Asymptotic Analysis

Formal Definitions

Big-O

Big-O

$$O(g(n)) = \{f(n) | \exists c > 0, n_0 > 0$$
$$(\forall n > n_0 (0 \leq f(n) \leq cg(n)))\}$$

Big-O

*"**Big -Oh**"* of some function $g(n)$ is the set of all functions $f(n)$, where there exists a positive constant c such that $0 \leq f(n) \leq cg(n)$ for sufficiently large values of n (for values of n above some positive value n_0 ($n \geq n_0 \geq 0$)).

Big-O

$$0 \leq f(n) \leq cg(n)$$

Big-O

- What this means is that, **this inequality must hold true for all values n that reach a threshold n_0**.

Big-O

- Therefore, for a function of n to be member of the set $O(g(n))$, **all images $f(n)$ must be not be greater than the corresponding image $g(n)$ times c.**

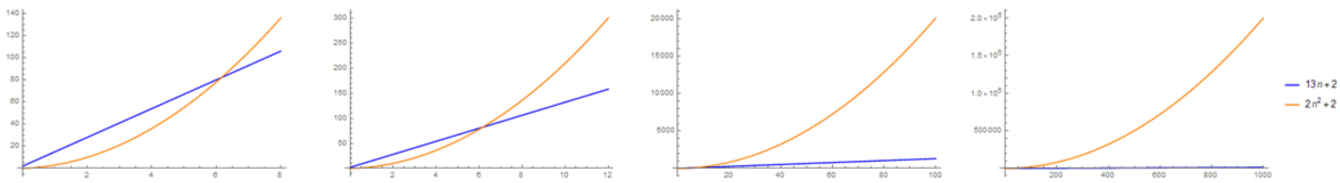
Big-O

- *This whole definition will make much more sense with a visual example like the one below. The function* $13n + 2$ *can be shown to be a member of the set* $O(2n^2 + n)$

Big-O

$$13n + 2 \in O(2n^2 + n)$$

Big-O



Big-O

- As shown in the plot, even though the function $13n + 2$ starts bigger than $2n^2 + n$, after some point, $2n^2 + n$ **overtakes** the function $13n + 2$ and leaves it behind so much that $13n + 2$ is only barely visible on the last plot.
- This shows that **there exists an n_0, a threshold for n, where, we are sure that for any n greater than n_0, the inequality $13n + 2 \leq 2n^2 + n$ is always true** In this example, the function $2n^2 + n$ starts to overtake between $n = 6$ and $n = 7$.

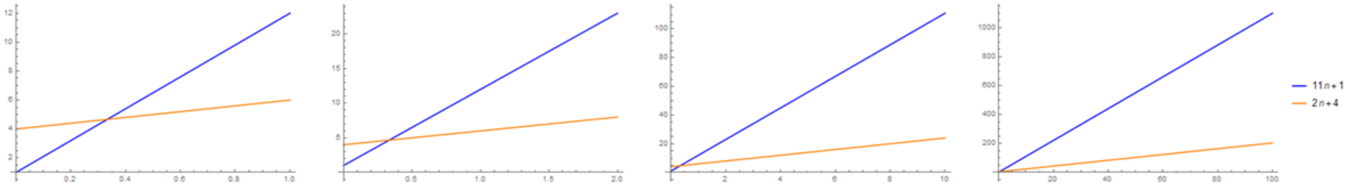
Big-O

- *Let's hammer it home with another example. One which forces us to explicitly choose a value for c*

Big-O

$$11n + 1 \in O(2n + 4)$$

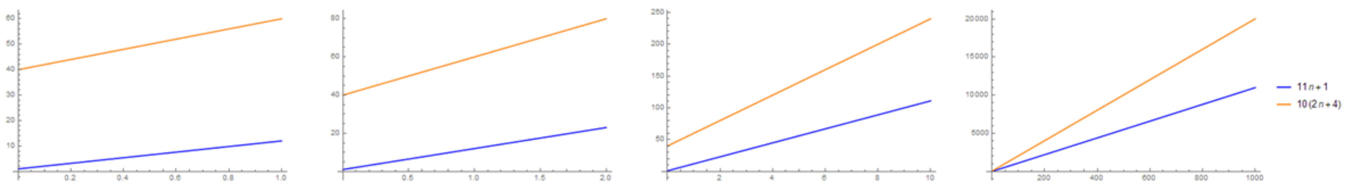
Big-O



Big-O

- From the plots above you can see that $2n + 4$**, starts bigger, but then** $11n + 1$ **overtakes it around** $n = 0.4$.

Big-O



Big-O

- This shows that f**or $c = 10$, the function $10(2n + 4)$, starts as bigger than $2n + 4$ and is always bigger than** $2n + 4$ (you can check that this is true if you notice that the difference between their values only gets bigger as n increases).

Big-O

- *The relationship* $13n + 2 \in O(2n^2 + n)$ *are usually read as the following phrases that you will repeatedly hear as computer scientists:*

Big-O

- $13n + 2$ **is upper bounded by** $O(2n^2 + n)$ - it's called an upper bound because as n increases to large values, $13n + 2$ can never be greater than $2n^2 + n$. There is a boundary above $13n + 2$ and that boundary is $2n^2 + n$.

Big-O

- $13n + 2$ **cannot be more complex than** $2n^2 + n$ - the term complexity is often used when contextualized with programming. When dealing with programming resources such as time, memory, and etc, CS often abstracts these resources as complexities (i.e. time complexity, space complexity). This will be discussed in detail later.

Big-O

When we say $f(n)$ cannot be more complex than $g(n)$, it means that it is still possible that $f(n)$ is equally as complex as $g(n)$,

Big-Ω

Big-Ω

$$\Omega(g(n)) = \{f(n) | \exists c > 0, n_0 > 0$$
$$(\forall n > n_0 (0 \leq cg(n) \leq f(n)))\}$$

Big-Ω

*"**Big -Omega**"* of some function $g(n)$ is the set of all functions $f(n)$, where there exists a positive constant c such that $0 \leq cg(n) \leq f(n)$ for sufficiently large values of n (for values of n above some positive value n_0 ($n \geq n_0 \geq 0$)).

Big-Ω

$$0 \leq cg(n) \leq f(n)$$

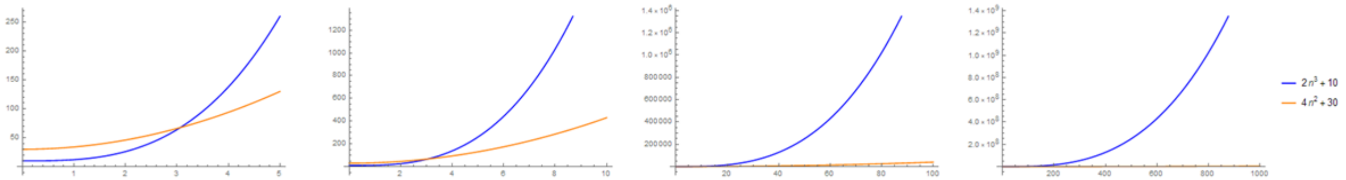
Big-Ω

- Here, $cg(n)$ **is now less than or equal to** $f(n)$.

Big-Ω

$$2n^3 + 10 \in \Omega(4n^2 + 30)$$

Big-Ω



Big-Ω

- $2n^3 + 10$ **is lower bounded by** $4n^2 + 30$**.**
- $2n^3 + 10$ **cannot be less complex than** $4n^2 + 30$**.**

Big-Ω

When we say $f(n)$ cannot be less complex than $g(n)$, it means that it is still possible that $f(n)$ is equally as complex as $g(n)$,

Big-θ

Big-θ

$$\Theta(g(n)) = \{f(n) | \exists c_1 > 0, c_2 > 0, n_0 > 0$$
$$(\forall n > n_0 (0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)))\}$$

Big-θ

*"**Big -Theta**"* of some function $g(n)$ is the set of all functions $f(n)$, where there exists a positive constant c_1 and c_2 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for sufficiently large values of n (for values of n above some positive value n_0 ($n \geq n_0 \geq 0$)).

Big-θ

- In this definition, **the inequality conditions for Big-O and Big-Ω are combined into one**.

Big-θ

$$4n^2 + 30 \in \Theta(2n^2 + n)$$

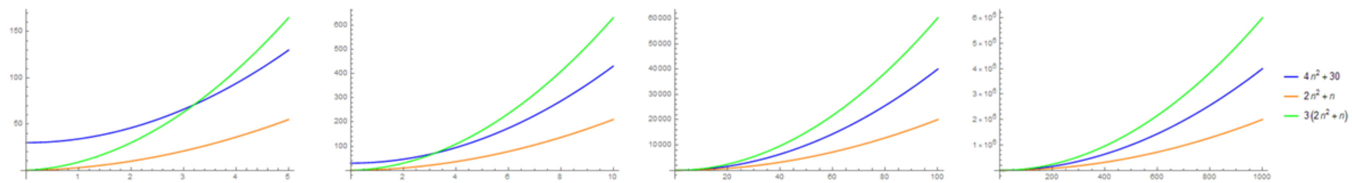
Big-θ

- *To demonstrate that this is indeed true, we just need to satisfy the inequality:*

Big-θ

$$c_1(2n^2 + n) \leq 4n^2 + 30 \leq c_2(2n^2 + n)$$

Big-θ



Big-θ

- $4n^2 + 30$ is tightly bounded by $2n^2 + n$.
- $4n^2 + 30$ is as complex as $2n^2 + n$.

Big-θ

Before we move on the next section, here are some questions to ponder upon: (don't worry, these questions will be answered and proven in the next sections)

- **Is it true that** $f(n) \in O(g(n)) \cup \Omega(g(n))$ **for any pair of functions** $f(n)$ **and** $g(n)$**?** (*I.e. Is* $f(n)$ *guaranteed to be a member of either* $O(g(n))$ *or* $\Omega(g(n))$*?*)
- **Is it true that if** $f(n) \in O(g(n))$**, then** $g(n) \in \Omega(f(n))$**?**
- **What is** $O(g(n)) \cap \Omega(g(n))$**?** (*What is the intersection of* $O(g(n))$ *and* $\Omega(g(n))$*?*)

Proving Asymptotic Relationships using the Formal Definition

Proving Asymptotic Relationships using the Formal Definition

- Questions such as: "**is** $n^2 + 5 \in O(n^2)$**?** ".

Proving Asymptotic Relationships using the Formal Definition

- *For example, let's prove that the relationship demonstrated visually in the previous sections, is indeed true:*

Proving Asymptotic Relationships using the Formal Definition

$$2n^3 + 10 \in \Omega(4n^2 + 30)$$

Proving Asymptotic Relationships using the Formal Definition

- *Recall that the definition for a* Ω *relationship is the following:*

Proving Asymptotic Relationships using the Formal Definition

$$\Omega(g(n)) = \{f(n) | \exists c > 0, n_0 > 0$$
$$(\forall n > n_0 (0 \leq cg(n) \leq f(n)))\}$$

Proving Asymptotic Relationships using the Formal Definition

$$\exists c > 0, n_0 > 0$$
$$(\forall n > n_0(0 \leq c(4n^2 + 30) \leq 2n^3 + 10)$$

Proving Asymptotic Relationships using the Formal Definition

- *We can satisfy the existential quantification above using proof by example with the following values:*

Proving Asymptotic Relationships using the Formal Definition

$$\text{let } c = 1, n_0 = 3.4$$

Proving Asymptotic Relationships using the Formal Definition

- *and a little bit of inequality algebra:*

Proving Asymptotic Relationships using the Formal Definition

$$
\begin{aligned}
4n^2 + 30 &\leq 2n^3 + 10 \\
4n^2 + 20 &\leq 2n^3 \\
2 + \frac{10}{n^2} &\leq n
\end{aligned}
$$

Proving Asymptotic Relationships using the Formal Definition

- [^1]: In fact this is connected to how limit definitions work (which will be discussed in the next section) and why this whole concept is called **asymptotic** analysis

Proving non-membership

- **What if we needed to prove that a function is not a member of O, Ω, or Θ**? For example, how do we prove the following?

Proving non-membership

$$2n^3 + 10 \notin O(4n^2 + 30)$$

Proving non-membership

$$
\begin{aligned}
&\neg(\exists c > 0, n_0 > 0 \\
&\quad (\forall n > n_0 (0 \leq (2n^3 + 10) \leq c(4n^2 + 30))))
\end{aligned}
$$

$$
\begin{aligned}
&\quad (\forall c > 0, n_0 > 0 \\
&\quad (\exists n > n_0 (0 > (2n^3 + 10) \vee (2n^3 + 10) > c(4n^2 + 30)))
\end{aligned}
$$

Proving non-membership

note: we do not actually care for cases where $n < 0$ or $f(n) < 0$ or in fact anything that is less than 0. In the context that we are using this in computer science, values are cannot be negative.

Proving non-membership

- With this negation we end up with a universal quantification that shows that **it is not enough to find a few values for c and n_0**.
- We should **prove that for all possible combinations of c and n_0, sometimes there is a value for n that is greater than n_0, that satisfies $(2n^3 + 10) > c(4n^2 + 30)$.**

Proving non-membership

$$\begin{aligned} 2n^3 + 10 &> c(4n^2 + 30) \\ 2n^3 + 10 &> 4n^2c + 30c \\ n + \frac{10}{2n^2} &> 2c + \frac{30c}{2n^2} \\ n &> 2c + \frac{30c}{2n^2} - \frac{10}{2n^2} \end{aligned}$$

Proving non-membership

- Now, **no matter what combination of c and n_0, you choose, there will always be a value of n that satisfies this because the left hand side (which is also n) increases as n increases.** On the other hand, **the right hand side decreases as n increases**.

Limit Definitions

Limit Definitions

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \to f(n) \in O(g(n))$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \to f(n) \in \Omega(g(n))$$

$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \to g(n) \in \Theta(f(n))$$

Limit Definitions

if the limit of the ratio $\frac{f(n)}{g(n)}$ as n approaches infinity…

Limit Definitions

- We're looking at the ratio of the two functions $f(n)$ and $g(n)$ and **we're trying to see what happens to the value of that ratio as n becomes very very large (as n approaches infinity).**

Approaching Infinity

Approaching Infinity

$$f(n) = 2n^3 + 10$$
$$g(n) = 4n^2 + 30$$

Approaching Infinity

Let $n = 5$

$$\frac{2(5)^3 + 10}{4(5)^2 + 30} = 5$$

Approaching Infinity

- But as we **increase** the value of n, we'll notice that the **value of the ratio becomes bigger and bigger**:

Approaching Infinity

Let $n = 100$

$$\frac{2(100)^3 + 10}{4(100)^2 + 30} \approx 50$$

Let $n = 1000$

$$\frac{2(1000)^3 + 10}{4(1000)^2 + 30} \approx 500$$

Approaching Infinity

Let $n = 100000$

$$\frac{2(100000)^3 + 10}{4(100000)^2 + 30} \approx 50000$$

Approaching Infinity

- We can expect this number to **increase** as we increase the value of n.
- When we reach really big values of n, (around infinity), we can expect that that number will be **really** big as well.
- When the value n becomes really big, **the value of the numerator will become extremely big compared to the denominator**.

Approaching Infinity

$$f(n) = 13n + 2$$
$$g(n) = 2n^2 + n$$

Approaching Infinity

Let $n = 100$

$$\frac{13(100)+2}{2(100)^2+100} \approx 0.0647$$

Let $n = 10000$

$$\frac{13(10000)+2}{2(10000)^2+10000} \approx 0.000649$$

Approaching Infinity

Let $n = 10000000$

$$\frac{13(10000000)+2}{2(10000000)^2+10000000} \approx 6.5 \times 10^{-7}$$

Approaching Infinity

- As you observe here, as the value of n increases, **the value of the ratio, becomes closer and closer to 0**.
- We can expect that when we reach really big values, **the value of the ratio will be very close to zero**.

Approaching Infinity

- We directly look at the value of the ratio when the value of n approaches to **infinity** (a very big value)

Approaching Infinity

$$\lim_{n \to \infty} \frac{2n^3 + 10}{4n^2 + 30}$$

Approaching Infinity

$$
\begin{aligned}
&= \frac{2(\infty)^3 + 10}{4(\infty)^2 + 30} \\
&= \frac{\infty}{\infty}
\end{aligned}
$$

Approaching Infinity

- The fraction infinity over infinity is an undefined value, so instead of automatically substituting infinity to n, we'll need to apply **L'Hopital's rule** first:

Approaching Infinity

$$\begin{aligned}\lim_{n\to\infty}\frac{2n^3+10}{4n^2+30} &= \lim_{n\to\infty}\frac{\frac{d}{dx}(2n^3+10)}{\frac{d}{dx}(4n^2+30)}\\ &= \lim_{n\to\infty}\frac{6n^2}{8n}\\ &= \lim_{n\to\infty}\frac{3n}{4}\end{aligned}$$

Approaching Infinity

- By applying L'Hopital's rule, we can now freely *substitute*, infinity to n, giving us:

Approaching Infinity

$$\lim_{n\to\infty} \frac{2n^3 + 10}{4n^2 + 30} = \lim_{n\to\infty} \frac{3n}{4} = \infty$$

Approaching Infinity

$$\lim_{n \to c} \frac{f(n)}{g(n)} = \lim_{n \to c} \frac{\frac{d}{dx} f(n)}{\frac{d}{dx} g(n)}$$

Approaching Infinity

$$\begin{aligned}\lim_{n\to\infty} \frac{13n+2}{2n^2+n} &= \lim_{n\to\infty} \frac{\frac{d}{dx}(13n+2)}{\frac{d}{dx}(2n^2+n)} \\ &= \lim_{n\to\infty} \frac{13}{4n+1} \\ &= 0\end{aligned}$$

$$\therefore 13n+2 \in O(2n^2+n)$$

Approaching Infinity

$$
\begin{aligned}
\lim_{n\to\infty} \frac{4n^3 + 3n^2}{6n^3 + 1} &= \lim_{n\to\infty} \frac{\frac{d}{dx}(4n^3 + 3n^2)}{\frac{d}{dx}(6n^3 + 1)} \\
&= \lim_{n\to\infty} \frac{12n^2 + 6n}{18n^2} \\
&= \lim_{n\to\infty} \frac{\frac{d}{dx}(12n^2 + 6n)}{\frac{d}{dx}(18n^2)} \\
&= \lim_{n\to\infty} \frac{24n + 6}{36n} \\
&= \lim_{n\to\infty} \frac{\frac{d}{dx}(24n + 6)}{\frac{d}{dx}(36n)} \\
&= \lim_{n\to\infty} \frac{24}{36}
\end{aligned}
$$

$4n^3 + 3n^2$

Approaching Infinity

$$\therefore 4n^3 + 3n^2 \in \Theta(6n^3 + 1)$$

Approaching Infinity

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n\log_5 n + 3n}{5n^2+4n+2} &= \lim_{n\to\infty} \frac{\frac{d}{dx}(n\log_5 n + 3n)}{\frac{d}{dx}(5n^2+4n+2)} \\
&= \lim_{n\to\infty} \frac{n\frac{1}{n\ln 5} + \log_5 n + 3}{10n+4} \\
&= \lim_{n\to\infty} \frac{\frac{d}{dx}(\frac{1}{\ln 5} + \log_5 n + 3)}{\frac{d}{dx}(10n+4)} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n\ln 5}}{10} \\
&= \lim_{n\to\infty} \frac{1}{10n(\ln 5)} \\
\lim_{n\to\infty} \frac{n\log_5 n + 3n}{5n^2+4n+2} &= 0
\end{aligned}
$$

Asymptotically Loose Relationships

Asymptotically Loose Relationships

- You might be wondering why, we read asymptotic notation with the qualifier "Big", why do we have to specify **Big**-O, or **Big**-Omega? Well, you might have guessed it, its because there are "little" versions of O, and Ω.

Little-o

Little-o

$$o(g(n)) = \{f(n) | \forall c > 0 (\exists n_0 > 0 (\forall n > n_0 (0 \leq f(n) \leq cg(n))))\}$$

Little-o

For all positive constants c, there exists a positive constant n_0, such that $0 \leq f(n) \leq cg(n)$ for all $n > n_0$, (for sufficiently large values of n)

Little-o

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \to f(n) \in o(g(n))$$

Little-o

$$5n^2 + 3n \in o(2n^3)$$

Little-ω

Little-ω

$$\omega(g(n)) = \{f(n) | \forall c > 0(\exists n_0 > 0(\forall n > n_0(0 \leq cg(n) \leq f(n))))\}$$

Little-ω

For all positive constants c, there exists a positive constant n_0, such that $0 \leq cg(n) \leq f(n)$ for all $n > n_0$, (for sufficiently large values of n)

Little-ω

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \to f(n) \in \omega(g(n))$$

Little-ω

$$2n^3 \in \omega(5n^2 + 3n)$$

Little-ω

Is there no such thing as little-θ? Try to form the definition of little-θ based on the definitions we have created earlier. Similar to how O and Ω conjunctively combine into Θ, the little relationships, o and ω must combine into θ. Once you manage to create this definition, you'll realize that for any function $g(n)$, $\theta(g(n)) = \emptyset$. No such function is a member of this set.

Asymptotic Tightness

Asymptotic Tightness

- If $f(n) \in O(g(n))$ but $f(n) \notin o(g(n))$, then we call their relationship **asymptotically tight**, meaning $f(n)$ is shown to be upper bounded by $g(n)$, but at the same time this bound is close/tight to $f(n)$, because they actually turn out to have the exact same complexity, $f(n) \in \Theta(g(n))$.

Asymptotic Tightness

- $n^2 + n \in O(n^2)$
- $n^2 + n \notin o(n^2)$
- $n^2 + n \in \Theta(n^2)$

Analogy to inequalities/equalities

Analogy to inequalities/equalities

- But **if we look at asymptotic relationships as something similar to inequalities/equalities it becomes easier to understand their concepts**

Analogy to inequalities/equalities

$$L(x) = \{y | y \leq x\}$$

Analogy to inequalities/equalities

- Therefore t**he set $L(x)$ is basically the set of all numbers less than or equal to x**.
- The set $L(x)$ **can be seen as the $O(g(n))$ but for numbers instead of functions**.

Analogy to inequalities/equalities

Algebraic Inequalities/Equalities

$L(x) = y|y \leq x$

$G(x) = y|y \geq x$

$E(x) = y|y = x$

$l(x) = y|y < x$

$g(x) = y|y > x$

Asymptotic Notation

$f(n) \in O(g(n))$

$f(n) \in \Omega(g(n))$

$f(n) \in \Theta(g(n))$

$f(n) \in o(g(n))$

$f(n) \in \omega(g(n))$

Asymptotically tight binding

Asymptotically tight binding

- If $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$, then $f(n) \in \Theta(g(n))$.
- If $f(n) \in \Theta(g(n))$ then $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$.
- Also, $O(g(n)) \cap \Omega(g(n)) = \Theta(g(n))$.

Asymptotic subsets

Asymptotic subsets

- If $f(n) \in o(g(n))$ then $f(n) \in O(g(n))$,
- $o(g(n)) \subset O(g(n))$.
- If $f(n) \in \omega(g(n))$ then $f(n) \in \Omega(g(n))$,
- $\omega(g(n)) \subset \Omega(g(n))$.

Asymptotic non-memberships

Asymptotic non-memberships

- If $f(n) \notin O(g(n))$ then $f(n) \in \omega(g(n))$.
- If $f(n) \notin \Omega(g(n))$ then $f(n) \in o(g(n))$.
- $\overline{O(g(n))} = \omega(g(n))$.
- $\overline{\Omega(g(n))} = o(g(n))$.

Application to Computer Science

Application to Computer Science

- By mastering this your focus shifts from writing algorithms that work to writing algorithms that work **efficiently**.
- It's not only important to write programs that work, it becomes important that these programs also work **quicker** and use **lesser resources**

Application to Computer Science

- **How do we judge which algorithm is more efficient**? Let's say we prefer the faster algorithm between the two.
- **How do we compare their speeds?**

Application to Computer Science

- One thing we can do is to look at empirical evidence and try to look at their **running times** when the algorithm is converted to actual code.
- For these algorithms, **the running time is dependent on the size of the given array**.
- If the given array is **shorter** in length then both algorithms have **less elements** to sort.
- Therefore, the algorithm will take less time time to run and the running time will be **faster**.
- The inverse is also true, if the given array is **longer**, then the running time would be **high**

Application to Computer Science

- This tells us that **the running time for an algorithm cannot be simply represented as one number** that we can easily compare against another algorithm's running time.
- Most of the time, an algorithm's running time is dependent on the **size of the input** [^4].

Application to Computer Science

- A more complete way of representing running time would be through a **mathematical function** in terms of the input size.
- This is why you'll usually see running times denoted as $T(n)$, where n is the input size

Application to Computer Science

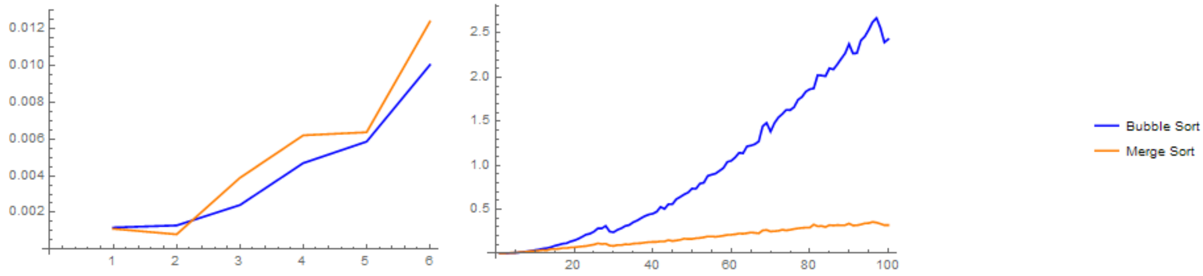
- But as it turns out, the process of figuring out the **exact** mathematical function to represent the running time is not an exact science as well.

Measuring running time through experiment

Measuring running time through experiment

Array Size	Runtime in s (Bubble Sort)	Runtime in s (Merge Sort)
1	0.001188	0.001119
2	0.001301	0.000818
10	0.035985	0.028135
20	0.146644	0.067852
30	0.239448	0.084049
50	0.735932	0.163289
70	1.382356	0.248158
100	2.432715	0.320786

Measuring running time through experiment



Measuring running time through experiment

- By comparing the plots of their running times, we can infer that, for **smaller** input sizes, both algorithm's have **similar** running times.
- As the input size becomes **bigger** and bigger, both algorithm's running times increase but **bubble sort's running time increases noticeably faster**.

Measuring running time through instruction approximation

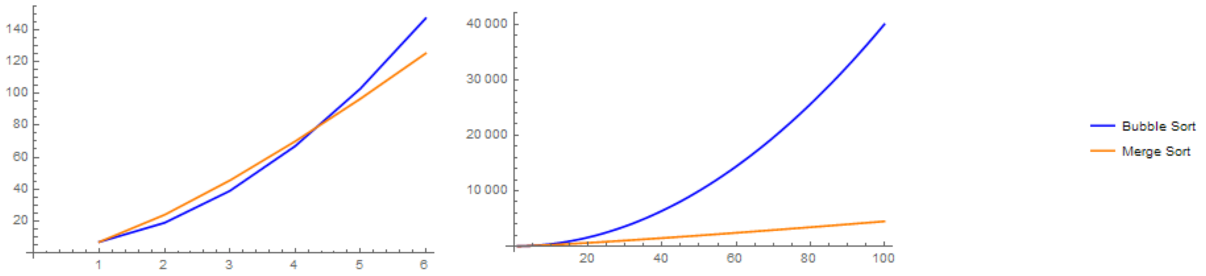
Measuring running time through instruction approximation

- Another method of approximating the running time functions is by approximating the number of **instructions** executed during the algorithms lifetime[^5].

Measuring running time through instruction approximation

- Bubble Sort: $T(n) = 4n^2 + 3$
- Merge Sort: $T(n) = 6n \log_2 n + 5n + 2$

Measuring running time through instruction approximation



Measuring running time through instruction approximation

- The running time functions that we ended up with also shows that for **smaller input sizes** both algorithms end up executing pretty much the same amount of instructions.
- For **larger input sizes**, it also shows that merge sort is clear winner, it ends up executing way fewer instructions compared to bubble sort

Measuring running time through instruction approximation

- **Merge sort is the preferred algorithm between the two**.
- Even though, bubble sort may be faster than merge sort for smaller input sizes, **for cases that matter the most (large input sizes), merge sort is way faster**.
- That's because **we expect that larger input sizes will take a longer time** to compute for.
- Because of this we barely care about how the algorithms compare for smaller input sizes, **we instead focus on the bigger picture that matters most**.
- **Which algorithm performs better in stressful situations (when the input sizes go big).**

Measuring running time through instruction approximation

- What we are comparing in these two representations (experimental running time and approximate instruction running time) are running times for the **written code** for the algorithms, not the algorithm's themselves.
- These running time measurements are attributed specifically to the Python **code** I wrote for the algorithm.

Finally, Asymptotic Notation

Finally, Asymptotic Notation

1. **Algorithm efficiency is highly dependent on input size, the best representations of efficiency is a function.**
2. **There is no reliable way to compare algorithm efficiency. The ways that we have always end up being approximations.**
3. **We care the most about how the algorithm performs in large input sizes**

Finally, Asymptotic Notation

1. **Since algorithm efficiency almost always ends up being functions of input sizes, we can makes use of asymptotic notation, which compare relationships between functions.** Remember that the member's of O, Ω, Θ, o, ω, are functions. We've even shown how asymptotic relationships are similar to inequality/equality relationships, but instead of comparing numerical values, we compare functions.

Finally, Asymptotic Notation

1. **You'll find that asymptotic notation relationships are good at calculating approximated values.** The asymptotic relationships between two functions still hold true even if you change the functions a little bit. For example $2n^2 + 5n \in \Theta(4n^2 + 3)$, even if we change the constant coefficients in these functions, their asymptotic relationship still stays the same: $an^2 + bn \in \Theta(cn^2 + d)$. This characteristic is useful in the case of running time comparisons because as we've shown, our methods of measurements are not precise enough or uniform enough. Asymptotic notation doesn't care about the small mistakes in measurement.

Finally, Asymptotic Notation

1. **Asymptotic notation by definition only cares about large values of n.** If we recall, asymptotic relationships only care about what happens on **sufficiently large values of** n, or on values of $n \to \infty$.

Finally, Asymptotic Notation

$$4n^2 + 3 \in \Omega(6n \log_2 n + 5n + 2)$$

Finally, Asymptotic Notation

- This leads us to the conclusion that bubble sort's **time complexity** is higher than merge sort's time complexity.

Reduction Rules and Complexity Classes

Reduction Rules

Dropping the coefficient

$$c > 0 \rightarrow cf(n) \in \Theta(f(n))$$

Dropping the coefficient

- **Given a function multiplied to some constant, $cf(n)$, its complexity is exactly the same as $f(n)$.** This property can easily be demonstrated using the limit definition of Θ

Proof

$$\lim_{n\to\infty} \frac{cf(n)}{f(n)} = \lim_{n\to\infty} c = c$$

Higher degree polynomial

$$c < d \to f(n)^c \in o(f(n)^d)$$

Higher degree polynomial

- **Given $f(n)^c$ and $f(n)^d$ where c and d are constants and $c < d$, $f(n)^c$ is less complex than $f(n)^d$**

Proof

Let $0 < c < d$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{f(n)^c}{f(n)^d} &= \lim_{n\to\infty} \frac{c f(n)^{c-1} \frac{d}{dn} f(x)}{d f(n)^{d-1} \frac{d}{dn} f(x)} \\
&= \lim_{n\to\infty} \frac{c f(n)^{c-1}}{d f(n)^{d-1}} \\
&= \lim_{n\to\infty} \frac{c(c-1) f(n)^{c-2}}{d(d-1) f(n)^{d-2}} \\
&\vdots \\
&= \lim_{n\to\infty} \frac{c(c-1)(c-2)(c-3)\cdots(c-(c-1)) f(n)^{c-c}}{d(d-1)(d-2)(d-3)\cdots(d-(c-1)) f(n)^{d-c}} \\
&= \lim \frac{c(c-1)(c-2)(c-3)\cdots(c-(c-1))}{}
\end{aligned}
$$

Proof

$$\lim_{n \to \infty} \frac{f(n)^c}{f(n)^d} = 0$$

Sum of functions

$$f(n) \in \Omega(g(n)) \rightarrow f(n) + g(n) \in \Theta(f(n))$$

Sum of functions

- **In a sum of functions, the most complex function dominates the sum therefore the overall complexity will be equal to the dominant function.** This can be shown using the set definitions

Proof

$$\text{Since} f(n) = \Omega(f(n)),$$

for any $n > n_0$

$$\begin{aligned} f(n) &\geq cf(n) \\ f(n) + g(n) &\geq cf(n) \end{aligned}$$

Proof

$$\therefore f(n) + g(n) = \Omega(f(n))$$

Proof

$$\text{Since } g(n) = O(f(n)),$$

$$\begin{aligned} g(n) &\leq cf(n) \\ f(n) + g(n) &\leq cf(n) + f(n) \\ f(n) + g(n) &\leq (c+1)f(n) \end{aligned}$$

Proof

$$\therefore f(n) + g(n) \in O(f(n))$$

Proof

$$\frac{\begin{array}{l} f(n) + g(n) = \Omega(f(n)) \\ \wedge f(n) + g(n) \in O(f(n)) \rightarrow \end{array}}{f(n) + g(n) \in \Theta(f(n))}$$

Different bases on log

- **Bases for logarithms doesn't matter in terms of asymptotic notation therefore** $\log_b n$ **for any base is of complexity** $\Theta(\log n)$**.** Because of this we can omit the base of the logarithm since its value is irrelevant from the complexity

Proof

$$\lim_{n\to\infty} \frac{\log_a n}{\log_b n} = \lim_{n\to\infty} \frac{\frac{\log_b n}{\log_b a}}{\log_b n}$$
$$= \lim_{n\to\infty} \frac{\log_b n}{\log_b n(\log_b a)}$$
$$= \lim_{n\to\infty} \frac{1}{\log_b a}$$
$$\lim_{n\to\infty} \frac{\log_a n}{\log_b n} = \frac{1}{\log_b a}$$

Different bases on exponential functions

$$0 < a < b \rightarrow a^n \in o(b^n)$$

Different bases on exponential functions

- **Bases for exponential functions on the other hand matter**, as shown in this proof:

Proof

$$
\begin{aligned}
\lim_{x \to \infty} \frac{a^n}{b^n} &= \lim_{x \to \infty} \frac{a^n}{(a^{\log_a b})^n} \\
&= \lim_{x \to \infty} \frac{a^n}{(a^n)^{\log_a b}} \\
&= \lim_{x \to \infty} (a^n)^{1-\log_a b}
\end{aligned}
$$

Proof

$$\lim_{x \to \infty} \frac{a^n}{b^n} = 0$$

Proof

- Based on these rules a complicated looking function can be **reduced** to the **simplest function** that matches its complexity.

Proof

$$6n \log_2 n + 5n + 2 \in \Theta(6n \log_2 n)$$

Proof

$$6n \log_2 n + 5n + 2 \in \Theta(n \log_2 n)$$

Proof

$$6n \log_2 n + 5n + 2 \in \Theta(n \log n)$$

Proof

- By finding this out, **we can use $n \log n$ as sort of the representative of $6n \log_2 n + 5n + 2$ on matters regarding its complexity**

Complexity Classes

Complexity Classes

- In the domain of computer science we know a few of these *representative* functions, we call these, **complexity classes**.

Complexity Classes

Complexity Class	**Name**	**Example**
$\Theta(1)$	Constant	converting Celsius to Fahrenheit
$\Theta(\log n)$	Logarithmic	binary search
$\Theta((\log n)^c)$	Polylogarithmic	dynamic planarity testing of a graph
$\Theta(n^c)$ where $0 < c < 1$	Fractional power	primality test
$\Theta(n)$	Linear	finding the smallest number in an array
$\Theta(n \log n)$	Quasilinear	merge sort
$\Theta(n^2)$	Quadratic	bubble sort
$\Theta(n^c)$ where $c > 1$	Polynomial	naïve matrix multiplication

Complexity Classes

Complexity Class	**Name**	**Example**
$\Theta(c^n)$	Exponential	dynamic travelling salesman
$\Theta(n!)$	Factorial	brute force travelling salesman

Using O versus using Θ

Using O versus using Θ

- When specifying time complexity, you'll find that it is usually expressed in terms of **O**.
- You might find this strange since **expressing the time complexity in terms of Θ is more specific**.

Using O versus using Θ

- You'll find that **computer scientists will end up expressing time complexities in terms of O as well because we generally want our algorithms to run on some acceptable standard of complexity**.
- Because of this **asymptotic notation in terms of O will usually suffice**