Modeling hw3

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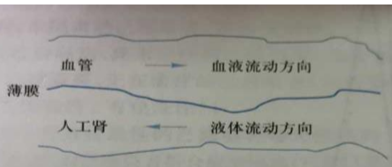
Problems

膜进入人工肾.



设血液和人工肾中液体的流速均为常数,废物进入人工肾的数量与它在这两种液体中的浓度差成正比,人工肾总长 l.建立单位时间内人工肾带走废物数量的模型[12].

5. 在鱼塘中投放 n_0 尾鱼苗,随着时间的增长,尾数将减少而每尾的质量将增加.

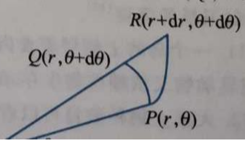
(1) 设尾数 $n(t)$ 的(相对)减少率为常数;由于喂养引起的每尾鱼质量的增加率与鱼表面积成正比,由于消耗引起的每尾鱼质量的减少率与质量本身成正比.分别建立尾数和每尾鱼质量的微分方

程,并求解.

(2) 用控制网眼的办法不捕小鱼,到时刻 T 才开始捕捞,捕捞能力用尾数的相对减少量 $|\dot{n}/n|$ 表示,记作 E,即单位时间捕获量是 $En(t)$.问如何选择 T 和 E,使从 T 开始的捕获量最大[12].

6. 侦察机搜索潜艇.设 $t=0$ 时艇在 O 点,飞机在 A 点,$OA=6$ n mile(1 n mile = 1.852 km).此时艇潜入水中并沿着飞机不知道的某一方向以直线形式逃去,艇速 20 n mile/h.飞机以速度 40 n mile/h 按照待定的航线搜索潜艇,当且仅当飞到艇的正上方时才可发现它.

(1) 以 O 为原点建立极坐标系 (r,θ),A 点位于 $\theta=0$ 的向径上,见下图.分析图中由 P,Q,R 组成的小三角形,证明在有限时间内飞机一定可以搜索到潜艇的航线,是先从 A 点沿直线飞到某点 P_0,再从 P_0 沿一条对数螺线飞行一周,而 P_0 是一个圆周上的任一点.给出对数螺线的表达式,并画出一条航线的示意图.



Solution

5. (1) 尾数 $n(t)$ 满足 $\dot{n}=-\lambda n(\lambda>0)$, $n(0)=n_0$ 得 $n(t)=n_0e^{-\lambda t}$. 每尾鱼重 $w(t)$ 满足 $\dot{w}=\alpha w^{2/3}-\beta w$, 不妨近似设 $w(0)=0$, 得 $w(t)=\left(\frac{\alpha}{\beta}\right)^3(1-e^{-\beta t/3})^3$.

(2) 设 $t=T$ 时开始捕捞,且单位时间捕捞率为 E,则 $t\geqslant T$ 时有 $\dot{n}=-(\lambda+E)n$,因此得 $n(t)=n_0e^{-\lambda T}e^{-(\lambda+E)(t-T)}$,单位时间捕捞鱼的尾数为 $En(t)$,每尾鱼重 $w(t)$,所以从 T 开始的总捕捞量是 $y=\int_T^{\infty}w(t)En(t)\,dt=\int_0^{\infty}\left(\frac{\alpha}{\beta}\right)^3[1-e^{-\beta(\tau+T)/3}]^3En_0e^{-\lambda T}e^{-(\lambda+E)\tau}d\tau$,问题为求 λ,E 使 y 最大,可用数值法求解.

Okay, now let's try to figure out how to numerically solve this problem.

$$y=\int_0^{\infty}\left(\frac{\alpha}{\beta}\right)^3\left[1-e^{-\frac{\beta(\tau+T)}{3}}\right]^3En_0e^{-\lambda T}e^{-(\lambda+E)\tau}d\tau,$$

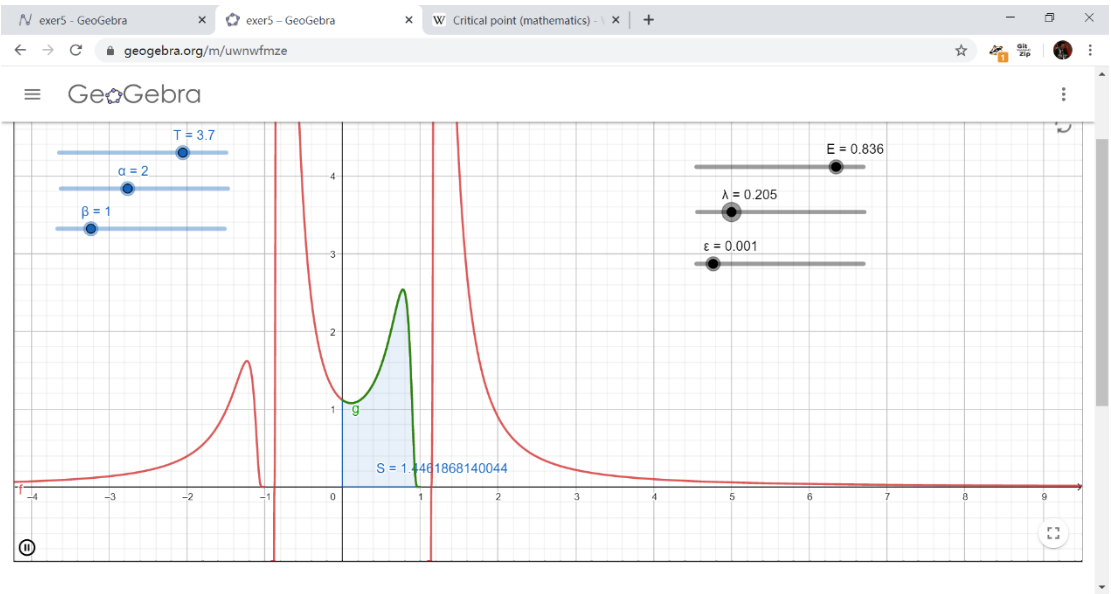
Let $\tau=\frac{t}{1-t^2}$, $0<t<1$, $d\tau=\frac{1+t^2}{(1-t^2)^2}dt$, thus

$$y=\int_0^{1}\left(\frac{\alpha}{\beta}\right)^3\left[1-e^{-\frac{\beta\left(\frac{t}{1-t^2}+T\right)}{3}}\right]^3En_0e^{-\lambda T}e^{-(\lambda+E)\frac{t}{1-t^2}}\frac{1+t^2}{(1-t^2)^2}dt.$$

Then we need to find out the critical point (λ,E) such that y_{max}.

Visualization

See https://www.geogebra.org/m/uwnwfmze for the dynamic graph.

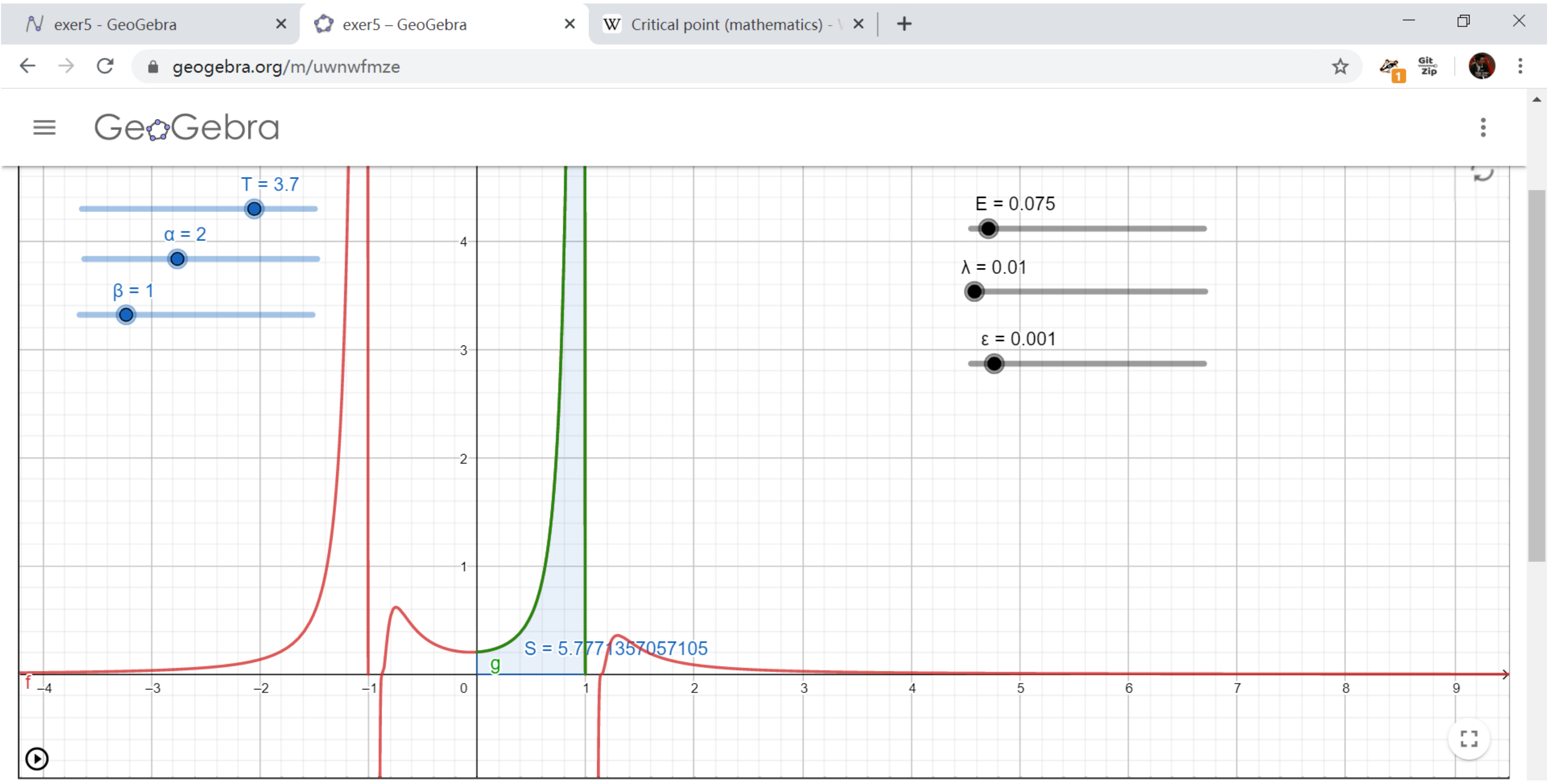


In this graph, g is the goal function to be integrated, i.e.

$$g = \left(\frac{\alpha}{\beta}\right)^3 \left[1 - e^{-\frac{\beta\left(\frac{t}{1-t^2}+T\right)}{3}}\right]^3 * E * e^{-\lambda T} e^{-(\lambda+E)\frac{t}{1-t^2}} \frac{1+t^2}{(1-t^2)^2}, \qquad 0 \le t < 1.$$

Note that we omit the positive constant n_0, for it has no contribution to the critical point. The blue sliders are constants as well, you can set those as you like. Since $t = 1$ is a discontinuous point, we can only numerically integrate g by interval $[0,\ 1-\varepsilon]$, you can set the ε as well at the right side of the graph.

Apparently, $g \ge 0, 0 \le x < 1$, and g decays as λ increases (you can verify it by dragging slider λ and keeping the rest static). In practical terms, λ is the rate of decrement of fry number that as time goes on (as they grow). The smaller λ is, the more fry will remain. Reasonable.



The critical point might be $(\lambda, E) = (0.075,\ 0.01)$ under the conditions that $T = 3.7, \alpha = 2, \beta = 1, \varepsilon = 0.001$

We can also approximate the critical point by writing a simple MATLAB program, letting λ and E be two vectors, consisting of a series of (evenly spaced) elements, say *linspace(0,1,100)*. Afterwards, get the maximum numerical integral by Simpson's method or Gaussian quadrature formula, and return this pair of values.