Markov Models

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Yahtzee is a game of chance, but is it properly weighted for each possibility? In order to investigate this question, we will be looking at Yahtzee through a mathematical lens. We will use Markov Probability Models to calculate the probability of each point value found in Yahtzee's point system, and compare our results to the values of points in Yahtzee. We will then be able to see how fair the weighting of points in Yahtzee is.

Our models and results show that the point values for a Yahtzee are not weighted according to the probability. Specifically, other options such as a full house have a much higher expected value than a Yahtzee.

Introduction

What is Yahtzee?

Yahtzee is a game where players roll six-sided dice. After each roll, the player has the option to roll any number of the five dice again (up to a total of 3 rolls). Then, the values of the dice are evaluated and scored. For example, a player can score a round based on three of a kind, sixes, or full house.

This process is continued for 13 rounds. However, the same categories for scoring cannot be used more than once per game (set of 13 rounds). For example, if three of a kind or used to score one round, then the player cannot user three of a kind to score another round. Finally, the scores are totaled from each round, with the player with the highest score winning the game.

Using Markov Models to Analyze Yahtzee

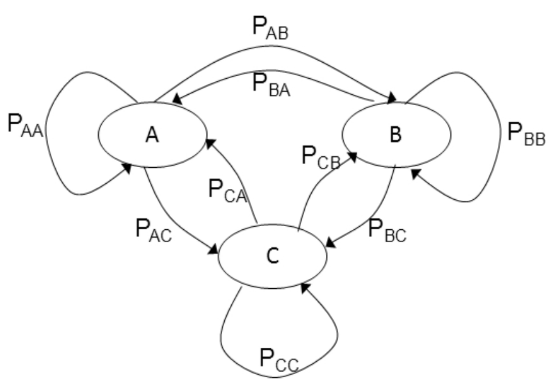
Markov Probability Models are state based conditional directed graphs that allow predictions on systems where independence cannot be assumed. Graphs can be represented as matrices, which are where Markov systems gain their power.

Markov Probability Models are a good fit for analyzing Yahtzee, since each die can be represented as having six states, ranging from one to six, and also because each new roll has some transition, whether that be to the same state or a new state. The probability of each type of set based on the scoring scale will be used to compare our findings to the values provided by the game of Yahtzee. Our goal is to examine how closely the point values match with the probability of rolling a certain pattern of dice.

Formulation

We will be using Markov Models to find the probability of the events. Markov models are graphs of events, that have weighted edges with the probability of their occurrence. This graph is then converted into a matrix where the values in row n column i represent the odds of moving from n to i. This is a useful formulation because if you take the product of two of these matrices then it shows the odds of going from one state to another after 2 changes of state. A proof, mostly to convince myself, is in Appendix I for the two by two case.

Graphical Example



$$\mathbf{P}=\begin{bmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{bmatrix}$$

[Figure 1: Probability transition diagram for a 3-state Markov chain]

Because we can use a matrix to represent a directed graph, we can use this matrix to apply a weight to the edits and predict the outcome of events. Figure 1 shows an example of a graph and the corresponding matrix.[1]

To find the odds of a transition, P can be multiplied by the column vector $\mathbf{x} = (P_A, P_B, P_C)^T$, where P_X is the probability that the system is in state

[1] http://www.intechopen.com/books/matlab-a-fundamental-tool-for-scientific-computing-and-engineering-applications-volume-2/wireless-channel-model-with-markov-chains-using-matlab

X. So, $P\mathbf{x}$ results in the probability after one set of changes, and $P(P\mathbf{x})$ results in the probability after two state changes. This can be simplified to $P(P\mathbf{x} = (PP)\mathbf{x} = P^2\mathbf{x}$. This can be further generalized to state that the probability after n state changes is: $P^n\mathbf{x}$. In the case of Yahtzee, we can use this to optimize how we play so that we can consistently beat other players.

Results

For the sake of brevity the derivations of the individual probabilities will be left to Appendix 2. In order to express the matrices in integers they will be written as $M = D^{-1}P$ where D is a diagonal matrix and P has elements $p_{i,j}$, the number of ways to move from state i to state j. Along with the M,P,D matrices we will define $E \in \mathbb{R}$ to be the expected value $\vec{s} \in \mathbb{R}^n$ to be the scores for the states and $\vec{x} \in \mathbb{R}^n$ to be incoming state distribution. The value of $E = \vec{x} \times M^3 \times \vec{s}$.

Probability of rolling a specific number

$$D = \begin{pmatrix} 7776 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1296 & 0 & 0 & 0 & 0 \\ 0 & 0 & 216 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 3125 & 3125 & 1250 & 250 & 25 & 1 \\ 0 & 625 & 500 & 150 & 20 & 1 \\ 0 & 0 & 125 & 75 & 15 & 1 \\ 0 & 0 & 0 & 25 & 10 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\vec{s} = \begin{pmatrix} 0n \\ 1n \\ 2n \\ 3n \\ 4n \\ 5n \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rolling for a specific number happens on the "top" of the scorecard and a score for the specific number is assigned by multiplying the number of dice of that number by the numbers. The expected value in terms of the die rolled for, n, is $2.1065 \times n$. The states are rolling 5,4,3,2,1,0 dice respectively.

Probability of getting a Yatzee

$$D = \begin{pmatrix} 7776 & 0 & 0 & 0 & 0 \\ 0 & 216 & 0 & 0 & 0 \\ 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 720 & 5400 & 1500 & 150 & 6 \\ 0 & 120 & 80 & 15 & 1 \\ 0 & 0 & 25 & 10 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\vec{s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 50 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rolling for a Yatzee gives an expected value of 2.3014 points. The states of the matrix are in oder rolling 5,3,2,1,0 dice while having the rest match.

Probability of getting a Full House

$$D = \begin{pmatrix} 7776 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 36 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 720 & 3600 & 6 & 1350 & 1800 & 300 \\ 0 & 12 & 0 & 4 & 8 & 2 \\ 0 & 0 & 1 & 30 & 0 & 5 \\ 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\vec{s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 25 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The expected value of going for a full house is 7.8431. The states of the full house are rolling all five dice, having a set of 2 and a set of 1, having a set of three (essentially a Yatzee), having a set of three and a set of one, having two sets of 2, and having a Full House, in that order.

Conclusions

Our results show that the point system for a game of Yahtzee is not necessarily based on the expected values for each of the possible scoring criterea. For example, while the expected value of a Yahtzee is about 2.3, the expected value for a full house (which the game rules give less points to than rolling a Yahtzee) is 7.8, slightly more than triple the expected value for a Yahtzee. This shows that while the game's scoring system encourages a certain style of play (namely, trying for Yahtzees), a much migher return can be expected from rolling for different combinations (such as a full house).

References

Appendix I: Proving Markov Models Work

2 × 2 matrices

Assumptions

Let $A \in \mathbb{R}^{2\times 2}$
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$a + b = 1$
$c + d = 1$
a is the probability of moving from state O to state O in one turn.
b is the probability of moving from state O to state T in one turn.
c is the probability of moving from state T to state O in one turn.
b is the probability of moving from state T to state T in one turn.

Let XY_n be the event of moving from state X to Y in n turns.
Let $P(X_n)$ be the probability of being in state X on turn n.
Let X,Y and Z be generically either states O or T.

Derivation for 2 × 2

(1) $P(XY_2) = P(OY_1) \times P(O_1|X_0) + P(TY_1) \times P(T_1|X_0)$
(2) $P(Z_1|X_0) = P(XZ_1)$
(3) $P(XY_2) = P(OY_1) \times P(XO_1) + P(TY_1) \times P(XT_1)$

This general formula can be used to find the odds of going from state to state in 2 turns.
$P(OO_2) = P(OO_1) \times P(OO_1) + P(OT_1) \times P(TO_1) = a \times a + b \times c$
$P(OT_2) = P(OO_1) \times P(OT_1) + P(OT_1) \times P(TT_1) = a \times b + b \times d$
$P(TO_2) = P(TO_1) \times P(OO_1) + P(TT_1) \times P(TO_1) = c \times a + d \times c$
$P(TT_2) = P(TO_1) \times P(OT_1) + P(TT_1) \times P(TT_1) = c \times b + d \times d$

$$A^2 = \begin{pmatrix} a \times a + b \times c & a \times b + b \times d \\ a \times c + c \times d & b \times c + d \times d \end{pmatrix} = \begin{pmatrix} P(OO_2) & P(OT_2) \\ P(TO_2) & P(TT_2) \end{pmatrix}$$

n × n matrices

Assumptions

Let $A \in \mathbb{R}^{n \times n}$ where $a_{i,j}$ is the entry of A in the i^{th} row and j^{th} column.

$$\sum_j a_{i,j} = 1$$

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Let $(O_iO_j)_n$ be the event of moving from state i to state j in n turns.
Let $P(T_{i,n})$ be the event of being in state i on turn n.

Derivation

$$P((O_iO_j)_n) = \sum_k P(O_kO_j) \times P(O_{k,turn}|O_{i,turn_0})$$