This Maple worksheet contains the computations for the paper "Two connected populations with Allee effect" where the parameter region is partitioned according to the number of non-negative steady states that the system (1) of the paper has.

Both way migration.

Consider the parametric system of polynomial equations given at equation (2) at the paper with the inequality constraints on the variables and the parameters, and \delta_{1,2}=\delta_{2,1}=1.

We define F to be the list of the two polynomial equations resulted by letting \dot{N_1}=\dot_{N_2}=0.

```
> F:=[N[1]*(1-N[1])*(N[1]-b)-a*N[1]+a*N[2],N[2]*(1-N[2])*(N[2]-b)+a*N[1]-a*N[2]]:
  <seq(i, i in F)>;
```

$$\begin{bmatrix} N_1\left(1-N_1\right)\left(N_1-b\right)-a\,N_1+a\,N_2 \\ N_2\left(1-N_2\right)\left(N_2-b\right)+a\,N_1-a\,N_2 \end{bmatrix} \qquad \textbf{(1.1)}$$

One can easily check that the number of nonnegative solutions for F=0 when "b=a=0" or "b=1 and a=0" is 4, and when "b=0 and a \neq 0" or "b=1 and a \neq 0" is 2.

If a,b > 0, then N_1=0 together with F=0 implies N_2=0 and vice versa, N_2=0 together with F=0 implies N_1=0. Therefore for every a,b>0, (0,0) is one nonnegative solution to the system, and the only boundary solution. Therefore we can reduce the constraints to N_1,N_2,a,b > 0 which are acceptable constraints for the maple package RootFinding[Parametric]. Note that this package uses an open CAD algorithm, therefore non-strict inequalities are not acceptable inputs for its predefined functions.

In the following "cadTwoSidedModel" stores the result of "CAD with respect to the discriminant variety" of the parametric system made by F=0 and the positivity constraints on the variables and the parameters.

```
> cadTwoSidedModel:=RootFinding:-Parametric:-CellDecomposition(
  [seq(i=0,i in F),seq(N[i]>0,i=1..2),a>0,b>0],[seq(N[i],i=1..2)
  ],[a,b]):
```

Up to here Maple has decomposed the positive orthant of the parameter space to a finite number of disjoint open connected sets so that the number of positive solutions to the system F=0 is invariant on each of these sets. Note that there might be more than one open region in this decomposition with the same number of solutions.

ListNumberSolutions is a list which its i-th member is the number of positive solutions to the system F=0 on the i-th open region in the above decomposition. Then we ask Maple to print the converted version of this list to a set.

```
> ListNumberSolutions:=RootFinding:-Parametric:-NumberOfSolutions
  (cadTwoSidedModel):
  {seq(i[2],i in ListNumberSolutions)};
```

$$\{2, 4, 8\} \tag{1.2}$$

Therefore the number of possible positive solutions to the system on open regions in the positive orthant of the parameter space are 2, 4 and 8. Together with the only boundary soultion, i.e. (0,0), the sytem can have 3, 5 or 9 nonnegative solutions.

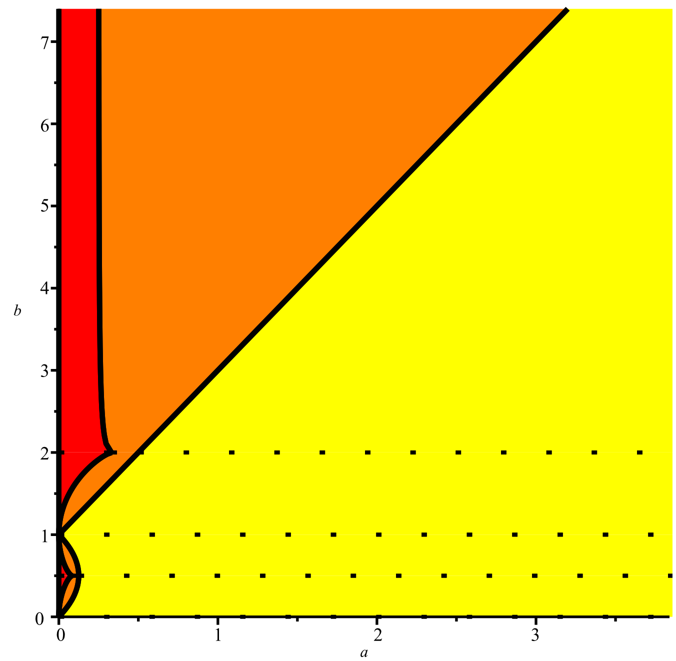
cellsIndexTable is a table (Maple's table data structure is similar to Python's dictionary data structure) storing index of the open regions in the decomposition with specific number of solutions.

```
> cellsIndexTable:=table([seq(i=RootFinding:-Parametric:-
  CellsWithSolutions(cadTwoSidedModel,i),i in (1.2))]);
```

$$cellsIndexTable := \mathrm{table}([2 = [3, 6, 9, 12], 4 = [2, 5, 8, 11], 8 = [1, 4, 7, 10]]) \tag{1.3}$$

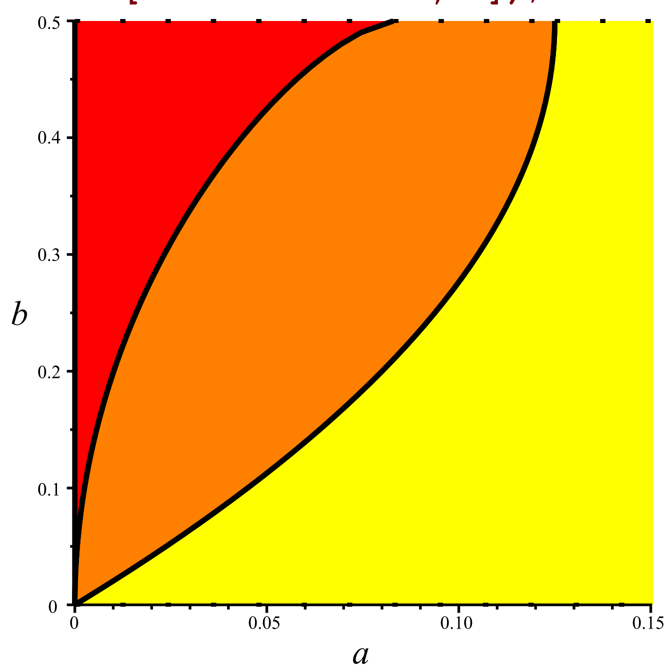
In below we ask Maple to plot these regions and color them with red, orange and yellow if the number of positive solutions on them are 8, 4 or 2 respectively.

```
> A2 := Array(1 .. numelems(cellsIndexTable[2]), 0):
  A4 := Array(1 .. numelems(cellsIndexTable[4]), 0):
  A8 := Array(1 .. numelems(cellsIndexTable[8]), 0):
  for i to numelems(cellsIndexTable[2]) do
      A2[i] := RootFinding:-Parametric:-CellPlot
  (cadTwoSidedModel, cellsIndexTable[2][i], color = yellow):
  end do:
  for i to numelems(cellsIndexTable[4]) do
      A4[i] := RootFinding:-Parametric:-CellPlot
  (cadTwoSidedModel, cellsIndexTable[4][i], color = coral):
  end do:
  for i to numelems(cellsIndexTable[8]) do
      A8[i] := RootFinding:-Parametric:-CellPlot
  (cadTwoSidedModel, cellsIndexTable[8][i], color = red):
  end do:
  PTwoSidedModel := plots:-display(seq(A2[i], i = 1 .. numelems
  (cellsIndexTable[2])), seq(A4[i], i = 1 .. numelems
  (cellsIndexTable[4])), seq(A8[i], i = 1 .. numelems
  (cellsIndexTable[8])));
```



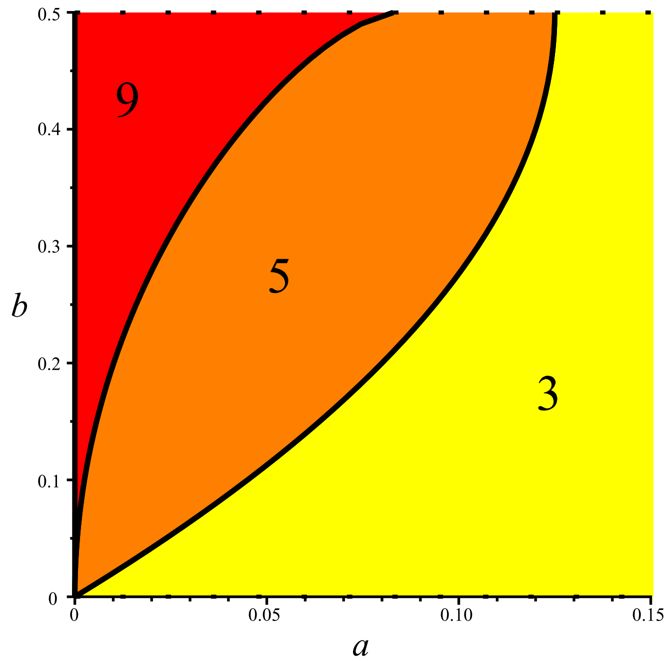
As one can see from this figure, the decomposition of the parameter space with respect to the number of non-negative solutions to the system F=0, has symmetries with respect to b=1/2 and b=1. Which we already were aware of it using Lemmas 2.1 and 2.2 from the reference [5] of the paper.

```
> plots:-display(PTwoSidedModel,view=[0..0.15,0..0.5],'labels'=
  [a,b],'labelfont'=["TimesNewRoman",20]);
```




```
> text1:=plots:-textplot([0.01,0.4,"9"],'align'={'above','right'}
  ,'color'='black','font'=["TimesNewRoman",30]):
  text2:=plots:-textplot([0.05,0.25,"5"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):
  text3:=plots:-textplot([0.12,0.15,"3"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):
  plots:-display(PTwoSidedModel,text1,text2,text3,view=[0..0.15,
```

```
0..0.5],'labels'=[a,b],'labelfont'=["TimesNewRoman",20]);
```



This is Figure 2 of the paper.

One way migration.

Consider the parametric system of polynomial equations given at equation (2) at the paper with the inequality constraints on the variables and the parameters ,and \delta_{1,2}=1, \delta_{2,1}=0.

We define F to be the list of the two polynomial equations resulted by letting \dot{N_1}=\dot_{N_2}=0.

```
> F:=[N[1]*(1-N[1])*(N[1]-b)-a*N[1],N[2]*(1-N[2])*(N[2]-b)+a*N[1]
  ]:
  <seq(i, i in F)>;
```

$$\begin{bmatrix} N_1 (1-N_1)(N_1-b) - a N_1 \\ N_2 (1-N_2)(N_2-b) + a N_1 \end{bmatrix} \tag{2.1}$$

One can easily check that the number of nonnegative solutions for F=0 when "b=a=0" or "b=1 and a=0" is 4, and when "b=0 and a \neq 0" or "b=1 and a \neq 0" is 2.

If a,b > 0, then N_1=0 together with F=0 implies N_2 = 0 or 1 or b. If a,b > 0, then N_2=0 together with F=0 implies N_1=0. Therefore for every a,b>0, (0,0), (0,b) and (0,1) are nonnegative solutions to the system, and the only boundary solutions. Therefore we can reduce the constraints to N_1,N_2, a,b > 0 which are acceptable constraints for the maple package RootFinding[Parametric]. Note that this package uses an open CAD algorithm, therefore non-strict inequalities are not acceptable inputs for its predefined functions.

In the following "cadOneSidedModel" stores the result of "CAD with respect to the discriminant variety" of the parametric system made by F=0 and the positivity constraints on the variables and the parameters.

```
> cadOneSidedModel:=RootFinding:-Parametric:-CellDecomposition(
  [seq(i=0,i in F),seq(N[i]>0,i=1..2),a>0,b>0],[seq(N[i],i=1..2)
  ],[a,b]):
```

Up to here Maple has decomposed the positive orthant of the parameter space to a finite number of disjoint open connected sets so that the number of positive solutions to the system F=0 is invariant on each of these sets. Note that there might be more than one open region in this decomposition with the same number of solutions.

ListNumberSolutions is a list which its i-th member is the number of positive solutions to the system F=0 on the i-th open region in the above decomposition. Then we ask Maple to print the converted version of this list to a set.

```
> ListNumberSolutions:=RootFinding:-Parametric:-NumberOfSolutions
  (cadOneSidedModel):
  {seq(i[2],i in ListNumberSolutions)};
```

$$\{0, 2, 4, 6\} \quad \textbf{(2.2)}$$

Therefore the number of possible positive solutions to the system on open regions in the positive orthant of the parameter space are 0, 2, 4 and 6. Together with the three boundary soultion, the sytem can have 3, 5, 7 or 9 nonnegative solutions.

cellsIndexTable is a table (Maple's table data structure is similar to Python's dictionary data structure) storing index of the open regions in the decomposition with specific number of solutions.

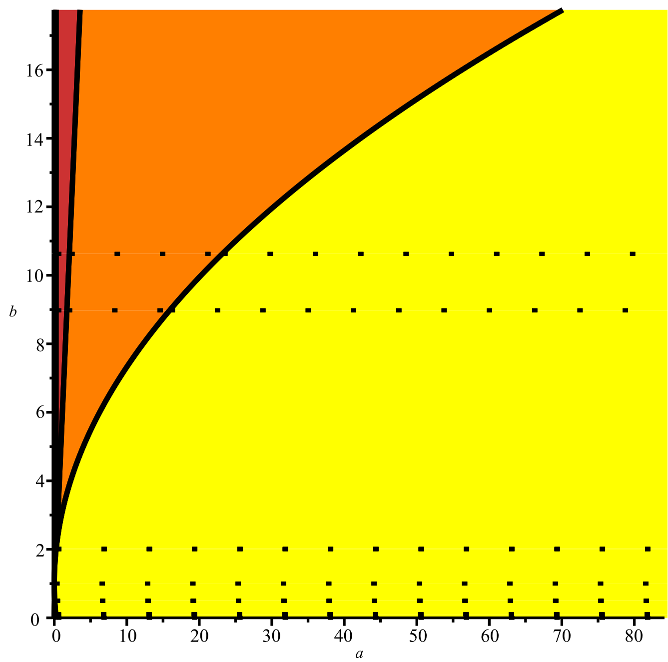
```
> cellsIndexTable:=table([seq(i=RootFinding:-Parametric:-
  CellsWithSolutions(cadOneSidedModel,i),i in (2.2))]);
```

$$cellsIndexTable := \mathrm{table}([0 = [4, 8, 12, 16, 18, 20, 24, 28, 32, 36], 2 = [3, 7, 11, 27, 31, 35], 4 = [2, 6, 10, 14, 22, 26, 30, 34], 6 = [1, 5, 9, 13, 15, 17, 19, 21, 23, 25, 29, 33]]) \quad \textbf{(2.3)}$$

In below we ask Maple to plot these regions and color them with red, brown, orange and yellow if the number of positive solutions on them are 6, 4, 2 or 0 respectively.

```
> A0 := Array(1 .. numelems(cellsIndexTable[0]), 0):
  A2 := Array(1 .. numelems(cellsIndexTable[2]), 0):
  A4 := Array(1 .. numelems(cellsIndexTable[4]), 0):
  A6 := Array(1 .. numelems(cellsIndexTable[6]), 0):
  for i to numelems(cellsIndexTable[0]) do
      A0[i] := RootFinding:-Parametric:-CellPlot
  (cadOneSidedModel, cellsIndexTable[0][i], color = yellow):
  end do:
  for i to numelems(cellsIndexTable[2]) do
      A2[i] := RootFinding:-Parametric:-CellPlot
```

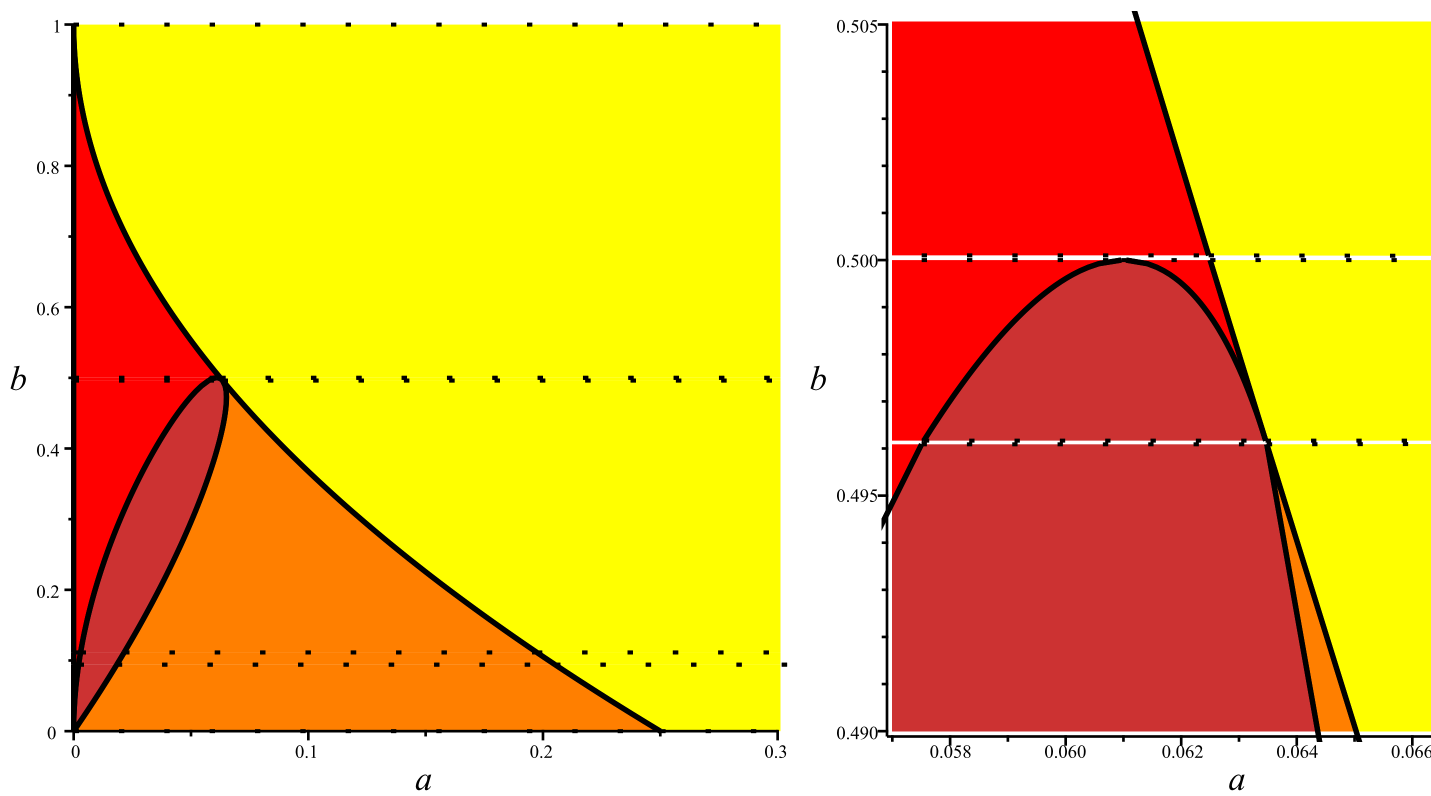
```
(cadOneSidedModel, cellsIndexTable[2][i], color = coral):
end do:
for i to numelems(cellsIndexTable[4]) do
    A4[i] := RootFinding:-Parametric:-CellPlot
(cadOneSidedModel, cellsIndexTable[4][i], color = orange):
end do:
for i to numelems(cellsIndexTable[6]) do
    A6[i] := RootFinding:-Parametric:-CellPlot
(cadOneSidedModel, cellsIndexTable[6][i], color = red):
end do:
POneSidedModel := plots:-display(seq(A0[i], i = 1 .. numelems
(cellsIndexTable[0])),seq(A2[i], i = 1 .. numelems
(cellsIndexTable[2])), seq(A4[i], i = 1 .. numelems
(cellsIndexTable[4])), seq(A6[i], i = 1 .. numelems
(cellsIndexTable[6])));
```



As one can see from this figure, the decomposition of the parameter space with respect to the number of non-negative solutions to the system F=0, has only one symmetry with respect to and b=1 and there is no symmetry with respect to b=1/2. Which we already were aware of it. The symmetry with resect to b=1 can be proven using Lemma 2.2 from the reference [5] of the paper.

```
> cadPlotFull:=plots:-display(POneSidedModel,view=[0..0.3,0..1],
'labels'=[a,b],'labelfont'=["TimesNewRoman",20]):
cadPlotZoomed:=plots:-display(POneSidedModel,view=[0.057 ..
0.069, 0.49 .. 0.505],'labels'=[a,b],'labelfont'=
["TimesNewRoman",20]):
resulto := DocumentTools:-Tabulate([cadPlotFull,
cadPlotZoomed], exterior = none, interior = none, weights =
[100, 100], widthmode = pixels, width = 600);
```

$$resulto := \text{"Tabulate"} \tag{2.4}$$

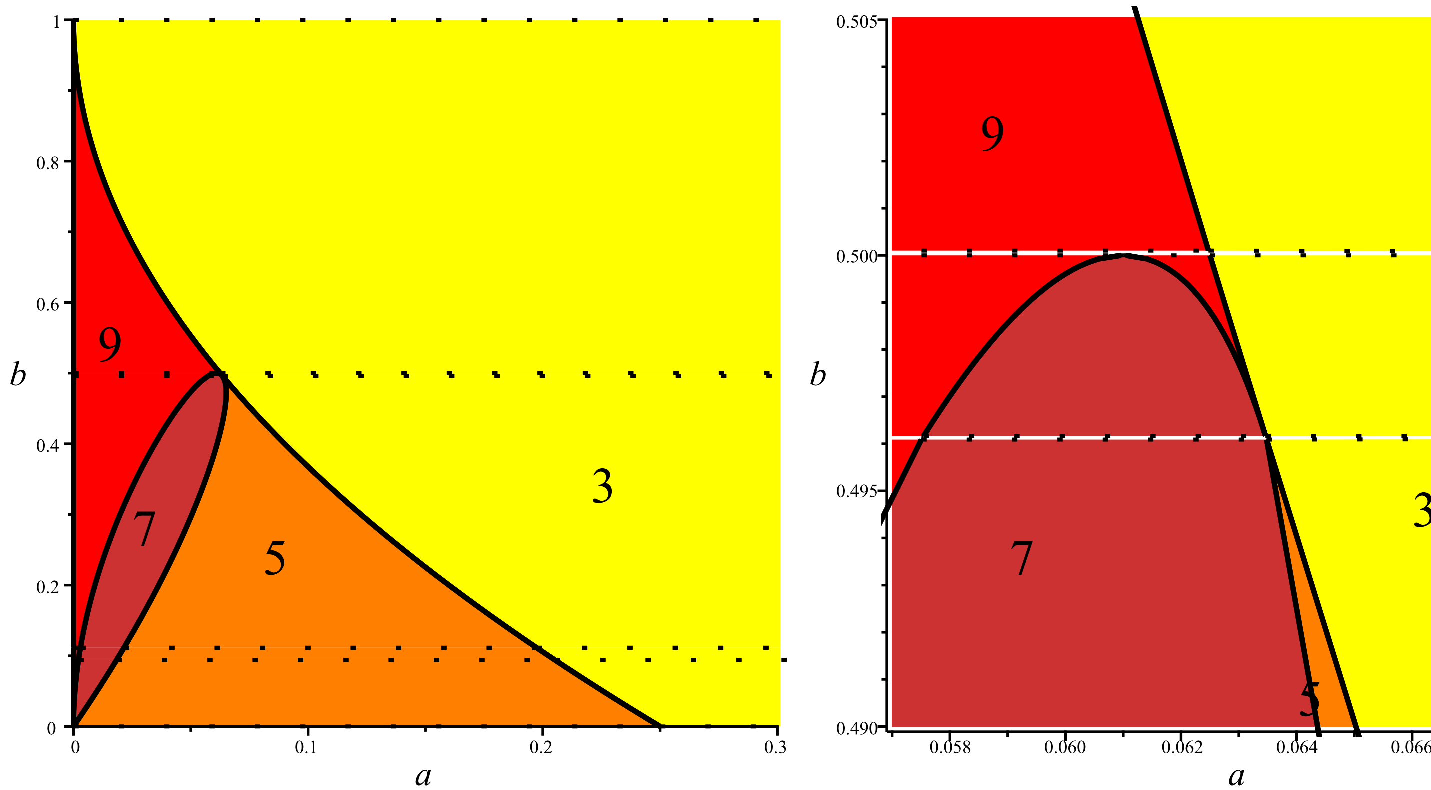



```
> textL1:=plots:-textplot([0.01,0.5,"9"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):
  textL2:=plots:-textplot([0.025,0.24,"7"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):
  textL3:=plots:-textplot([0.08,0.2,"5"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):
  textL4:=plots:-textplot([0.22,0.3,"3"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):

  textR1:=plots:-textplot([0.0585,0.502,"9"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):
  textR2:=plots:-textplot([0.059,0.493,"7"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):
  textR3:=plots:-textplot([0.064,0.490,"5"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):
  textR4:=plots:-textplot([0.066,0.494,"3"],'align'={'above',
  'right'},'color'='black','font'=["TimesNewRoman",30]):

  cadPlotFull:=plots:-display(POneSidedModel,textL1,textL2,
  textL3,textL4,view=[0..0.3,0..1],'labels'=[a,b],'labelfont'=
  ["TimesNewRoman",20]):
  cadPlotZoomed:=plots:-display(POneSidedModel,textR1,textR2,
  textR3,textR4,view=[0.057 .. 0.069, 0.49 .. 0.505],'labels'=[a,
  b],'labelfont'=["TimesNewRoman",20]):
  DocumentTools:-Tabulate([cadPlotFull, cadPlotZoomed], exterior
  = none, interior = none, weights = [100, 100], widthmode =
  pixels, width = 600);
```

"Tabulate11" (2.5)



This is Figure 3 of the paper.

End of the file.

Source paper: Gergely Rost and AmirHosein Sadeghimanesh, Two connected populations with Allee effect, 2021.