SUPPLEMENTARY MATERIAL: PROOFS OF THEOREMS 4.15–4.18

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Given two finite and tropically separable sets P and Q in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$, assume $|P| = |Q| = n > 0$ (remark that all our results can be directly extended when $|P| \neq |Q|$). For any $\xi \in P \cup Q$, we denote by $i(\xi)$ and $j(\xi)$ two indices in terms of ξ, which are integers in the set $\{1, \ldots, d\}$. We denote the two sets of indices $\{i(\xi)|\xi \in P \cup Q\}$ and $\{j(\xi)|\xi \in P \cup Q\}$ by $\mathcal{I}$ and $\mathcal{J}$, respectively. We also assume that

(1) $\quad \forall \xi \in P \cup Q, i(\xi) \neq j(\xi), \quad$ and $\quad\quad$ (2) $\quad \forall p \in P,\ \forall q \in Q,\ \ i(p) \neq i(q).$

We formulate an optimization problem for solving the normal vector ω of an optimal tropical separating hyperplane H_ω for P and Q:

$$\max_{z \in \mathbb{R}} \ z \tag{3}$$

$$\text{s.t. } \forall \xi \in P \cup Q,\ z + \xi_{j(\xi)} + \omega_{j(\xi)} - \xi_{i(\xi)} - \omega_{i(\xi)} \leq 0, \tag{4}$$

$$\forall \xi \in P \cup Q,\ \omega_{j(\xi)} - \omega_{i(\xi)} \leq \xi_{i(\xi)} - \xi_{j(\xi)}, \tag{5}$$

$$\forall \xi \in P \cup Q, \forall l \neq i(\xi), j(\xi),\ \omega_l - \omega_{j(\xi)} \leq \xi_{j(\xi)} - \xi_l. \tag{6}$$

Theorems 0.1–0.4 below are respectively Theorems 4.15–4.18 in the manuscript.

THEOREM 0.1. *Suppose P and Q are two finite sets in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. For all $p \in P$, assume $i(p)$ and $j(p)$ are constants, say i_P and j_P. For all $q \in Q$, assume $i(q)$ and $j(q)$ are constants, say i_Q and j_Q. If the four numbers i_P, j_P, i_Q and j_Q are pairwise distinct, then the linear programming (3)–(6) has a feasible solution if and only if*

$$\max\{-F,\ -A - E\} \ \leq\ \min\{D + B,\ C\}. \tag{7}$$

If a feasible solution exists, then the optimal value z is given by

$$\min\left\{\ A + C + E,\ D + B + F,\ \frac{1}{2}(A + B + D + E)\ \right\}, \tag{8}$$

where

$$\begin{aligned} A &= \min_{p \in P}\{p_{i_P} - p_{j_P}\}, C = \min_{p \in P}\{p_{j_P} - p_{j_Q}\}, E = \min_{q \in Q}\{q_{j_Q} - q_{i_P}\}, \\ B &= \min_{p \in P}\{p_{j_P} - p_{i_Q}\}, D = \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\}, F = \min_{q \in Q}\{q_{j_Q} - q_{j_P}\}. \end{aligned} \tag{9}$$

PROOF. For any $p \in P$, by the assumptions $i(p) = i_P$ and $j(p) = j_P$, and by (5)–(6), we have:

$$\omega_{j_P} - \omega_{i_P} \leq p_{i_P} - p_{j_P}, \text{ and } \omega_l - \omega_{j_P} \leq p_{j_P} - p_l,\ \ \forall l \neq i_P, j_P.$$

By the assumption that the four numbers i_P, j_P, i_Q and j_Q are pairwise distinct, there exists $l \neq i_P, j_P$ such that $l = i_Q$ or $l = j_Q$. Similarly, for any $q \in Q$, by (5)–(6), we have

$$\omega_{j_Q} - \omega_{i_Q} \leq q_{i_Q} - q_{j_Q}, \text{ and } \omega_l - \omega_{j_Q} \leq q_{j_Q} - q_l, \ \ \forall l \neq i_Q, j_Q,$$

and there exists $l \neq i_Q, j_Q$ such that $l = i_P$ or $l = j_P$. So for any $p \in P$, we have

(10) $$\omega_{j_P} - \omega_{i_P} \leq p_{i_P} - p_{j_P},$$
(11) $$\omega_{i_Q} - \omega_{j_P} \leq p_{j_P} - p_{i_Q},$$
(12) $$\omega_{j_Q} - \omega_{j_P} \leq p_{j_P} - p_{j_Q}.$$

and for any $q \in Q$,

(13) $$\omega_{j_Q} - \omega_{i_Q} \leq q_{i_Q} - q_{j_Q},$$
(14) $$\omega_{i_P} - \omega_{j_Q} \leq q_{j_Q} - q_{i_P},$$
(15) $$\omega_{j_P} - \omega_{j_Q} \leq q_{j_Q} - q_{j_P}.$$

By adding (11) and (13), we have

(16) $$\omega_{j_Q} - \omega_{j_P} \ \leq \ \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} + \min_{p \in P}\{p_{j_P} - p_{i_Q}\}.$$

By adding (10) and (14), we have

(17) $$\max_{p \in P}\{p_{j_P} - p_{i_P}\} + \max_{q \in Q}\{q_{i_P} - q_{j_Q}\} \ \leq \ \omega_{j_Q} - \omega_{j_P}.$$

By (12), (15), (16) and (17), the inequality (7) holds. Therefore, if the linear programming (3)–(6) has a feasible solution, then we have (7).

On the other hand, if we have (7), then there exist real numbers ω_{j_Q} and ω_{j_P} such that the inequalities (12), (15), (16) and (17) hold. Notice that the inequality (16) is equivalent to

$$\omega_{j_Q} - \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} \leq \omega_{j_P} + \min_{p \in P}\{p_{j_P} - p_{i_Q}\}.$$

So, there exists a number ω_{i_Q} such that

(18) $$\omega_{j_Q} - \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} \leq \omega_{i_Q},$$
(19) $$\omega_{i_Q} \leq \omega_{j_P} + \min_{p \in P}\{p_{j_P} - p_{i_Q}\}.$$

Symmetrically, the inequality (17) is equivalent to

$$\max_{p \in P}\{p_{j_P} - p_{i_P}\} + \omega_{j_P} \leq \omega_{j_Q} - \max_{q \in Q}\{q_{i_P} - q_{j_Q}\}.$$

So, there exists a number ω_{i_P} such that

(20) $$\max_{p \in P}\{p_{j_P} - p_{i_P}\} + \omega_{j_P} \leq \omega_{i_P},$$
(21) $$\omega_{i_P} \leq \omega_{j_Q} - \max_{q \in Q}\{q_{i_P} - q_{j_Q}\}.$$

The inequality (20) can be rewritten as

(22) $$\max_{p \in P}\{p_{j_P} - p_{i_P}\} \leq \omega_{i_P} - \omega_{j_P} \ \Leftrightarrow \ \omega_{j_P} - \omega_{i_P} \leq \min_{p \in P}\{p_{i_P} - p_{j_P}\}.$$

The inequality (21) can be rewritten as

(23) $$\omega_{i_P} - \omega_{j_Q} \leq -\max_{q \in Q}\{q_{i_P} - q_{j_Q}\} \ \Leftrightarrow \ \omega_{i_P} - \omega_{j_Q} \leq \min_{q \in Q}\{q_{j_Q} - q_{i_P}\}.$$

By (18) and (22), the inequality (5) holds. By (12), (15), (19) and (23), the inequality (6) holds for $l = i_Q, j_Q$ when $\xi \in P$, or for $l = i_P, j_P$ when $\xi \in Q$. For $l \neq i_Q, j_Q$ when $\xi \in P$, or for $l \neq i_P, j_P$ when $\xi \in Q$, there always exist sufficiently small numbers for ω_l such that the inequality (6) holds. So the inequality (7) guarantees the feasibility of the inequalities (5) and (6). Notice that once (5) and (6) are feasible, there is always a non-negative number z such that (4) holds. So, if we have (7), then the linear programming (3)–(6) has a feasible solution.

If a feasible solution exists, then by (4), for any feasible solution $(\omega; z)$,

$$z \le \min_{p\in P}\{p_{i_P} - p_{j_P}\} + \omega_{i_P} - \omega_{j_P}, \;\; z \le \min_{q\in Q}\{q_{i_Q} - q_{j_Q}\} + \omega_{i_Q} - \omega_{j_Q}. \tag{24}$$

So, by (11) and (14), and by summing up the above two inequalities, we have

$$\begin{aligned}
2z &\le \min_{p\in P}\{p_{i_P} - p_{j_P}\} + \min_{q\in Q}\{q_{i_Q} - q_{j_Q}\} + \omega_{i_P} - \omega_{j_P} + \omega_{i_Q} - \omega_{j_Q} && (25)\\
&= \min_{p\in P}\{p_{i_P} - p_{j_P}\} + \min_{q\in Q}\{q_{i_Q} - q_{j_Q}\} + (\omega_{i_Q} - \omega_{j_P}) + (\omega_{i_P} - \omega_{j_Q})\\
&\le \min_{p\in P}\{p_{i_P} - p_{j_P}\} + \min_{q\in Q}\{q_{i_Q} - q_{j_Q}\} + \min_{p\in P}\{p_{j_P} - p_{i_Q}\} + \min_{q\in Q}\{q_{j_Q} - q_{i_P}\} && (26)\\
&= A + D + B + E.
\end{aligned}$$

Also, by (12), (14) and the first inequality in (24), we have

$$\begin{aligned}
z &\le \min_{p\in P}\{p_{i_P} - p_{j_P}\} + \omega_{i_P} - \omega_{j_P} && (27)\\
&= \min_{p\in P}\{p_{i_P} - p_{j_P}\} + \omega_{i_P} - \omega_{j_Q} + \omega_{j_Q} - \omega_{j_P}\\
&= \min_{p\in P}\{p_{i_P} - p_{j_P}\} + (\omega_{j_Q} - \omega_{j_P}) + (\omega_{i_P} - \omega_{j_Q})\\
&\le \min_{p\in P}\{p_{i_P} - p_{j_P}\} + \min_{p\in P}\{p_{j_P} - p_{j_Q}\} + \min_{q\in Q}\{q_{j_Q} - q_{i_P}\} \;=\; A + C + E. && (28)
\end{aligned}$$

Symmetrically, by (11), (15) and the second inequality in (24), we have

$$\begin{aligned}
z &\le \min_{q\in Q}\{q_{i_Q} - q_{j_Q}\} + \omega_{i_Q} - \omega_{j_Q} && (29)\\
&= \min_{q\in Q}\{q_{i_Q} - q_{j_Q}\} + (\omega_{i_Q} - \omega_{j_P}) + (\omega_{j_P} - \omega_{j_Q})\\
&\le \min_{q\in Q}\{q_{i_Q} - q_{j_Q}\} + \min_{p\in P}\{p_{j_P} - p_{i_Q}\} + \min_{q\in Q}\{q_{j_Q} - q_{j_P}\} \;=\; D + B + F. && (30)
\end{aligned}$$

Hence, all the values $\frac{1}{2}(A + B + D + E)$, $A + C + E$ and $B + D + F$ are upper bounds for z.

If $\frac{1}{2}(A + B + D + E) \le \min\{A + C + E, B + D + F\}$, then we can choose a feasible ω such that

$$\omega_{i_Q} - \omega_{j_P} = -B,\; \omega_{i_P} - \omega_{j_Q} = -E,$$

$$\omega_{j_P} - \omega_{i_P} = \frac{1}{2}(A - B - D - E),\; \omega_{j_Q} - \omega_{i_Q} = \frac{1}{2}(D - A - B - E)$$

which imply $\omega_{j_Q} - \omega_{j_P} = \frac{1}{2}(B + D - E - A)$. For this ω, the equalities in (24), (25) and (26) hold, and z reaches its optimal value $\frac{1}{2}(A + B + D + E)$.

If $A + C + E < \min\{\frac{1}{2}(A + B + D + E),\, B + D + F\}$, then we can choose a feasible ω such that

$$\omega_{j_Q} - \omega_{j_P} = C, \text{ and } \omega_{i_P} - \omega_{j_Q} = E,$$

which imply $\omega_{j_P} - \omega_{i_P} = -C - E$. For this ω, the equalities in (27) and (28) hold, and z reaches its optimal value $A + C + E$. Symmetrically, if

$$B + D + F < \min\{\frac{1}{2}(A + B + D + E),\ A + C + E\},$$

then we can choose a feasible ω such that

$$\omega_{j_P} - \omega_{j_Q} = F, \text{ and } \omega_{i_Q} - \omega_{j_P} = B,$$

which imply $\omega_{j_Q} - \omega_{i_Q} = -B - F$. For this ω, the equalities in (29) and (30) hold, and z reaches its optimal value $B + D + F$. □ □

THEOREM 0.2. *Suppose P and Q are two finite sets in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. For all $p \in P$, assume $i(p)$ and $j(p)$ are constants, say i_P and j_P. For all $q \in Q$, assume $i(q)$ and $j(p)$ are constants, say i_Q and j_Q.*

(i) *If $i_P = j_Q$ and $i_Q \neq j_P$, then linear programming (3)–(6) has a feasible solution if and only if*

$$A + B + C \geq 0. \tag{31}$$

If a feasible solution exists, then the optimal value z is given by

$$\min\{\, A + B + C,\ \frac{1}{2}\big(A' + B + C\big)\},$$

where

$$A' = \min_{p \in P}\{p_{i_P} - p_{j_P}\},\ A = \min_{\xi \in P \cup Q}\{\xi_{i_P} - \xi_{j_P}\},$$
$$B = \min_{p \in P}\{p_{j_P} - p_{i_Q}\},\quad C = \min_{q \in Q}\{q_{i_Q} - q_{i_P}\}.$$

(ii) *If $i_Q = j_P$ and $i_P \neq j_Q$, then linear programming (3)–(6) has a feasible solution if and only if*

$$A + B + C \geq 0. \tag{32}$$

If a feasible solution exists, then the optimal value z is given by

$$\min\{\, A + B + C,\ \frac{1}{2}\big(A' + B + C\big)\,\},$$

where

$$A' = \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\},\ A = \min_{\xi \in P \cup Q}\{\xi_{i_Q} - \xi_{j_Q}\},$$
$$B = \min_{q \in Q}\{q_{j_Q} - q_{i_P}\},\quad C = \min_{p \in P}\{p_{i_P} - p_{i_Q}\}.$$

PROOF. We only need to prove part (i) since part (ii) can be symmetrically argued. For any $p \in P$, by the assumptions $i(p) = i_P$ and $j(p) = j_P$, and by (5)–(6), we have:

$$\omega_{j_P} - \omega_{i_P} \leq p_{i_P} - p_{j_P}, \text{ and } \omega_l - \omega_{j_P} \leq p_{j_P} - p_l,\ \ \forall l \neq i_P, j_P.$$

By (1), $i_P \neq i_Q$. Note that we assume $i_Q \neq j_P$. So, there exists $l \neq i_P, j_P$ such that $l = i_Q$. For any $q \in Q$, by (5)–(6), we have

$$\omega_{j_Q} - \omega_{i_Q} \leq q_{i_Q} - q_{j_Q}, \text{ and } \omega_l - \omega_{j_Q} \leq q_{j_Q} - q_l,\ \ \forall l \neq i_Q, j_Q.$$

By the definition of $i(p)$ and $j(p)$, we have $i_P \neq j_P$. So we have $j_Q \neq j_P$ since we assume that $i_P = j_Q$. Notice again that we assume $i_Q \neq j_P$. Hence, there exists $l \neq i_Q, j_Q$ such that $l = j_P$. So, for any $p \in P$,

(33) $\omega_{j_P} - \omega_{i_P} \le p_{i_P} - p_{j_P}$, (34) $\omega_{i_Q} - \omega_{j_P} \le p_{j_P} - p_{i_Q}$,

and for any $q \in Q$,

(35) $\omega_{j_Q} - \omega_{i_Q} \le q_{i_Q} - q_{j_Q}$, (36) $\omega_{j_P} - \omega_{j_Q} \le q_{j_Q} - q_{j_P}$.

If $i_P = j_Q$, then by (33) and (36), we have

$$\max_{\xi \in P \cup Q}\{\xi_{j_P} - \xi_{i_P}\} \le \omega_{i_P} - \omega_{j_P}. \tag{37}$$

If $i_P = j_Q$, then by adding (34) and (35), we have

$$\omega_{i_P} - \omega_{j_P} \le \min_{p \in P}\{p_{j_P} - p_{i_Q}\} + \min_{q \in Q}\{q_{i_Q} - q_{i_P}\}. \tag{38}$$

So, if $i_P = j_Q$, then by (37) and (38), we have (31).

On the other hand, if we have (31), then there exist real numbers ω_{i_P} and ω_{j_P} such that (37) and (38) hold. By (37), we have (33) and (36). Let $\omega_{i_Q} = \omega_{i_P} + \min_{p \in P}\{p_{j_P} - p_{i_Q}\}$. Then we have the inequality (34), and by (38), we have

$$\max_{q \in Q}\{q_{i_P} - q_{i_Q}\} \le \omega_{i_Q} - \omega_{i_P},$$

which is equivalent to (35). By (33), (34), (35) and (36), the inequality (5) holds, and the the inequality (6) holds for $l = i_Q$ when $\xi \in P$, or for $l = j_P$ when $\xi \in Q$. For $l \ne i_Q$ when $\xi \in P$, or for $l \ne j_P$ when $\xi \in Q$, there always exist sufficiently small numbers for ω_l such that the inequality (6) holds. So the inequality (31) guarantees the feasibility of the inequalities (5) and (6). Notice that once (5) and (6) are feasible, there is always a non-negative number z such that (4) holds. So, if we have (31), then the linear programming (3)–(6) has a feasible solution.

If a feasible solution exists, then by (4),

$$z \le \min_{p \in P}\{p_{i_P} - p_{j_P}\} + \omega_{i_P} - \omega_{j_P}, \quad z \le \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} + \omega_{i_Q} - \omega_{j_Q}. \tag{39}$$

Note $i_P = j_Q$. So, by summing up the above two inequalities and by (34),

$$2z \le \min_{p \in P}\{p_{i_P} - p_{j_P}\} + \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} + \omega_{i_Q} - \omega_{j_P} \tag{40}$$

$$\le \min_{p \in P}\{p_{i_P} - p_{j_P}\} + \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} + \min_{p \in P}\{p_{j_P} - p_{i_Q}\} = A' + C + B. \tag{41}$$

Also, by (34), (37) and the second inequality in (39), we have

$$\begin{aligned} z &\le \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} + \omega_{i_Q} - \omega_{j_Q} && (42)\\ &= \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} + \omega_{i_Q} - \omega_{j_P} + \omega_{j_P} - \omega_{j_Q} \\ &= \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} + (\omega_{i_Q} - \omega_{j_P}) + (\omega_{j_P} - \omega_{i_P}) \\ &\le \min_{q \in Q}\{q_{i_Q} - q_{j_Q}\} + \min_{p \in P}\{p_{j_P} - p_{i_Q}\} + \min_{\xi \in P \cup Q}\{\xi_{i_P} - \xi_{j_P}\} = C + B + A. && (43) \end{aligned}$$

Hence, both values $\frac{1}{2}(A' + B + C)$ and $A + B + C$ are upper bounds for z.

If $\frac{1}{2}(A'+B+C) \leq A+B+C$, then we can choose a feasible ω such that

$$\omega_{i_P} - \omega_{j_P} = \frac{1}{2}\left(B+C-A'\right), \text{ and } \omega_{i_Q} = \omega_{j_P} + B,$$

which imply $\omega_{i_Q} - \omega_{j_Q} = \frac{1}{2}(A'+B-C)$. For this ω, the equalities in (39)–(41) hold, and z reaches its optimal value $\frac{1}{2}(A'+B+C)$.

If $A+B+C < \frac{1}{2}(A'+B+C)$, then we can choose a feasible ω such that

$$\omega_{i_Q} - \omega_{j_Q} = A+B, \text{ and } \omega_{i_Q} = \omega_{j_P} + B,$$

which imply $\omega_{i_P} - \omega_{j_P} = -A$. For this ω, the equalities in (42)–(43) hold, and z reaches its optimal value $A+B+C$. □ □

THEOREM 0.3. *Suppose P and Q are two finite sets in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. If for all $p \in P$, we have $i(p) = j(q) = k_1$, and for all $q \in Q$, $i(q) = j(p) = k_2$, then the linear programming (3)–(6) has a feasible solution if and only if*

$$\max_{p\in P}\{p_{k_2} - p_{k_1}\} \leq \min_{q\in Q}\{q_{k_2} - q_{k_1}\} \tag{44}$$

If a feasible solution exists, then the optimal value z is given by

$$\frac{1}{2}(\min_{p\in P}\{p_{k_1} - p_{k_2}\} + \min_{q\in Q}\{q_{k_2} - q_{k_1}\}) \tag{45}$$

PROOF. For any $p \in P$, and for any $q \in Q$, by the assumptions $i(p) = j(q) = k_1$ and $i(q) = j(p) = k_2$, and by (5), we have:

$$\omega_{k_2} - \omega_{k_1} \leq p_{k_1} - p_{k_2}, \text{ and } \omega_{k_1} - \omega_{k_2} \leq q_{k_2} - q_{k_1}$$

Therefore,

$$\max_{p\in P}\{p_{k_2} - p_{k_1}\} \leq \omega_{k_1} - \omega_{k_2} \leq \min_{q\in Q}\{q_{k_2} - q_{k_1}\}. \tag{46}$$

So if the linear programming (3)–(6) has a feasible solution, then we have (44).

On the other hand, if we have (44), then there exist real numbers ω_{k_1} and ω_{k_2} such that (46), and hence (5) holds. For $l \neq k_1, k_2$, there always exist sufficiently small numbers for ω_l such that the inequality (6) holds. So inequalities (44) guarantees the feasibility of the inequalities (5) and (6). Notice that once (5) and (6) are feasible, there is always a non-negative number z such that (4) holds. So, if we have (44), then the linear programming (3)–(6) has a feasible solution.

If a feasible solution exists, then by (4),

$$z \leq \min_{p\in P}\{p_{k_1} - p_{k_2}\} + \omega_{k_1} - \omega_{k_2}, \;\; z \leq \min_{q\in Q}\{q_{k_2} - q_{k_1}\} + \omega_{k_2} - \omega_{k_1}. \tag{47}$$

By summing up the above two inequalities,

$$2z \leq \min_{p\in P}\{p_{k_1} - p_{k_2}\} + \min_{q\in Q}\{q_{k_2} - q_{k_1}\}.$$

So the value (45) is an upper bound for z. Note that we can choose a feasible ω such that

$$\omega_{k_1} - \omega_{k_2} = \frac{1}{2}\left(\min_{q\in Q}\{q_{k_2} - q_{k_1}\} + \max_{p\in P}\{p_{k_2} - p_{k_1}\}\right),$$

which satisfies the inequality (46). For this ω, the equalities in (47) hold, and z reaches its the optimal value (45). □ □

THEOREM 0.4. *Suppose P and Q are two finite sets in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. If for all $p \in P$ and for all $q \in Q$, the indices $i(p)$ and $i(q)$ are respectively constants, say i_P and i_Q, and for all $\xi \in P \cup Q$, $j(\xi)$ is a constant, say j, then the linear programming (3)–(6) has a feasible solution if and only if*

$$\max_{q \in Q}\{q_{i_P} - q_j\} \;\le\; \min_{p \in P}\{p_{i_P} - p_j\} \tag{48}$$

and

$$\max_{p \in P}\{p_{i_Q} - p_j\} \;\le\; \min_{q \in Q}\{q_{i_Q} - q_j\}. \tag{49}$$

If a feasible solution exists, then the optimal value z is given by

$$\min\{\,\min_{p \in P}\{p_{i_P} - p_j\} + \min_{q \in Q}\{q_j - q_{i_P}\},\ \min_{q \in Q}\{q_{i_Q} - q_j\} + \min_{p \in P}\{p_j - p_{i_Q}\}\,\}. \tag{50}$$

PROOF. For any $p \in P$, by the assumptions $i(p) = i_P$ and $j(p) = j$, and by (5)–(6), we have:

$$\omega_j - \omega_{i_P} \le p_{i_P} - p_j, \text{ and } \omega_l - \omega_j \le p_j - p_l,\ \ \forall l \ne i_P, j.$$

By (1), $i_P \ne i_Q$. By the definition of $i(q)$ and $j(q)$, $i(q) \ne j(q)$, and hence $i_Q \ne j$. So, there exists $l \ne i_P, j$ such that $l = i_Q$. Similarly, for any $q \in Q$, we have

$$\omega_j - \omega_{i_Q} \le q_{i_Q} - q_j, \text{ and } \omega_l - \omega_j \le q_j - q_l,\ \ \forall l \ne i_Q, j,$$

and there exists $l \ne i_Q, j$ such that $l = i_P$. So we have

$$\forall p \in P,\ \ \omega_j - \omega_{i_P} \le p_{i_P} - p_j, \text{ and } \omega_{i_Q} - \omega_j \le p_j - p_{i_Q}, \tag{51}$$

and

$$\forall q \in Q,\ \ \omega_j - \omega_{i_Q} \le q_{i_Q} - q_j, \text{ and } \omega_{i_P} - \omega_j \le q_j - q_{i_P}. \tag{52}$$

Therefore,

$$\max_{q \in Q}\{q_{i_P} - q_j\} \le \omega_j - \omega_{i_P} \le \min_{p \in P}\{p_{i_P} - p_j\} \tag{53}$$

and

$$\max_{p \in P}\{p_{i_Q} - p_j\} \le \omega_j - \omega_{i_Q} \le \min_{q \in Q}\{q_{i_Q} - q_j\}. \tag{54}$$

So, if the linear programming (3)–(6) has a feasible solution, then we have (48) and (49).

On the other hand, if we have (48) and (49), then there exist real numbers ω_j, ω_{i_P} and ω_{i_Q} such that (53) and (54) hold, and hence (51) and (52) hold. For $l \ne i_P, j, i_Q$, there always exist sufficiently small numbers for ω_l such that the inequality (6) holds. So inequalities (48) and (49) guarantee the feasibility of the inequalities (5) and (6). Notice that once (5) and (6) are feasible, there is always a non-negative number z such that (4) holds. So, if we have (48) and (49), then the linear programming (3)–(6) has a feasible solution.

If a feasible solution exists, then by (4) and the first inequality in (53),

$$z \le \min_{p \in P}\{p_{i_P} - p_j\} + \omega_{i_P} - \omega_j \le \min_{p \in P}\{p_{i_P} - p_j\} + \min_{q \in Q}\{q_j - q_{i_P}\}, \text{ and}$$

by (4) and the first inequality in (54),

$$z \le \min_{q \in Q}\{q_{i_Q} - q_j\} + \omega_{i_Q} - \omega_j \le \min_{q \in Q}\{q_{i_Q} - q_j\} + \min_{p \in P}\{p_j - p_{i_Q}\}.$$

So the maximum z is given by (50), and this optimal value is reached when either $\omega_{i_P} - \omega_j = \min_{q \in Q}\{q_j - q_{i_P}\}$, or $\omega_{i_Q} - \omega_j = \min_{p \in P}\{p_j - p_{i_Q}\}$. □ □