This Mathematica notebook contains the computations for the paper "Two connected populations with Allee effect" where the sequence of nonnegative steady state of the system (1) of the paper for fixed values of the Allee threshold parameter, but varying spatial dispersal rate is computed in the one way migration case.
Part 1: Left side of Figure 4.

This section contains the functions that are used in the next section.

Consider the parametric system of polynomial equations given at equation (2) at the paper with the inequality constraints on the variables and the parameters, and $\delta_{1,2}=1$, $\delta_{2,1}=0$.

We define systemEquations to be the list of the two polynomial equations resulted by letting $\dot{N_1}=$ $\dot{N_2}=0$.

In[1]:= `systemEquations = {N₁ * (1 - N₁) * (N₁ - b) - a * N₁, N₂ * (1 - N₂) * (N₂ - b) + a * N₁}`

Out[1]= $\left\{- a\, N_1 + \left(1 - N_1\right) N_1 \left(-b + N_1\right),\ a\, N_1 + \left(1 - N_2\right) N_2 \left(-b + N_2\right)\right\}$

The following function solves the system of equations systemEquations=0 for a given a and b values and returns the nonnegative real solutions.

In[2]:=
```
solvor[aa_, bb_] := (
   equations = ReplaceAll[{a → aa, b → bb}][systemEquations];
   solutions =
    NSolve[equations[[1]] == 0 && equations[[2]] == 0 && N₁ ≥ 0 && N₂ ≥ 0, {N₁, N₂}, Reals];
   points = ConstantArray[{0, 0}, Length[solutions]];
   For[i = 1, i ≤ Length[solutions], i++,
    points[[i]] = ReplaceAll[solutions[[i]]][{N₁, N₂}];
    ];
   Return[points];
  )
```

The following function finds the critical values of a for a fixed given value of b. This is done by solving f(a,b)=0 after substituting the given choice of b. Where f is any of the two polynomials defining the boundary between the open regions in Figure 3 of the paper. They are accessibly by asking the RootFinding[Parametric] package of Maple to show the algebraic description of the cells.

```
In[3]:= criticalValuesA[bb_] := (
    equation1 = ReplaceAll[{b -> bb}][
      16 * a^3 * b^6 + 16 * a^2 * b^7 - 4 * a * b^8 + 108 * a^4 * b^4 + 60 * a^3 * b^5 - 75 * a^2 * b^6 +
        10 * a * b^7 + b^8 - 54 * a^4 * b^3 - 12 * a^3 * b^4 + 42 * a^2 * b^5 + 4 * a * b^6 - 4 * b^7 +
        729 * a^6 + 1458 * a^5 * b + 405 * a^4 * b^2 - 328 * a^3 * b^3 + 50 * a^2 * b^4 - 20 * a * b^5 +
        6 * b^6 - 54 * a^4 * b - 12 * a^3 * b^2 + 42 * a^2 * b^3 + 4 * a * b^4 - 4 * b^5 + 108 * a^4 +
        60 * a^3 * b - 75 * a^2 * b^2 + 10 * a * b^3 + b^4 + 16 * a^3 + 16 * a^2 * b - 4 * a * b^2];
    equation2 = ReplaceAll[{b -> bb}][4 * a - b^2 + 2 * b - 1];
    solutions1 = Solve[equation1 == 0 && a ≥ 0, {a}, Reals];
    points1 = ConstantArray[{0, 0}, Length[solutions1]];
    For[i = 1, i ≤ Length[solutions1], i++,
     points1[[i]] = ReplaceAll[solutions1[[i]]][a];
    ];
    solutions2 = Solve[equation2 == 0 && a ≥ 0, {a}, Reals];
    points2 = ConstantArray[{0, 0}, Length[solutions2]];
    For[i = 1, i ≤ Length[solutions2], i++,
     points2[[i]] = ReplaceAll[solutions2[[i]]][a];
    ];
    criticalValues = Union[points1, points2];
    Return[Sort[criticalValues, Less]];
   )
```

The following function receives a choice of b and the list of values as critical values of a and a list specifying how many sample points between two critical values must be considered and then returns a list of solutions for the given value of b and different values of a.

```
In[4]:= solvorSequencePoints[bb_, alist_, NNL_] := (
    criticalN = Length[alist] - 1;
    pointsL = {};
    (* a=alist[[1]] *)
    AppendTo[pointsL, solvor[alist[[1]], bb]];
    (* alist[[i]] until alist[[i+1]] *)
    For[l1 = 1, l1 ≤ criticalN, l1++,
     NN = NNL[[l1]];
     For[l2 = 1, l2 ≤ NN, l2++,
      aa = ((NN - l2) * alist[[l1]] + l2 * alist[[l1 + 1]]) / NN;
      AppendTo[pointsL, solvor[aa, bb]];
     ];
    ];
    Return[pointsL];
   )
```

The following function decides the color of each solution point in the returned list of the previous function according the the a value with respect to the critical values of a.

```
In[5]:= pointsColorGen[alist_, NNL_] := (
        criticalN = Length[alist] - 1;
        pointsColorL = {};
        colorsL = {{128/255, 0, 128/255}, {1, 0, 0}, {1, 128/255, 0},
          {1, 1, 0}, {128/255, 1, 0}, {0, 102/255, 0}, {0, 0, 1}, {0, 1, 1}};
        (* a=alist[[1]] *)
        AppendTo[pointsColorL, Directive[RGBColor[colorsL[[1]]], PointSize[0.02]]];
        (* alist[[i]] until alist[[i+1]] *)
        For[l1 = 1, l1 ≤ criticalN, l1++,
         NN = NNL[[l1]];
         For[l2 = 1, l2 ≤ NN, l2++,
          AppendTo[pointsColorL, Directive[RGBColor[
               ((NN - l2) * colorsL[[2 * l1 - 1]] + l2 * colorsL[[2 * l1]]) / NN], PointSize[0.02]]];
         ];
        ];
        Return[pointsColorL];
       )
```

The following function is a sub-function for the next function.

```
In[25]:= plotSubTask[inputos_, alist_] := (
         (*listo1=Reverse[inputos[[1]]];
         listo2=Reverse[inputos[[2]]];*)
         listo1 = inputos[[1]];
         listo2 = inputos[[2]];
         Return[ListPlot[listo1, PlotStyle → listo2, AxesLabel →
            {Style["N₁", FontSize → 16, Bold, Black], Style["N₂", FontSize → 16, Bold, Black]},
           ColorFunctionScaling → False, PlotLegends → Switch[
            Length[alist],
            5, BarLegend[{Blend[{{alist[[1]], RGBColor[128/255, 0, 128/255]},
                  {alist[[2]], RGBColor[1, 0, 0]}, {alist[[2]], RGBColor[1, 128/255, 0]},
                  {alist[[3]], RGBColor[1, 1, 0]}, {alist[[3]], RGBColor[128/255, 1, 0]},
                  {alist[[4]], RGBColor[0, 102/255, 0]}, {alist[[4]], RGBColor[0, 0, 1]},
                  {alist[[5]], RGBColor[0, 1, 1]}}, #] &, {alist[[1]], alist[[5]]}}],
            4, BarLegend[{Blend[{{alist[[1]], RGBColor[128/255, 0, 128/255]},
                  {alist[[2]], RGBColor[1, 0, 0]}, {alist[[2]], RGBColor[1, 128/255, 0]},
                  {alist[[3]], RGBColor[1, 1, 0]}, {alist[[3]], RGBColor[128/255, 1, 0]},
                  {alist[[4]], RGBColor[0, 102/255, 0]}}, #] &, {alist[[1]], alist[[4]]}}],
            3, BarLegend[{Blend[{{alist[[1]], RGBColor[128/255, 0, 128/255]},
                  {alist[[2]], RGBColor[1, 0, 0]}, {alist[[2]], RGBColor[1, 128/255, 0]},
                  {alist[[3]], RGBColor[1, 1, 0]}}, #] &, {alist[[1]], alist[[3]]}}]
           ]]];
        )
```

The following function plots the steady states for a fixed value of b, but varying values of a and color them with respect to position of the a value to the critical values of a as in the left side of Figure 4 of the paper.

```
In[26]:= plotTask[bb_, alist_, NNL_] := (
           SSlist = solvorSequencePoints[bb, alist, NNL];
           colorlist = pointsColorGen[alist, NNL];
           Return[plotSubTask[{SSlist, colorlist}, alist]]
          )
```

This section contains the plots used in Figure 4 of the paper.

Sequences for 0 < b < 0.4961661.

b = 0.1

```
In[27]:= N[criticalValuesA[1/10]]
```
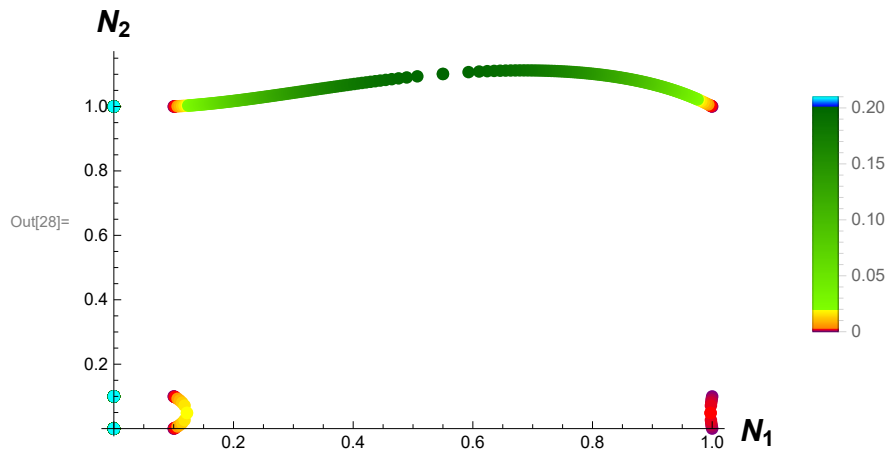
```
Out[27]= {0.00238297, 0.0194548, 0.2025}
```

```
In[28]:= plotTask[1/10,
          {0, 0.002382970637023988`, 0.019454811603742996`, 0.2025`, 0.21}, {5, 5, 100, 1}]
```


Out[28]=

b = 0.3

```
In[•]:= N[criticalValuesA[3/10]]
```

```
Out[•]= {0.01986, 0.0504942, 0.1225}
```

```
In[13]:= plot = plotTask1[3/10,
          {0, 0.01986001392249426`, 0.050494173879918135`, 0.1225`, 0.13}, {20, 20, 90, 1}];
         Show[
          plot]
```

Out[14]=
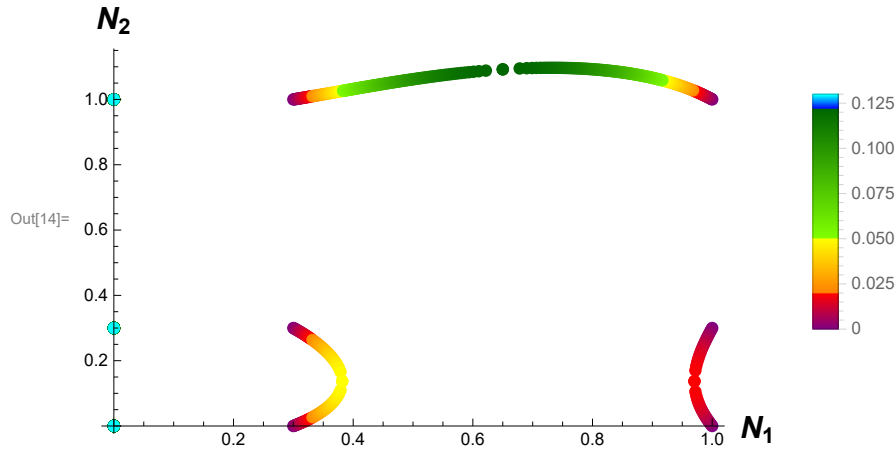
In[15]:=
```
s1 = Graphics[{Text[Style[ToExpression["S_1", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.05, 0.075}]}];
s2 = Graphics[{Text[Style[ToExpression["S_2", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.25, 0.075}]}];
s3 = Graphics[{Text[Style[ToExpression["S_3", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.925, 0.075}]}];
s4 = Graphics[{Text[Style[ToExpression["S_4", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.95, 0.375}]}];
s5 = Graphics[{Text[Style[ToExpression["S_5", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.25, 0.375}]}];
s6 = Graphics[{Text[Style[ToExpression["S_6", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.05, 0.375}]}];
s7 = Graphics[{Text[Style[ToExpression["S_7", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.05, 0.95}]}];
s8 = Graphics[{Text[Style[ToExpression["S_8", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.25, 0.95}]}];
s9 = Graphics[{Text[Style[ToExpression["S_9", TeXForm, HoldForm],
        Black, Bold, FontSize → 16], {0.95, 0.95}]}];
Show[plot, s1, s2, s3, s4, s5, s6, s7, s8, s9]
```
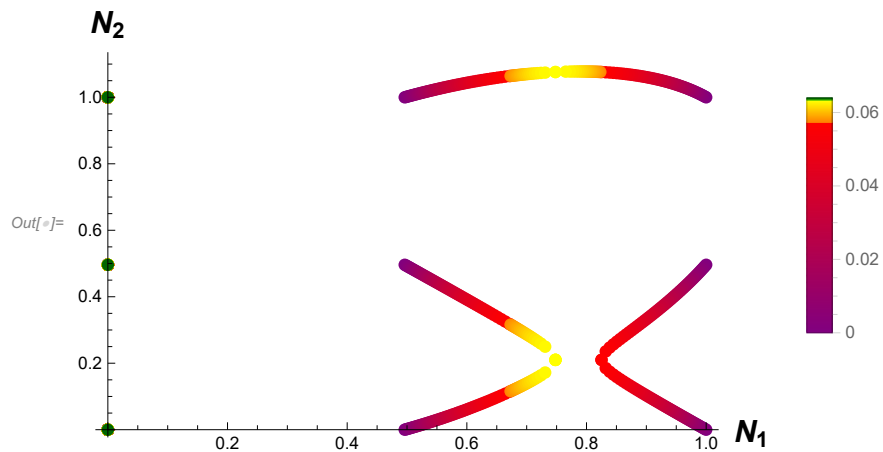
Out[24]=
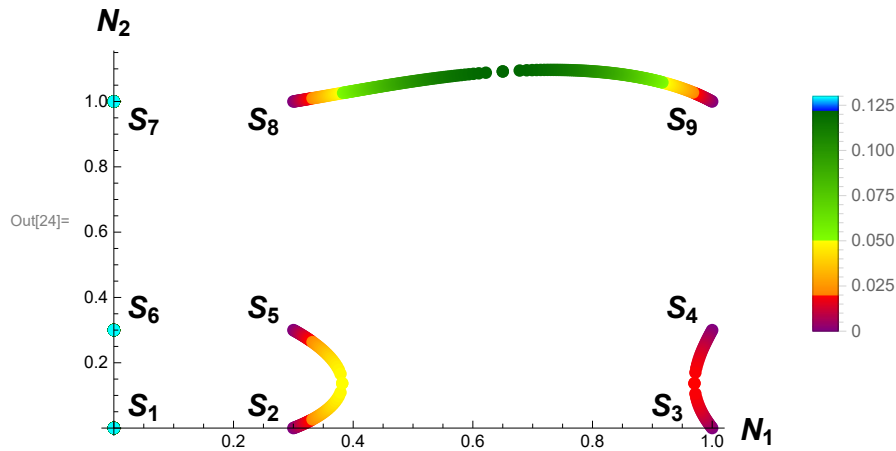
Sequence for b=0.4961661. **(Critical Value)**

In[◦]:= **N[criticalValuesA[4 961 661 * 10^(-7)]]**
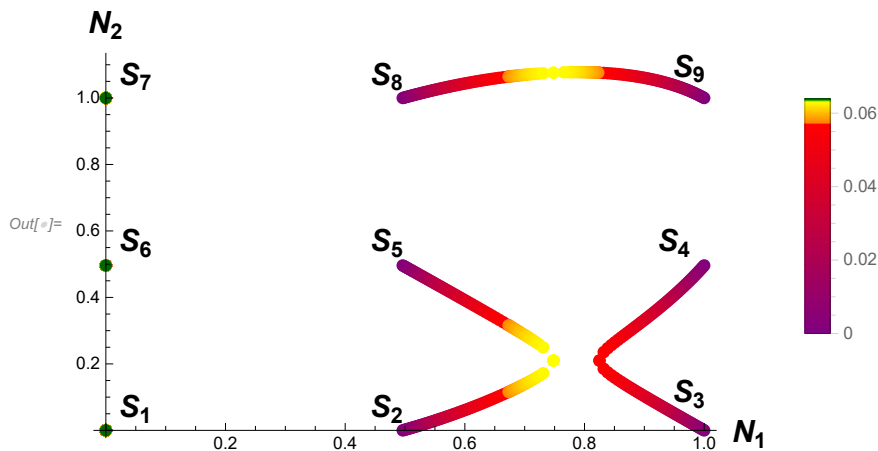
Out[◦]= {0.057545, 0.0634621, 0.0634621}

In[◦]:= **plot = plotTask[4 961 661 * 10^(-7),**
{0, 0.057544964217530525`, 0.06346214969730232`, 0.064}, {50, 20, 1}];
Show[
 plot]

```
In[ ]:= s1 = Graphics[{Text[Style[ToExpression["S_1", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.05, 0.075}]}];
     s2 = Graphics[{Text[Style[ToExpression["S_2", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.47, 0.075}]}];
     s3 = Graphics[{Text[Style[ToExpression["S_3", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.975, 0.12}]}];
     s4 = Graphics[{Text[Style[ToExpression["S_4", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.95, 0.575}]}];
     s5 = Graphics[{Text[Style[ToExpression["S_5", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.47, 0.575}]}];
     s6 = Graphics[{Text[Style[ToExpression["S_6", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.05, 0.575}]}];
     s7 = Graphics[{Text[Style[ToExpression["S_7", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.05, 1.075}]}];
     s8 = Graphics[{Text[Style[ToExpression["S_8", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.47, 1.075}]}];
     s9 = Graphics[{Text[Style[ToExpression["S_9", TeXForm, HoldForm],
          Black, Bold, FontSize → 16], {0.975, 1.1}]}];
     Show[plot, s1, s2, s3, s4, s5, s6, s7, s8, s9]
```


Sequence for 0.4961661 < b < 1/2.

b = 0.948

```
In[ ]:= N[criticalValuesA[498 / 1000]]
```
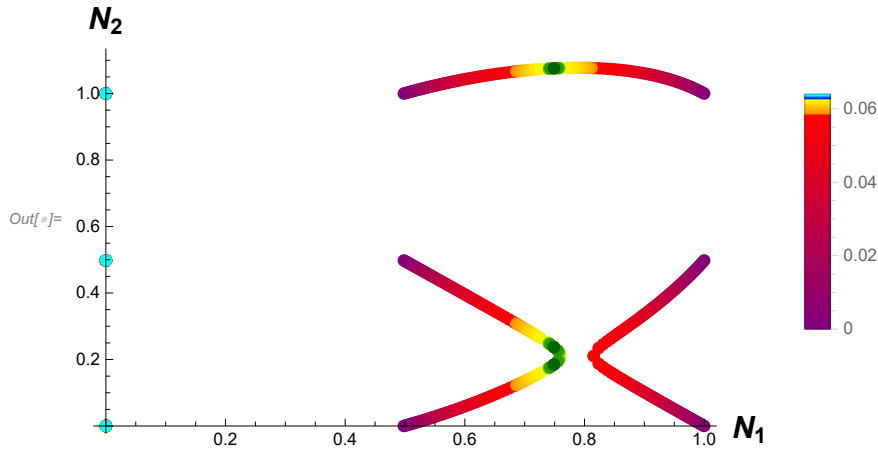
```
Out[ ]= {0.0586017, 0.0628843, 0.063001}
```

```
In[ ]:= plot = plotTask[498 / 1000,
       {0, 0.05860173612241946`, 0.06288434240012102`, 0.063001`, 0.064}, {50, 10, 3, 1}];
     Show[
      plot]
```
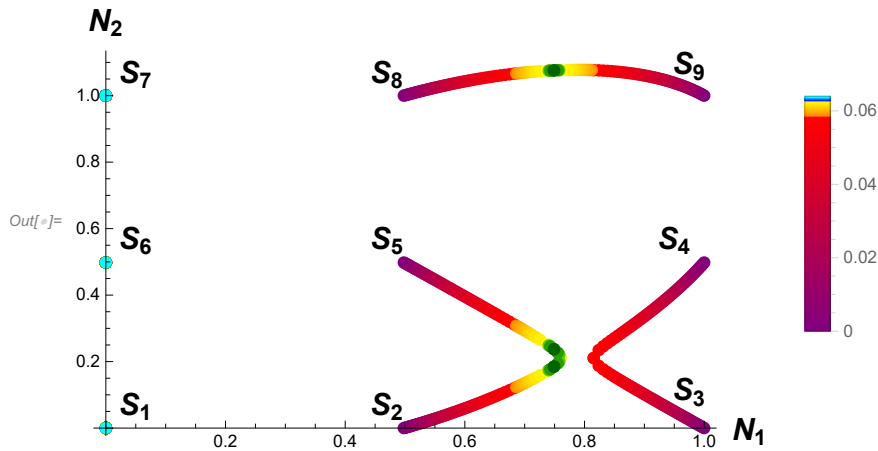
```
In[•]:= s1 = Graphics[{Text[Style[ToExpression["S_1", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 0.075}]}];
      s2 = Graphics[{Text[Style[ToExpression["S_2", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.47, 0.075}]}];
      s3 = Graphics[{Text[Style[ToExpression["S_3", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.975, 0.12}]}];
      s4 = Graphics[{Text[Style[ToExpression["S_4", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.95, 0.575}]}];
      s5 = Graphics[{Text[Style[ToExpression["S_5", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.47, 0.575}]}];
      s6 = Graphics[{Text[Style[ToExpression["S_6", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 0.575}]}];
      s7 = Graphics[{Text[Style[ToExpression["S_7", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 1.075}]}];
      s8 = Graphics[{Text[Style[ToExpression["S_8", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.47, 1.075}]}];
      s9 = Graphics[{Text[Style[ToExpression["S_9", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.975, 1.1}]}];
      Show[plot, s1, s2, s3, s4, s5, s6, s7, s8, s9]
```
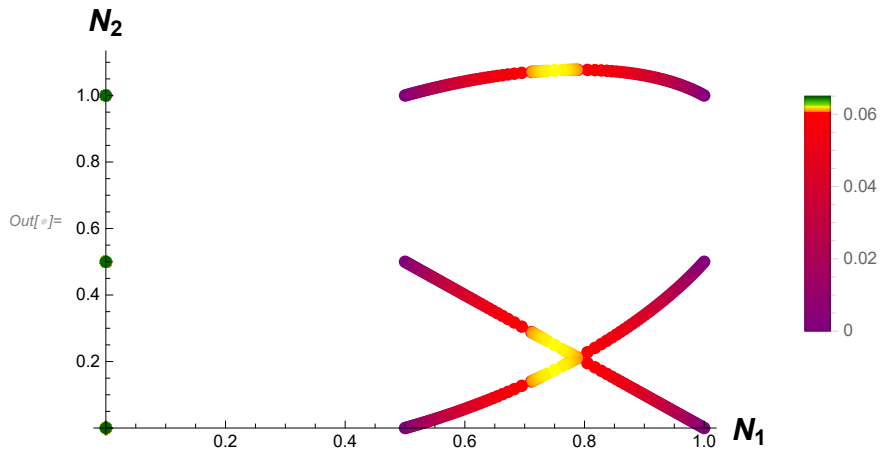
Sequence for b=1/2. **(Critical value)**

In[ ]:= **N[criticalValuesA[1/2]]**
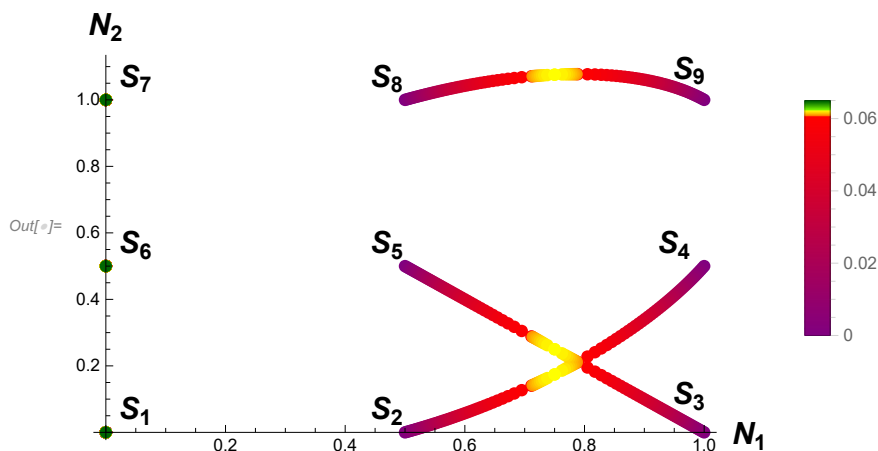
Out[ ]= {0.0610042, 0.0625}

In[ ]:= **plot = plotTask[1/2, {0, 0.0610042339640731`, 0.0625`, 0.065}, {40, 10, 1}];**
Show[plot]

```
In[ ]:= s1 = Graphics[{Text[Style[ToExpression["S_1", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 0.075}]}];
        s2 = Graphics[{Text[Style[ToExpression["S_2", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.47, 0.075}]}];
        s3 = Graphics[{Text[Style[ToExpression["S_3", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.975, 0.12}]}];
        s4 = Graphics[{Text[Style[ToExpression["S_4", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.95, 0.575}]}];
        s5 = Graphics[{Text[Style[ToExpression["S_5", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.47, 0.575}]}];
        s6 = Graphics[{Text[Style[ToExpression["S_6", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 0.575}]}];
        s7 = Graphics[{Text[Style[ToExpression["S_7", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 1.075}]}];
        s8 = Graphics[{Text[Style[ToExpression["S_8", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.47, 1.075}]}];
        s9 = Graphics[{Text[Style[ToExpression["S_9", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.975, 1.1}]}];
        Show[plot, s1, s2, s3, s4, s5, s6, s7, s8, s9]
```
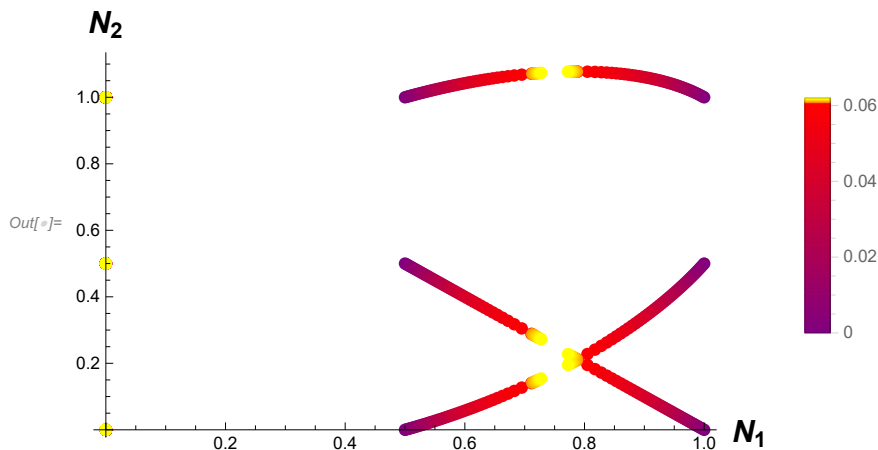

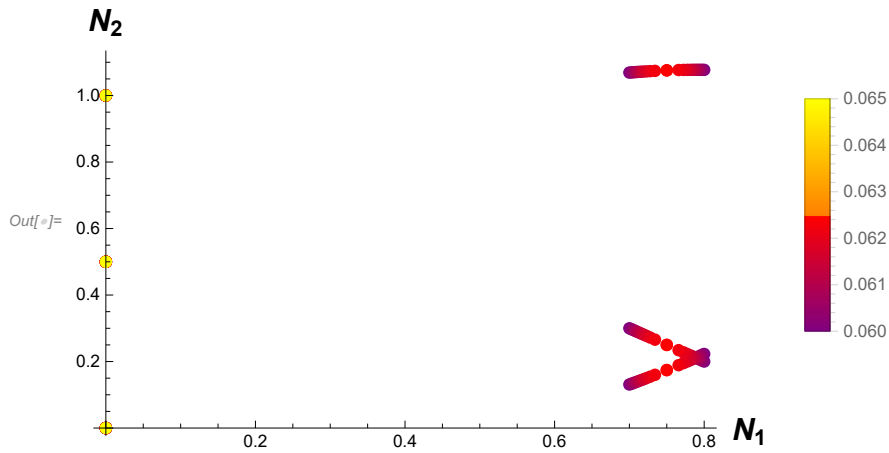Restricting a value to smaller interval for a better observation.

```
In[ ]:= plotTask[1 / 2, {0, 0.0610042339640731`, 0.062}, {40, 10}]
```

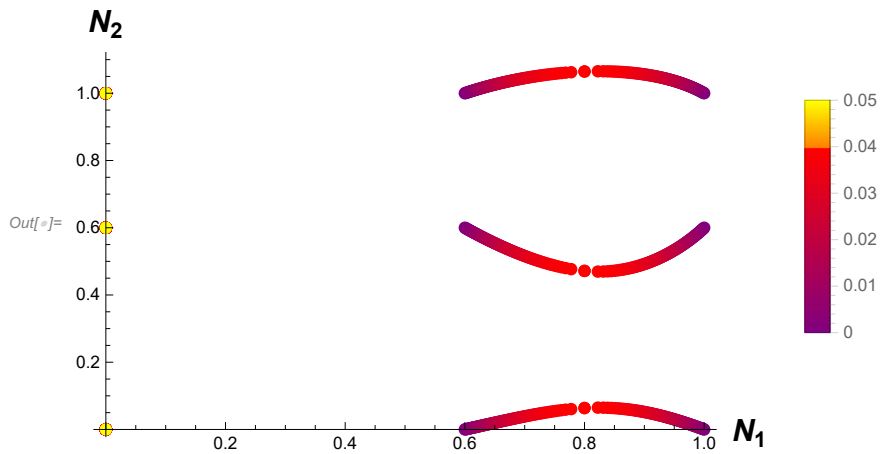*In[ ]:=* **plotTask$[1/2, \{0.06, 0.0625`, 0.065\}, \{10, 1\}]$**

Out[ ]=

Sequences for b > 1/2.

b = 0.6

In[ ]:= **N$[\text{criticalValuesA}[6/10]]$**

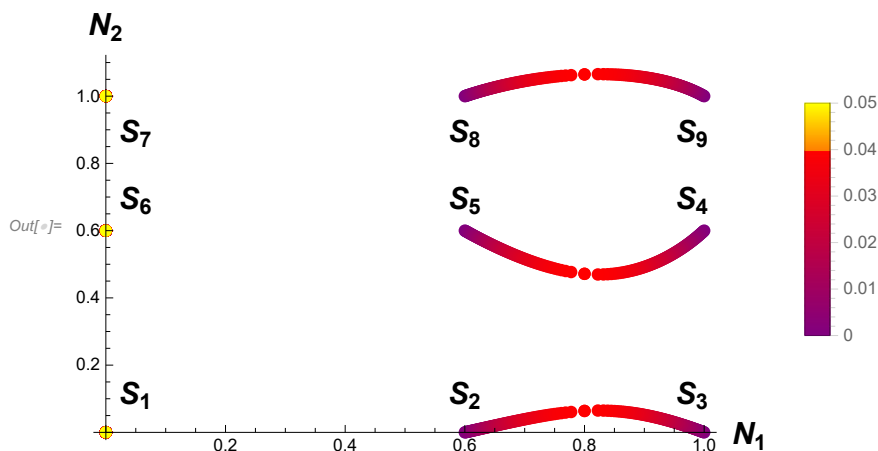*Out[ ]=* $\{0.04\}$

In[ ]:= **plot = plotTask$[6/10, \{0, 0.04`, 0.05\}, \{80, 1\}]$;**
Show[plot]

Out[ ]=

```
In[ ]:= s1 = Graphics[{Text[Style[ToExpression["S_1", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 0.12}]}];
        s2 = Graphics[{Text[Style[ToExpression["S_2", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.6, 0.12}]}];
        s3 = Graphics[{Text[Style[ToExpression["S_3", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.98, 0.12}]}];
        s4 = Graphics[{Text[Style[ToExpression["S_4", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.98, 0.7}]}];
        s5 = Graphics[{Text[Style[ToExpression["S_5", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.6, 0.7}]}];
        s6 = Graphics[{Text[Style[ToExpression["S_6", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 0.7}]}];
        s7 = Graphics[{Text[Style[ToExpression["S_7", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.05, 0.9}]}];
        s8 = Graphics[{Text[Style[ToExpression["S_8", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.6, 0.9}]}];
        s9 = Graphics[{Text[Style[ToExpression["S_9", TeXForm, HoldForm],
            Black, Bold, FontSize → 16], {0.98, 0.9}]}];
        Show[plot, s1, s2, s3, s4, s5, s6, s7, s8, s9]
```


$b = 0.9$
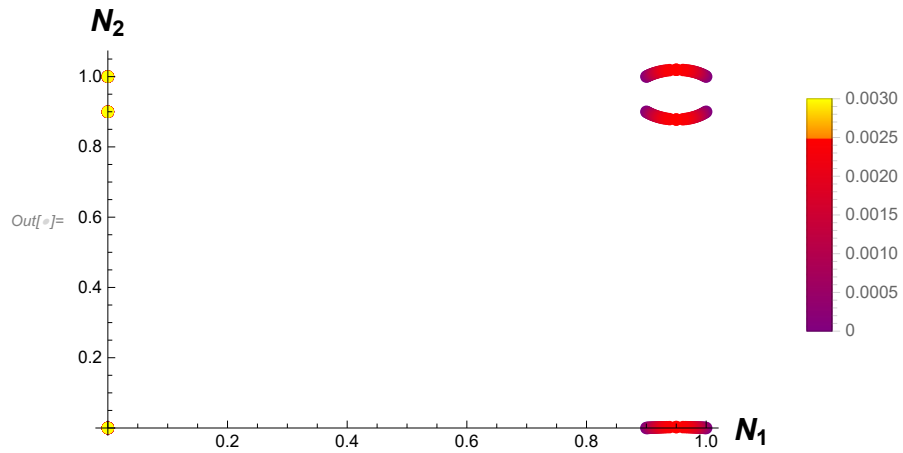
```
In[ ]:= N[criticalValuesA[9/10]]

Out[ ]= {0.0025}
```

```
In[ ]:= plotTask[9/10, {0, 0.0025`, 0.003}, {20, 1}]
```

End of the file.

Source paper: Gergely Rost and AmirHosein Sadeghimanesh, Two connected populations with Allee effect, 2021.