Assignment 1 - Math 201

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Question 1

Solve the Equation: $x' = x^2 - 4$

Separating Variables:

$$\begin{aligned} g(x) &= x^2 - 4 \\ h(t) &= 1 \end{aligned} \Rightarrow \frac{dx}{dy} = g(x) * h(t)$$

Integrating:

$$\int \frac{dx}{x^2 - 4} = \int 1dt$$

Via Partial Fractions:

$$\frac{1}{x^2 - 4} = \frac{A(x-2) + B(x+2)}{(x+2)(x-2)}$$

$$A = -\frac{1}{4}, B = \frac{1}{4}$$

Substituting:

$$\int \frac{-\frac{1}{4}}{x+2}\, dx + \int \frac{\frac{1}{4}}{x-2}\, dx = \int 1dt$$

The Implicit Solution:

$$-\frac{1}{4}\ln|x+2| + \frac{1}{4}\ln|x-2| = t + c$$

Solving for x:

Via log rules:

$$\ln\frac{x-2}{x+2} = 4(t+c)$$
$$\frac{x-2}{x+2} = e^{4t+4c}$$
$$x - 2 = xe^{4t+4c} + 2e^{4t+4c}$$
$$x = \frac{2e^{4t+4c} + 2}{1 - e^{4t+4c}}$$

Which simplifies to our **explicit solution**:

$$x = \frac{-2(e^{4t+4c} + 1)}{e^{4t+4c} - 1}$$

However since e^{4c} is simply a constant, we can simplify further to:

$$x = \frac{-2(e^{4t} * k + 1)}{e^{4t} * k - 1}$$

Singular Solutions

Already accounted for in the general solution.

$$x = 2 \Rightarrow x' = 2^2 - 4 = 0$$
$$\gamma(t) = 2, \gamma : \mathbb{R} \to \mathbb{R}$$

Required singular solution.

$$x = -2 \Rightarrow x' = (-2)^2 - 4 = 0$$
$$\gamma(t) = -2, \gamma : \mathbb{R} \to \mathbb{R}$$

Question 2

What is the half-life of Krypton 85, knowing that it's decay rate is 6.3% per year?

$k = \dfrac{-6.3}{100} = -0.063$
$C = 1$
$x(t) = .5$

The general solution: $x(t) = Ce^{kt}$

Substituting:

$0.5 = 1e^{-0.063t}$

Solving:

$\dfrac{\ln 0.5}{-0.063} = t$
$t = 11.0023$

$\therefore$ The half life is 11.0023 years.

Question 3

Solve the initial value problem: $y' = [(t-1)y]^{-\frac{1}{3}}$

When $y(0) = 1$.

Expanding:
$y' = (t-1)^{-\frac{1}{3}}y^{-\frac{1}{3}}$

Separating Variables:

$$\begin{aligned} g(y) &= y^{-\frac{1}{3}} \\ h(t) &= (t-1)^{-\frac{1}{3}} \end{aligned} \quad \Rightarrow \frac{dy}{g(y)} = h(t)dt$$

Integrating:

$$\int \frac{dy}{y^{\frac{1}{3}}} = \int \frac{1}{(t-1)^{\frac{1}{3}}}\,dt$$

The implicit solution:
$$\frac{3}{4}y^{\frac{4}{3}} = \frac{3}{2}(t-1)^{\frac{2}{3}} + c$$

Solving for y:

$$y^{\frac{4}{3}} = 2(t-1)^{\frac{2}{3}} + c$$

The explicit solution:
$$y = [2(t-1)^{\frac{2}{3}} + c]^{\frac{3}{4}}$$

Finding the c:

Using the initial value: $y(0) = 1$

$$y(0) = 1 = [2(0-1)^{\frac{2}{3}} + c]^{\frac{3}{4}}$$

Solving for c:
$$1 = [2(-1)^{\frac{2}{3}} + c]^{\frac{3}{4}}$$
$$c = 1^{\frac{4}{3}} - 2(-1)^{\frac{2}{3}}$$
$$c = 1 - 2(1) = -1$$

Substituting Back

$$y = (2(t-1)^{\frac{2}{3}} - 1)^{\frac{3}{4}}$$

Question 4

Solve the equation $x' = tx + 6te^{-t^2}$

Noting that this is a linear non-homogeneous equation.

$$\begin{array}{r} F(t) = t \\ g(t) = 6te^{-t^2} \end{array} \Rightarrow x' = F(t)x + g(t)$$

Considering $x' = F(t)x$:

$$\frac{dx}{dt} = f(t)x = tx$$
$$x = ce^{\frac{1}{2}t^2}$$

Making c a variable:
$$x(t) = c(t)e^{\frac{1}{2}t^2}$$

$$x'(t) = c'(t)e^{\frac{1}{2}t^2} + tc(t)e^{\frac{1}{2}t^2}$$

Substituting Back:

$$t(c(t)e^{\frac{1}{2}t^2}) + 6te^{-t^2} = c'(t)e^{\frac{1}{2}t^2} + tc(t)e^{\frac{1}{2}t^2}$$

Noting how the terms containing $c(t)$ cancel out:
$$6te^{-t^2} = c'(t)e^{\frac{1}{2}t^2}$$

Solving for $c'(t)$:
$$c'(t) = 6te^{-t^2}e^{-\frac{1}{2}t^2}$$

Noting that $6te^{-t^2} = g(t)$ and $(-\frac{1}{2}t) = \phi$:
$$c(t) = \int g(t)e^{\phi t}dt$$
$$c(t) = \int 6te^{-t^2}e^{-\frac{1}{2}t^2}dt$$
$$c(t) = -2e^{-\frac{3}{2}t^2} + k$$

Substituting $c(t)$:

$$x = (-2e^{-\frac{3}{2}t^2} + k)e^{\frac{1}{2}t^2}$$

Question 5

Solve the initial value problem: $r' = [\cot 2\phi]r - \cos 2\phi$

When $r(\frac{\pi}{4}) = 0.$

Noting that this is a linear non-homogeneous equation.

$$\begin{matrix} F(\phi) = \cot 2\phi \\ g(\phi) = \cos 2\phi \end{matrix} \Rightarrow r' = F(\phi)r + g(\phi)$$

Considering $r' = F(\phi)r$:

$$\frac{dr}{d\phi} = F(\phi)r = \cot 2\phi r$$
$$\int \frac{dr}{r} = \int \cot 2\phi d\phi$$
$$r = ce^{\frac{1}{2}\ln \sin 2\phi}$$
$$r = c\sin^{\frac{1}{2}}(2\phi)$$

Making c a variable:

$$r(\phi) = c(\phi)\sin^{\frac{1}{2}} 2\phi$$
$$r'(\phi) = c'(\phi)\sin^{\frac{1}{2}}(2\phi) + c(\phi)\sin^{-\frac{1}{2}}(2\phi) * (-\frac{1}{2})(-2\cos(2\phi))$$

Simplifying:

$$r'(\phi) = c'(\phi)\sin^{\frac{1}{2}}(2\phi) + \frac{\cos(2\phi)c(\phi)}{\sin^{\frac{1}{2}}(2\phi)}$$

Substituting Back

$$F(\phi)r + g(\phi) = r'$$
$$c(\phi)\cot(2\phi)\sin^{\frac{1}{2}}(2\phi) - \cos(2\phi) = c'(\phi)\sin^{\frac{1}{2}}(2\phi) + \frac{\cos(2\phi)c(\phi)}{\sin^{\frac{1}{2}}(2\phi)}$$

Observing how terms containing $c(\phi)$ cancel out:

$$-\cos(2\phi) = c'(\phi)\sin^{\frac{1}{2}}(2\phi)$$

Solving for $c'(\phi)$:
$$c'(\phi) = \frac{-\cos(2\phi)}{\sin^{\frac{1}{2}} 2\phi}$$

Integrating to find $c(\phi)$:
$$c(\phi) = -\sqrt{\sin 2\phi} + k$$

Substituting $c(\phi)$:

$$r = (-\sqrt{\sin 2\phi} + k)\sin^{\frac{1}{2}}(2\phi)$$

Finding k:

$$r(-\frac{\pi}{4}) = 0 = (\sqrt{\sin 2\frac{\pi}{4}} + k)\sin^{\frac{1}{2}}(2\frac{\pi}{4})$$
$$k = 1$$

Giving us the **solution**:
$$r = (-\sqrt{\sin 2\phi} + 1)\sin^{\frac{1}{2}}(2\phi)$$