Making Dumplings

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Background

I think the assumptions in our text (see appendix) are a little inaccurate & unpractical, especially the shapes. After searching the Internet, I found some images (see below) of dumplings in real life, labeling them with upper-left, upper-right, lower-left, lower-right (***ul***, ***ur***, ***ll***, ***lr***), respectively.

Dumplings in real life

As we shall see, the real dumplings are not **sphere-like** actually, but more in a way to be **ellipsoid-like**[*] (or some other particular shape) & with a considerable area of **fin** on their body (which is kinda like dinosaurs with fins on their backs but without any legs😂), and their bottom bases like **oval** shapes. Besides, we may consider the thickness of the flour as well. Thus, we can rebuild the model by taking above parameters into consideration and classify them into two cases as follows:

- None-thickness & with fin
- Thickness & with fin

[*] Though in reality they both can be seen as ball-like, and a sphere is a specialization of an ellipsoid ($i.e.$ $a = b = c = r$). Here we wanna generalize it and see how results look.

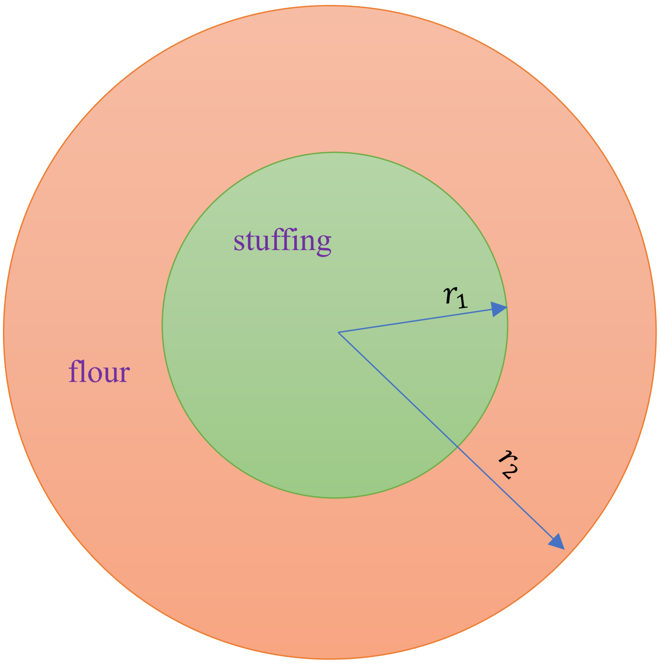
Before proceeding, I'd like to discuss **Tangyuan-like** dumplings, of which the thickness must be taken into consideration, and the stuffing and flour are virtually concentric balls. Let inner radius be r_1, and outer radius be r_2. Hence, it's very easy to see that the consumption of stuffing and flour depends on the ratio of their volumes, and it should be

$$k_{ball\_like} = \frac{V_{stuffing}}{V_{flour}} = \frac{\left(\frac{r_1}{r_2}\right)^3}{1 - \left(\frac{r_1}{r_2}\right)^3}$$

Instantiate it with $\frac{r_1}{r_2} = \frac{4}{5}$ and we can get $k_{ball\_like} = \frac{64}{61}$, (almost $1:1$) and it only depends on the ratio r_1/r_2. Let's find the critical point that is exactly $1:1$, all we have to is to let $k_{ball\_like} = \frac{1}{2}$, and we can therefore obtain critical point $\frac{r_1}{r_2} = \sqrt[3]{\frac{1}{2}} \approx 0.7937$.

If **flour is insufficient**, we just need to adjust the r_1/r_2 such that $\frac{r_1}{r_2} > \sqrt[3]{\frac{1}{2}}$, otherwise let $\frac{r_1}{r_2} < \sqrt[3]{\frac{1}{2}}$.

Note that even if $r_1 : r_2 = 9:10$ (relatively very thin layer of flour), the volume ratio of flour to stuffing isn't that low, it's 371:729, and if we slightly diminish r_1/r_2, say into $7:10$, the ratio becomes $657:343$, flour even accounts more!

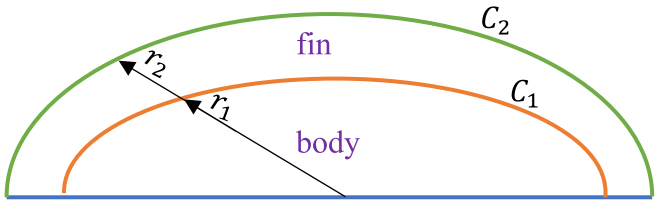


Tangyuan-like dumpling diagram

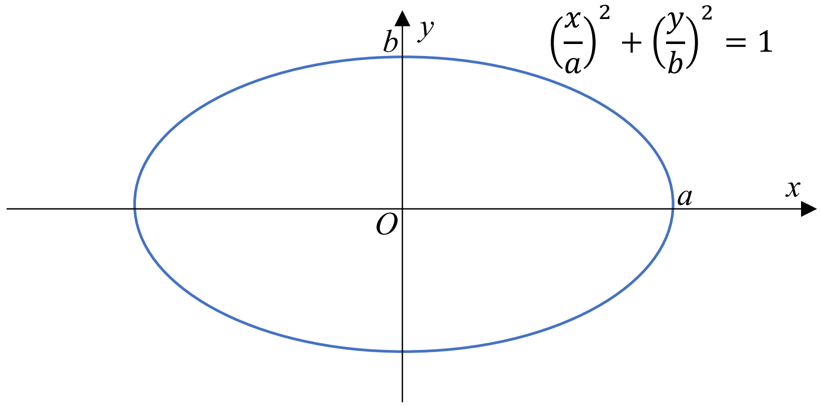
None-thickness & With Fin

Now, look back at dumpling ur, one with fin on its back (~cute).
OK, let us obstruct the cutaway view (see *dumpling cutaway view diagram*). It seems

we need to add more assumptions to move on.



dumpling cutaway view diagram



Bottom base elliptic curve diagram

rugby football

Assumptions

There are several reasonable assumptions that will make us proceed easier.

- The body of a dumpling is virtually a hemisphere elliptic ball (see above *rugby football*, but the dumpling only have the upper half), satisfying $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ $(a, b, c > 0)$. Hence, C_1: $\left(\frac{x}{a}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$, and the bottom base curve: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

- Associated dependencies of radii of these two curves C_1 & C_2 might have following two possible idealized cases:
  - Ratio r_1/r_2 is invariable everywhere, i.e. $r_1/r_2 = const.$
  - Or alternatively, $r_2 - r_1 = const\ that\ dependes\ on\ a, b, \&\ c.$

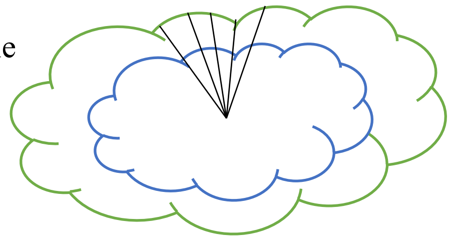
Model Establishing

By first case of assumption 2:

Building

Let $\frac{r_1}{r_2} = k_e$, where k_e is a constant (here subscript *e* means ellipse-like).

Then via the similarity principles (we can divide the random curves into many small segments or partitions, then each of them can be processed by elementary mathematics – triangle similarity theorems), we can determine some useful results.



Concentric random shape of curves

Since C_1 is the upper curve of an ellipse, we can directly get the area of the body by those known formulas, generating

$$Area\ of\ body = \frac{1}{2} * \pi * a * c,$$

And therefore,

$$Area\ of\ fin = \frac{1-\left(\frac{r_1}{r_2}\right)^2}{\left(\frac{r_1}{r_2}\right)^2} * Area\ of\ body$$

$$= \frac{1 - {k_e}^2}{{k_e}^2} * \frac{1}{2}\pi ac$$

The surface area of body (including bottom base) is

$$S_{body} = S_{lateral} + S_{bottom}$$

$$= \frac{1}{2} * \frac{4\pi}{3}(\mathrm{ab} + \mathrm{bc} + \mathrm{ca}) \ + \ \pi ab$$

$$= \frac{2\pi}{3}\left(\frac{5}{2}\mathrm{ab} + \mathrm{bc} + \mathrm{ca}\right)$$

So, the total area of each dumpling will be

$$S_{flour} = S_{body} + S_{fin}$$

$$= \frac{2\pi}{3}\left(\frac{5}{2}\mathrm{ab} + \mathrm{bc} + \frac{3 + {k_e}^2}{4{k_e}^2}ac\right)$$

As for its volume, it simply is

$$V_{stuffing} = \frac{1}{2} * \frac{4\pi}{3} abc = \frac{2\pi}{3} abc$$

Thus we obtain

$$g = \frac{V_{stuffing}}{S_{flour}} = \frac{abc}{\frac{5}{2}\mathrm{ab} + \mathrm{bc} + \frac{3 + {k_e}^2}{4{k_e}^2} ac}$$

From the expression, we can see that if $\boldsymbol{a}, \boldsymbol{b}$, & $\boldsymbol{c}$ **scale with the same proportion**[†], say ω ($\omega > 0$), then $g = g_0 * \omega$, where the coefficient g_0 is a **positive** constant depending on initial values a, b, c, & k_e.

Analysis

- ✧ Qualitatively, as ω grows, ratio of stuffing to flour of each dumpling ($i.e.\ g$) goes up as well, which indicates **unit area of flour can contain more stuffing as we increase the size of the dumplings**. And hence, with the same amount of flour, if we enlarge the size of each dumpling, then we will consume more stuffing ($ as well😂, but may bring better flavor).
- ✧ Quantitatively, we need to delve deeper into details.

  Let $R = f(a, b, c)$, and $a = k_a R, b = k_b R, c = k_c R$, once parameters a, b, c are determined, R, k_a, k_b, k_c will be fixed. In addition, if a, b, c scale with the **same proportion** ω, the coefs k_a, k_b, k_c remain unchanged. In other words, if $R' = f(a', b', c') = f(\omega a, \omega b, \omega c)$, then

  $$k_a = \frac{a}{R} = \frac{a'}{R'}, k_b = \frac{b}{R} = \frac{b'}{R'}, k_c = \frac{c}{R} = \frac{c'}{R'}$$

  This is reasonable, we can imagine the R as a virtual, equivalent radius of a round-like 3D object, which is also the exact case when the ellipsoid is

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[†] Well, you might ask what if they are out of tune with each other, say expanding with proportions $a \times 2, b \times 1.5, c \times 1.2$. Yeah, it happens, and it **will influence the results quantitatively**. In the terms of qualitativeness, however, it remains unchanged as that of with the same proportion. It can be explained like this:

$$\frac{D \times 3.6}{N_1 \times 3 + N_2 \times 1.8 + N_3 \times 2.4} > \frac{D \times 3.6}{N_1 \times 3.6 + N_2 \times 3.6 + N_3 \times 3.6} = \frac{D}{N_1 + N_2 + N_3}$$

where D represents denominator, N_i represents the ith item of numerator. In general, we just need to substitute *3.6* with *the total product of each parameter*, but yielding identical result – if we enlarge the size, g goes up; and if we decrease the size, g falls off.

Oops, wait a minute, some might ask what if it's a compound, I mean, the scaling coefs are not all larger than 1 or not all smaller than 1 (e.g., $a \times 0.9, b \times 0.8, c \times 1.5$). Well, it's sounds possible, and it requires further discussion (usually complicated). But, in such cases, where the dumplings attempt to get slimmer and taller so that they girls can become more good-looking and thus attracting boys, I dunno what the hell the dumpling maker is thinking about, maybe he's an ingenious artist. (Goddamn it), why must we this trouble ourselves, huh?

specialized as a sphere.

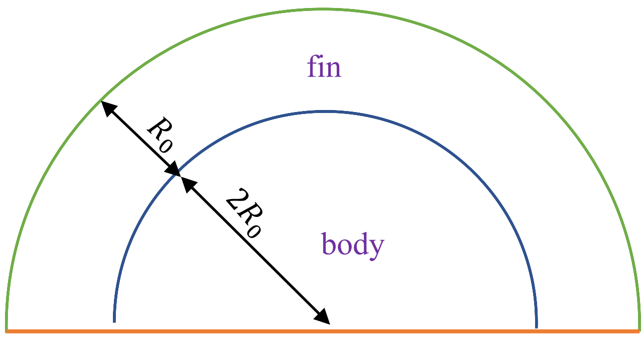
Hence we have

$$V_{stuffing} = \frac{2\pi}{3} k_a k_b k_c R^3 \qquad (1)$$

$$S_{flour} = \frac{2\pi}{3}\left(\frac{5}{2}\mathrm{k_a k_b} + \mathrm{k_b k_c} + \frac{3 + {k_e}^2}{4{k_e}^2} k_a k_c\right) R^2 \qquad (2)$$

Well, this looks so like the expressions in our text if we substitute the stuff before R^3 with k_1, and replace the long scary coef before R^2 by k_2. Then it can yield exactly the same results as our text. But, uh, wait a minute, what the hell is the **fin** doing then, isn't it supposed to have some effect on the results?

Actually, replacing the coefs before R^3, R^2 with constants k_1, k_2 respectively is reasonable under the big **precondition** that $\boldsymbol{a}, \boldsymbol{b}, \& \, \boldsymbol{c}$ **scale with the same proportion**, for as we have claimed before – the coefs $k_a, k_b, \& \, k_c$ remain unchanged under this context.



Concentric circles that with ratio of fin to body $1:2$

Let's verify it with an extreme example – supposing $a = b = c = R$, then naturally $k_a = k_b = k_c = 1$, and let ratio of fin to body of fin be $1:2$. Without any difficulties, we can get

$$S_{body} = S_{bottom} + S_{lateral} = \pi R^2 + \frac{1}{2} * 4\pi R^2 = 3\pi R^2$$

$$S_{fin} = \frac{\left(1 - \left(\frac{2}{3}\right)^2\right)}{\left(\frac{2}{3}\right)^2} * \frac{1}{2}\pi R^2 = \frac{5}{8}\pi R^2$$

$$S_{flour} = S_{body} + S_{fin} = \frac{29\pi}{8} R^2 \qquad (3)$$

$$V_{stuffing} = \frac{1}{2} * \frac{4\pi}{3} R^3 = \frac{2\pi}{3} R^3 \qquad (4)$$

which totally agree with results of equation (1) & (2) if we initialize k_e with $2/3$.

We can see that, by the special case, $V_{stuffing} = k_1 R^3$, $S_{flour} = k_2 R^2$, the results are consistent with our text, even though we have additionally considered the **fin** on the dumpling's back. So, processing in a duplicate way as our text does, we can get the final answer $\sqrt{n}$ times.

Note that if each guy has their tempers, I mean, they refuse to scale in tune, then $k_a, k_b,$ & k_c can be variable, and have their own particular relationships with R, depending on $a, b, \& \ c$. So, we cannot regard they as constants that has nothing to do with R in such case. See also prior footnote.

(Actually, I made a mistake at first, and I wanna note it down here.)
[Since the flour and stuffing are used in a specific proportion, the number of dumplings of flour and stuffing should be the same, i.e.,

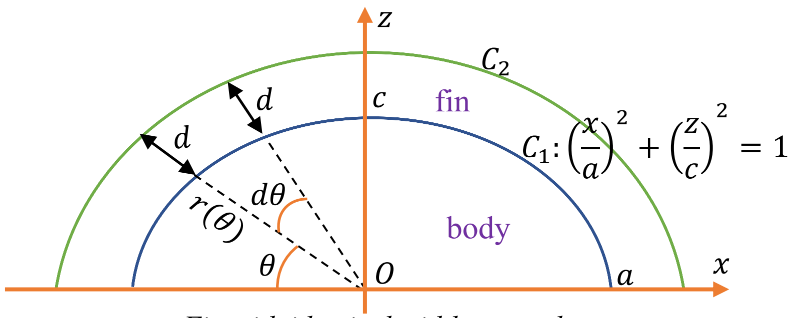
$$no.of dum = \frac{V}{V_{stuffing}} = \frac{S}{S_{flour}} \xrightarrow{yields} \frac{V}{S} = \frac{V_{stuffing}}{S_{flour}} = g_0 * \omega \qquad (*)$$

By maintaining the amount of flour, and varying ω , we have

$$V' = S * g_0 * \omega = V * \omega \qquad (\#)$$

So, the answer we get is ω times, i.e. the proportion we scale. If we double the dumpling's virtual radius R (or a, b, c), we will consume twice stuffing as much as before under the invariable amount of flour.
Of course, the answer is flawed. The issue is that if we change the practical volume V into V', both parts of equation $(*)$ will also be changed from V to V', thus rendering equation $(\#)$ nonsense.**]**

By second case of assumption 2:



Fin with identical width everywhere

In this section, we will discuss the situation where the width of the fin is virtually identical everywhere, denoted by d (see diagram above). The width d, however, is determined by $a, b, \& \ c$.

Building

From the discuss in previous section, we have known that

$$V_{stuffing} = \frac{2\pi}{3}abc, \qquad S_{body} = \frac{2\pi}{3}(\frac{5}{2}ab + bc + ca)$$

Now let's think about how to calculate the area of fin.
Let the radius of C_1 be $r(\theta)$, then the area under curve C_2 is

$$S_2 = 2 * \int_0^{\frac{\pi}{2}} \frac{1}{2}[r(\theta) + d]^2 d\theta$$
$$= \frac{\pi}{2}d^2 + 2d * \int_0^{\frac{\pi}{2}} r(\theta)d\theta + \int_0^{\frac{\pi}{2}} [r(\theta)]^2 d\theta$$

Note that the area under C_1 is

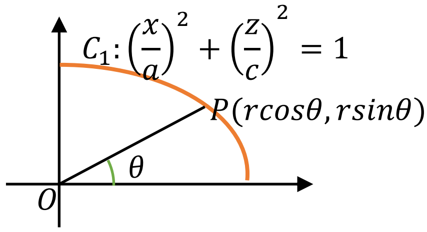
$$S_1 = 2 * \int_0^{\frac{\pi}{2}} \frac{1}{2}[r(\theta)]^2 d\theta = \int_0^{\frac{\pi}{2}} [r(\theta)]^2 d\theta$$

Hence, the area of fin is

$$S_{fin} = S_2 - S_1 = \frac{\pi}{2}d^2 + 2d * \int_0^{\frac{\pi}{2}} r(\theta)d\theta$$

We'd better figure out a way of determining $\int_0^{\frac{\pi}{2}} r(\theta)d\theta$[‡].

Let P be a point on C_1, then its coordinate is $P = (rcos\theta, rsin\theta)$. Inserting point P into the ellipse equation C_1, we have



$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{c^2} = 1$$

$$\Rightarrow r = \frac{ac}{\sqrt{c^2\cos^2\theta + a^2\sin^2\theta}}$$

Therefore,

$$\int_0^{\frac{\pi}{2}} r(\theta)d\theta = ac * \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{c^2\cos^2\theta + a^2\sin^2\theta}}$$
$$= ac * \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{c^2 + (a^2 - c^2)\sin^2\theta}}$$
$$= a * \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + \frac{a^2 - c^2}{c^2}\sin^2\theta}}$$

Let

‡ In fact, this integral is a quarter of the circumference of the ellipse C_1. See more information at WolframMathWorld, and Wikipedia.

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

which is known as the **Complete Elliptic Integral of the First Kind.**

Then we have,

$$\int_0^{\frac{\pi}{2}} r(\theta)d\theta = a\,K\left(-\frac{a^2-c^2}{c^2}\right).$$

For simplicity of writing, we denote $K\left(-\frac{a^2-c^2}{c^2}\right)$ by K_0 (which is a constant, and can be numerically calculated by some mathematical software, for example, $@ellipticK$ provided by $MATLAB$ is a good tool for numeric computation of elliptic integral).
Thus,

$$\begin{aligned} S_{flour} &= S_{fin} + S_{body} \\ &= \frac{\pi}{2}d^2 + 2adK_0 + \frac{2\pi}{3}\left(\frac{5}{2}ab + bc + ca\right) \end{aligned} \tag{5}$$

Similar to the approaches we adopted before, let

$$g = \frac{V_{stuffing}}{S_{flour}} = \frac{\frac{2\pi}{3}abc}{\frac{\pi}{2}d^2 + 2adK_0 + \frac{2\pi}{3}\left(\frac{5}{2}ab + bc + ca\right)} \tag{6}$$

Note that d is a constant depending on $a, b, \& \, c$.
Let $d = \varphi(a, b, c)$.

Analysis

✧ Qualitatively

Let's think about the problem from a new perspective – unit. We know that $a, b, \& \, c$ are units of length, so does d, it could be something like, $(a + 1.2c)/3$, $\sqrt{(3a + 2c)^2 - 7bc} + 0.1a$, $\sqrt[3]{(2a - c)^3 + abc - 3ab^2}$, or whatever. All they have to follow is that guaranteeing their final result be measurable by length & in the process, each term must be manipulative (can do operations, e.g. In second one, term $(3a + 2c)^2$ & term $7bc$ are both based on unit of area, so they two can do subtraction operation, after square root, it becomes unit of length, then it can be added by operand with the same unit of type, i.e. $0.1a$).

By observing these examples, we can see that the manipulativeness insures the homogeneity[§] of function $d = \varphi(a, b, c)$. Further, φ is of first order, for width d is one-dimensional. Therefore, we have

[§] I looked up in the dictionary for the word homogeneous, and surprisingly found that the meaning in the dict agrees so much well with the example I'd given. Here is the meaning in the dict: consisting of things or people that are all the same or all of the same type.

$$\varphi(\omega a, \omega b, \omega c) = \omega\varphi(a, b, c) = \omega d \tag{7}$$

So, if we try to scale the size with proportion of ω, then

$$g' = g * \omega$$

In fact, by equation (6) & (7), we have

$$g' = \frac{\frac{2\pi}{3}abc * \omega^3}{\frac{\pi}{2}(\omega d)^2 + 2(\omega a)(\omega d)K_0 + \frac{2\pi}{3}\left(\frac{5}{2}ab + bc + ca\right) * \omega^2} = g * \omega$$

Hence, the conclusion is identical to that of in previous qualitative analysis, provided, of course, that $a, b, \& \ c$ **scale with the same proportion.**

✧ Quantitatively

The result varies depending on the width d, and there are a couple of possible situations as follows (Again, we only discuss the situations where $a, b, \& \ c$ scale with the same proportion).

① $d = \varphi(a, b, c)$ is a linear function of $a, b, \& \ c$.
This case may be the simplest & most intuitive, for example, it can be $d = \frac{a+2c}{8}$, which suggests width d must be larger than one eighth of a, and larger than a quarter of c.

To be complemented …

② $d = \varphi(a, b, c)$ is a function of $V_{stuffing}$, i.e., $d = \varphi(a, b, c) = \lambda(\frac{2\pi}{3}abc)$
This case is also kinda possible.

Summery

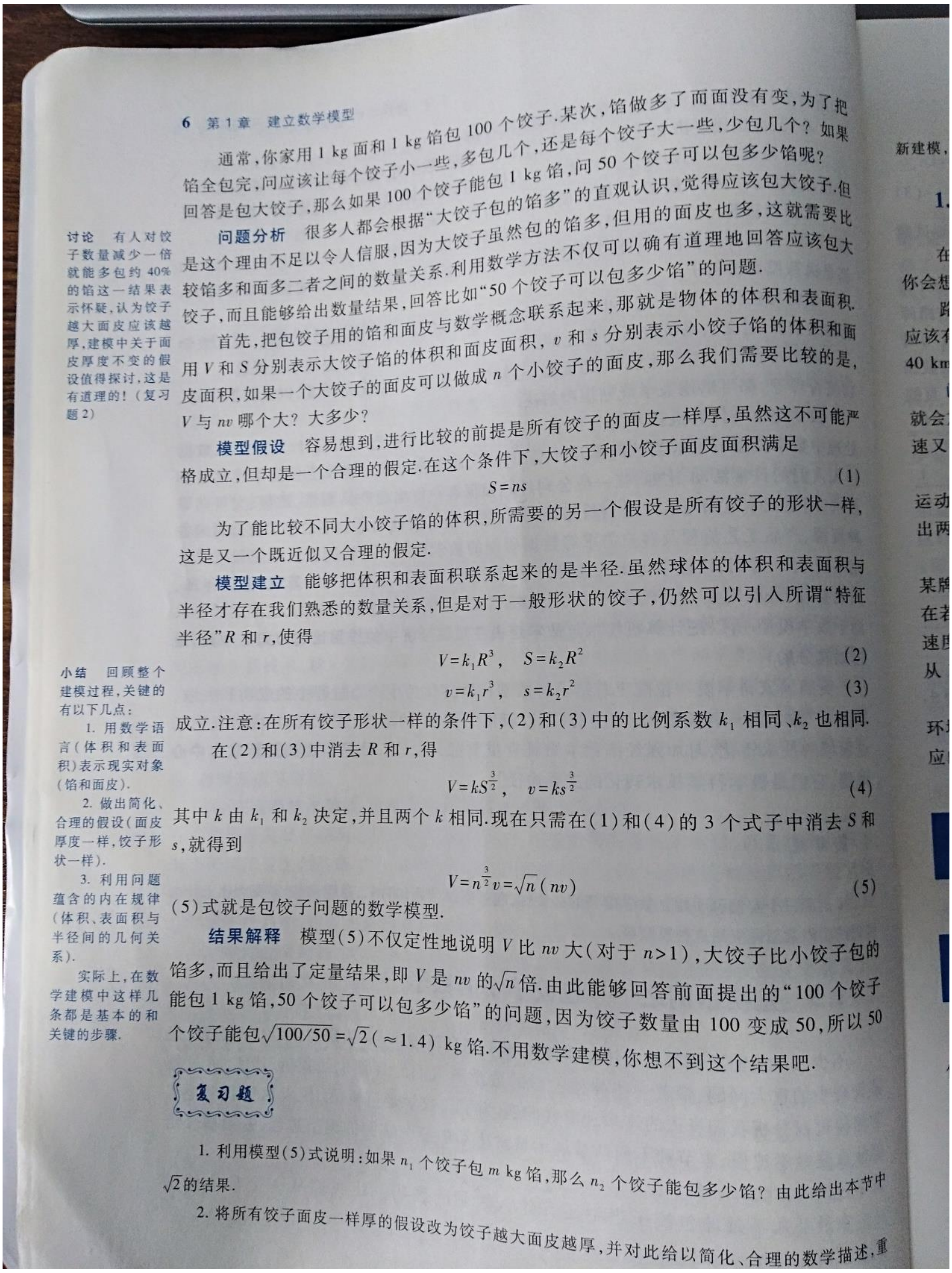
See analysis part of each section:

- Case 1, $r_1/r_2 = const$
- Case 2, $r_2 - r_1 = const\ that\ dependes\ on\ a, b, \& \ c$

Thickness & With Fin

To be complemented …

Appendix



6 第1章 建立数学模型

通常，你家用 1 kg 面和 1 kg 馅包 100 个饺子.某次，馅做多了而面没有变，为了把馅全包完，问应该让每个饺子小一些，多包几个，还是每个饺子大一些，少包几个？如果回答是包大饺子，那么如果 100 个饺子能包 1 kg 馅，问 50 个饺子可以包多少馅呢？

问题分析 很多人都会根据"大饺子包的馅多"的直观认识，觉得应该包大饺子.但是这个理由不足以令人信服，因为大饺子虽然包的馅多，但用的面皮也多，这就需要比较馅多和面多二者之间的数量关系.利用数学方法不仅可以确有道理地回答应该包大饺子，而且能够给出数量结果，回答比如"50 个饺子可以包多少馅"的问题.

首先，把包饺子用的馅和面皮与数学概念联系起来，那就是物体的体积和表面积.用 V 和 S 分别表示大饺子馅的体积和面皮面积，v 和 s 分别表示小饺子馅的体积和面皮面积，如果一个大饺子的面皮可以做成 n 个小饺子的面皮，那么我们需要比较的是，V 与 nv 哪个大？大多少？

模型假设 容易想到，进行比较的前提是所有饺子的面皮一样厚，虽然这不可能严格成立，但却是一个合理的假定.在这个条件下，大饺子和小饺子面皮面积满足

$$S=ns \tag{1}$$

为了能比较不同大小饺子馅的体积，所需要的另一个假设是所有饺子的形状一样，这是又一个既近似又合理的假定.

模型建立 能够把体积和表面积联系起来的是半径.虽然球体的体积和表面积与半径才存在我们熟悉的数量关系，但是对于一般形状的饺子，仍然可以引入所谓"特征半径"R 和 r，使得

$$V=k_1R^3, \quad S=k_2R^2 \tag{2}$$

$$v=k_1r^3, \quad s=k_2r^2 \tag{3}$$

成立.注意：在所有饺子形状一样的条件下，(2)和(3)中的比例系数 k_1 相同、k_2 也相同.

在(2)和(3)中消去 R 和 r，得

$$V=kS^{\frac{3}{2}}, \quad v=ks^{\frac{3}{2}} \tag{4}$$

其中 k 由 k_1 和 k_2 决定，并且两个 k 相同.现在只需在(1)和(4)的 3 个式子中消去 S 和 s，就得到

$$V=n^{\frac{3}{2}}v=\sqrt{n}\,(nv) \tag{5}$$

(5)式就是包饺子问题的数学模型.

结果解释 模型(5)不仅定性地说明 V 比 nv 大(对于 $n>1$)，大饺子比小饺子包的馅多，而且给出了定量结果，即 V 是 nv 的 $\sqrt{n}$ 倍.由此能够回答前面提出的"100 个饺子能包 1 kg 馅，50 个饺子可以包多少馅"的问题，因为饺子数量由 100 变成 50，所以 50 个饺子能包 $\sqrt{100/50}=\sqrt{2}\,(\approx 1.4)$ kg 馅.不用数学建模，你想不到这个结果吧.

讨论 有人对饺子数量减少一倍就能多包约 40% 的馅这一结果表示怀疑，认为饺子越大面皮应该越厚，建模中关于面皮厚度不变的假设值得探讨，这是有道理的！(复习题 2)

小结 回顾整个建模过程，关键的有以下几点：

1. 用数学语言(体积和表面积)表示现实对象(馅和面皮).
2. 做出简化、合理的假设(面皮厚度一样，饺子形状一样).
3. 利用问题蕴含的内在规律(体积、表面积与半径间的几何关系).

实际上，在数学建模中这样几条都是基本的和关键的步骤.

复习题

1. 利用模型(5)式说明：如果 n_1 个饺子包 m kg 馅，那么 n_2 个饺子能包多少馅？由此给出本节中 $\sqrt{2}$ 的结果.
2. 将所有饺子面皮一样厚的假设改为饺子越大面皮越厚，并对此给以简化、合理的数学描述，重

Model in our text