Introduction to Linear Algebra

Matrices and Matrix Operations

Matrices and Matrix Operations

- A matrix is a **rectangular collection** of numbers.

Matrices and Matrix Operations

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 11 & \pi & 12 \end{bmatrix}$$

Matrices and Matrix Operations

$$A = \begin{bmatrix} 0.1 & 0.9 & 0.3 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.7 & 0.9 \end{bmatrix}$$

Matrices and Matrix Operations

Matrices and Matrix Operations

- An **element** of a matrix is denoted by a_{rc} which corresponds the element of matrix a in the r^{th} row and c^{th} column.

Matrices and Matrix Operations

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Matrices and Matrix Operations

- Matrices are also generally used to represent **systems of linear equations** for example:

Matrices and Matrix Operations

$$\begin{aligned} 4x + 2y &= 12 \\ 3x - 2y &= 8 \end{aligned}$$

Matrices and Matrix Operations

$$\begin{bmatrix} 4 & 2 \\ 3 & -28 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

Matrices and Matrix Operations

how this works will be explained later

Matrix Arithmetic

Matrix Addition

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Matrix Addition

- To get each element$(A+B){*ij*}*$, you simply* ***add the corresponding elements****,*
*$a*{11}+b_{11}$.

Matrix Addition

$$A + x = \begin{bmatrix} a_{11} + x & a_{12} + x & \dots & a_{1n} + x \\ a_{21} + x & a_{22} + x & \dots & a_{2n} + x \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + x & a_{m2} + x & \dots & a_{mn} + x \end{bmatrix}$$

Matrix Multiplication

- **Matrix times matrix** multiplication works differently, let say A is an $m \times k$ matrix and B is a $k \times n$ matrix.
- Their **cross product** $C = A \times B$ defined to be the matrix with each element:

Matrix Multiplication

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Matrix Multiplication

$$A \times B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{1j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{1j} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

Matrix Multiplication

Matrix Multiplication

- **Scalar times matrix** multiplication works similar to scalar plus matrix addition:

Special Matrices

Identity Matrix

- The **identity matrix** $Inofordernisaspecialn \times n (square)matrixsuchthatitselements\iota{ij}=1ifi=jotherwise\iota_{ij}=0$

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

This matrix is special because, for any $m \times n$ matrix A,

$$AI_n = I_m A = A$$

Identity Matrix

- **Powers of square matrices** can be defined such that:

Identity Matrix

$$A^0 = I_n$$
$$A^r = A \times A \times \cdots \times A$$

Transpose of a Matrix

- The elements of A^T**, \$t{*ij*}=*a*{ji}\$**.
- Therefore the transpose of a matrix is the same matrix but the rows and columns are **interchanged**

Transpose of a Matrix

$$\begin{bmatrix} 4 & 2 & -3 \\ 11 & \pi & 12 \end{bmatrix}^T = \begin{bmatrix} 4 & 11 \\ 2 & \pi \\ -3 & 12 \end{bmatrix}$$

Transpose of a Matrix

- A, square matrix A is said to be **symmetric** if $A^T = A$

Linear Algebra

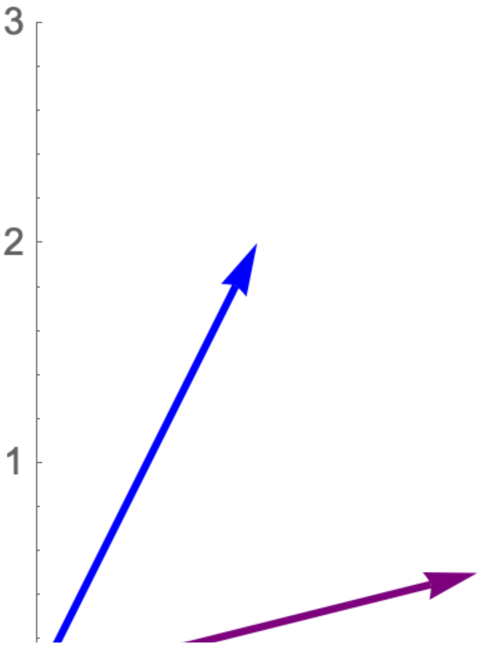
Vectors

- The foundation of linear algebra is the concept of a **vector**.

Vectors

- For a physics student, a vector is defined by **direction** and **magnitude**:

Vectors



Vectors

- For a computer science student, a vector is just an **ordered collection of numbers**.

Vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Vectors

- A vector in a CS students point of view can be thought of as a **coordinate list** specifying the destination of an arrow.
- In this case the vector $\vec{v}$ can be thought of as the **arrow** pointing from the origin, $(0, 0, 0)$ to the point $(1, 3, 2)$

Vectors

To differentiate vectors and coordinates, vectors are written as single-column matrices while coordinates are written as ordered tuples.

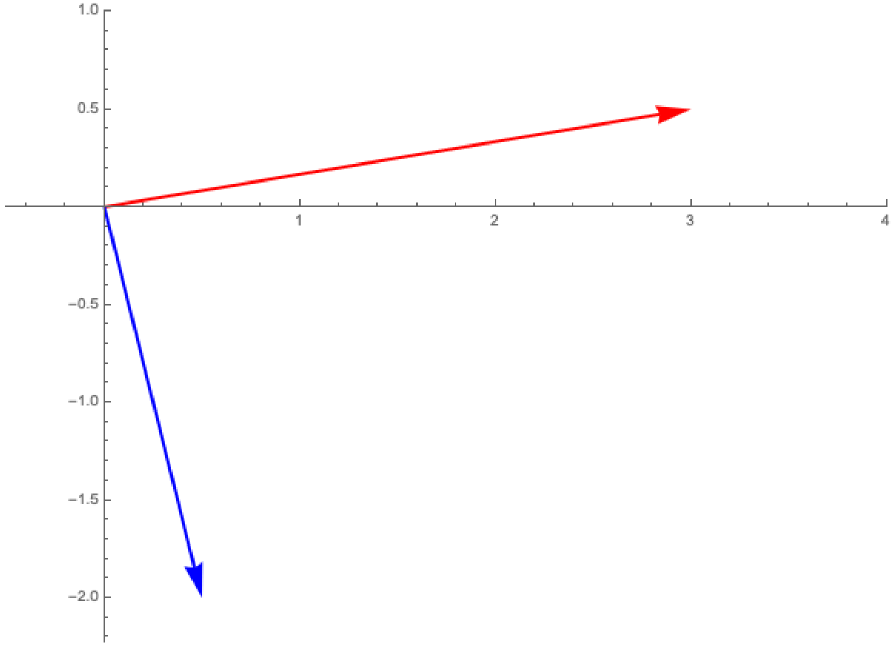
Vector Addition

- Vectors can be added to each other which also **add** their corresponding arrows

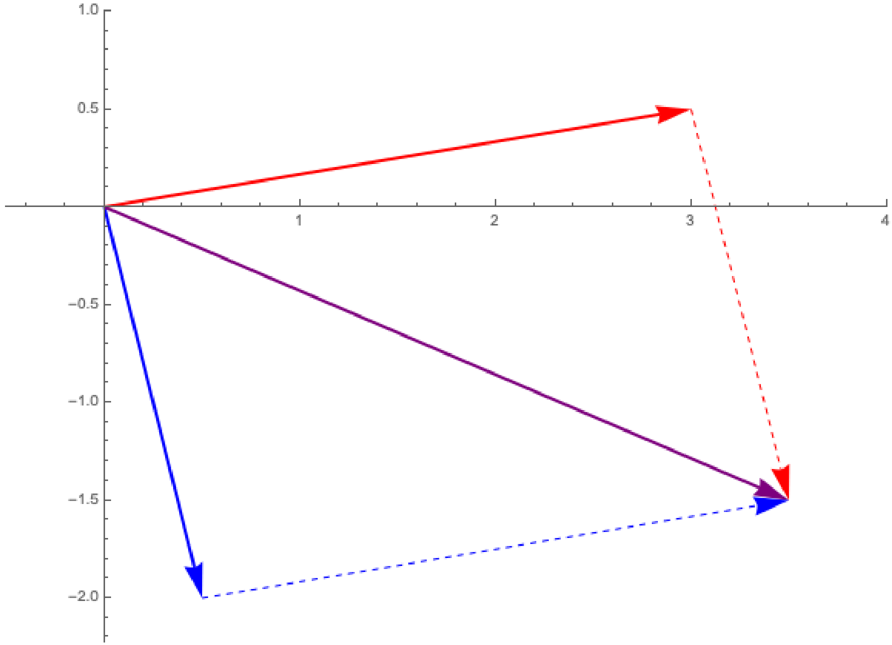
Vector Addition

$$\begin{bmatrix} 3 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -2 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -1.5 \end{bmatrix}$$

Vector Addition



Vector Addition



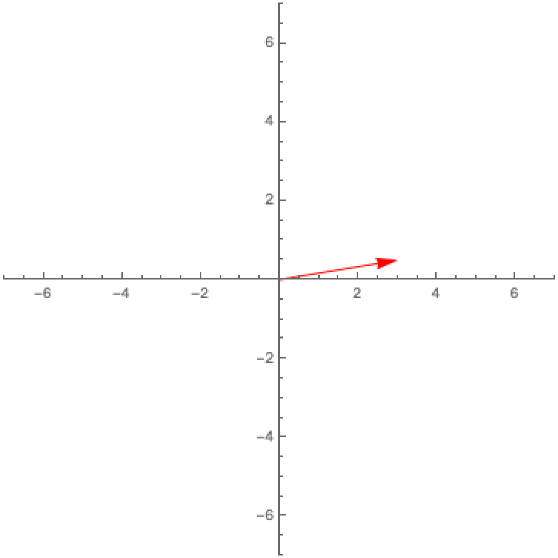
Vector Multiplication

- Vectors can also be multiplied with **scalar** values which will scale their corresponding arrows.

Vector Multiplication

$$\vec{v} = \begin{bmatrix} 3 \\ 0.5 \end{bmatrix}$$

Vector Multiplication



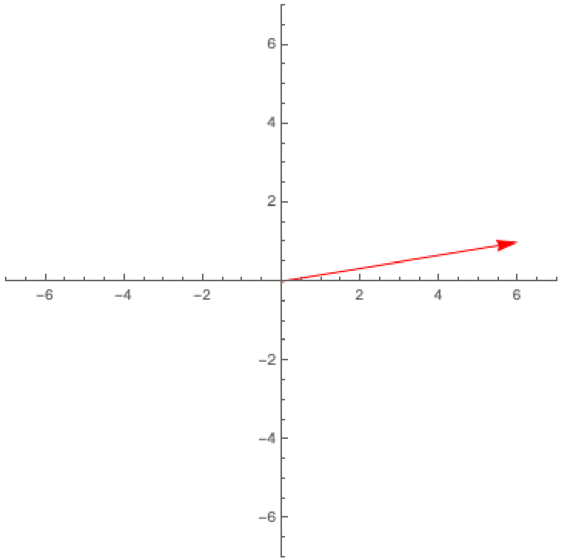
Vector Multiplication

- The vector **scaled** by a factor of **2**:

Vector Multiplication

$$2\vec{v} = 2\begin{bmatrix} 3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Vector Multiplication



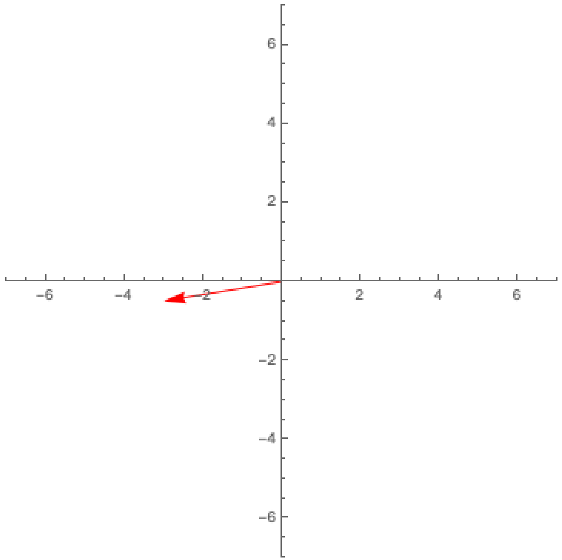
Vector Multiplication

- Scaled by a factor of -1

Vector Multiplication

$$-1\vec{v} = -1 \begin{bmatrix} 3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -3 \\ -.5 \end{bmatrix}$$

Vector Multiplication



Vector Multiplication

Scalars are called scalars because they **scale** the vectors.

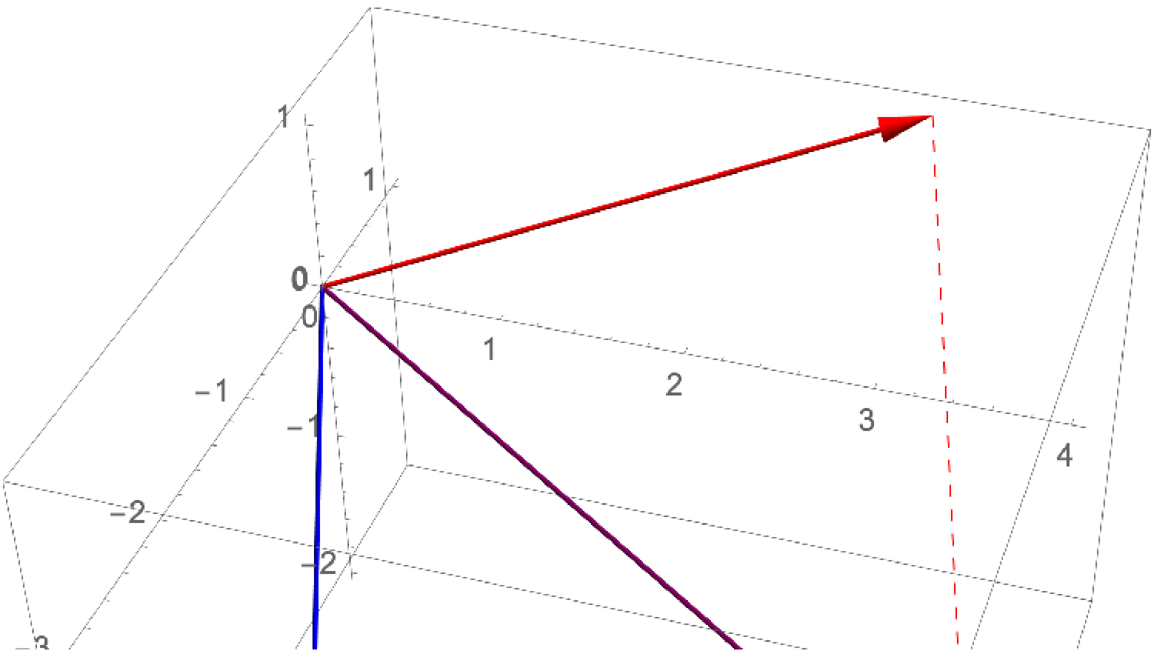
Vector Multiplication

- These operations also work on **3 dimensional vectors**

Vector Multiplication

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

Vector Multiplication



Basis Vectors

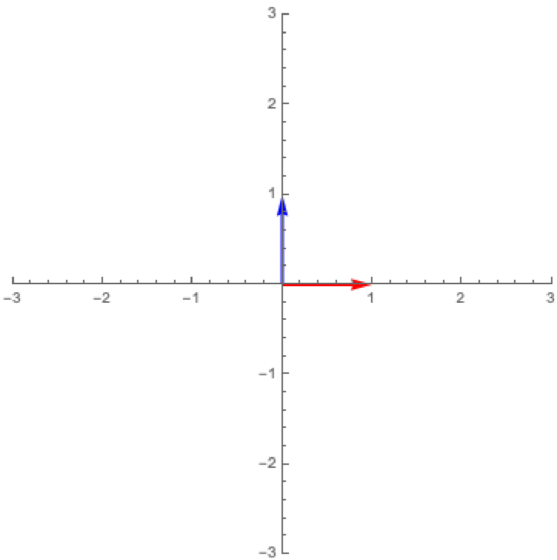
Basis Vectors

- To generalize vectors and vector operations, linear algebra makes use of **basis vectors** which are unit vectors along the x and y axis of a Cartesian plane.
- We call these vectors, $\hat{\imath}$ and $\hat{\jmath}$ respectively.

Basis Vectors

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Basis Vectors



Basis Vectors

- Basis vectors are special because you can **define** new vectors vectors based on the the definitions of the basis.

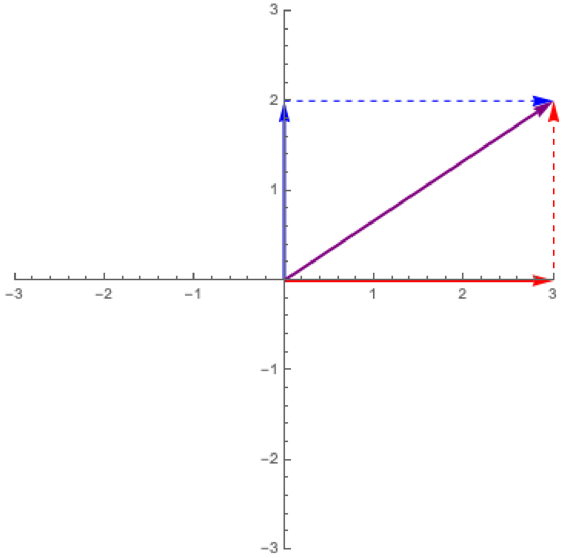
Basis Vectors

$$\vec{v} = 3\hat{\imath} + 2\hat{\jmath}$$

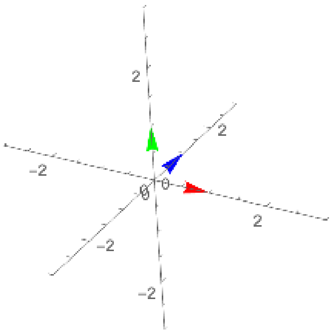
Basis Vectors

$$\vec{v} = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

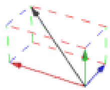
Basis Vectors

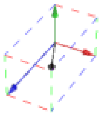


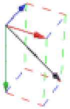
Basis Vectors



Basis Vectors



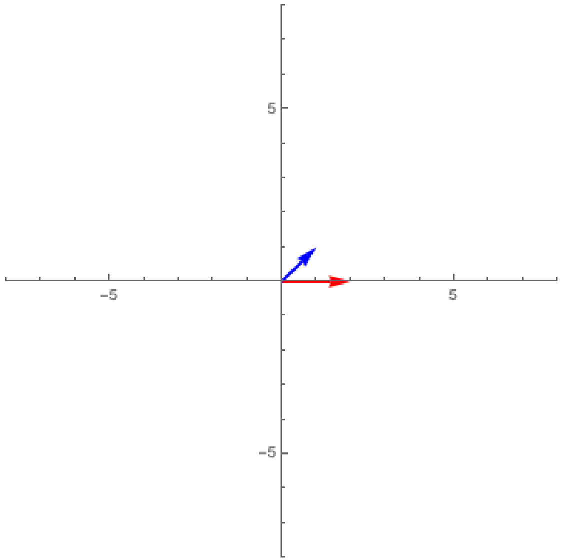




Basis Vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

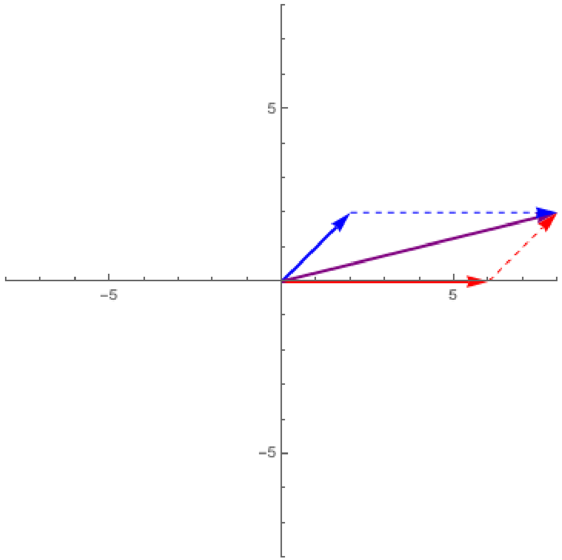
Basis Vectors



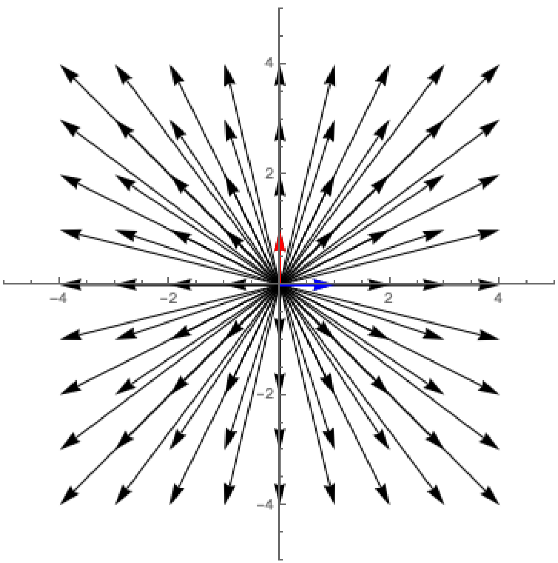
Basis Vectors

$$\vec{v} = 3\begin{bmatrix}2\\0\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}8\\2\end{bmatrix}$$

Basis Vectors



Span



Span

- For the set of vectors $\hat{\imath}, \hat{\jmath}$, you can generate **all** of the possible vectors in the 2 dimensional vector space

Span

- You can also generate **all** of the possible vectors in the 2-dimensional vector space using the basis vectors:

Span

$$\left\{\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

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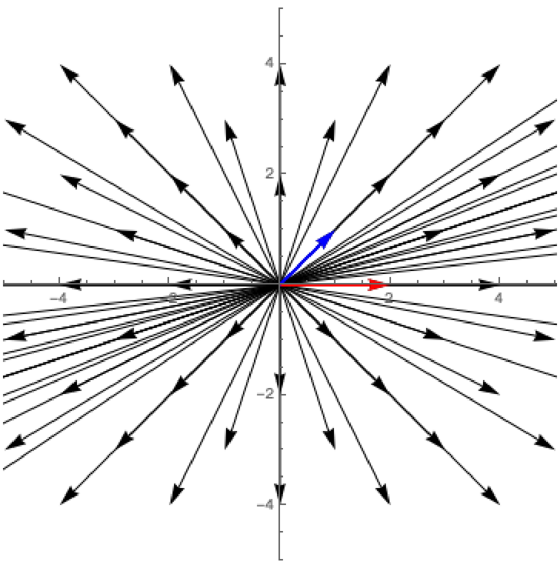

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$$\{\vec{r} = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$$

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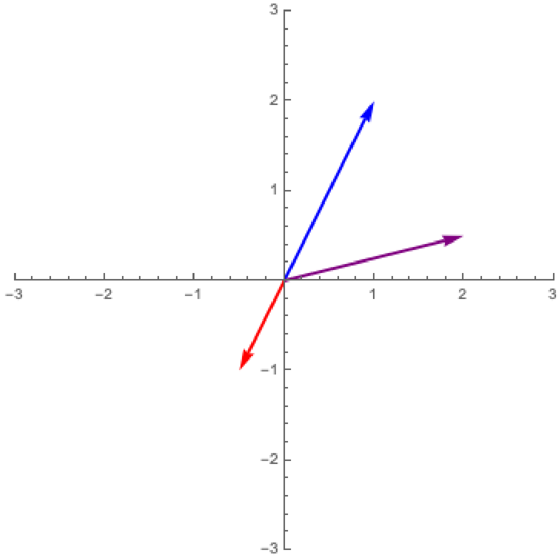

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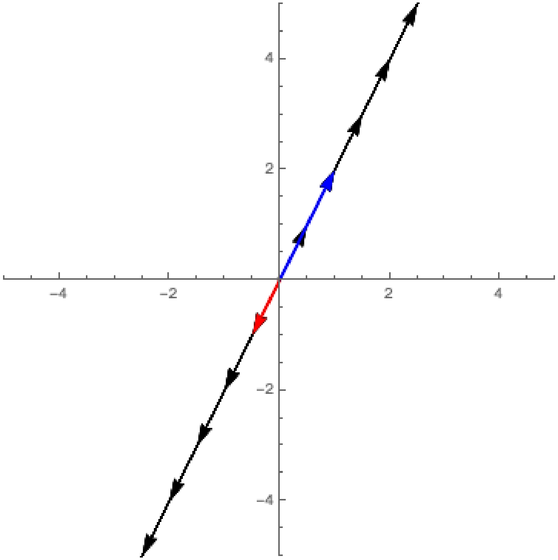
- You **can't** generate the purple vector since one of the basis vectors you are using is **redundant**.
- It is redundant in the sense that the $\vec{r}$ is just a **scaled version** on of $\vec{b}$ and vice versa.

Span

$$\vec{r} = -0.5\vec{b}$$
$$\vec{b} = -2\vec{r}$$

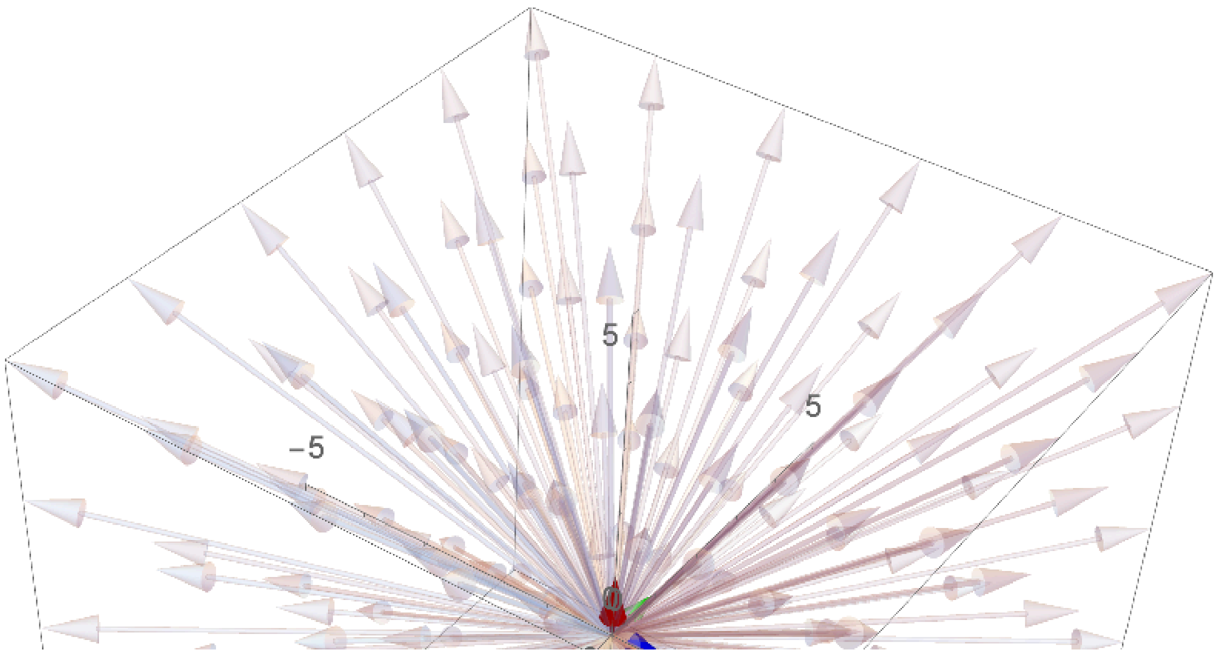
Span



Span

- We call $\vec{r}$ and $\vec{b}$ and any set of vectors that has some kind of redundancy as **linearly dependent**.
- The basis vectors defined earlier, $\hat{\imath}, \hat{\jmath}$, and $\vec{u}, \vec{w}$ as **linearly independent**

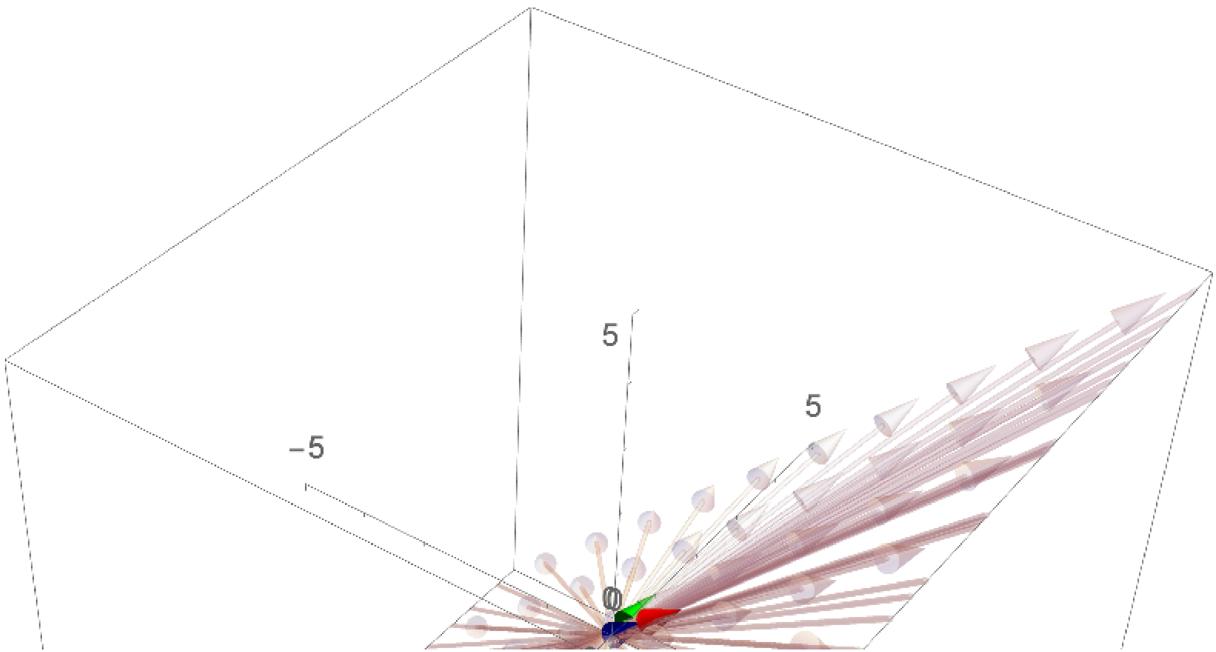
Span



Span

- Linear dependence in three dimensions can result to spans that describe a **plane**:

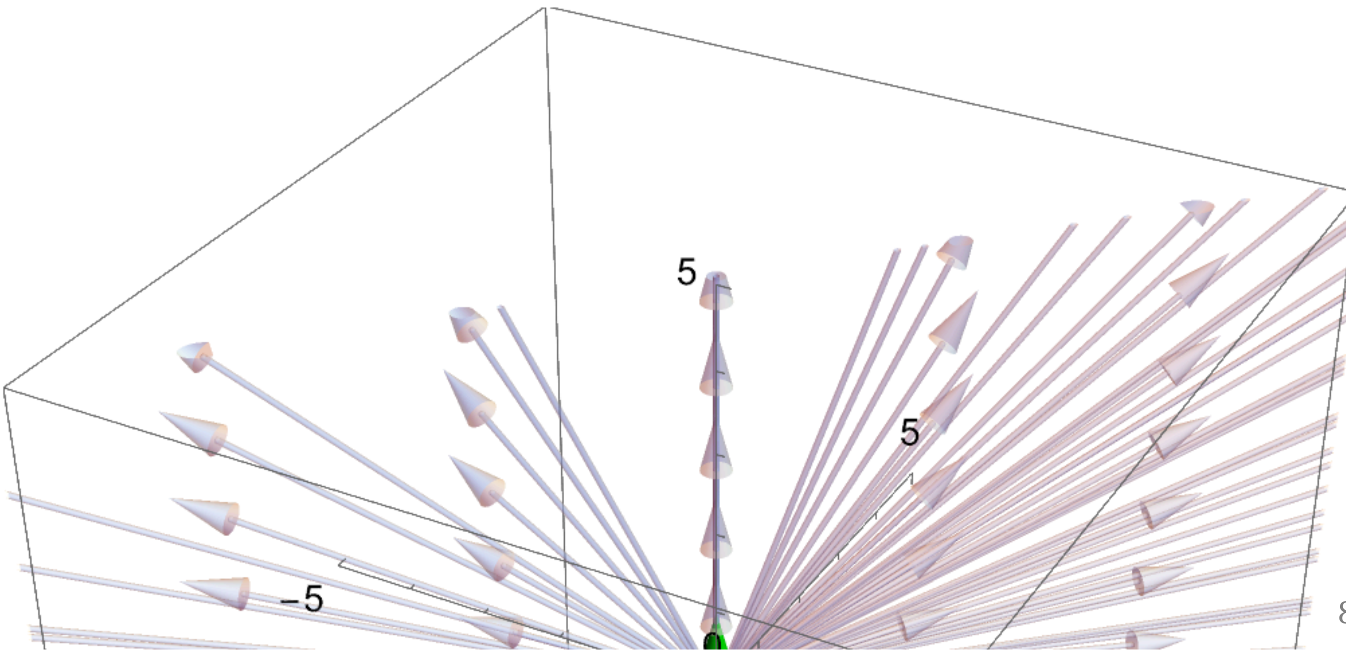
Span



Span

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0.5 \end{bmatrix} \right\}$$

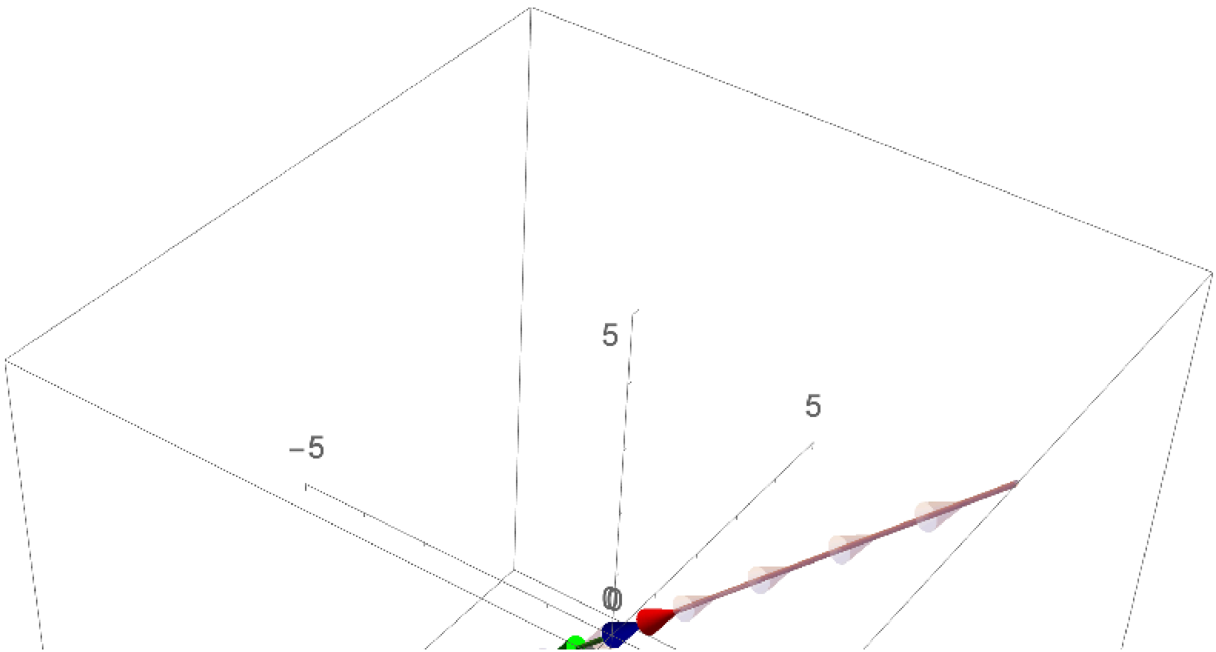
Span



Span

- And spans that describe a **line**:

Span

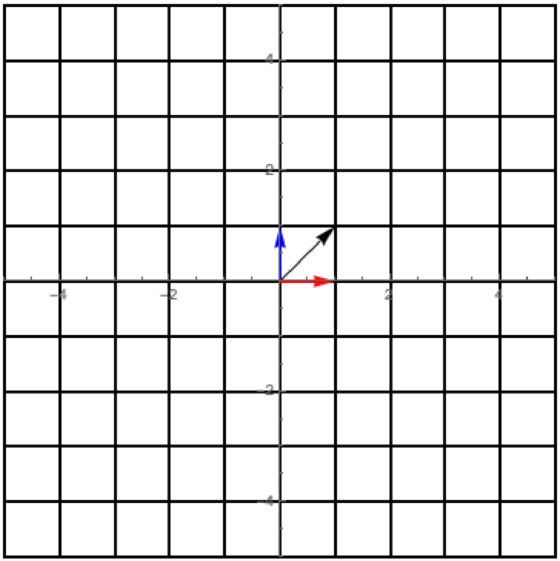


Linear Transformations and Matrix Multiplication

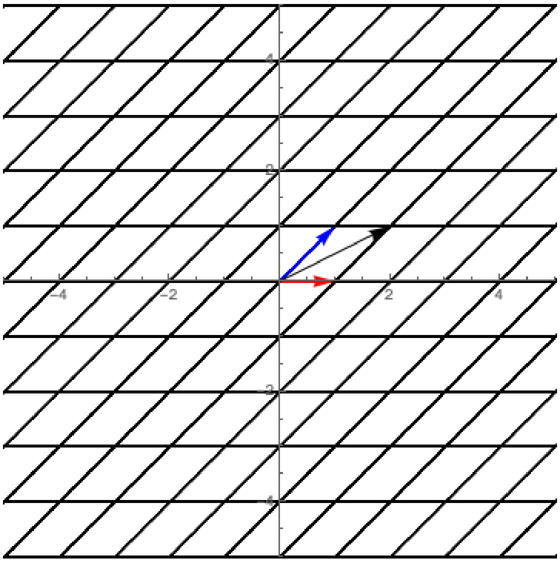
Linear Transformations and Matrix Multiplication

- A **transformation** is basically a function that converts one vector to another vector.
- For example the transformation f can be defined as $f(\vec{x}) = 3\vec{x}$.
- You can think of a transformation visually as the **distortion** of the entire vector space.

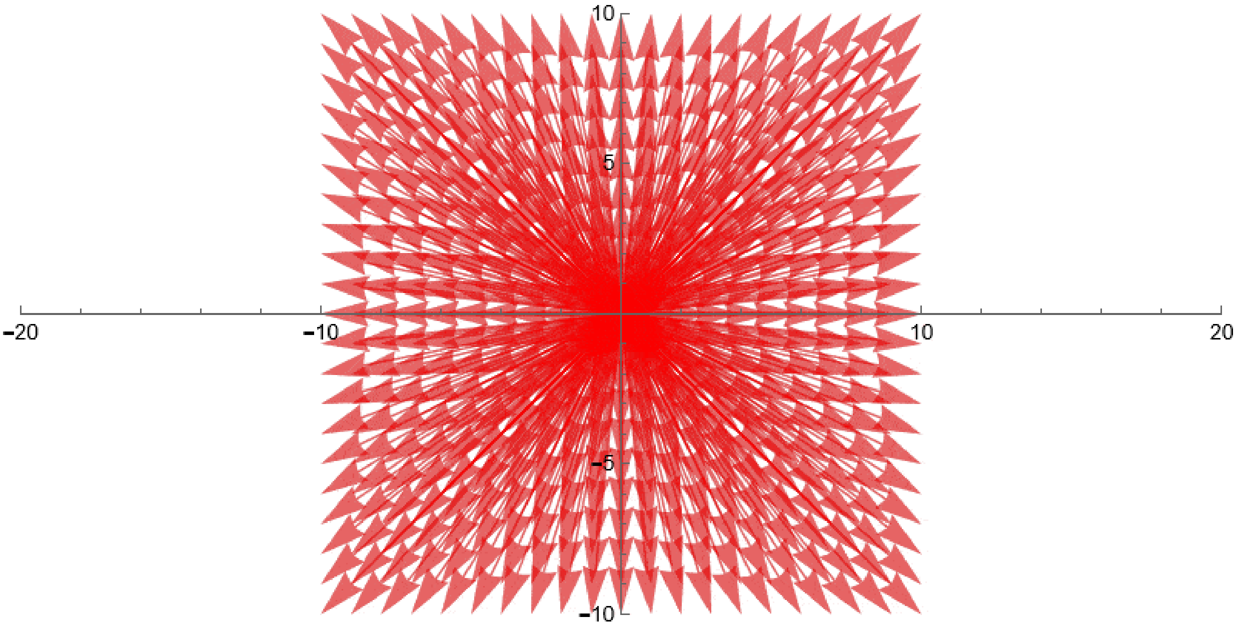
Linear Transformations and Matrix Multiplication



Linear Transformations and Matrix Multiplication



Linear Transformations and Matrix Multiplication



Linear Transformations

- **Linear Transformations** are special transformations where the distortion of the vector space follows these rules:

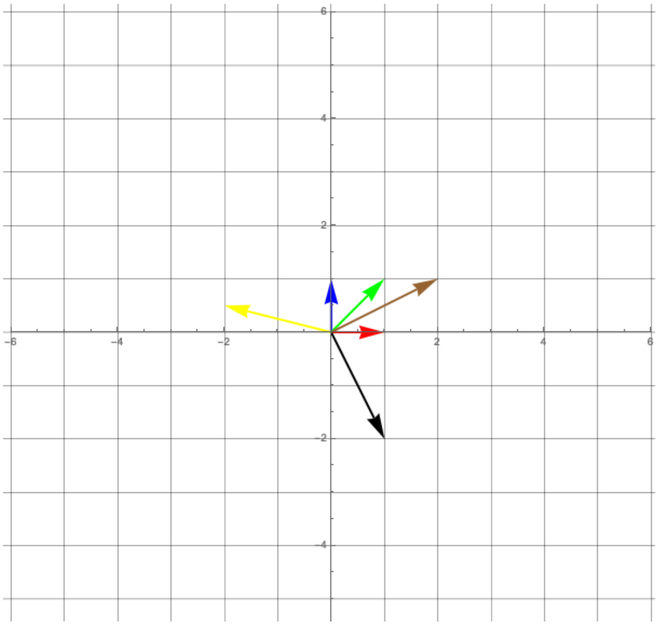
Linear Transformations

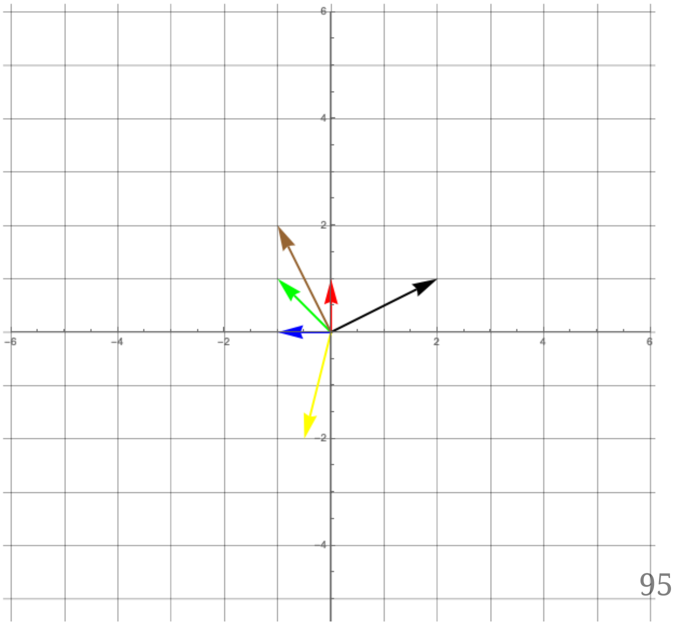
1. The origin should not move
2. Parallel lines stay parallel
3. Straight lines stay straight

Linear Transformations

- It turns out all transformations that satisfy the above rules can be perfectly described by watching how the **basis vectors** are transformed.

Linear Transformations





Linear Transformations

- new $\hat{\imath}$ (red vector) :$\hat{\imath}' = [0\backslash 1]$
- new $\hat{\jmath}$ (blue vector): $\hat{\jmath}' = [-1\backslash 0]$
- Green vector: $1\,[0\backslash 1] + 1\,[-1\backslash 0] = [-1\backslash 1]$
- Brown vector: $2\,[0\backslash 1] + 1\,[-1\backslash 0] = [-1\backslash 2]$
- Yellow vector: $-2\,[0\backslash 1] + 0.5\,[-1\backslash 0] = [-0.5\backslash\text{-}2]$

Linear Transformations

- Black vector: $1\,[0\backslash 1] + -2\,[-1\backslash 0] = [2\backslash 1]$

Linear Transformations

- All of the other vector values after the transformation is basically **scaled versions** of the new basis vectors in the same way that the pretransformed vector values are combinations of the original basis vectors.
- This means that any 2-dimensional linear transformation can be represented by 4 numbers, which we can write as a **matrix**, where each column corresponds to a basis vector.

Linear Transformations

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Linear Transformations

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Linear Transformation and Matrix Multiplication

$$x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Linear Transformation and Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

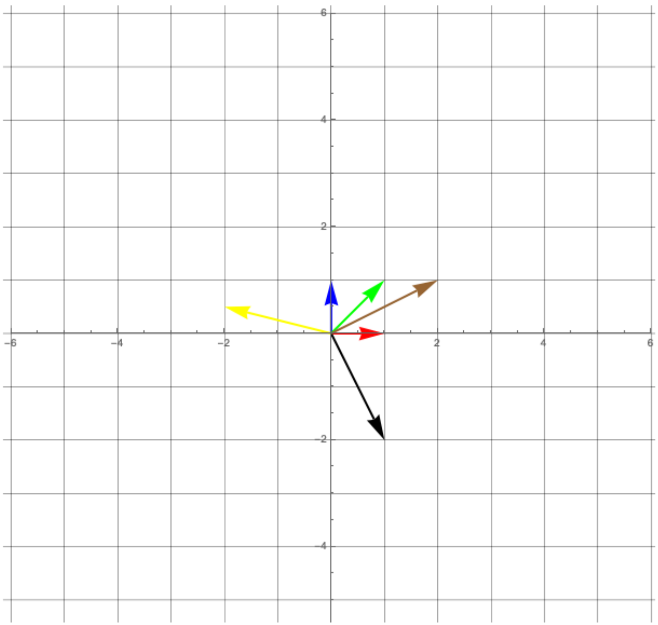
Linear Transformation and Matrix Multiplication

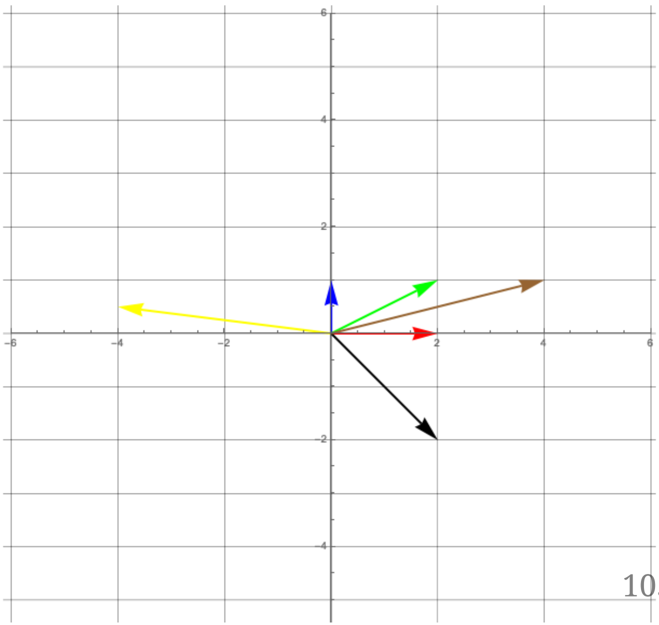
$$f(\vec{v}) = T\vec{v}$$

Horizontal Stretch

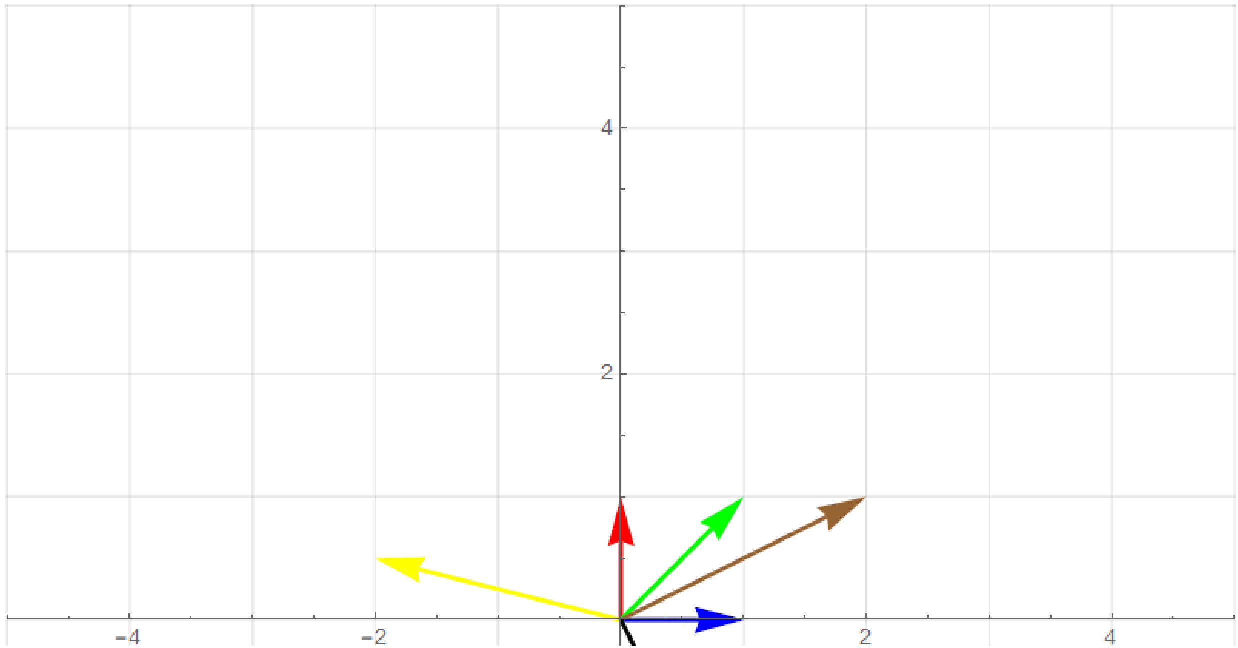
$$T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Horizontal Stretch





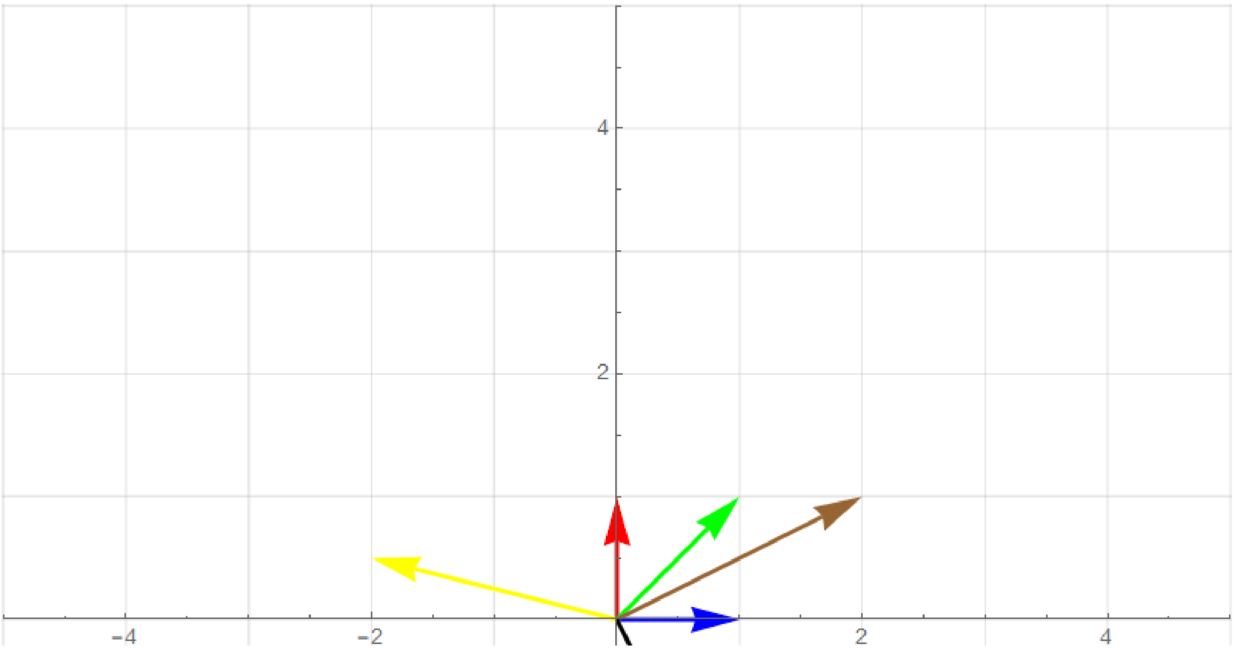
Horizontal Stretch



Vertical Stretch

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

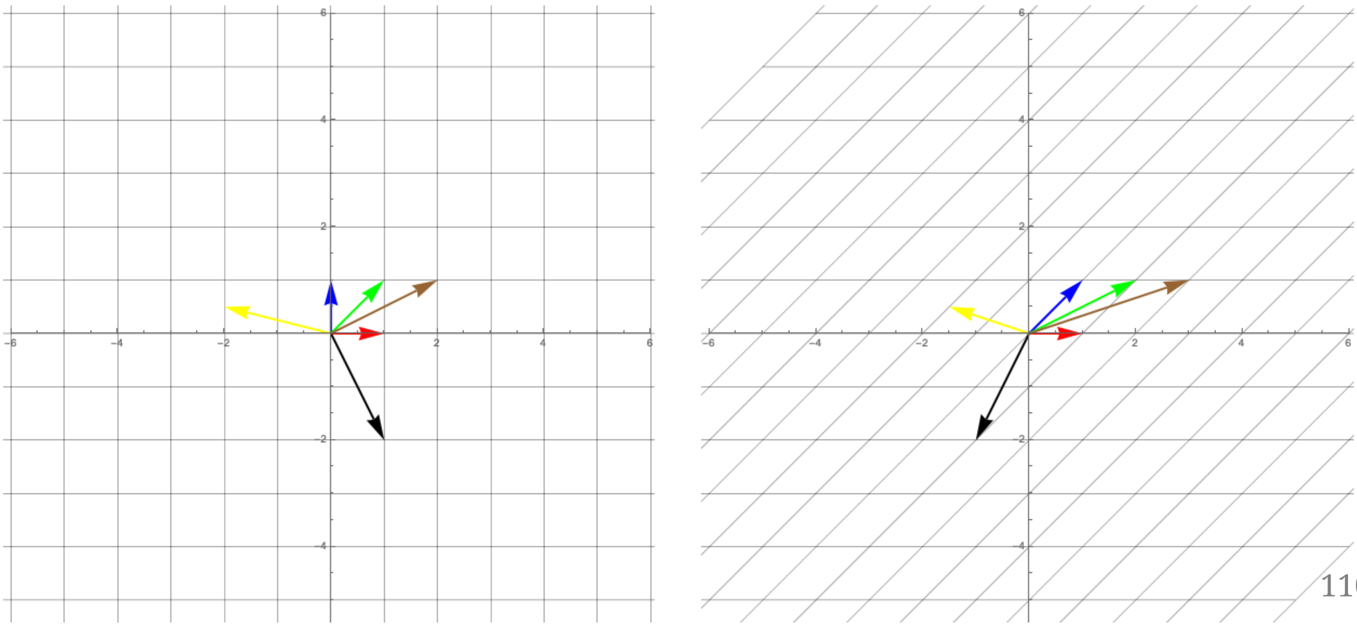
Vertical Stretch



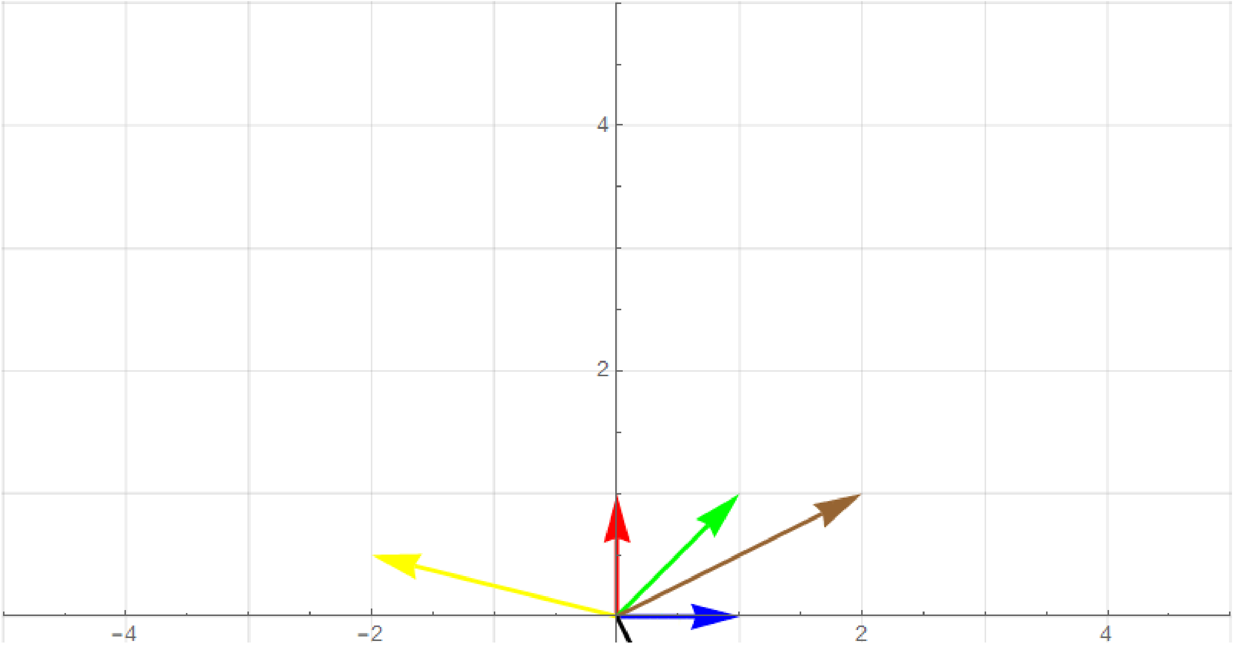
Horizontal Shear

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Horizontal Shear



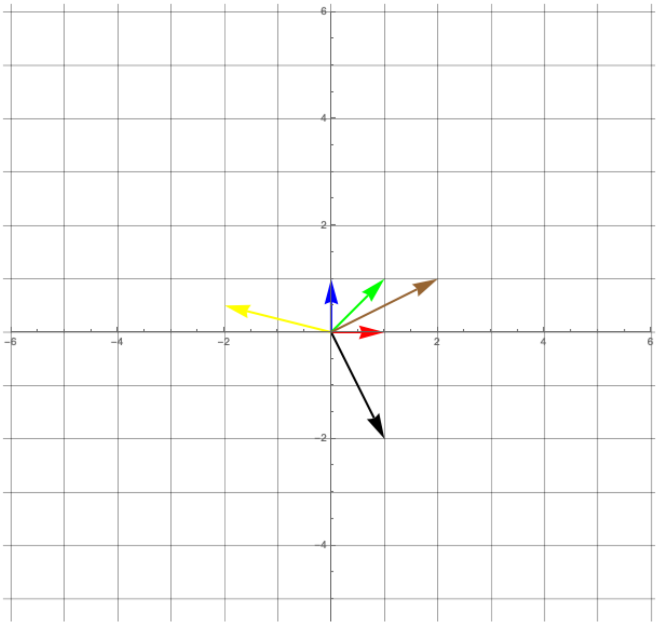
Horizontal Shear

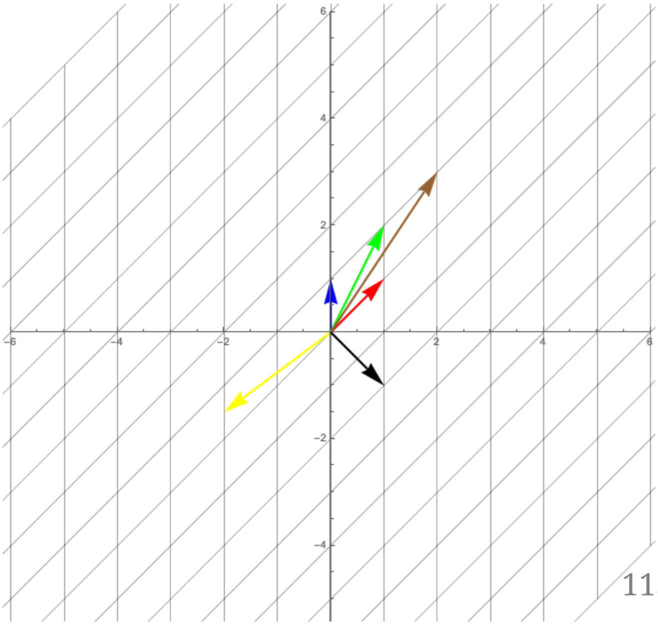


Vertical Shear

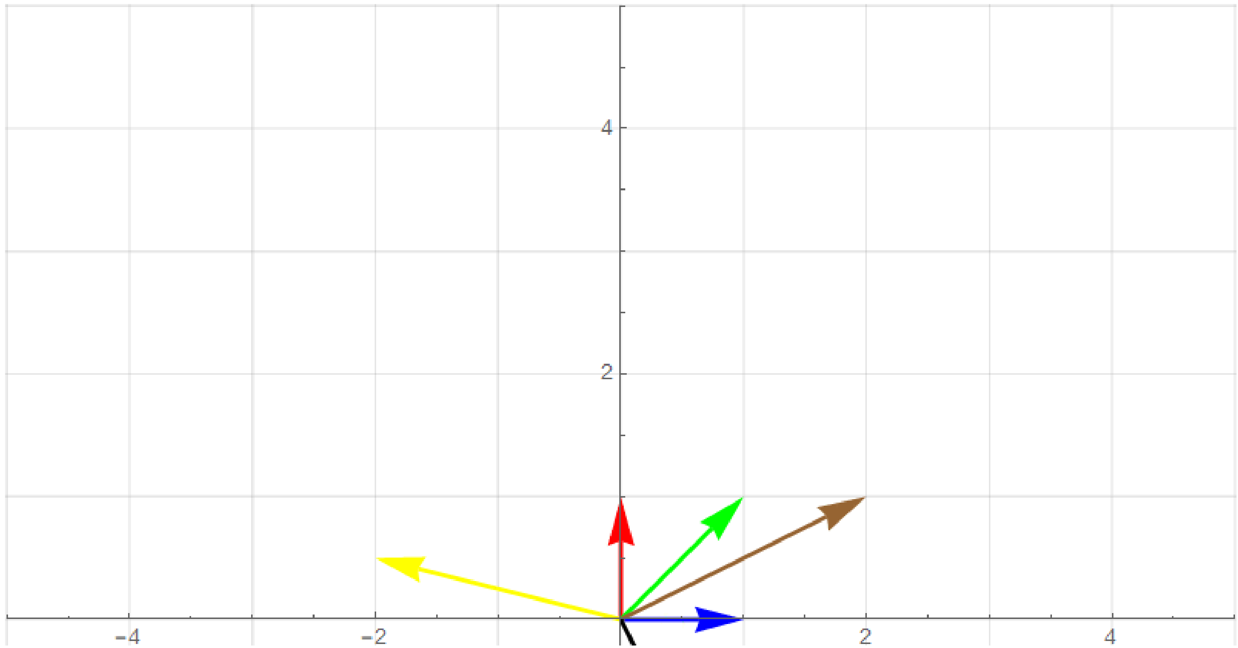
$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Vertical Shear





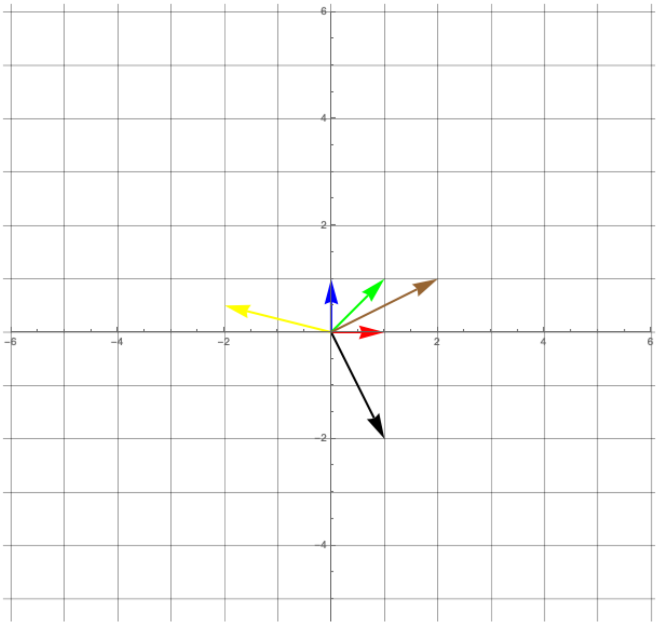
Vertical Shear

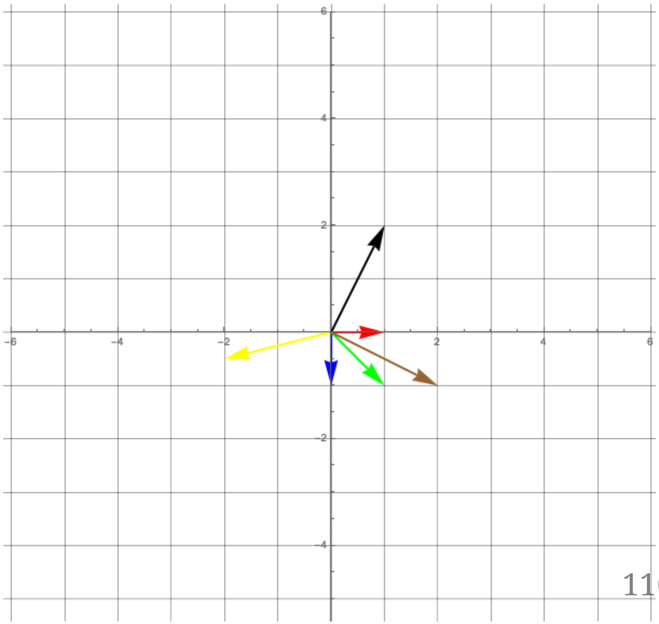


flip with respect to x-axis

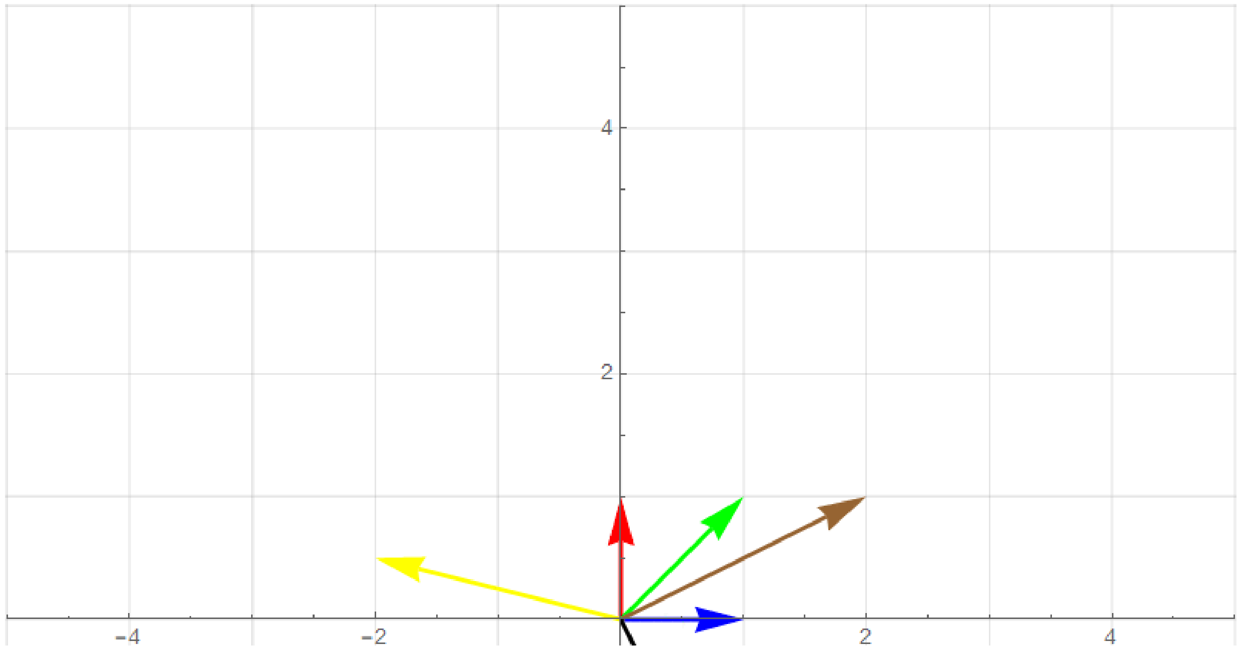
$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

flip with respect to x-axis





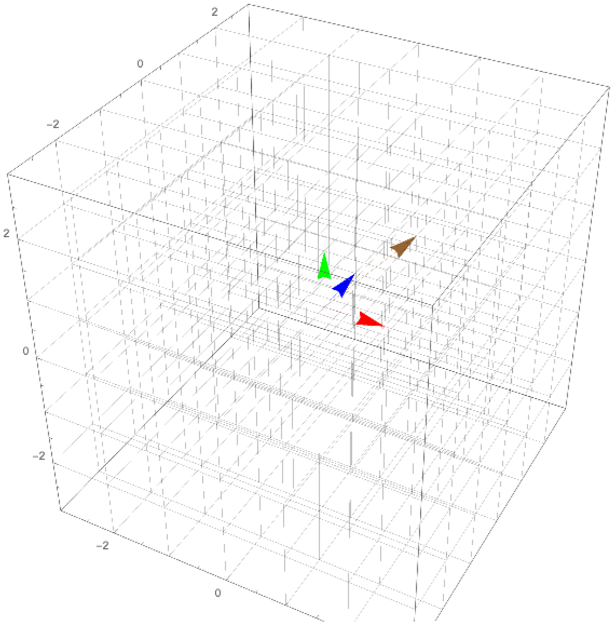
flip with respect to x-axis

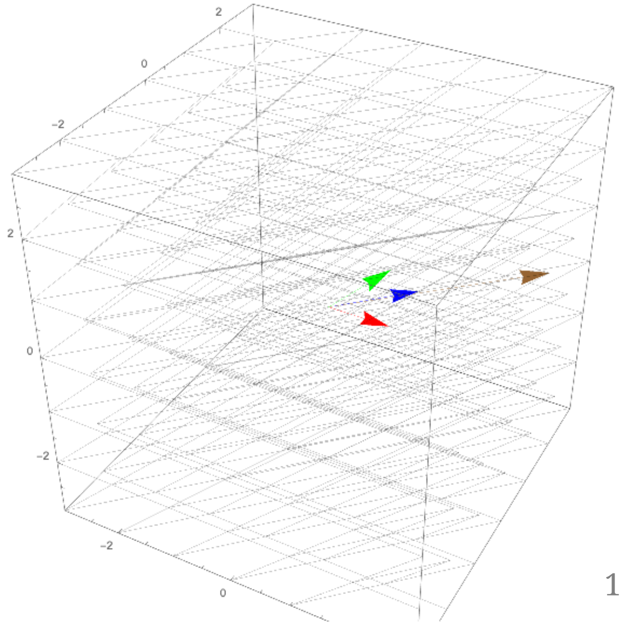


Three Dimensional Transformations

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Three Dimensional Transformations

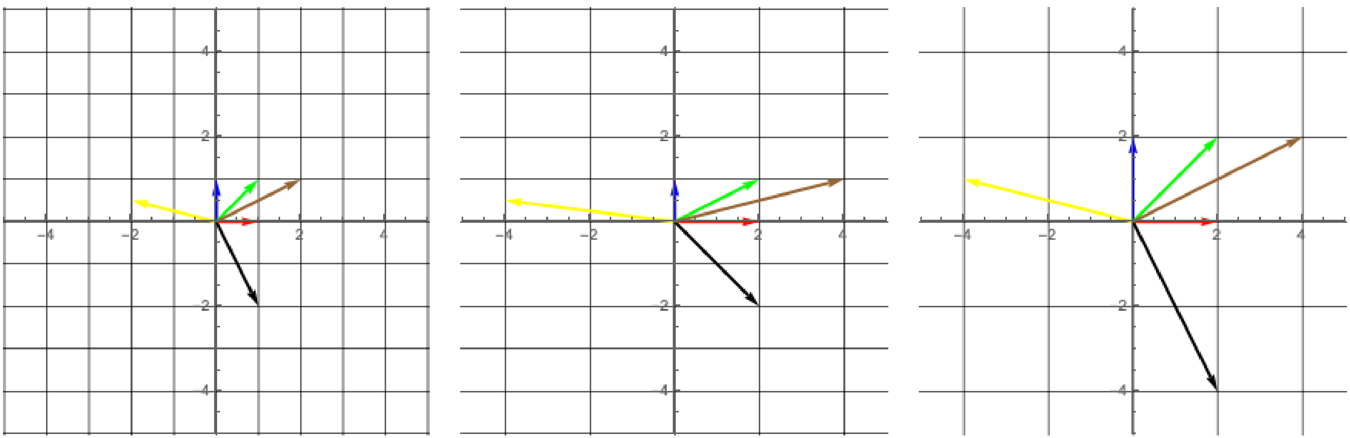




Composing transformations

- You can combine transformations by performing transformations in a **sequence**.
- For example, you can **combine** a horizontal stretch with a vertical stretch which can be defined as a new transformation

Composing transformations



Composing transformations

- Since you are applying one transformation after the other, the overall transformation can be defined as a **composition** of the horizontal stretch transformation inside a vertical stretch transformation

Composing transformations

$$f(\vec{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$$

$$g(\vec{v}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{v}$$

$$g(f(\vec{v})) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{v} \right)$$

Composing transformations

$$g(f(\vec{v})) = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\left(\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\vec{v}\right) = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\vec{v}$$

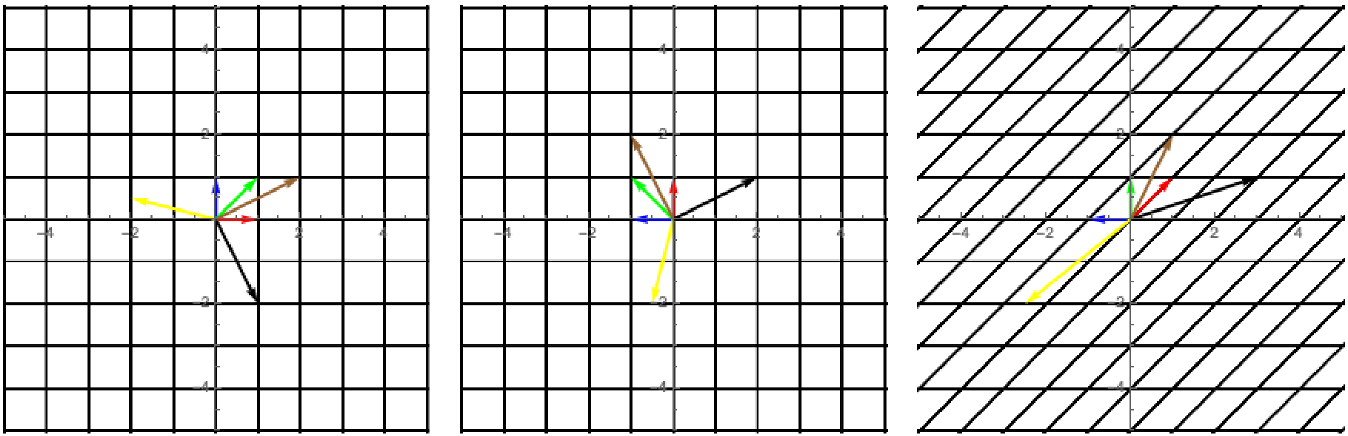
Composing transformations

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

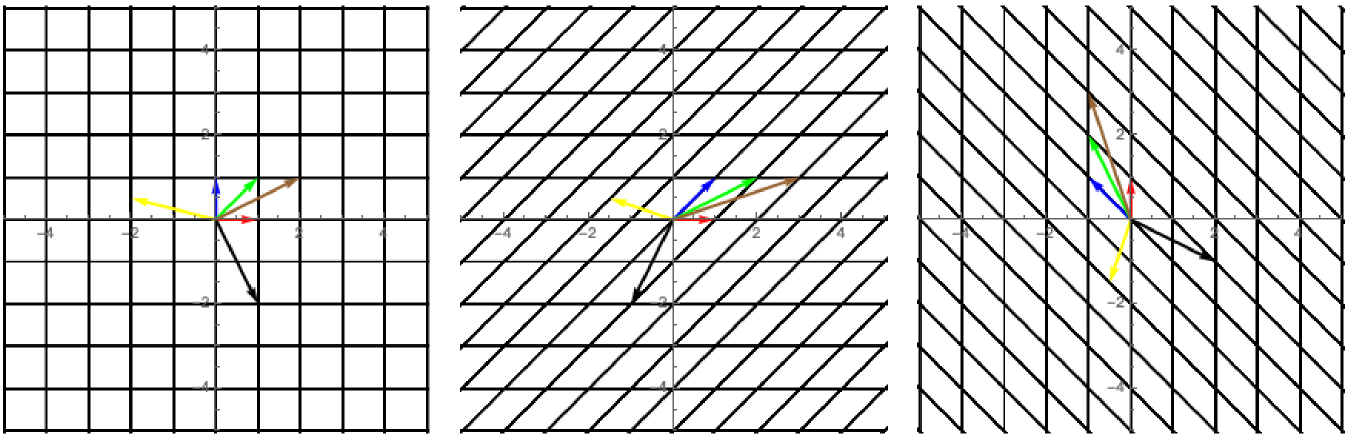
Composing transformations

$$
\begin{aligned}
f(\vec{v}) &= T_1\vec{v} \\
g(\vec{v}) &= T_2\vec{v} \\
g(f(\vec{v})) &= T_2T_1\vec{v}
\end{aligned}
$$

Composing transformations



Composing transformations



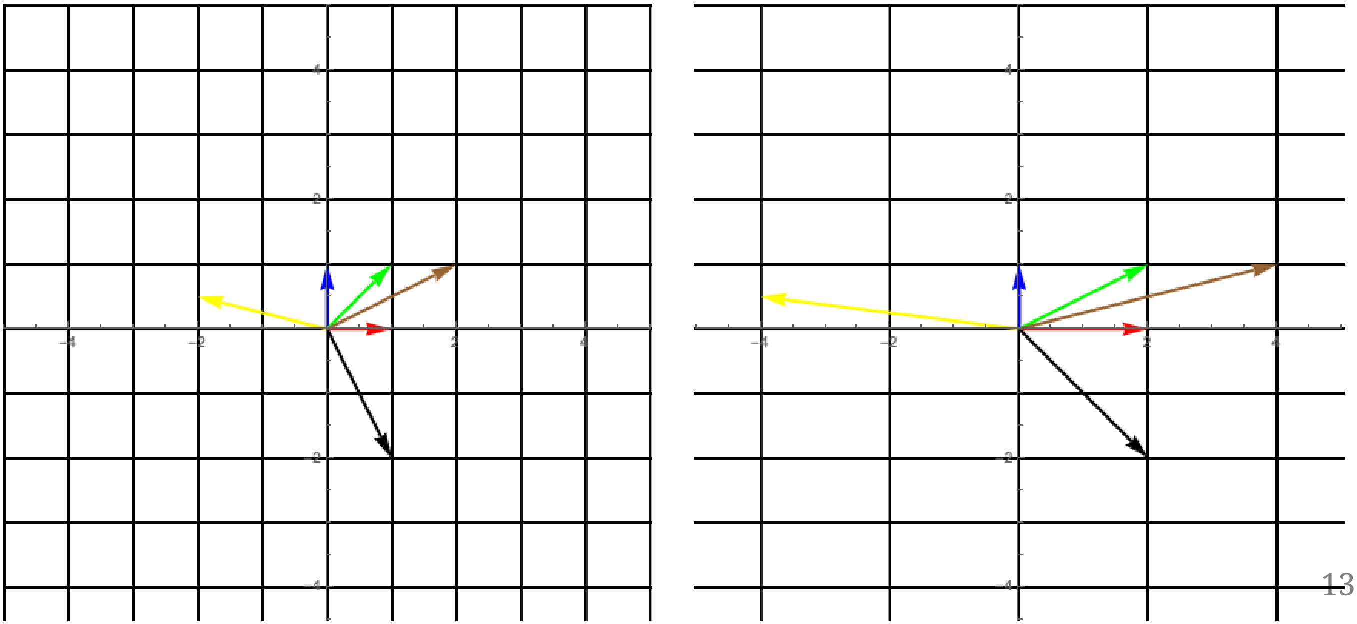
Determinant

- One of the important things you can study about a given linear transformation is how it generally **stretches** or **compresses** the space.

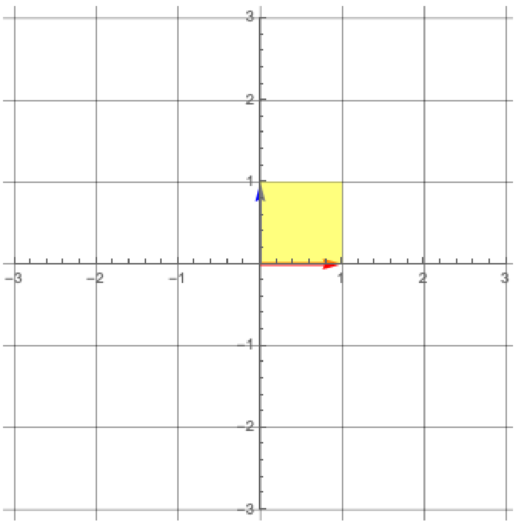
Determinant

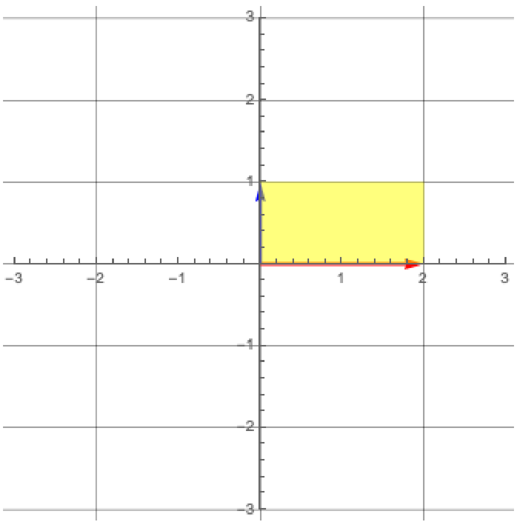
$$T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Determinant

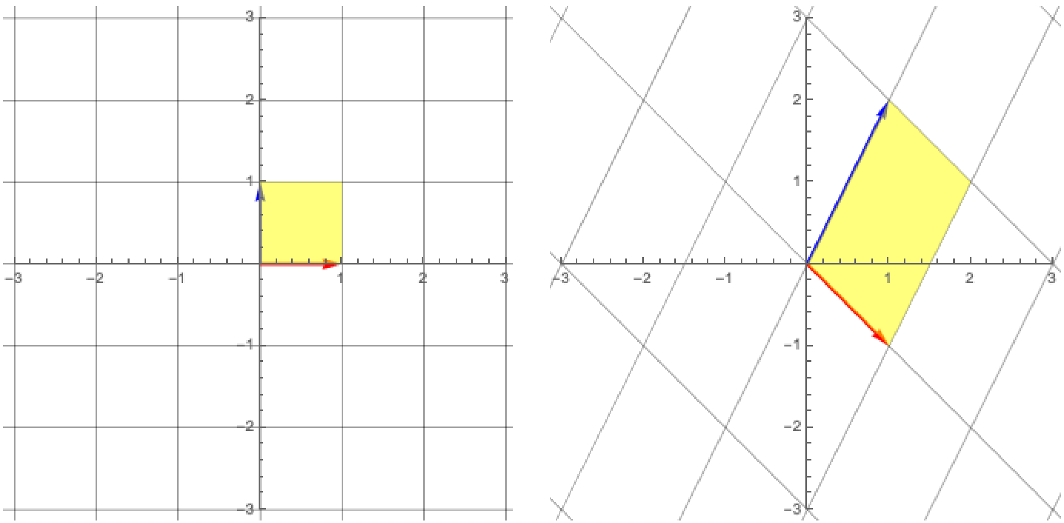


Determinant





Determinant



Determinant

- The factor stretching/compression of the entire space that occurs during a transformation has a special name called the **determinant** of a transformation.

Determinant

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

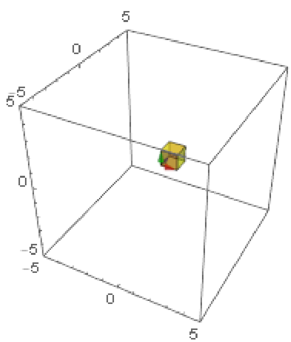
Determinant

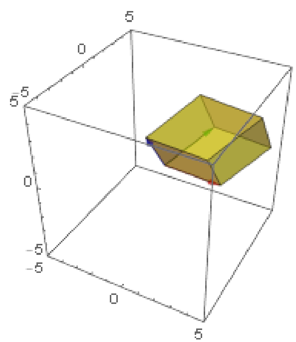
- This formula can be derived by calculating the **area** of the resulting yellow parallelogram, which is the transformed version of the yellow square.

Determinant

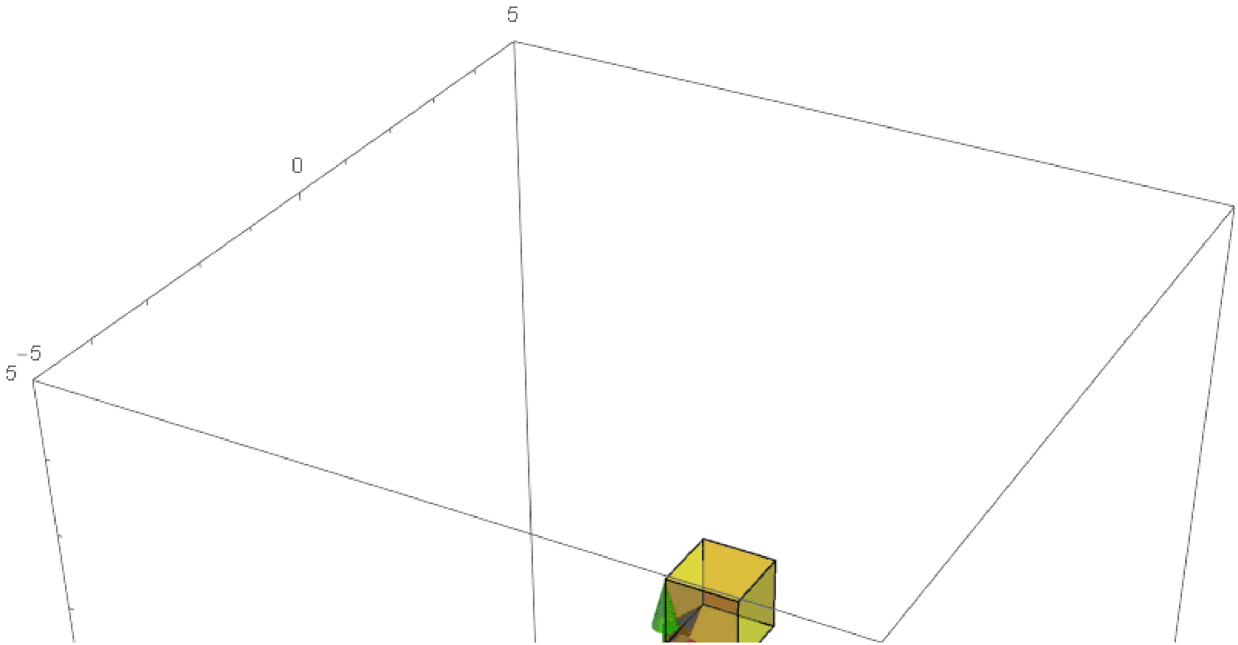
- The determinant of a 3 dimensional transformation is the factor of stretching/compression of the **volume** of the unit cube to the parallelepiped:

Determinant

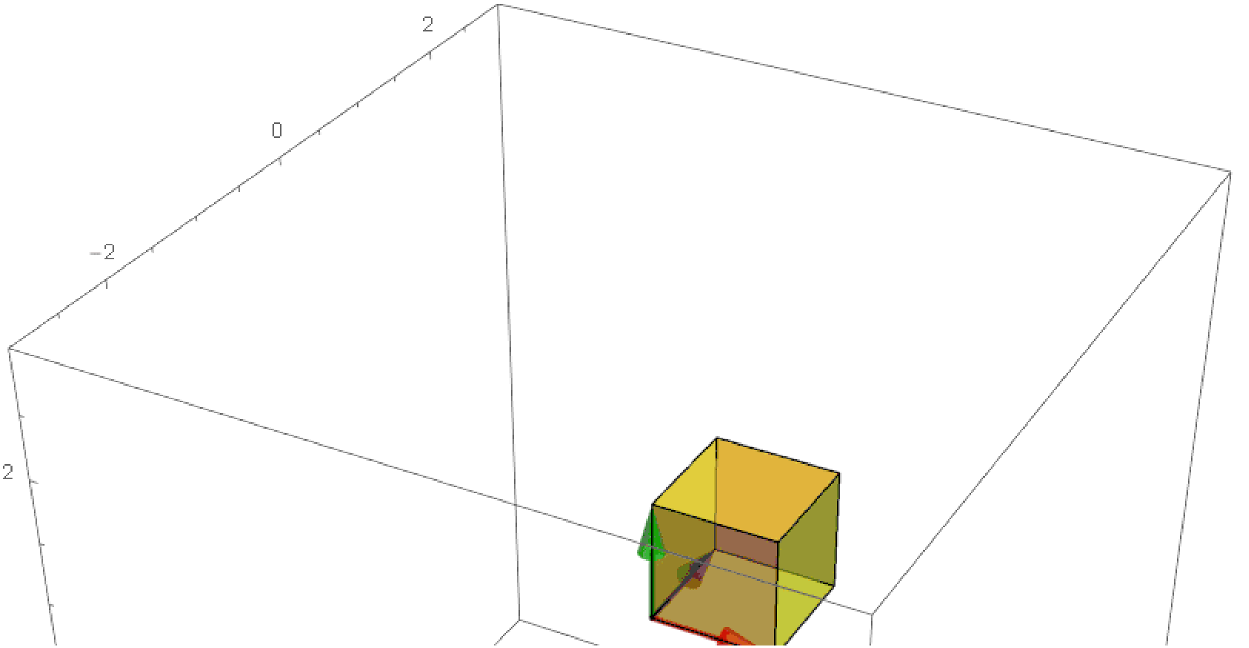




Determinant



Determinant



Determinant

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

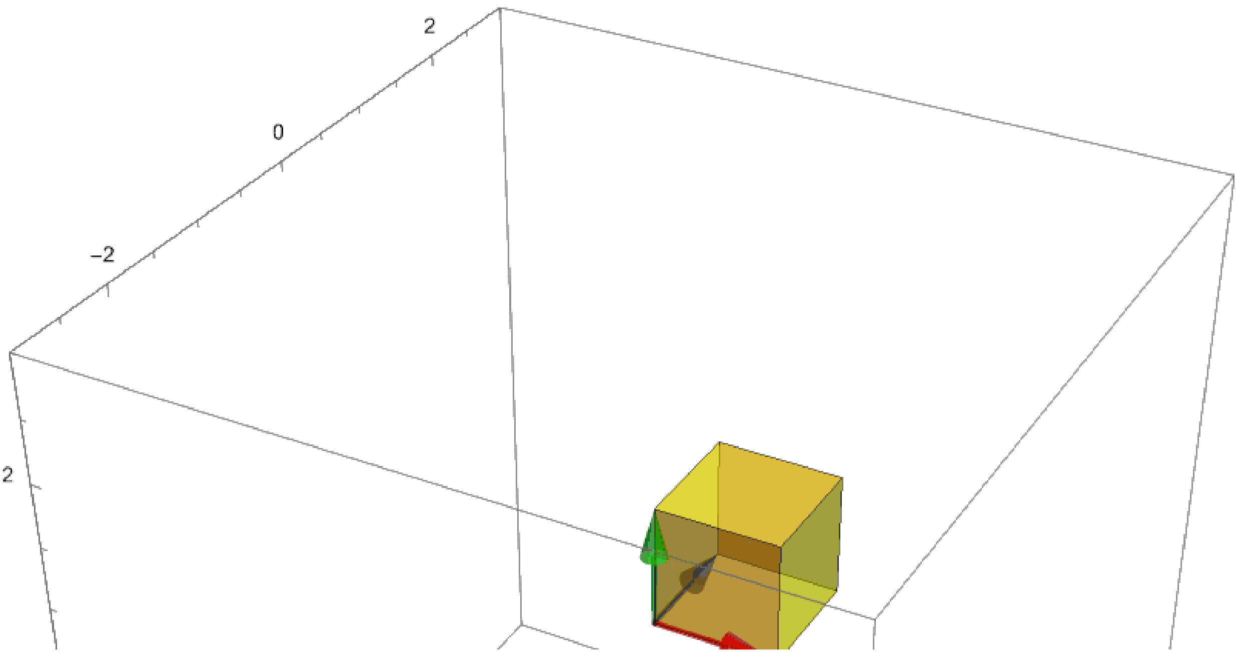
Determinant

- If the determinant of the transformation is **zero**, it means that the transformation has **compressed** the vector space to the point where the area or volume is zero:

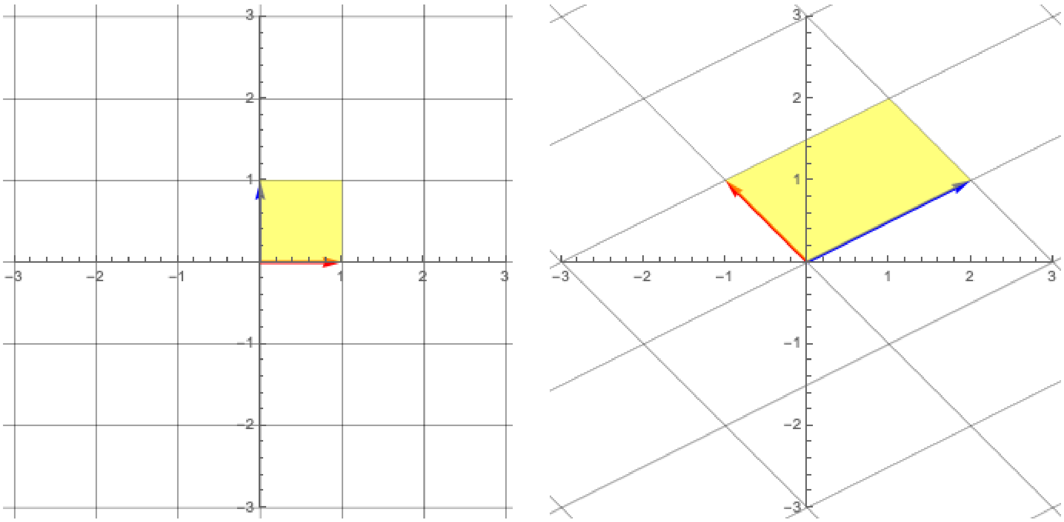
Determinant

$$\begin{bmatrix} 3 & 0 & -1.5 \\ -1 & -2 & 2.5 \\ 2 & 1 & -2 \end{bmatrix}$$

Determinant



Determinant



Determinant

$$\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

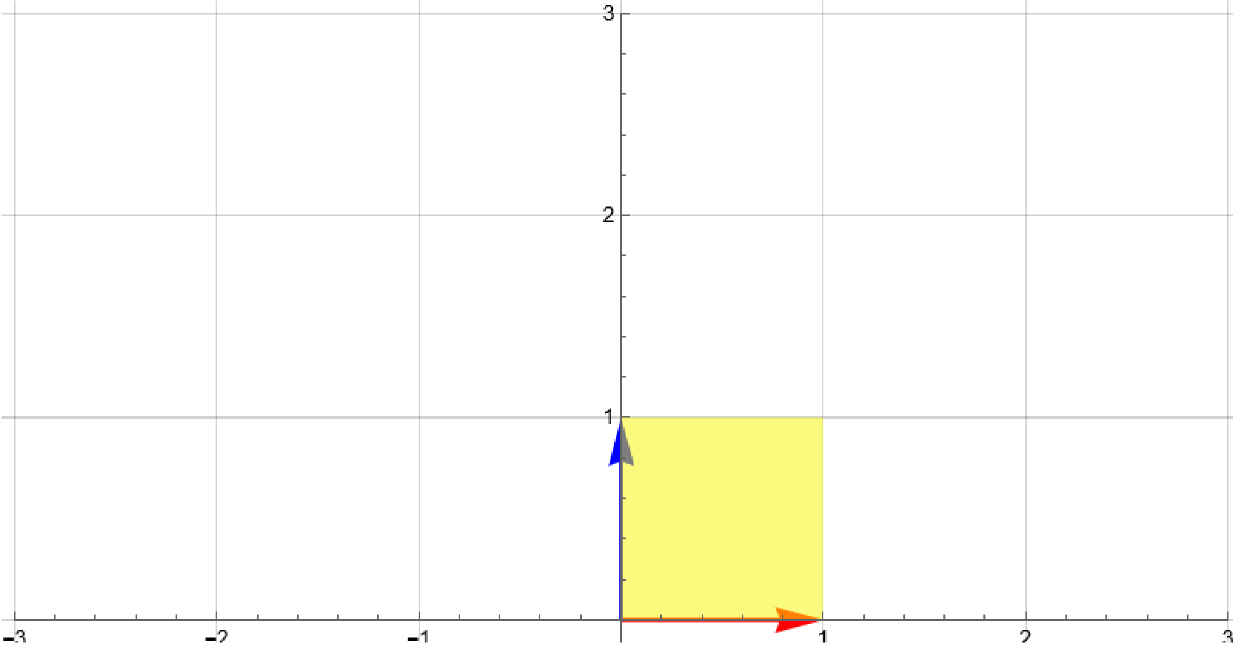
Determinant

$$\det\left(\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\right) = -1(1) - 2(1) = -3$$

Determinant

- This means that the factor of stretching/compression for the transformation is -3.
- The **sign** of the determinant has meaning.
- If the determinant is negative it means that the space is **compressed beyond 0** to the point that the space is flipped

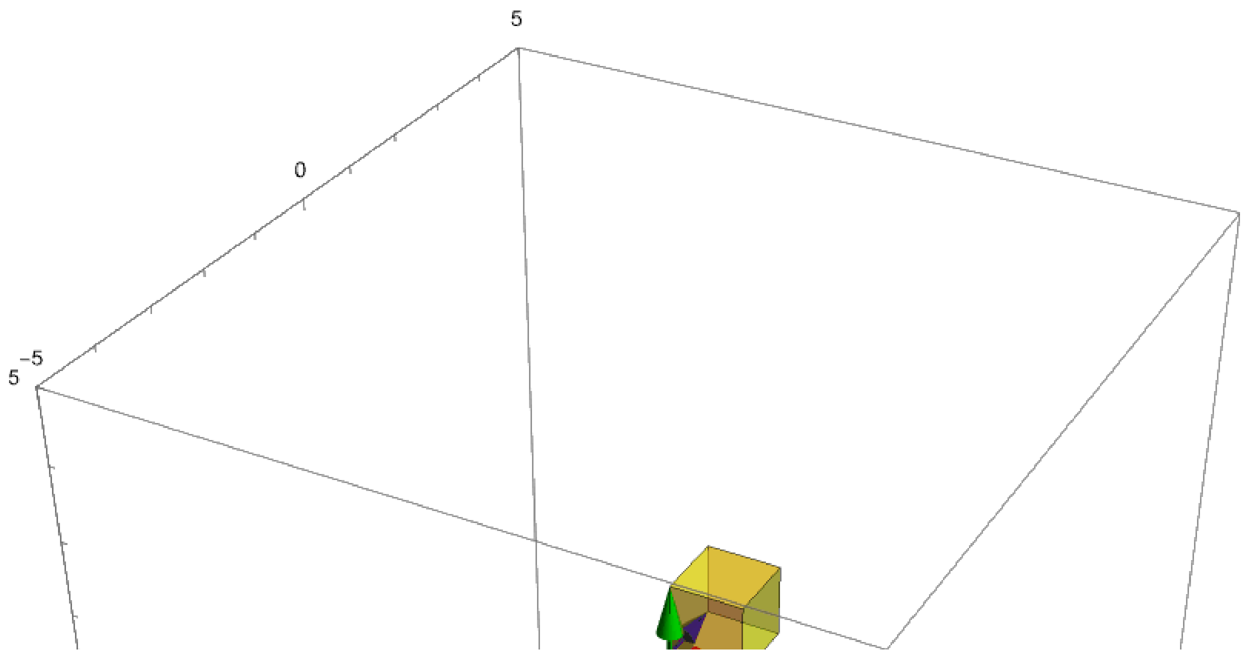
Determinant



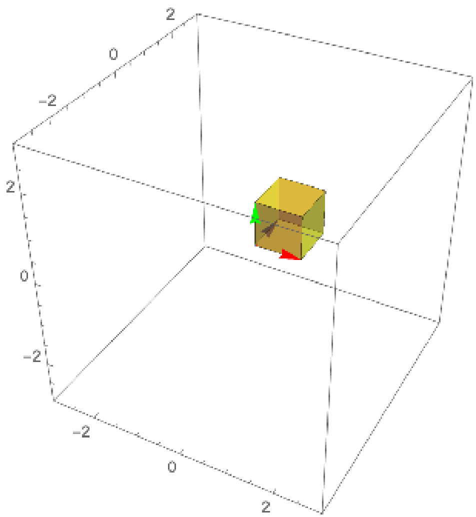
Determinant

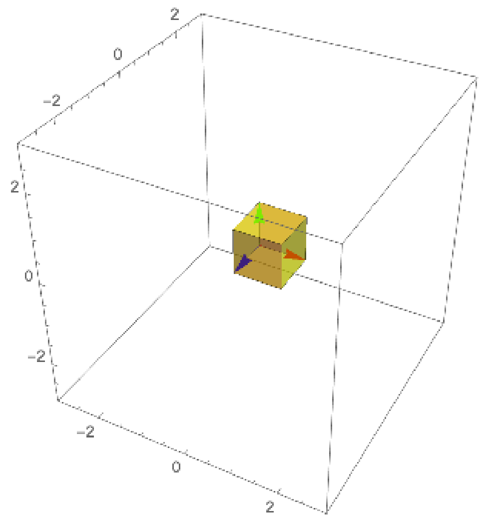
- Flipping space in the 3 dimensions means that the basis vectors cannot follow **right hand rule**

Determinant



Determinant





Inverse Matrices

$$\begin{aligned} 2x + 0y &= 4 \\ 0x + y &= 3 \end{aligned}$$

Inverse Matrices

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Inverse Matrices

- You can think of the **coefficients** for x and y as the numbers that make up the **transformation matrix**, T.

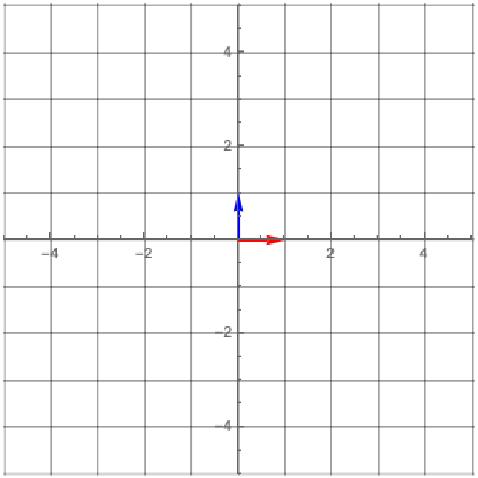
Inverse Matrices

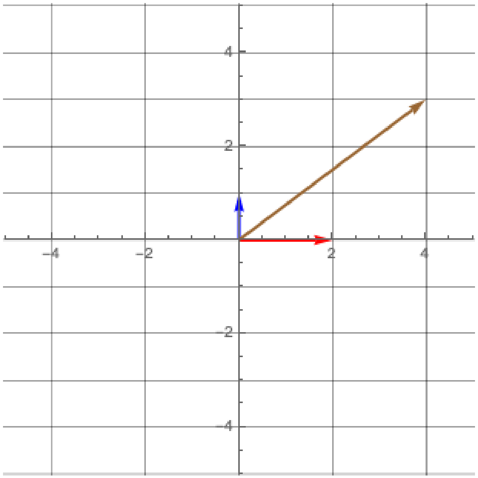
$$T\vec{v} = \vec{v}'$$

Inverse Matrices

- To find the solutions of this system of linear equations, you need to look for the **original value** of $\vec{v}'$ **before** the transformation

Inverse Matrices





Inverse Matrices

- To do this you need to make use of a special matrix related to the transformation matrix T, called the **inverse matrix**, T^{-1}.
- This matrix serves as the inverse of the transformation T such that, **combining T and T^{-1}** results to the original locations of the basis, or the identity matrix.

Inverse Matrices

$$T^{-1}T = I$$

Inverse Matrices

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

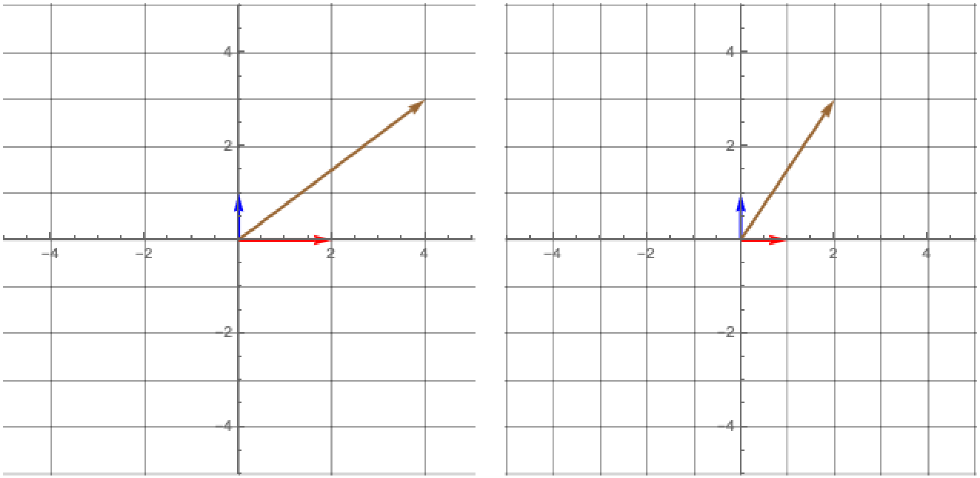
Inverse Matrices

$$
\begin{aligned}
T^{-1}T\vec{v} &= T^{-1}\vec{v}' \\
\vec{v} &= T^{-1}\vec{v}'
\end{aligned}
$$

Inverse Matrices

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Inverse Matrices



Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

Finding the inverse matrix

1. You can change the values of a row by multiplying all of the numbers in the row by a constant
2. You can change rows by adding the elements of other rows to it.

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} (3) & 0 & 2 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] R_1 \to \frac{1}{3} R_1$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -2 & \frac{4}{3} & -\frac{1}{3} & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] R_2 \to -R_1 + R_2$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -2 & \frac{4}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 3 & \frac{8}{3} & \frac{1}{3} & 0 & 1 \end{array}\right] R_3 \to R_1 + R_3$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 3 & \frac{8}{3} & \frac{1}{3} & 0 & 1 \end{array}\right] R_2 \to -\frac{1}{2} R_2$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{14}{3} & -\frac{1}{6} & \frac{3}{2} & 1 \end{array}\right] R_3 \to -3R_2 + R_3$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{array}\right] R_3 \to \frac{3}{14} R_3$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{14} & -\frac{3}{14} & -\frac{1}{7} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{array}\right] R_1 \to -\frac{2}{3}R_3 + R_1$$

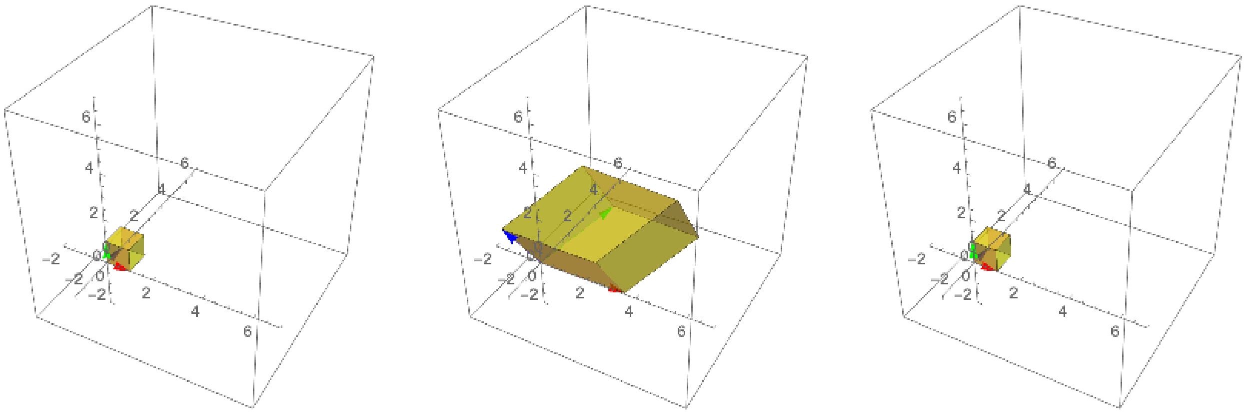
Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{14} & -\frac{3}{14} & -\frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & -\frac{2}{7} & \frac{1}{7} \\ 0 & 0 & 1 & -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{array}\right] R_2 \to \frac{2}{3} R_3 + R_2$$

Finding the inverse matrix

$$\begin{bmatrix} \frac{5}{14} & -\frac{3}{14} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} & \frac{1}{7} \\ -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{bmatrix}$$

Finding the inverse matrix



Minors and Cofactors

- The **minor** of some element $a{ij}ofasquarematrixA, denotedbyM{ij} isthedeterminantofthesubmatrixformedbydeletingtheithrowandjthcolumnofA$.

Minors and Cofactors

- The **cofactor** of some element $a{ij}ofasquarematrixA, denotedbyC{ij}$ is calculated as:

Minors and Cofactors

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Minors and Cofactors

- The **cofactor matrix** C of some matrix A is the matrix of A's cofactors C_{ij}:

Minors and Cofactors

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Minors and Cofactors

$$A^{-1} = \frac{1}{\det(A)} C^T$$

Minors and Cofactors

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} C^T$$

Minors and Cofactors

$$A^{-1} = \frac{1}{\det(A)} C^T$$

$$A^{-1} = \frac{1}{-28} \begin{bmatrix} -10 & -4 & 1 \\ 6 & 8 & -9 \\ 4 & -4 & -6 \end{bmatrix}^T$$

Minors and Cofactors

$$A^{-1} = \begin{bmatrix} \frac{5}{14} & -\frac{3}{14} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} & \frac{1}{7} \\ -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{bmatrix}$$

Singular Matrices

1. There is one solution
2. There are infinitely many solutions
3. There are no solutions

Singular Matrices

- These happen when the span of the transformed basis vectors (also called **column space**) has reduced dimension.

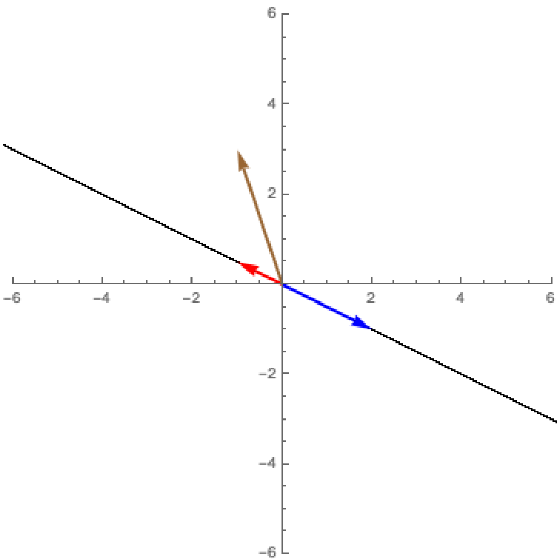
Singular Matrices

$$T = \begin{bmatrix} -1 & 2 \\ 0.5 & -1 \end{bmatrix}$$

Singular Matrices

$$\begin{bmatrix} -1 & 2 \\ 0.5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Singular Matrices



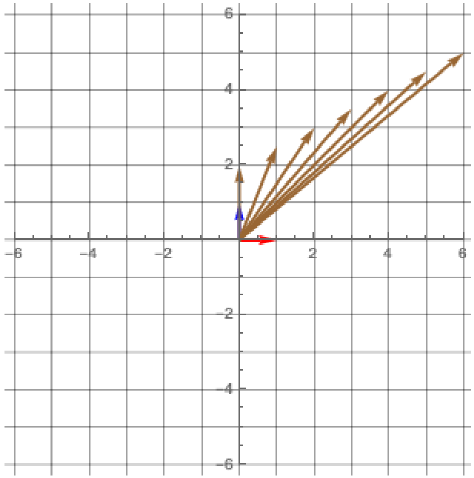
Singular Matrices

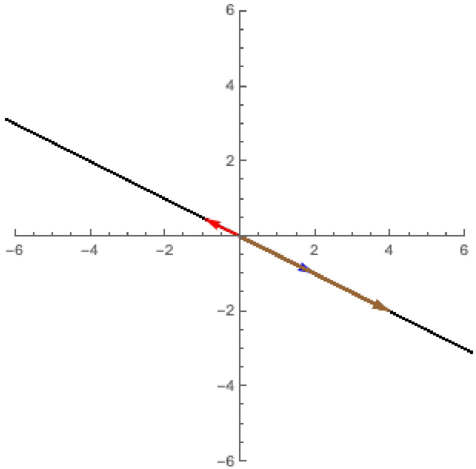
- Since a singular transformation **collapses** the space from an n-dimensional span to m-dimensional span (where $m < n$), the space is shrunk to 0.
- Therefore, any singular matrix will have a **determinant of zero**.

Singular Matrices

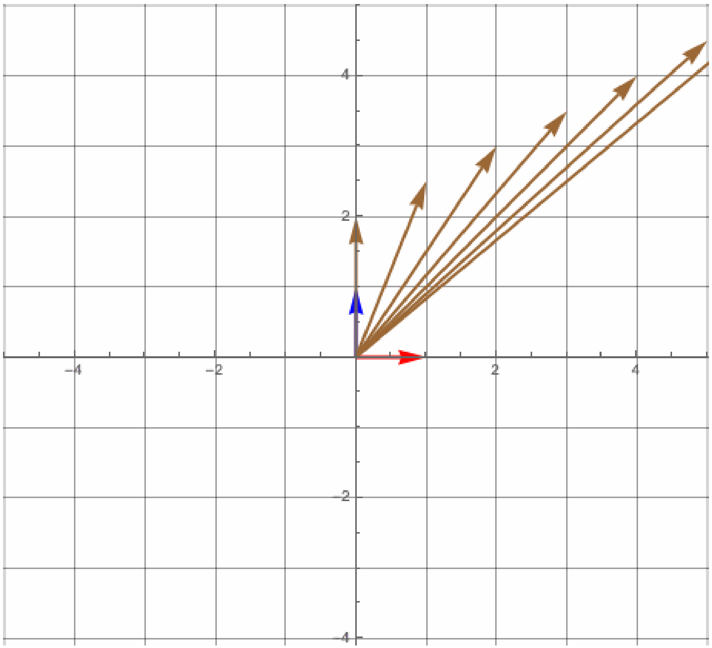
$$\det \begin{bmatrix} -1 & 2 \\ 0.5 & -1 \end{bmatrix} = (-1)(-1) - (2)(0.5) = 0$$

Singular Matrices





Singular Matrices



Singular Matrices

- We call non-invertible matrices such as the aforementioned transformation above, **singular**

Matrices in the Perspective of Linear Algebra

- The main thesis of this series of lectures should not just be on the definitions of **matrices and their concepts**.
- The focus of this lecture is the essence of these concepts both **numerically** and **visually**.

Matrices in the Perspective of Linear Algebra

- A **vector** - an arbitrary member of some vector space, presented as an arrow from origin to a corresponding point in some n-dimensional space.
- An n-dimensional **square matrix** - defines some linear transformation, its values correspond to the new location of the basis vectors after the transformation.
- **matrix multiplication** - composition of linear transformations. The product summarizes the linear transformations into one.
- **determinant** - the factor of scaling of the vector space during a transformation
- **matrix inverse** - the functional inverse of a matrix, reverses the transformation

Non-square Matrices

- Let's talk about a class of matrices we haven't talked about before, a **non-square matrix**:

Non-square Matrices

$$\begin{bmatrix} 2 & 2 & -3 \\ -2 & 1 & 0 \end{bmatrix}$$

Non-square Matrices

- But the closest related concept for these matrices are **linear transformations**.
- If these are indeed linear transformations, these matrices are meant to be **multiplied** to other vectors to apply some transformation

Non-square Matrices

$$\begin{bmatrix} 2 & 2 & -3 \\ -2 & 1 & 0 \end{bmatrix} \vec{v} = \vec{v}'$$

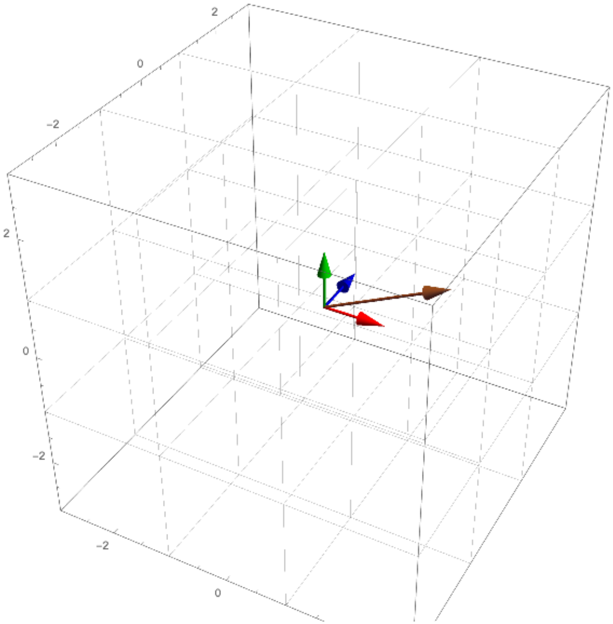
Non-square Matrices

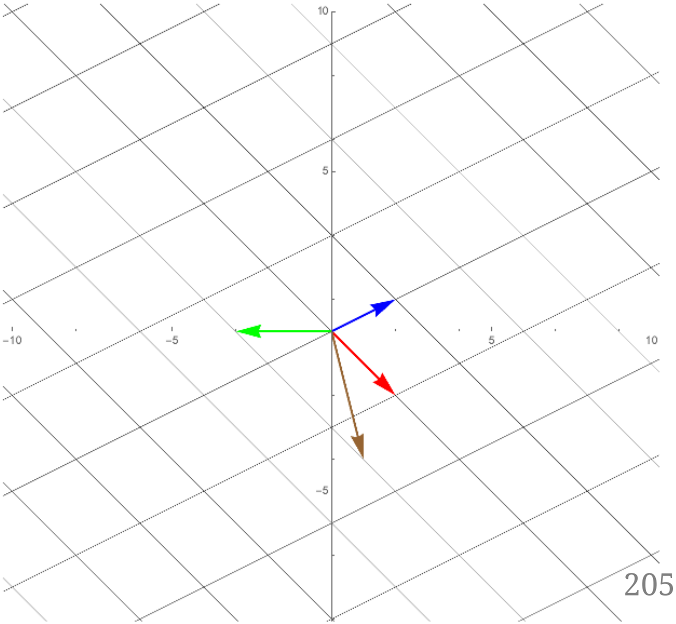
- Since this is a 2×3 matrix, you can only multiply these with vectors of size 3×1.
- But the interesting thing about this transformation is that it produces a vector of size 2×1

Non-square Matrices

$$\begin{bmatrix} 2 & 2 & -3 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Non-square Matrices





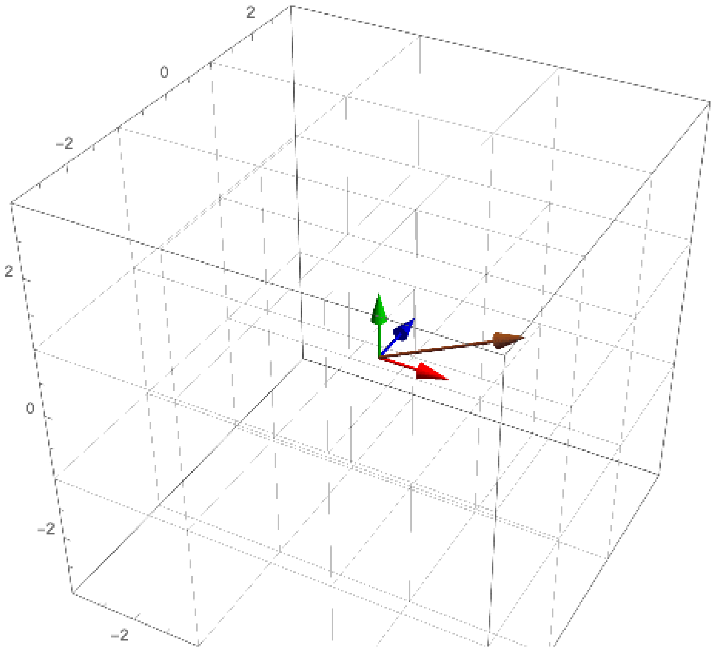
Non-square Matrices

- It is also a linear transformation, but it is specifically a transformation that **changes** the number of dimensions of the vector space.

Non-square Matrices

$$\begin{bmatrix} 2 & 2 & -3 & \\ -2 & 1 & 0 & \\ 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$$

Non-square Matrices



Non-square Matrices

$$\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$$

Non-square Matrices

$$\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Non-square Matrices

$$\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

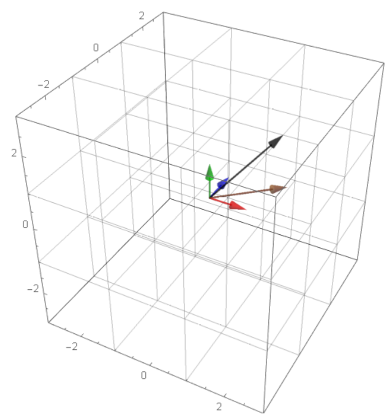
Non-square Matrices

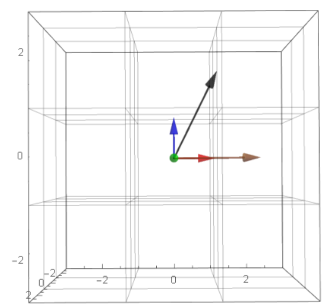
- The transformation of any n-dimensional vector to less than n-dimensional space, is similar to producing the **projection** or shadow of the vector.

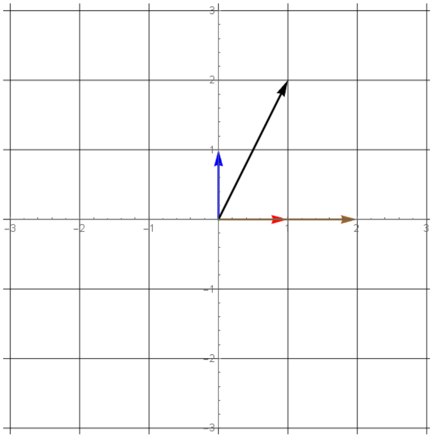
Non-square Matrices

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Non-square Matrices



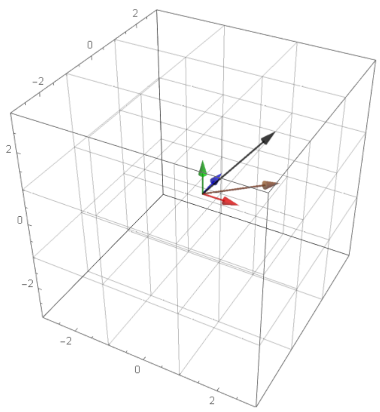


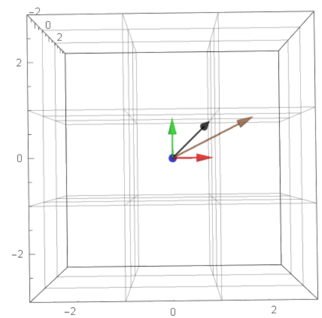


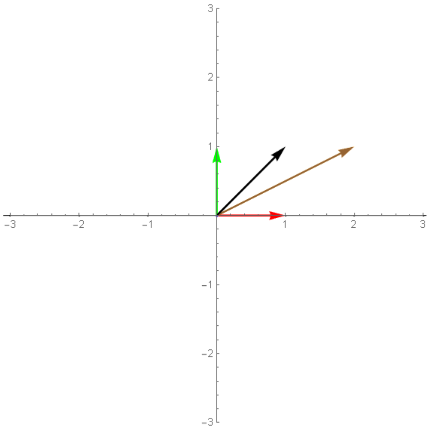
Non-square Matrices

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Non-square Matrices



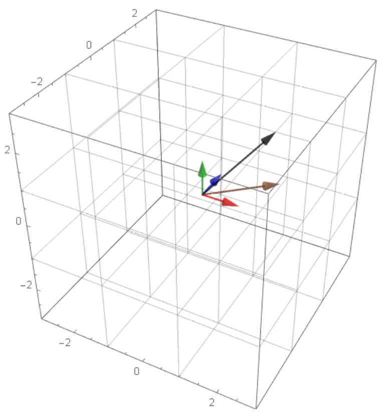


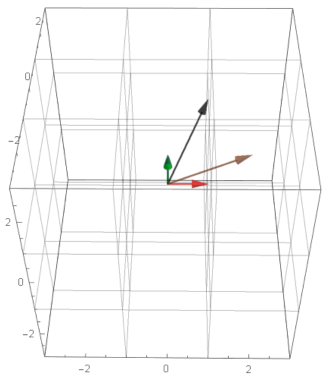


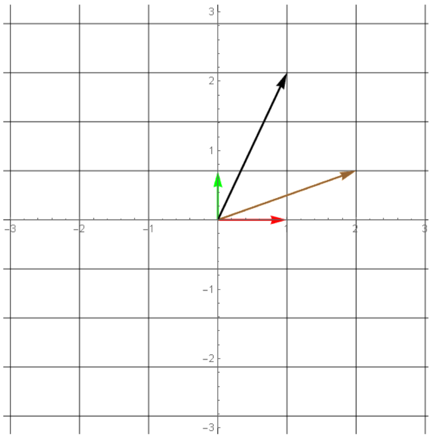
Non-square Matrices

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Non-square Matrices







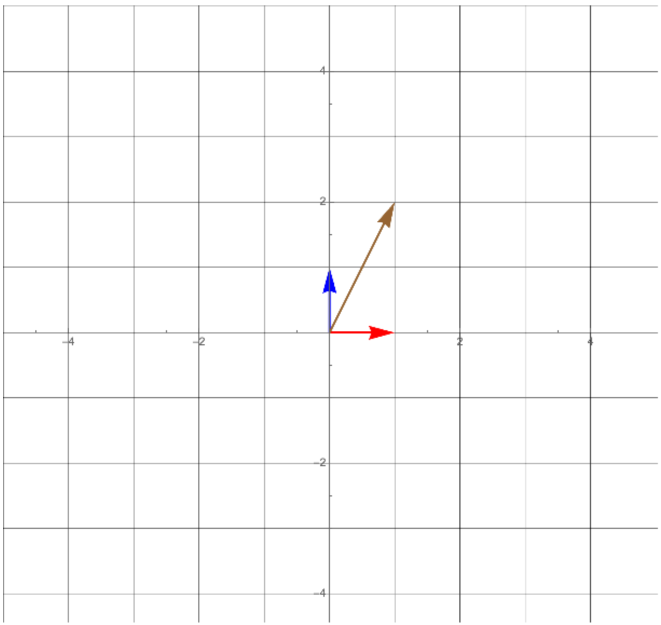
Non-square Matrices

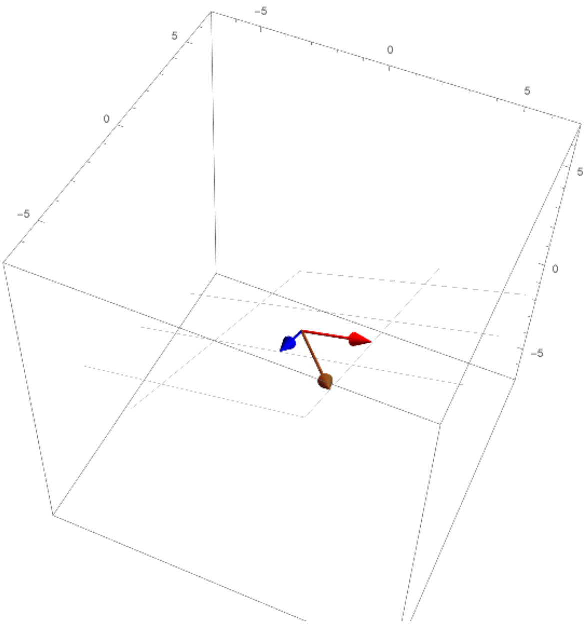
- You can also transform a **2-dimensional** vector into a **3-dimensional vector**, to do this you need a 3×2 transformation matrix

Non-square Matrices

$$\begin{bmatrix} 3 & 0 \\ -1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$

Non-square Matrices





Non-square Matrices

- As you can see above, even though the two dimensional vectors now reside in the three dimensional vector space, **the span that the basis vectors produce is still 2 dimensional**.
- This is because there is no way for the vectors to be transformed with **extra dimensionality**

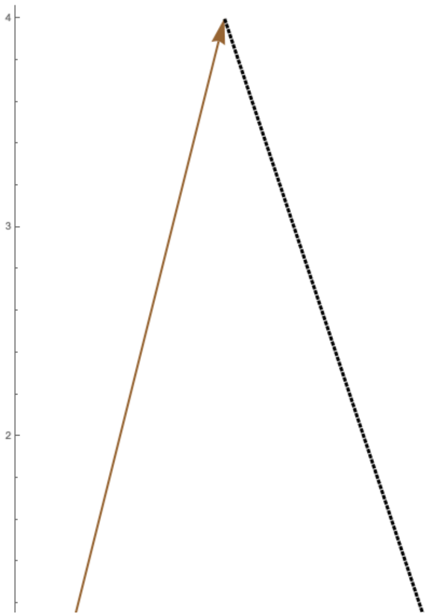
Non-square Matrices

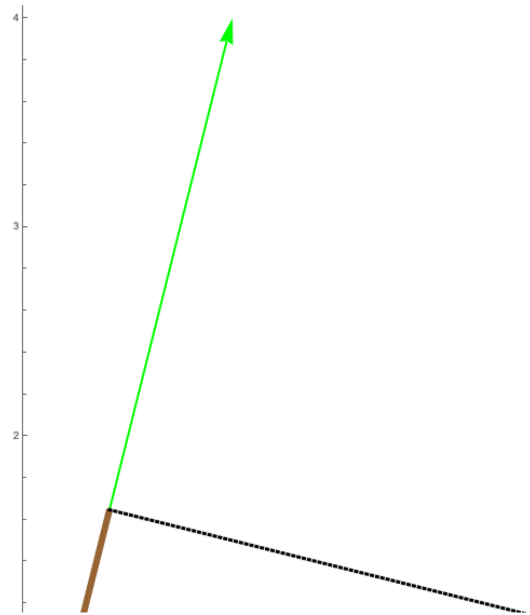
- As a general rule, you can think of any $m \times n$ matrix as a transformation that transforms an n**-dimensional** vector into an m**-dimensional vector**

Scalar products

The scalar product of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$ is the product of the magnitude of $\vec{a}$ and the magnitude of the projected version of $\vec{b}$ onto $\vec{a}$.

Scalar products





Scalar products

$$\begin{bmatrix} a \\ b \\ \vdots \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ \vdots \\ z \end{bmatrix} = ax + by + \cdots + cz$$

Scalar products

- It is the measure of **similarity** between two vectors, For example, given three vectors,

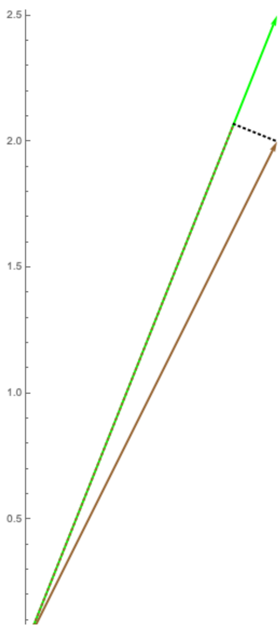
Scalar products

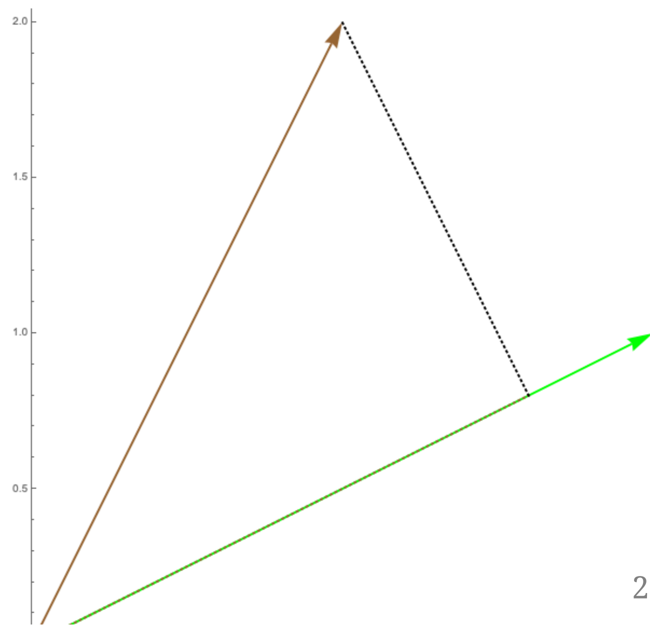
$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}, \vec{c} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Scalar products

- Is $\vec{a}$ **more similar** to $\vec{b}$ or to $\vec{c}$?

Scalar products





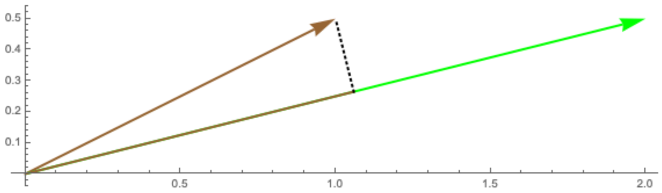
Scalar products

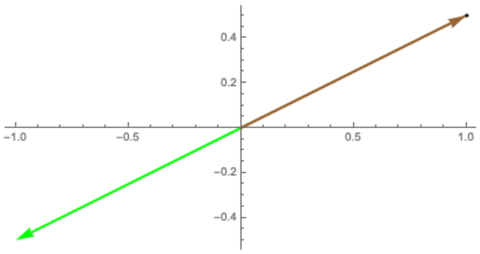
$$\vec{a} \cdot \vec{b} = 6$$
$$\vec{a} \cdot \vec{c} = 4$$

Scalar products

- This means that two vectors perpendicular to each other will have a scalar product of **zero** and two vectors pointing in the opposite direction will have a **negative** scalar product:

Scalar products





Scalar products

- How does the scalar product relate to **linear transformations**?

Scalar products

- And it turns out, a scalar product is merely a transformation of any vector to **one-dimensional space**:

Scalar products

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = u_x a + u_y b$$

Scalar products

$$\begin{bmatrix} u_x & u_y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = u_x a + u_y b$$

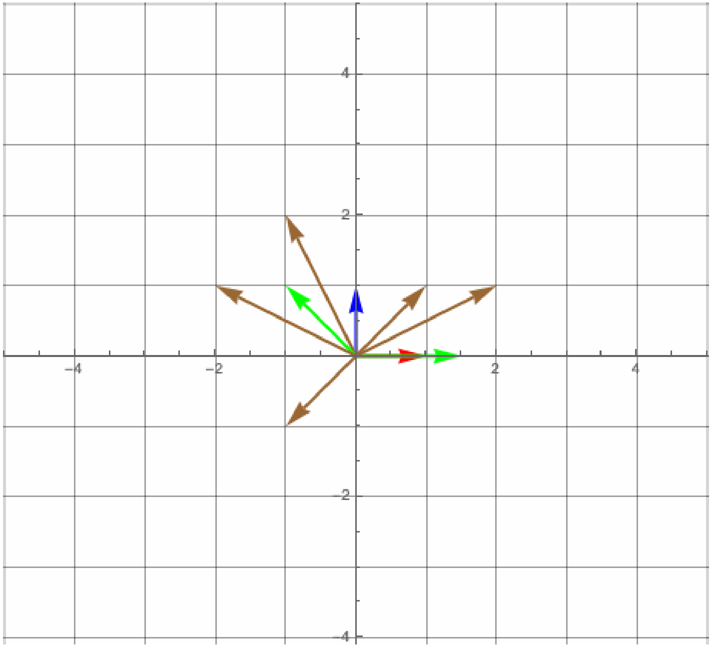
Scalar products

- When looking at, the scalar product of two $n \times 1$ vectors, you can imagine that one vector is **reduced** into the **one dimensional vector space**.

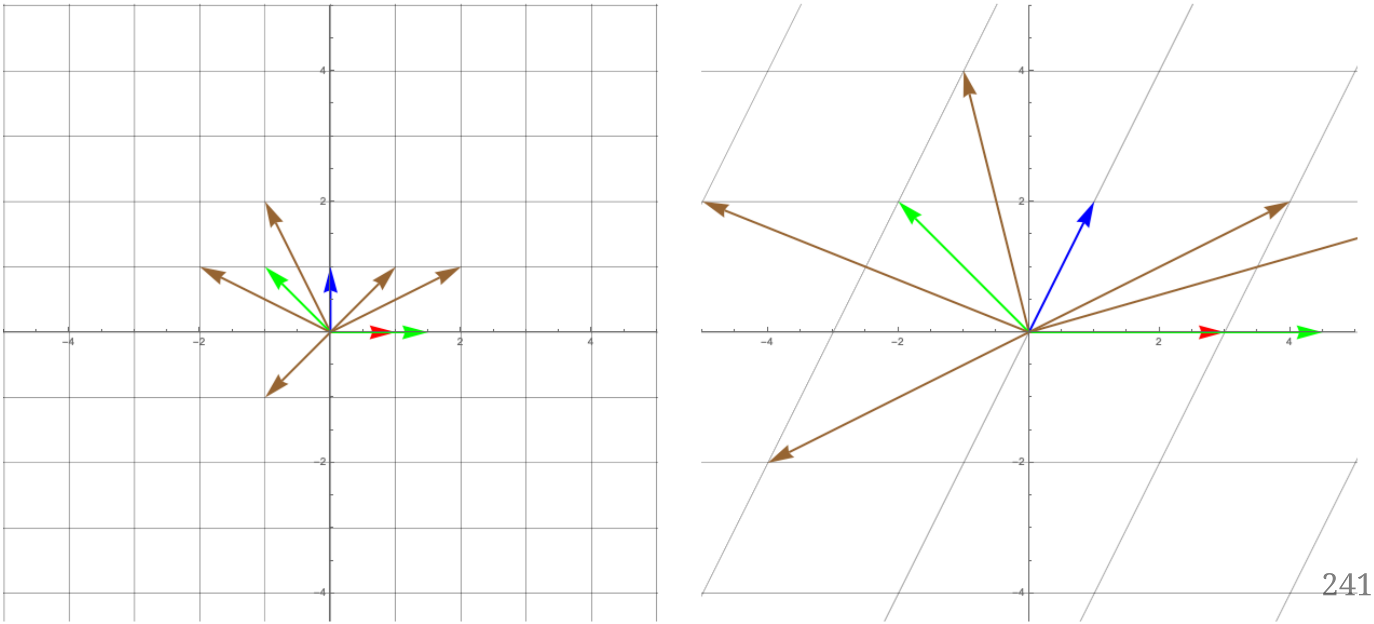
Eigenvectors and Eigenvalues

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenvectors and Eigenvalues

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

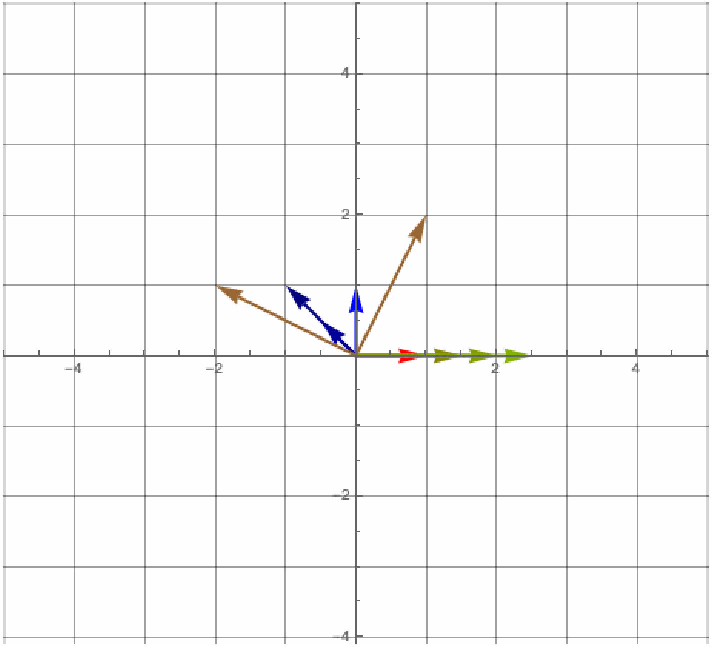
Eigenvectors and Eigenvalues

- These special vectors are called **eigenvectors**.
- Each eigenvector is always accompanied by special scalar values called **eigenvalues**, which correspond to the factor of scaling for the transformation

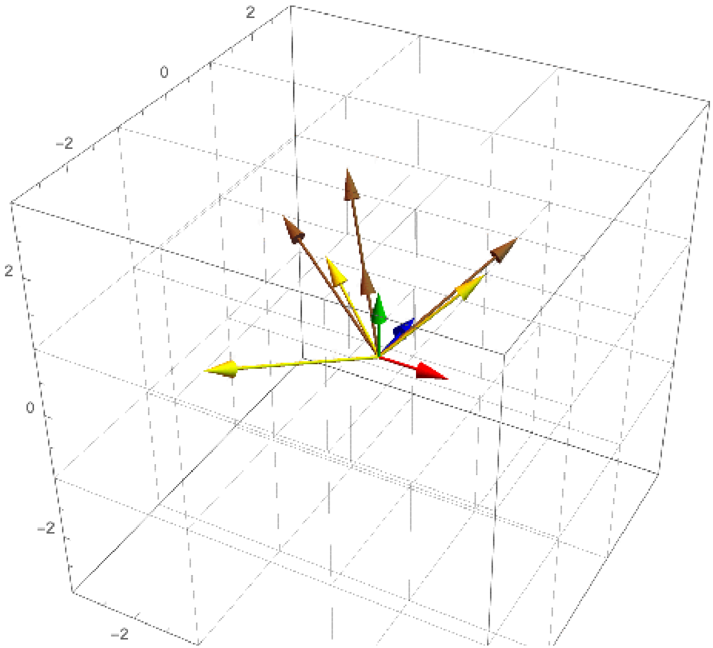
Eigenvectors and Eigenvalues

- In fact **all** the vectors along the **span of the green vectors** are eigenvectors as well

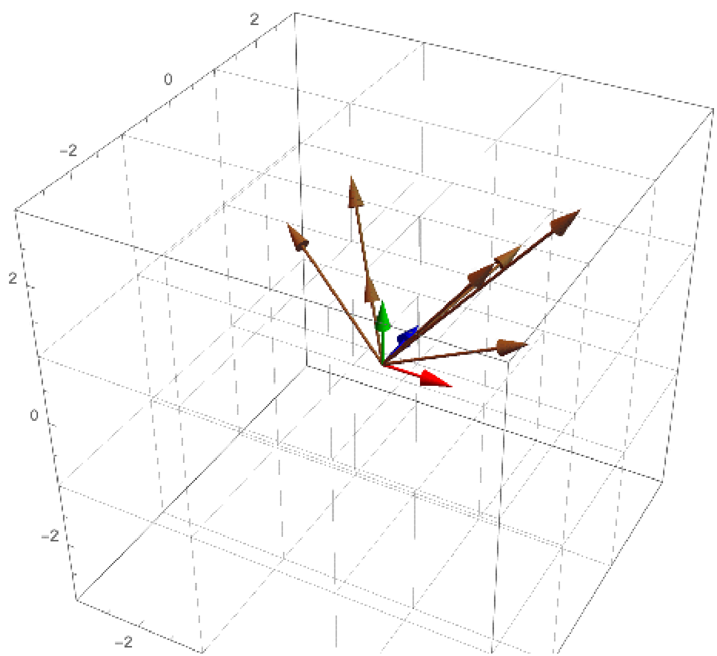
Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues

- Since eigenvectors are vectors that are **only** scaled as a result of the transformation, we can solve for $\vec{e}$ in the following equality.
- The scalar value λ refers to the unknown eigenvalue

Eigenvectors and Eigenvalues

$$T\vec{e} = \lambda\vec{e}$$

Eigenvectors and Eigenvalues

- Scalar times vector multiplication $\lambda\vec{e}$ can be written as a **linear transformation** instead.
- The effect of multiplying, λ to $\vec{e}$, is exactly the same as **multiplying λI to $\vec{e}$**

Eigenvectors and Eigenvalues

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \vec{e}$$

Eigenvectors and Eigenvalues

- To solve for the unknowns, we can rewrite $T\vec{e} = \lambda\vec{e}$ into a s**olution from zero**:

Eigenvectors and Eigenvalues

$$
\begin{aligned}
T\vec{e} &= \lambda I\vec{e} \\
T\vec{e} - \lambda I\vec{e} &= \vec{0} \\
(T - \lambda I)\vec{e} &= \vec{0}
\end{aligned}
$$

Eigenvectors and Eigenvalues

- If you recall, a non-zero vector can only be transformed to zero if and only if the whole vector space has been **squished to zero itself**.
- And this can only happen when the **determinant** of transformation is **zero**

Eigenvectors and Eigenvalues

$$\det(T - \lambda I) = \det(\begin{bmatrix} t_{11} - \lambda & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} - \lambda & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda \end{bmatrix}) = 0$$

Eigenvectors and Eigenvalues

- This means that we can find the eigenvalues of any transformation by finding the **lambdas** that reduces the determinant to 0.

Eigenvectors and Eigenvalues

$$
\begin{aligned}
\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) &= 0 \\
(a-\lambda)(d-\lambda) - bc &= 0
\end{aligned}
$$

Eigenvectors and Eigenvalues

$$
\begin{aligned}
\det(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}) &= 0 \\
(3-\lambda)(2-\lambda) - (1)(0) &= 0 \\
(3-\lambda)(2-\lambda) &= 0 \\
\\
\lambda &= 2 \\
\lambda &= 3
\end{aligned}
$$

Eigenvectors and Eigenvalues

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{e} = 2\vec{e}$$
$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{e} = 3\vec{e}$$

Eigenvectors and Eigenvalues

$$\begin{aligned} \text{Eigenvectors for } \lambda &= 2 \\ 3x + y &= 2x \\ 2y &= 2y \\ x + y &= 0 \\ x &= -y \\ \begin{bmatrix} u \\ -u \end{bmatrix} \end{aligned}$$

Eigenvectors and Eigenvalues

$$\begin{aligned}\text{Eigenvectors for } \lambda = 3 \\ 3x + y = 3x \\ 2y = 3y \\ y = 0 \\ \begin{bmatrix} v \\ 0 \end{bmatrix}\end{aligned}$$

Eigenvectors and Eigenvalues

- As you can see, the solutions for $\vec{e}$ is infinitely many, any vector of the form $[u \backslash \text{-} u]$ and any vector of the form $[v \backslash 0]$ is an eigenvector

Positive Definite Matrices and Positive Semidefinite Matrices

- A symmetric matrix A is said to be positive **semidefinite** if the scalar value $v^T Av$ is a positive number