Exploring Markov Models with Yahtzee

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Abstract

Yahtzee is a game of chance, but is it properly weighted for each possibility? In order to investigate this question, we will be looking at Yahtzee through a mathematical lens. We will use Markov Probability Models to calculate the probability of each point value found in Yahtzee's point system, and compare our results to the values of points in Yahtzee. We will then be able to see how fair the weighting of points in Yahtzee is.

Our models and results show that the point values for a Yahtzee are not weighted according to the probability. Specifically, other options such as a full house have a much higher expected value than a Yahtzee.

1 Introduction

1.1 What is Yahtzee?

Yahtzee is a game where players roll six-sided dice. After each roll, the player has the option to roll any number of the five dice again (up to a total of 3 rolls). Then, the values of the dice are evaluated and scored. For example, a player can score a round based on three of a kind, sixes, or full house.

This process is continued for 13 rounds. However, the same categories for scoring cannot be used more than once per game (set of 13 rounds). For example, if three of a kind or used to score one round, then the player cannot user three of a kind to score another round. Finally, the scores are totaled from each round, with the player with the highest score winning the game.

1.2 Using Markov Models to Analyze Yahtzee

Markov Probability Models are state based conditional directed graphs that allow predictions on systems where independence cannot be assumed. Graphs can be represented as matrices, which are where Markov systems gain their power.

Markov Probability Models are a good fit for analyzing Yahtzee, since each die can be represented as having six states, ranging from one to six, and also because each new roll has some transition, whether that be to the same state or a new state. The probability of each type of set based on the scoring scale will be used to compare our findings to the values provided by the game of Yahtzee. Our goal is to examine how closely the point values match with the probability of rolling a certain pattern of dice.

2 Formulation

We will be using Markov Models to find the probability of the events. Markov models are graphs of events, that have weighted edges with the probability of their occurrence. This graph is then converted into a matrix where the values in row n column i represent the odds of moving from state n to i. This is a useful formulation because if you take the product of two of these matrices then it shows the odds of going from one state to another after 2 changes of state. A proof, mostly to convince myself, is in Appendix I for the two by two case.

We use this to represent the probability to get certain outcome from a starting state when the outcome doesn't depend on path. For example to path to getting a Yahtzee doesn't matter, all that matters is the outcome, and the hopefully 50 points that come along with it.

Now rather than having to find the odds for each possibility and generalize those for n changes of state, we can find the odds of changing from each state to another and the generalsation to n state changes is done for us.

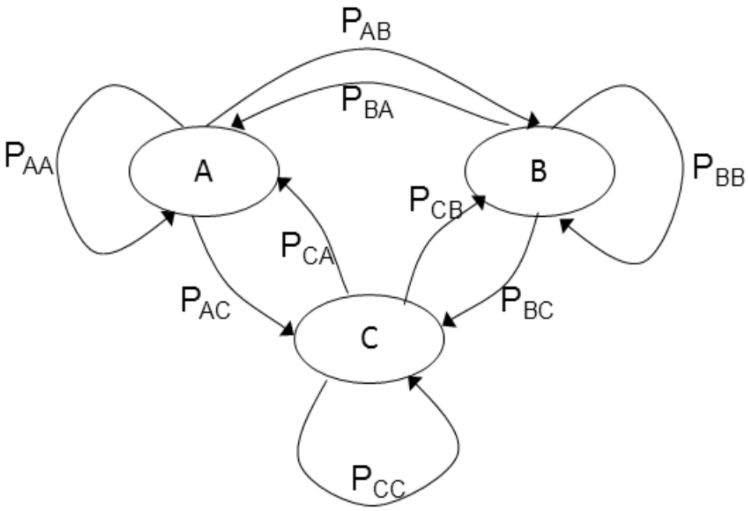
We also express the markov matrix as the product of two matrices, that we denote as D and P. D is a diagonal matrix that has the integer number of possible ways to move from the state of the row/column that it belongs to

2.1 Graphical Example

Because we can use a matrix to represent a directed graph, we can use this matrix to apply a weight to the edits and predict the outcome of events. Figure 1 shows an example of a graph and the corresponding matrix.[1]

To find the odds of a transition, P can be multiplied by the column vector $\mathbf{x} = (P_A, P_B, P_C)^T$, where P_X is the probability that the system is in state X. So, $P\mathbf{x}$ results in the probability after one set of changes, and $P(P\mathbf{x})$ results in the probability after two state changes. This can be simplified to $P(P\mathbf{x}) = (PP)\mathbf{x} = P^2\mathbf{x}$. This can be further generalized to state that the probability after n state changes is: $P^n\mathbf{x}$. In the case of Yahtzee, we can use this to optimize how we play so that we can consistently beat other players.

[1] http://www.intechopen.com/books/matlab-a-fundamental-tool-for-scientific-computing-and-engineering-applications-volume-2/wireless-channel-model-with-markov-chains-using-matlab



$$\mathbf{P}=\begin{bmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{bmatrix}$$

3 Results

For the sake of brevity the derivations of the individual probabilities will be left to Appendix 2. In order to express the matrices in integers they will be written as $M = D^{-1}P$ where D is a diagonal matrix and P has elements $p_{i,j}$, the number of ways to move from state i to state j. Along with the M,P,D matrices we will define $E \in \mathbb{R}$ to be the expected value $\vec{s} \in \mathbb{R}^n$ to be the scores for the states and $\vec{x} \in \mathbb{R}^n$ to be incoming state distribution. The value of $E = \vec{x} \times M^3 \times \vec{s}$.

3.1 Probability of rolling a specific number

$$D = \begin{pmatrix} 7776 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1296 & 0 & 0 & 0 & 0 \\ 0 & 0 & 216 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 3125 & 3125 & 1250 & 250 & 25 & 1 \\ 0 & 625 & 500 & 150 & 20 & 1 \\ 0 & 0 & 125 & 75 & 15 & 1 \\ 0 & 0 & 0 & 25 & 10 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\vec{s} = \begin{pmatrix} 0n \\ 1n \\ 2n \\ 3n \\ 4n \\ 5n \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rolling for a specific number happens on the "top" of the scorecard and a score for the specific number is assigned by multiplying the number of dice of that number by the numbers. The expected value in terms of the die rolled for, n, is $2.1065 \times n$. The states are rolling 5,4,3,2,1,0 dice respectively.

3.2 Probability of getting a Yatzee

$$D = \begin{pmatrix} 7776 & 0 & 0 & 0 & 0 \\ 0 & 216 & 0 & 0 & 0 \\ 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 720 & 5400 & 1500 & 150 & 6 \\ 0 & 120 & 80 & 15 & 1 \\ 0 & 0 & 25 & 10 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\vec{s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 50 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rolling for a Yatzee gives an expected value of 2.3014 points. The states of the matrix are in oder rolling 5,3,2,1,0 dice while having the rest match.

3.3 Probability of getting a Full House

$$D = \begin{pmatrix} 7776 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 36 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 720 & 3600 & 6 & 1350 & 1800 & 300 \\ 0 & 11 & 0 & 10 & 12 & 3 \\ 0 & 0 & 1 & 30 & 0 & 5 \\ 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\vec{s} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 25 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The expected value of going for a full house is 9.0249. The states of the full house are rolling all five dice, having a set of 2 and a set of 1, having a set of three (essentially a Yatzee), having a set of three and a set of one, having two sets of 2, and having a Full House, in that order.

4 Conclusions

Our results show that the point system for a game of Yahtzee is not necessarily based on the expected values for each of the possible scoring criterea. For example, while the expected value of a Yahtzee is about 2.3 points, the expected value for a full house (which the game rules give less points to than rolling a Yahtzee) is about 9 points, slightly more than triple the expected value for a Yahtzee. This shows that while the game's scoring system encourages a certain style of play (namely, trying for Yahtzees), a much migher return can be expected from rolling for different combinations (such as a full house).

The part of the game that really produces points, more than either the Yahtzee or the Full House is the top, which gives an expected value 2.1 times the number rolled for as an expected value, so going for sixes gives an expected value of almost 13 points, being the biggest bang for your buck in Yahtzee.

5 References

Sites we used to build a mental basis for Markov Methods:

`http://ethanmarkowitz.com/index.php/2015/05/24/hacking-chutes-and-ladders-using-r/`
`http://www.datagenetics.com/blog/january42012/`

6 Appendix I: Proving Markov Models Work on 2 by 2 Matricies

6.0.1 Assumptions

Let $A \in \mathbb{R}^{2\times 2}$
$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$a + b = 1$
$c + d = 1$
a is the probability of moving from state O to state O in one turn.
b is the probability of moving from state O to state T in one turn.
c is the probability of moving from state T to state O in one turn.
b is the probability of moving from state T to state T in one turn.

Let XY_n be the event of moving from state X to Y in n turns.
Let $P(X_n)$ be the probability of being in state X on turn n.
Let X,Y and Z be generically either states O or T.

6.1 Derivation

(1) $P(XY_2) = P(OY_1) \times P(O_1|X_0) + P(TY_1) \times P(T_1|X_0)$
(2) $P(Z_1|X_0) = P(XZ_1)$
(3) $P(XY_2) = P(OY_1) \times P(XO_1) + P(TY_1) \times P(XT_1)$

This general formula can be used to find the odds of going from state to state in 2 turns.
$P(OO_2) = P(OO_1) \times P(OO_1) + P(OT_1) \times P(TO_1) = a \times a + b \times c$
$P(OT_2) = P(OO_1) \times P(OT_1) + P(OT_1) \times P(TT_1) = a \times b + b \times d$
$P(TO_2) = P(TO_1) \times P(OO_1) + P(TT_1) \times P(TO_1) = c \times a + d \times c$
$P(TT_2) = P(TO_1) \times P(OT_1) + P(TT_1) \times P(TT_1) = c \times b + d \times d$

$$A^2 = \begin{pmatrix} a \times a + b \times c & a \times b + b \times d \\ a \times c + c \times d & b \times c + d \times d \end{pmatrix} = \begin{pmatrix} P(OO_2) & P(OT_2) \\ P(TO_2) & P(TT_2) \end{pmatrix}$$

7 Appedix II: Getting the odds

7.1 Yatzee

7.1.1 rolling 0 dice

This now is nice and easy there is ony one way for the dice to do anything, and thats to stay the same. So there is a 1 in column 5.

Total rolls: $6^0 = 1$

7.1.2 rolling 1 die

Five of the six sides do not match the other 4 dice, so for column 4 there's 5 possibilities, and there's one way to match the other dice, so for column 5 theres 1 possibility.

Total rolls: $6^1 = 6$

7.1.3 rolling 2 dice

There's 5^2 ways to not match, $\binom{2}{1} * 5$ ways to match one of the two, and 1 way to match both.

Total rolls: $6^2 = 36$

7.1.4 rolling 3 dice

Three dice is the one state that's a little tricky, because if the three dice match and aren't the number I was saving I should switch numbers.

There's 1 way to roll a Yahtzee, $\binom{3}{2} * 5$ ways to move to rolling 1 die, $\binom{3}{1} * 5^2$ ways to rolling 2 dice plus an additional 5 ways where the dice match but don't match the two that were saved, to a total of $\binom{3}{1} * 5^2 + 5$. Finally theres $5^3 - 5$ ways to stay at rolling 3 dice.

Total rolls: $6^3 = 216$

7.1.5 rolling 5 dice

Five dice is quite nice, because all the cases are special in the same way, simply that there isn't a number that it going for.

There are 6 ways to straight roll a Yahtzee.

There are 6 ways to pick a number times $5 choose 4$ ways to pick what dice will be that number times 5 things the other die could be, so $6 * \binom{5}{4} * 5 = 150$.

There are two ways to get a three of a kind, a full house which occurs $6 * \binom{5}{3} * 5 * \binom{2}{2} = 300$ ways,and three of a kind and two non matching numbers which occurs $6 * \binom{5}{3} * 5 * 4 = 1200$ ways, so in all it can occur 1500 ways.

There are also two ways to get two of a kind, two pairs and another number $6 * \binom{5}{2} * 5 * \binom{4}{2} = 1800$ and then there's the two pair and no other matching number case so $6 * \binom{5}{2} * 5 * 4 * 3 = 3600$ so there's $1800 + 3600 = 5400$ ways

There are $6 * 5 * 4 * 3 * 2 = 720$ ways to have no matching numbers and then to stay in the current state

Total rolls: $6^5 = 7776$

7.1.6 Resulting matricies

Plugging all of these numbers into their places results in the following matrix.

$$\begin{pmatrix} 720 & 5400 & 1500 & 150 & 6 \\ 0 & 120 & 80 & 15 & 1 \\ 0 & 0 & 25 & 10 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This is then multiplied by the inverse of the diagonal scaling matrix:

$$\begin{pmatrix} 7776 & 0 & 0 & 0 & 0 \\ 0 & 216 & 0 & 0 & 0 \\ 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

7.2 Rolling for a specific number

This one can be fully described in a simple formula, moving from rolling n dice to r dice, given that $n \geq r$ is $\binom{n}{r} * 5^r$. This makes getting the matrix much easier. Also the number to scale by is 6^n

7.2.1 Resulting matricies

$$\begin{pmatrix} 3125 & 3125 & 1250 & 250 & 25 & 1 \\ 0 & 625 & 500 & 150 & 20 & 1 \\ 0 & 0 & 125 & 75 & 15 & 1 \\ 0 & 0 & 0 & 25 & 10 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 7776 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1296 & 0 & 0 & 0 & 0 \\ 0 & 0 & 216 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

7.3 Full House

7.3.1 Holding no dice

The only way to stay rolling no dice is if all are different so $6 * 5 * 4 * 3 * 2 = 720$

The way to go to a single is only if the first two match and the restare unique, so $\binom{5}{2} * 6 * 5 * 4 * 3 = 3600$.

To get 3 of a number and 0 of any other is a Yahtzee, and there's 6 ways to do that.

The 3 of one and 1 of another has $\binom{5}{3} * 6 * 5 * 4 = 1200$ ways the first ones can match and the last two don't, and another $\binom{5}{4} * 6 * 5 = 150$ ways the four dice will match with one odd ball, for a total of 1350 possibilities.

Then there's $\binom{5}{2} * 6 * \binom{3}{2} * 5 * 2 = 1800$ ways to get exactly 2 pair.

Finally there's $\binom{5}{3} * 6 * 5 = 300$ ways to staright roll a Yahtzee.

7.3.2 Holding a Pair and a single

There are 2 ways to move to a Full House from here, getting a pair that matches the single, 1 possiblity, or having each die match one of the numbers, 2 possibilities, for a total of 3 possibilities.

There are two wayst o move to two pair, one is matching the single and not the double, $\binom{2}{1} * 1 * 4 = 8$ possibilities, and the other is having the pair match a number other than the single so 4 possibilities, bringing it to a total of 12.

Then there's $\binom{2}{1} * 1 * 5 = 10$ ways to move to the 3 and 1 case.

Finally there's 11ways to stay in the current state.

7.3.3 Holding 3 of a kind

To move to a full house one of 5 pairs is needed, so there's 5 ways to move to full house, and then to stay an pair that matches the three is needed, so 1 way to stay. In addition there are $5^2 = 25$ ways to moving to holding 3 of a number and 1 of another.

7.3.4 Holding 3 of one number and 1 of another

To change state a specific number is needed, so 1 chance to move up, 5 to stay

7.3.5 Holding 2 pair

If you are holding two pairs then the die can be any of two of six, so 4 chances to stay and 2 to move.

7.3.6 Full House!

Easy case, if you have a full house stick with it!!

7.3.7 Resulting matricies

$$\begin{pmatrix} 720 & 3600 & 6 & 1350 & 1800 & 300 \\ 0 & 11 & 0 & 10 & 12 & 3 \\ 0 & 0 & 1 & 30 & 0 & 5 \\ 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 7776 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 36 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

8 Appendix III: Code

```
function yatzeeProb
        disp('Expected value when rolling for 1')
        disp(expectedVal([1,0,0,0,0,0],[0;1;2;3;4;5],NumberM(),3))
        disp('Expected value when rolling for a Yatzee')
        disp(expectedVal([1,0,0,0,0],[0;0;0;0;50],YatzeeM(),3))
        disp('Expected value when rolling for a Full House')
        disp(expectedVal([1,0,0,0,0,0],[0;0;0;0;0;25],FullHouseM(),3))
end

function num = expectedVal(in,vals,Mat,times)
        %{
        Params:
        in: row vector of the odds going in
        vals: value in terms of points for the vector coming out column
        Mat: Probability distribution matrix
        times: number of rolls left
        %}
        Mat = Mat ^ times;
        num = in*Mat*vals;
end

function out = YatzeeM
        D=diag([6^5,6^4,6^3,6^2,6^1]);
        P=[
                720,5400,1500,150,6;
                0,720,480,90,6;
                0,0,150,60,6;
                0,0,0,30,6;
                0,0,0,0,6
        ];
        out = D^-1*P;
end

function out = NumberM
        D=diag([6^5,6^4,6^3,6^2,6^1,6^0]);
        P=[
                3125,3125,1250,250,25,1;
                0,625,500,150,20,1;
                0,0,125,75,15,1;
                0,0,0,25,10,1;
```

```
                0,0,0,0,5,1;
                0,0,0,0,0,1
        ];
        out = D^-1*P;
end

function out = FullHouseM
        D=diag([6^5,6^2,6^2,6^1,6^1,6^0]);
        P=[
                720,3600,6,1350,1800,300;
                0,11,0,10,12,3;
                0,0,1,30,0,5;
                0,0,0,5,0,1;
                0,0,0,0,4,2;
                0,0,0,0,0,1;
        ];
        out = D^-1*P;
end
```