AdaCUR: Efficient Low-rank Approximation of Parameter-dependent matrices $A(t)$ via CUR decomposition

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Abstract

Let $A(t) \in \mathbb{R}^{m\times n}$ be a parameter-dependent matrix and suppose we want to compute its low-rank approximation at a finite number of parameter values $t_1, t_2, ..., t_q$. This problem arises in several applications including the compression of a series of images, dynamical systems, and Gaussian process regression, where low-rank approximations are needed for the sequence $A(t_1), A(t_2), ..., A(t_q)$. While existing methods such as dynamical low-rank approximation [6] and random embedding techniques [7] offer solutions, they typically incur a complexity of $\mathcal{O}(r_i T_{A(t_i)})$ for each parameter t_i, with r_i as the target rank and $T_{A(t_i)}$ as the cost of a matrix-vector product with $A(t)$. We propose an alternative approach using the CUR decomposition, which can accelerate low-rank approximation to an average complexity of $\mathcal{O}(T_{A(t_i)})$ while addressing key challenges, such as rank-adaptivity and error control, often missing in other methods.

The CUR decomposition [3, 5, 8, 11] approximates a matrix A using subsets of its rows and columns:

$$A \approx CU^{\dagger}R,$$

where C and R are subsets of A's columns and rows, and U is their intersection. This decomposition, in contrast to methods like the truncated SVD, preserves properties such as sparsity and aids in data interpretation by identifying significant columns and rows. For $A(t)$, recomputing row and column indices for each t_i is inefficient, as indices derived for one parameter value may still provide useful information for nearby parameters. Building on this insight, we introduce an algorithm, AdaCUR [12], which maximizes the reuse of row and column indices across parameter values.

AdaCUR computes low-rank approximations of parameter-dependent matrices via CUR decomposition:

$$A(t) \approx C(t)U(t)^{\dagger}R(t),$$

where $C(t)$ and $R(t)$ are subsets of the columns and rows of $A(t)$, and $U(t)$ is their intersection. Starting from an initial CUR decomposition, AdaCUR reuses row and column indices until the error exceeds a specified threshold, at which point the indices are recomputed. To achieve this efficiently and reliably, we rely on a variety of tools from randomized numerical linear algebra [9]. Specifically, we use pivoting on a random sketch [1, 2] to obtain a reliable set of row and column indices, randomized rank estimation [10] to adapt to rank changes across parameter values, and randomized norm estimation [4] to approximate the relative error, ensuring effective error control. The resulting algorithm is efficient, rank-adaptive, and incorporates error control.

Additionally, we present FastAdaCUR, a variation that prioritizes speed over precision. FastAdaCUR achieves linear complexity in m and n after an initial index computation phase. Although highly efficient and rank-adaptive, it lacks rigorous error control, as it emphasizes speed over accuracy by only examining a subset of rows and columns of the matrix.

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