}{\text{distance on the planet}} \quad (33)$$

Thus we could refine the variables as

$$\begin{cases} U = \frac{\mu_d u}{m_y} \\ V = \frac{\mu_d v}{m_x} \\ W = \frac{\mu_d w}{m_y} \\ \Omega = \frac{\mu_d u}{m_y} \end{cases} \quad (34)$$

By applying these redifined variables, the governing prognostic equations including map factors could be written as

$$\begin{aligned} &\partial_t U + m_x(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V u)) + \frac{1}{r\sin\theta}\frac{\partial W u}{\partial\psi} + \\ &\frac{m_x}{m_y}\mu_d\alpha(\sin\theta\cos\psi\frac{\partial p}{\partial r} + \frac{\cos\theta\cos\psi}{r}\frac{\partial p}{\partial\theta} - \frac{\sin\psi}{r\sin\theta}\frac{\partial p}{\partial\psi}) + \\ &\frac{m_x}{m_y}(\frac{\alpha}{\alpha_d}\partial_\mu p(\sin\theta\cos\psi\frac{\partial\phi}{\partial r} + \frac{\cos\theta\cos\psi}{r}\frac{\partial\phi}{\partial\theta} - \frac{\sin\psi}{r\sin\theta}\frac{\partial\phi}{\partial\psi})) \\ &= F_U \end{aligned} \quad (35)$$

$$\begin{aligned} &\partial_t V + m_y(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 U v) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V v)) + \frac{m_y}{m_x}\frac{1}{r\sin\theta}\frac{\partial W v}{\partial\psi} + \\ &\frac{m_y}{m_x}\mu_d\alpha(\sin\theta\sin\psi\frac{\partial p}{\partial r} + \frac{\cos\theta\sin\psi}{r}\frac{\partial p}{\partial\theta} + \frac{\cos\psi}{r\sin\theta}\frac{\partial p}{\partial\psi}) + \\ &\frac{m_y}{m_x}(\frac{\alpha}{\alpha_d}\partial_\mu p(\sin\theta\cos\psi\frac{\partial\phi}{\partial r} + \frac{\cos\theta\cos\psi}{r}\frac{\partial\phi}{\partial\theta} - \frac{\sin\psi}{r\sin\theta}\frac{\partial\phi}{\partial\psi})) \\ &= F_V \end{aligned} \quad (36)$$