Introduction to Linear Algebra

Matrices and Matrix Operations

A matrix is a **rectangular collection** of numbers

Matrices and Matrix Operations

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 11 & \pi & 12 \end{bmatrix}$$

Matrices and Matrix Operations

$$A = \begin{bmatrix} 0.1 & 0.9 & 0.3 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.7 & 0.9 \end{bmatrix}$$

Matrices and Matrix Operations

Matrices and Matrix Operations

An **element** of a matrix is denoted by a_{rc} which corresponds the element of matrix a in the r^{th} row and c^{th} column

Matrices and Matrix Operations

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Matrices and Matrix Operations

Matrices are also generally used to represent **systems of linear equations** for example:

Matrices and Matrix Operations

$$4x + 2y = 12$$
$$3x - 2y = 8$$

Matrices and Matrix Operations

$$\begin{bmatrix} 4 & 2 \\ 3 & -28 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

how this works will be explained later

Matrix Addition

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Matrix Addition

To get each element $(A+B)_{ij}$, you simply **add the corresponding elements**, $a_{11}+b_{11}$

Matrix Addition

$$A + x = \begin{bmatrix} a_{11} + x & a_{12} + x & \dots & a_{1n} + x \\ a_{21} + x & a_{22} + x & \dots & a_{2n} + x \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + x & a_{m2} + x & \dots & a_{mn} + x \end{bmatrix}$$

Matrix Multiplication

- **Matrix times matrix** multiplication works differently, let say A is an $m \times k$ matrix and B is a $k \times n$ matrix
- Their **cross product** $C = A \times B$ defined to be the matrix with each element:

Matrix Multiplication

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Matrix Multiplication

$$A \times B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{1j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{m1} & b_{m2} & \dots & b_{1j} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

Matrix Multiplication

Scalar times matrix multiplication works similar to scalar plus matrix addition:

$$Ax = \begin{bmatrix} a_{11}x & a_{12}x & \dots & a_{1n}x \\ a_{21}x & a_{22}x & \dots & a_{2n}x \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}x & a_{m2}x & \dots & a_{mn}x \end{bmatrix}$$

Identity Matrix

The **identity matrix** I_n of order n is a special $n \times n$ (square) matrix such that its elements $\iota_{ij} = 1$ if $i = j$ otherwise $\iota_{ij} = 0$

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

This matrix is special because, for any $m \times n$ matrix A,

$$AI_n = I_m A = A$$

Identity Matrix

Powers of square matrices can be defined such that:

Identity Matrix

$$A^0 = I_n$$
$$A^r = A \times A \times \cdots \times A$$

Transpose of a Matrix

- The elements of A^T, $t_{ij} = a_{ji}$
- Therefore the transpose of a matrix is the same matrix but the rows and columns are **interchanged**.

Transpose of a Matrix

$$\begin{bmatrix} 4 & 2 & -3 \\ 11 & \pi & 12 \end{bmatrix}^T = \begin{bmatrix} 4 & 11 \\ 2 & \pi \\ -3 & 12 \end{bmatrix}$$

Transpose of a Matrix

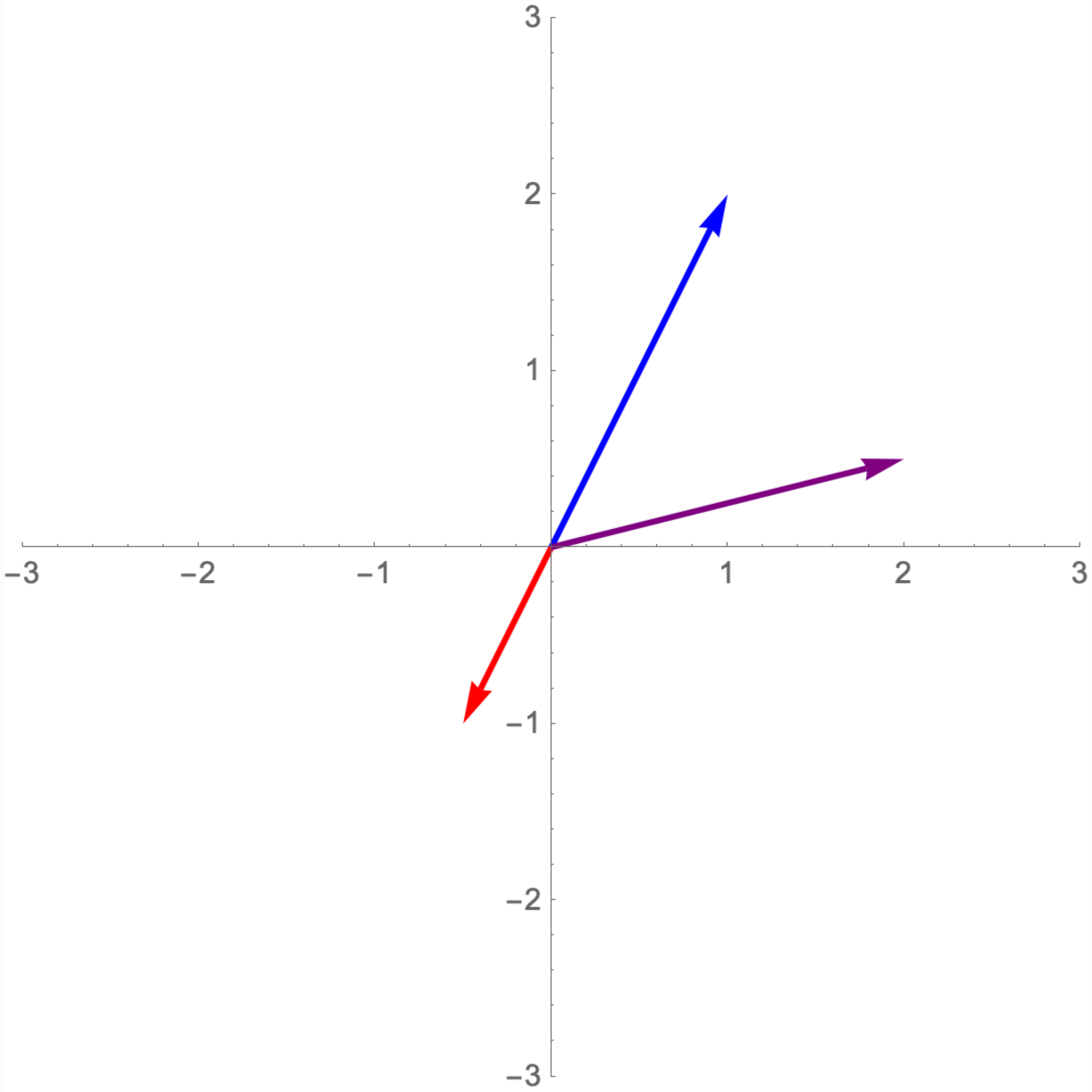
A, square matrix A is said to be **symmetric** if $A^T = A$

Vectors

The foundation of linear algebra is the concept of a **vector**

Vectors

For a physics student, a vector is defined by **direction** and **magnitude**:



Vectors

Vectors

For a computer science student, a vector is just an **ordered collection of numbers**

Vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Vectors

- A vector in a CS students point of view can be thought of as a **coordinate list** specifying the destination of an arrow
- In this case the vector $\vec{v}$ can be thought of as the **arrow** pointing from the origin, $(0,0,0)$ to the point $(1,3,2)$.

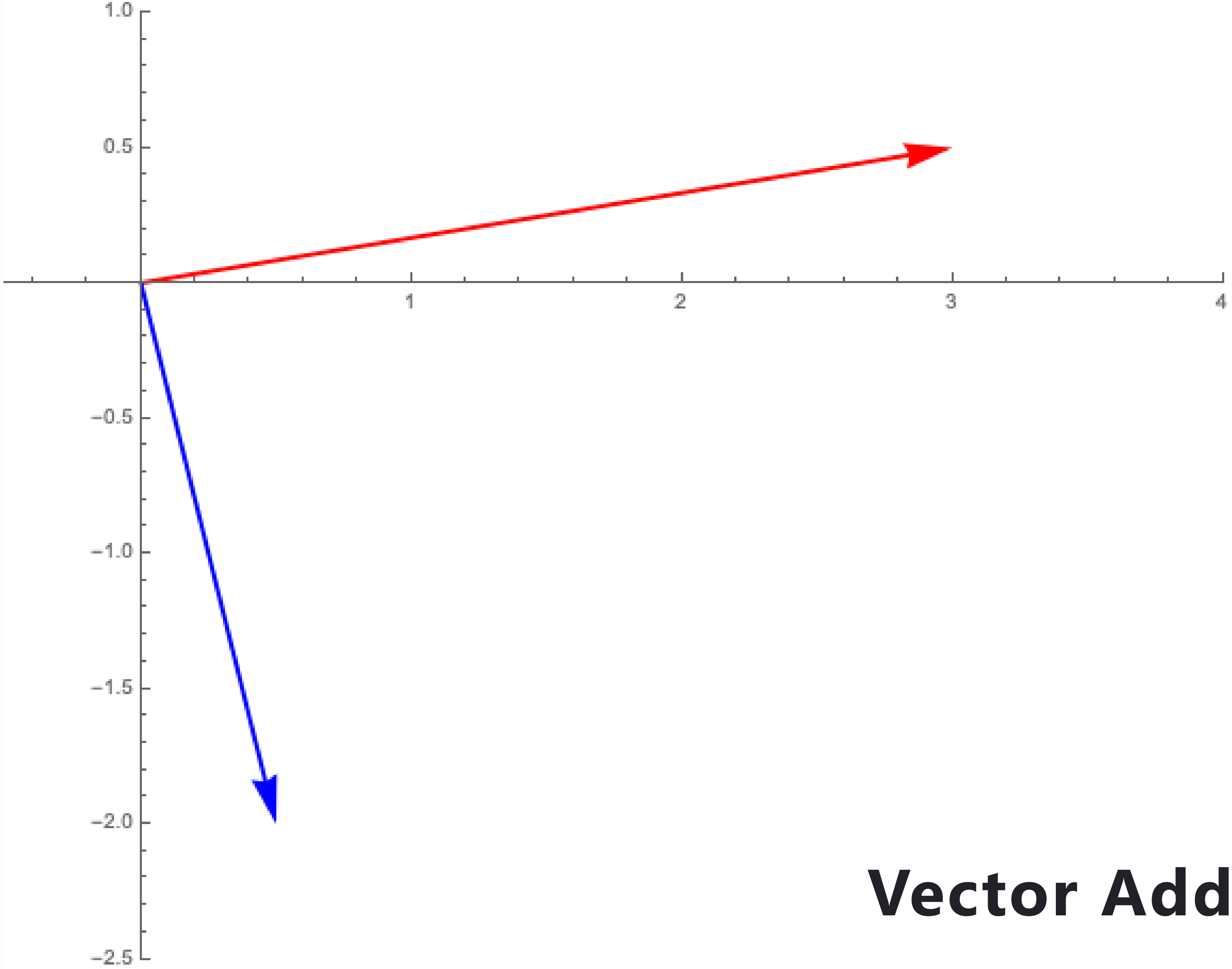
To differentiate vectors and coordinates, vectors are written as single-column matrices while coordinates are written as ordered tuples.

Vector Addition

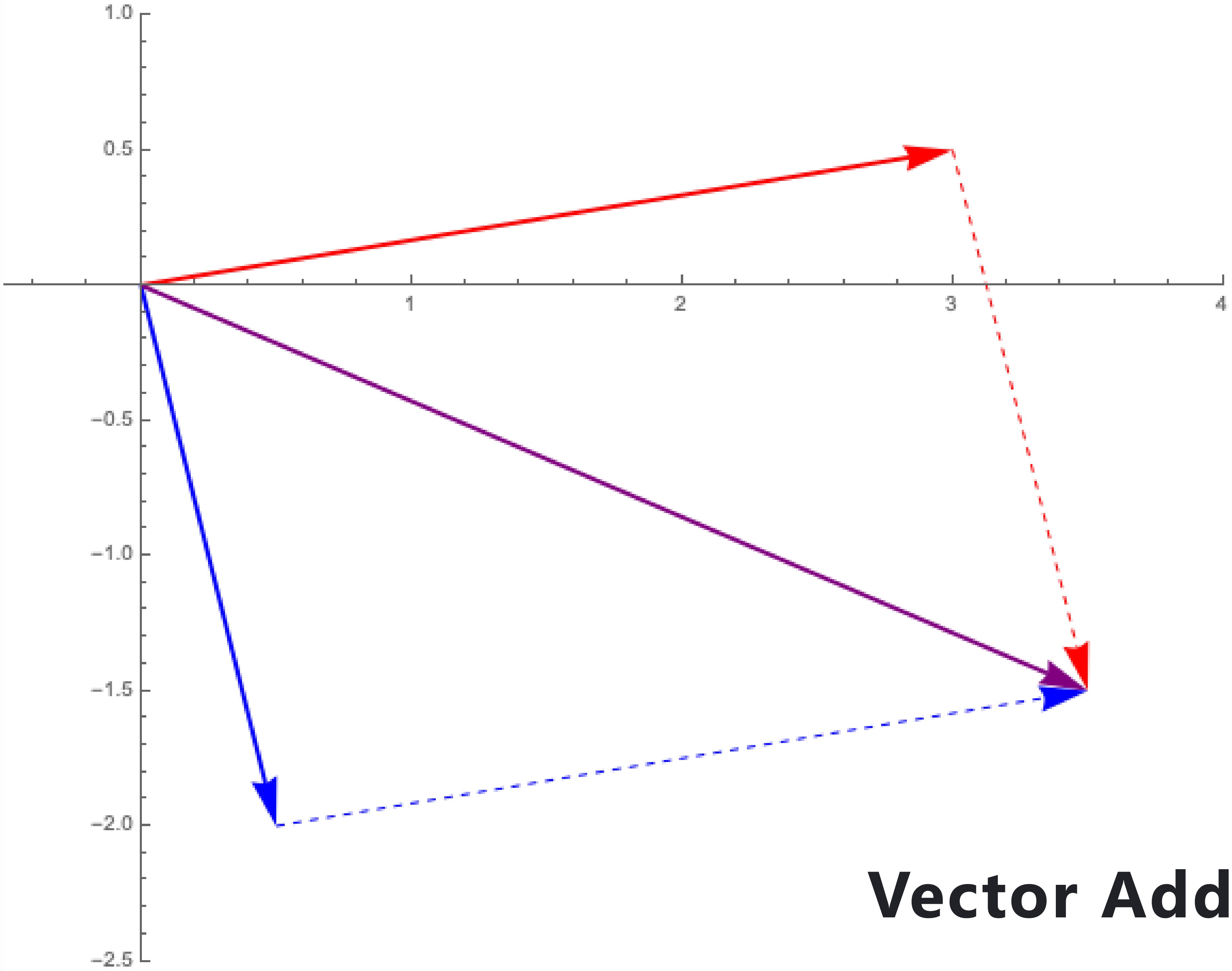
Vectors can be added to each other which also **add** their corresponding arrows.

Vector Addition

$$\begin{bmatrix} 3 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -2 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -1.5 \end{bmatrix}$$



Vector Addition



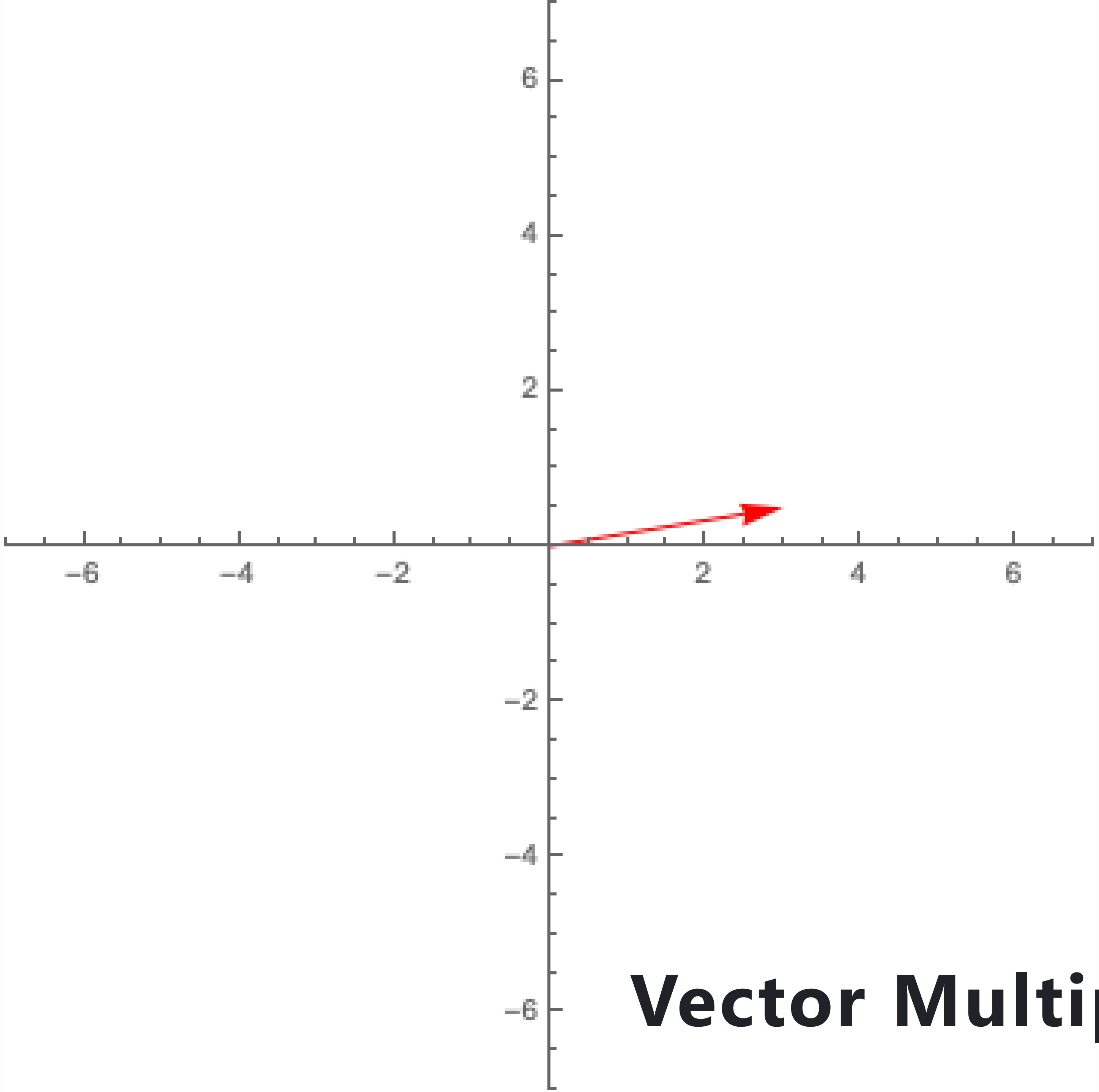
Vector Addition

Vector Multiplication

Vectors can also be multiplied with **scalar** values which will scale their corresponding arrows

Vector Multiplication

$$\vec{v} = \begin{bmatrix} 3 \\ 0.5 \end{bmatrix}$$



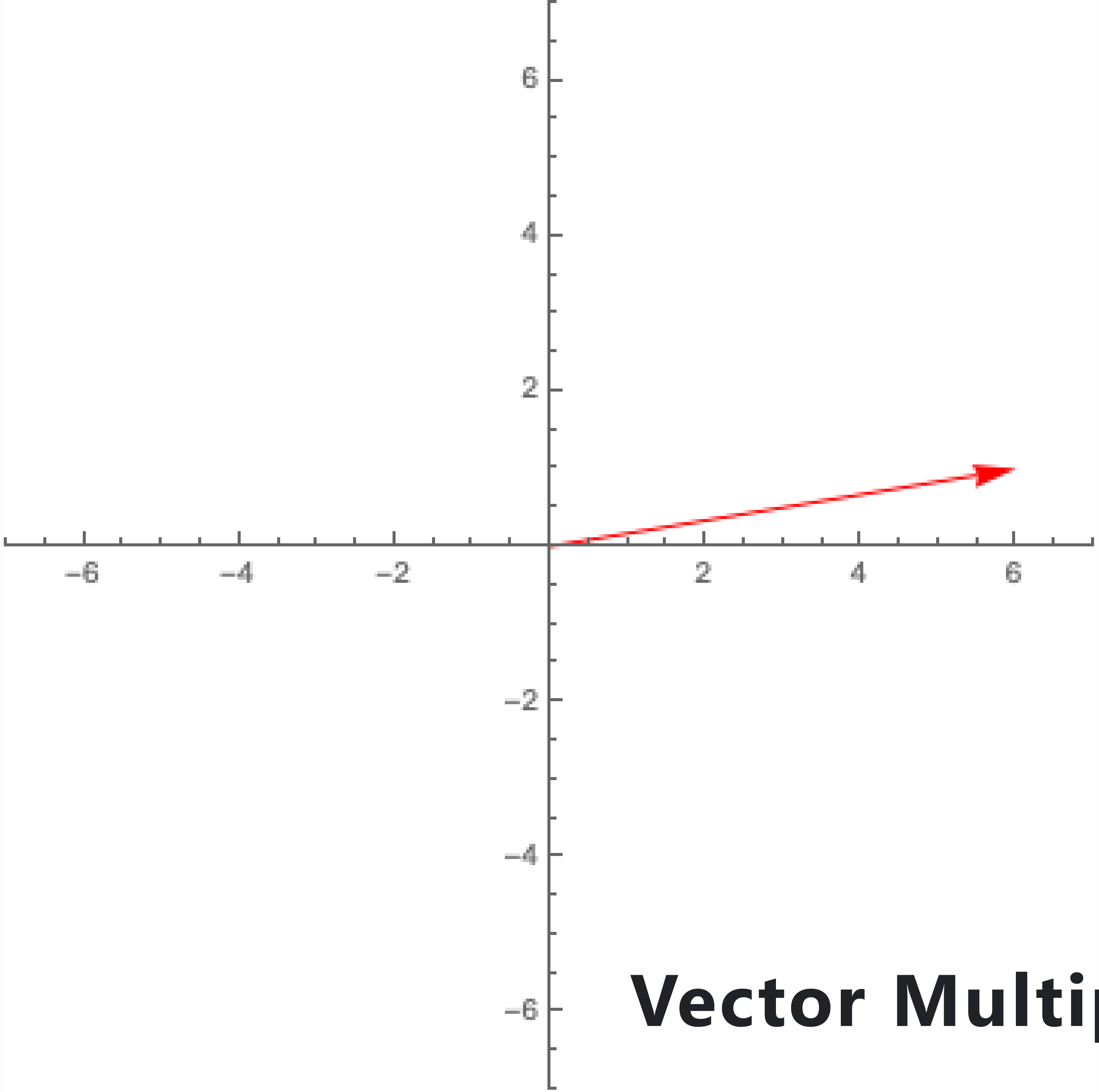
Vector Multiplication

Vector Multiplication

The vector **scaled** by a factor of **2**:

Vector Multiplication

$$2\vec{v} = 2\begin{bmatrix} 3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$



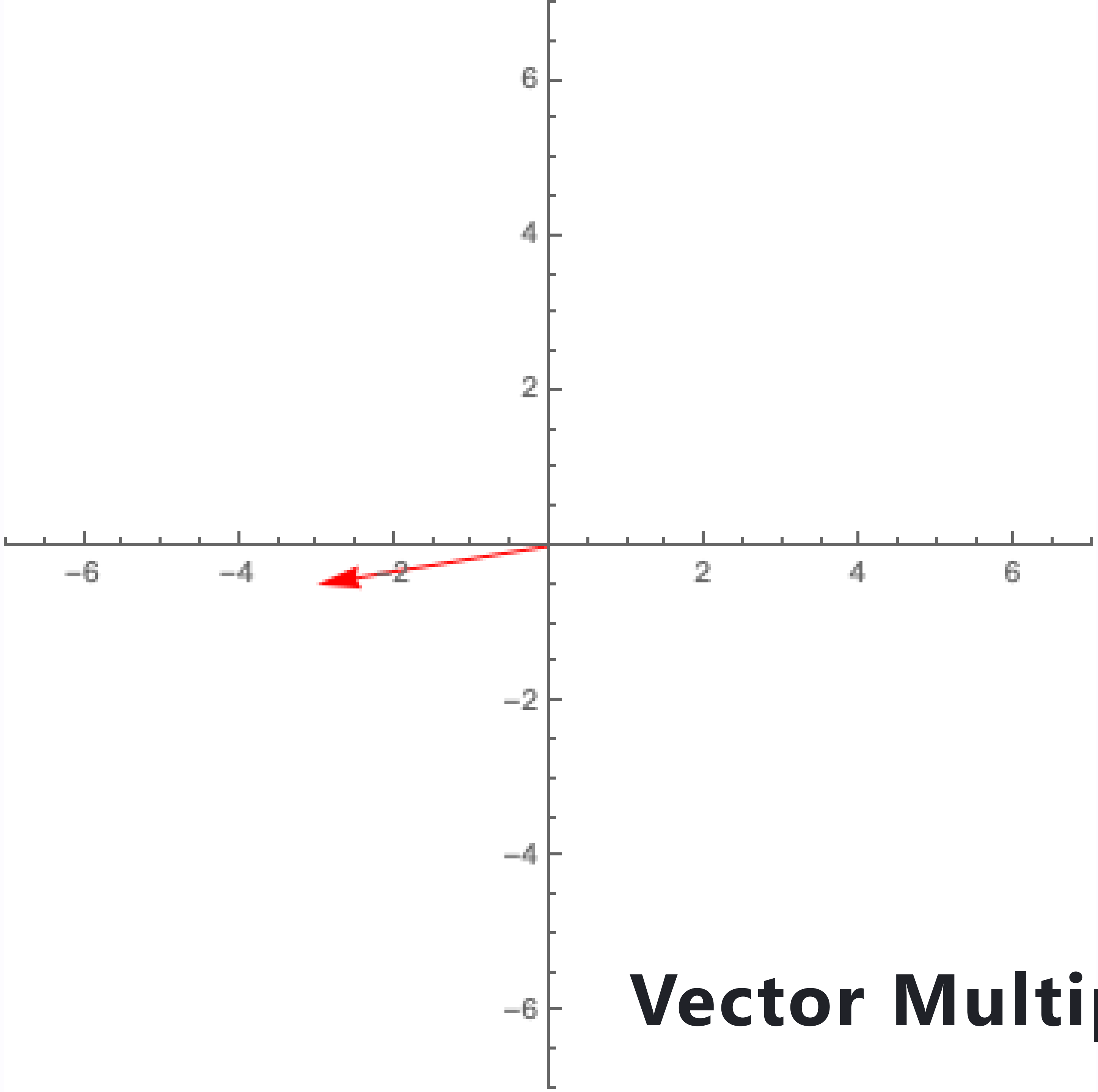
Vector Multiplication

Vector Multiplication

Scaled by a factor of -1.

Vector Multiplication

$$-1\vec{v} = -1\begin{bmatrix} 3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -3 \\ -.5 \end{bmatrix}$$



Vector Multiplication

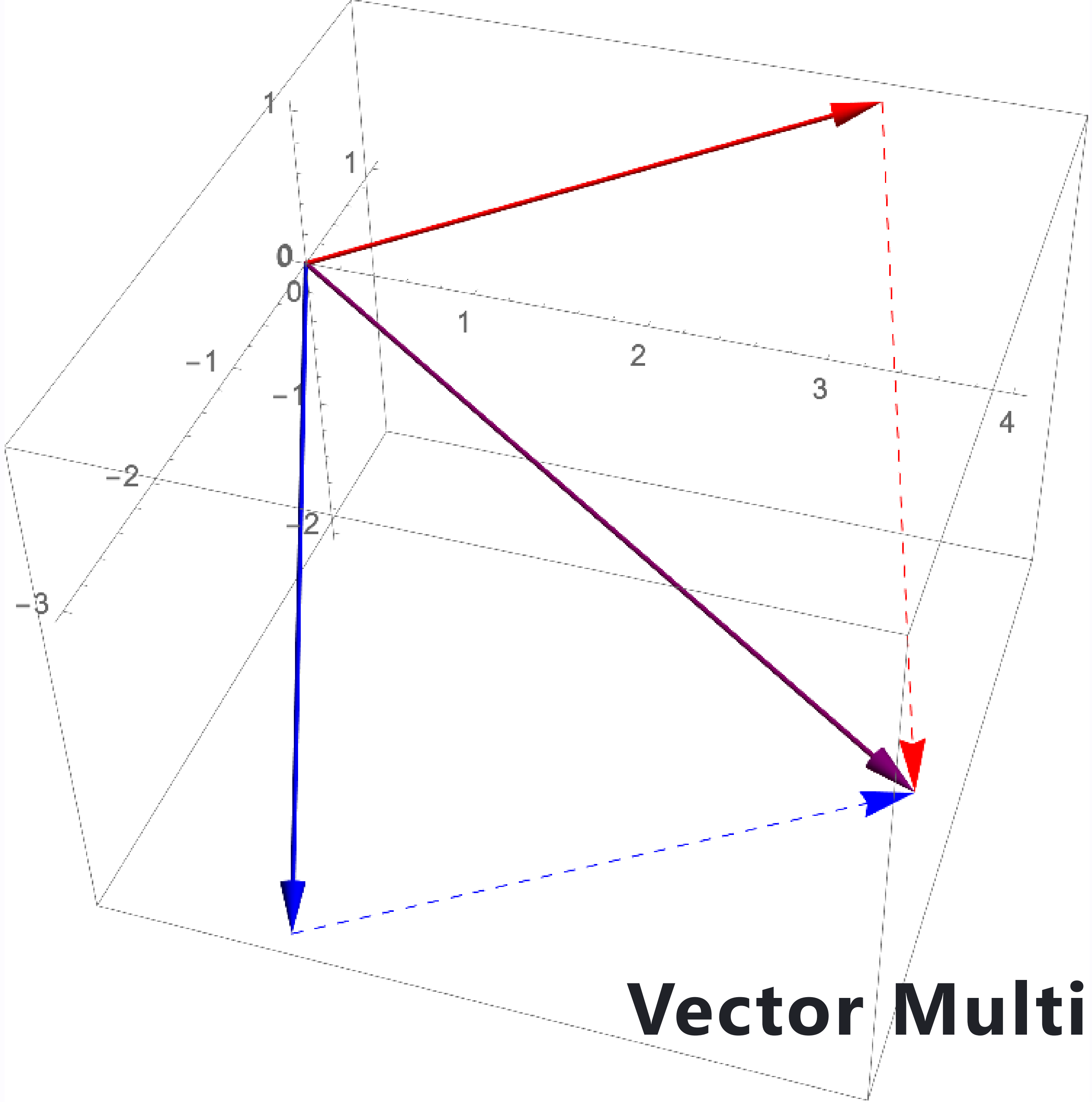
Scalars are called scalars because they **scale** the vectors.

Vector Multiplication

These operations also work on **3 dimensional vectors**

Vector Multiplication

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$



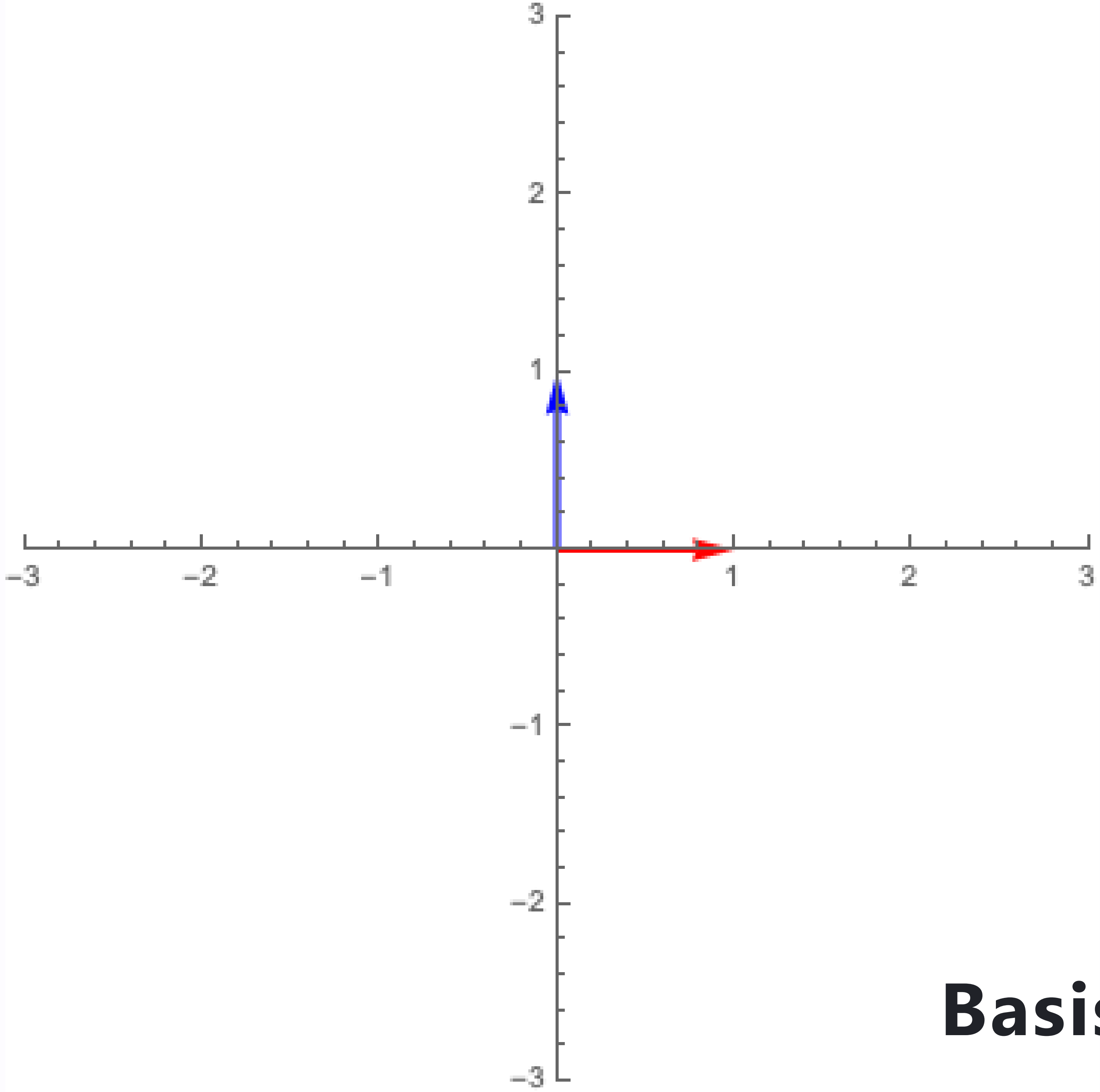
Vector Multiplication

Basis Vectors

- To generalize vectors and vector operations, linear algebra makes use of **basis vectors** which are unit vectors along the x and y axis of a Cartesian plane
- We call these vectors, $\hat{\imath}$ and $\hat{\jmath}$ respectively

Basis Vectors

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$



Basis Vectors

Basis Vectors

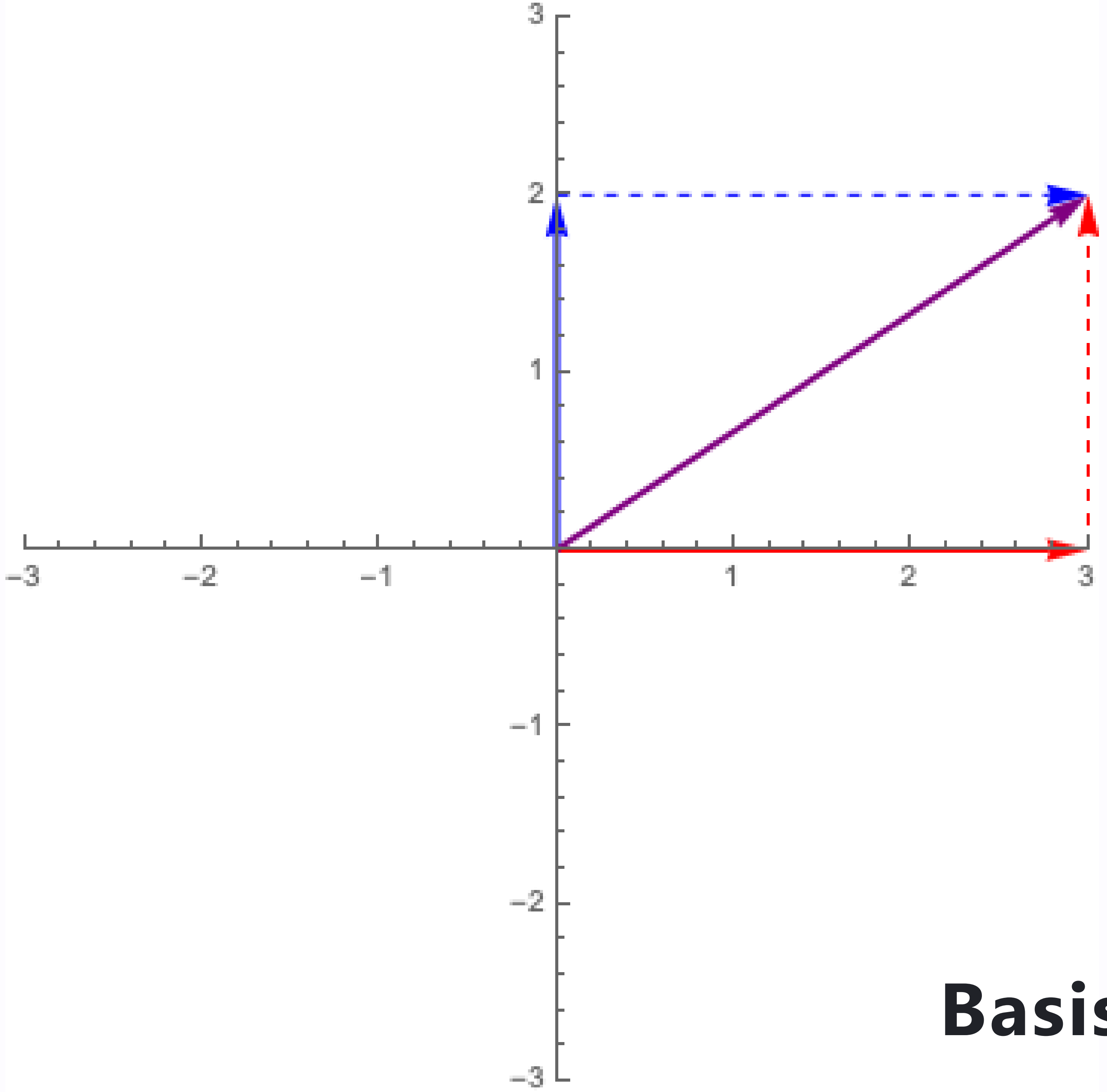
Basis vectors are special because you can **define** new vectors vectors based on the the definitions of the basis

Basis Vectors

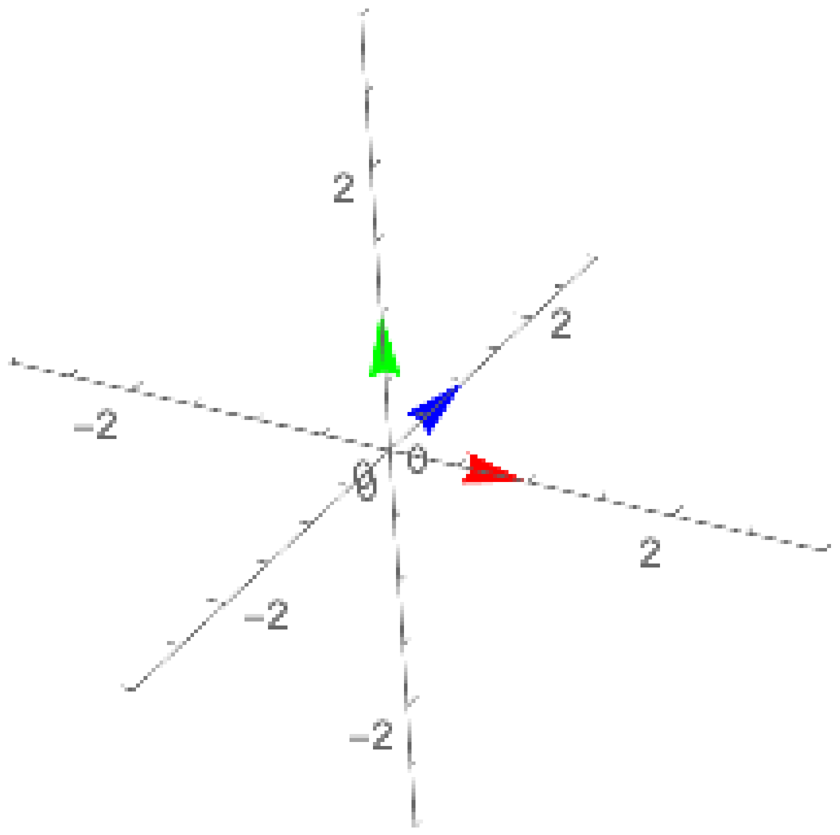
$$\vec{v} = 3\hat{\imath} + 2\hat{\jmath}$$

Basis Vectors

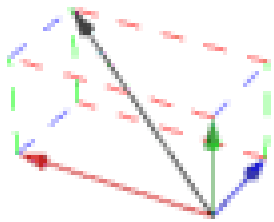
$$\vec{v} = 3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

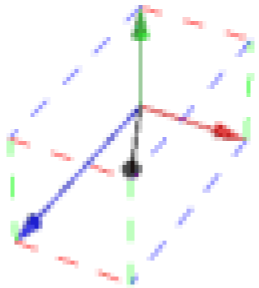


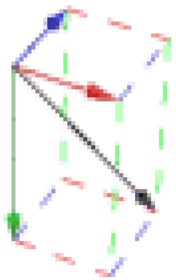
Basis Vectors



Basis Vectors



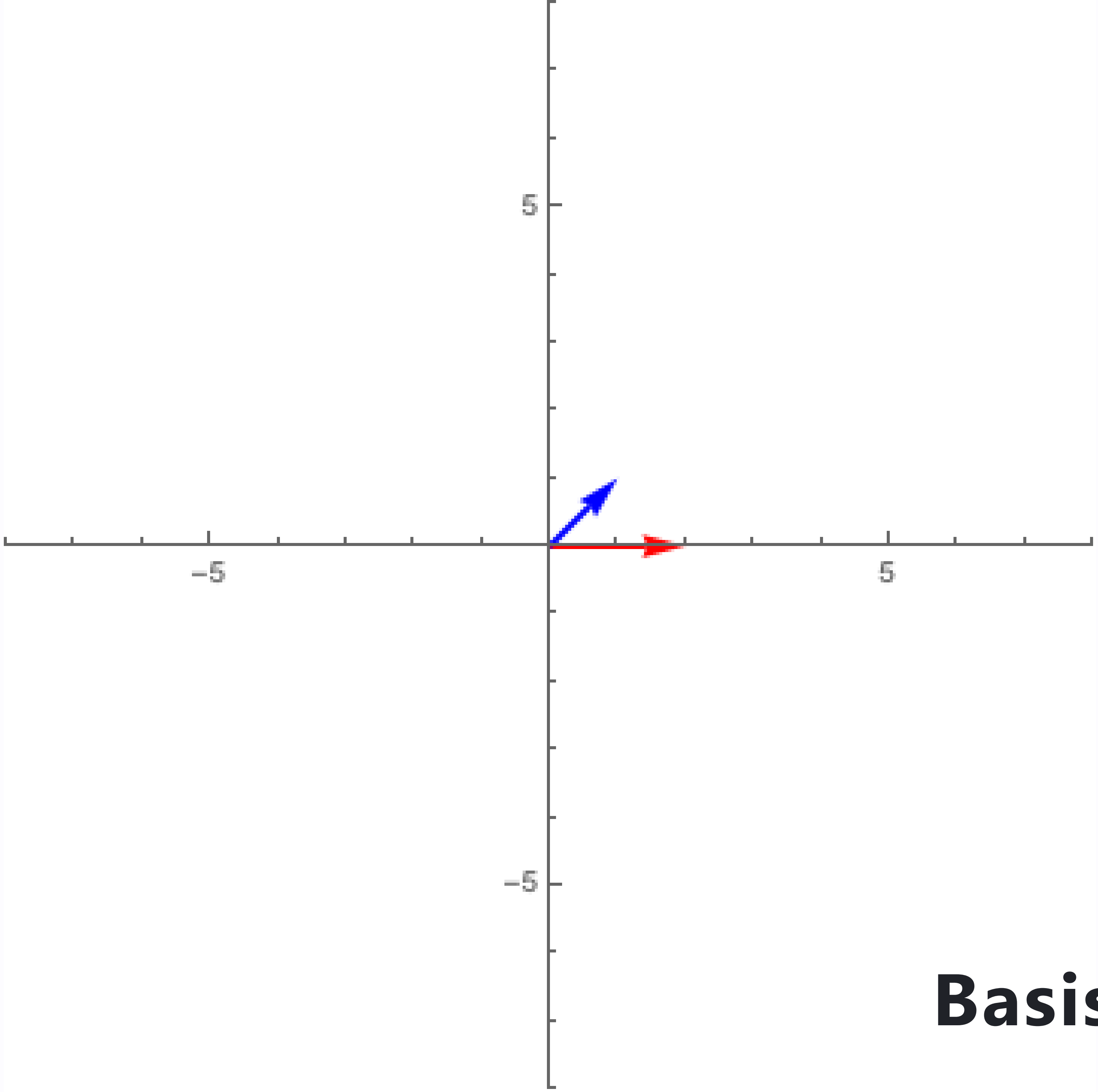




Basis Vectors

Basis Vectors

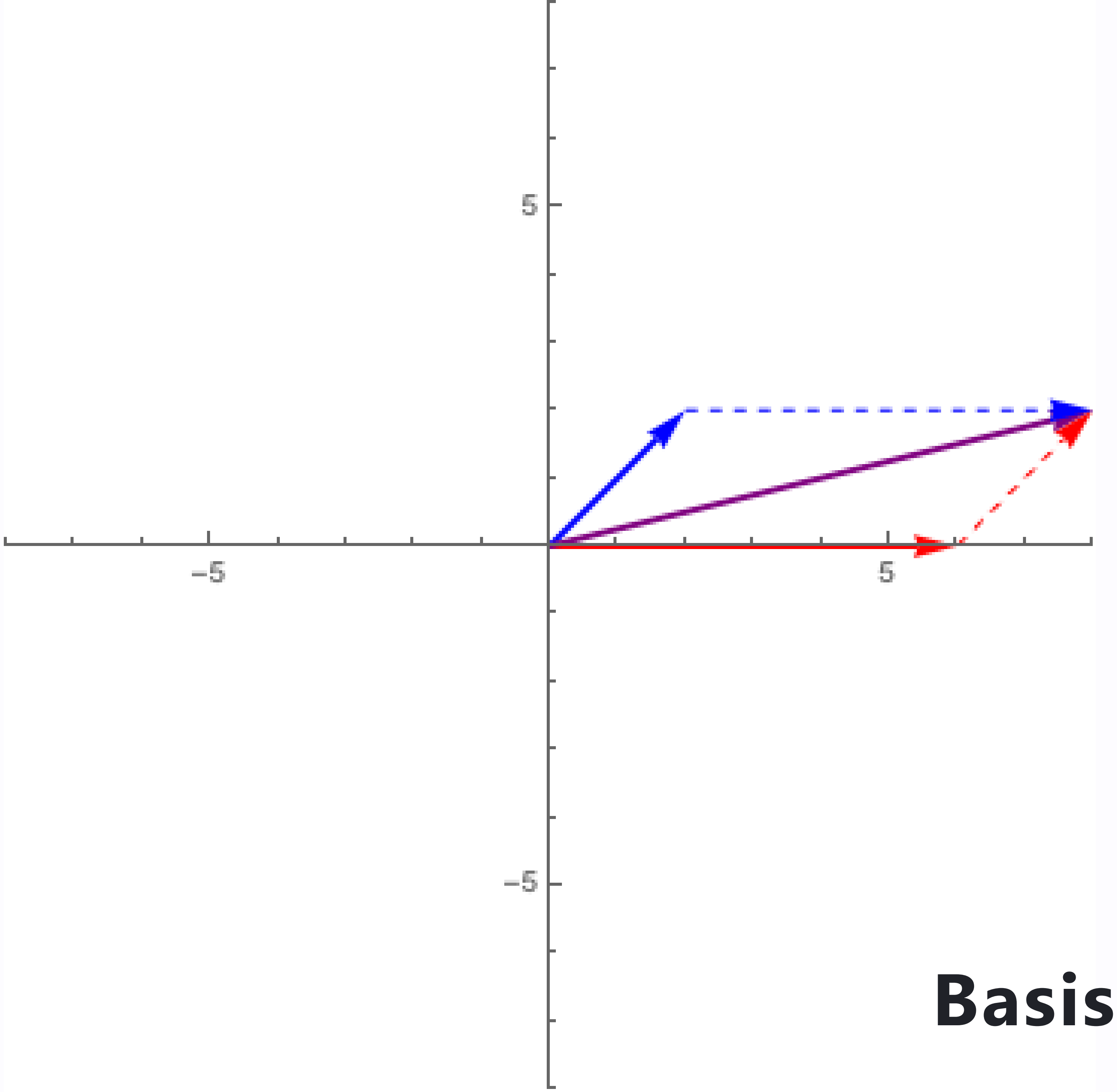
$$\vec{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



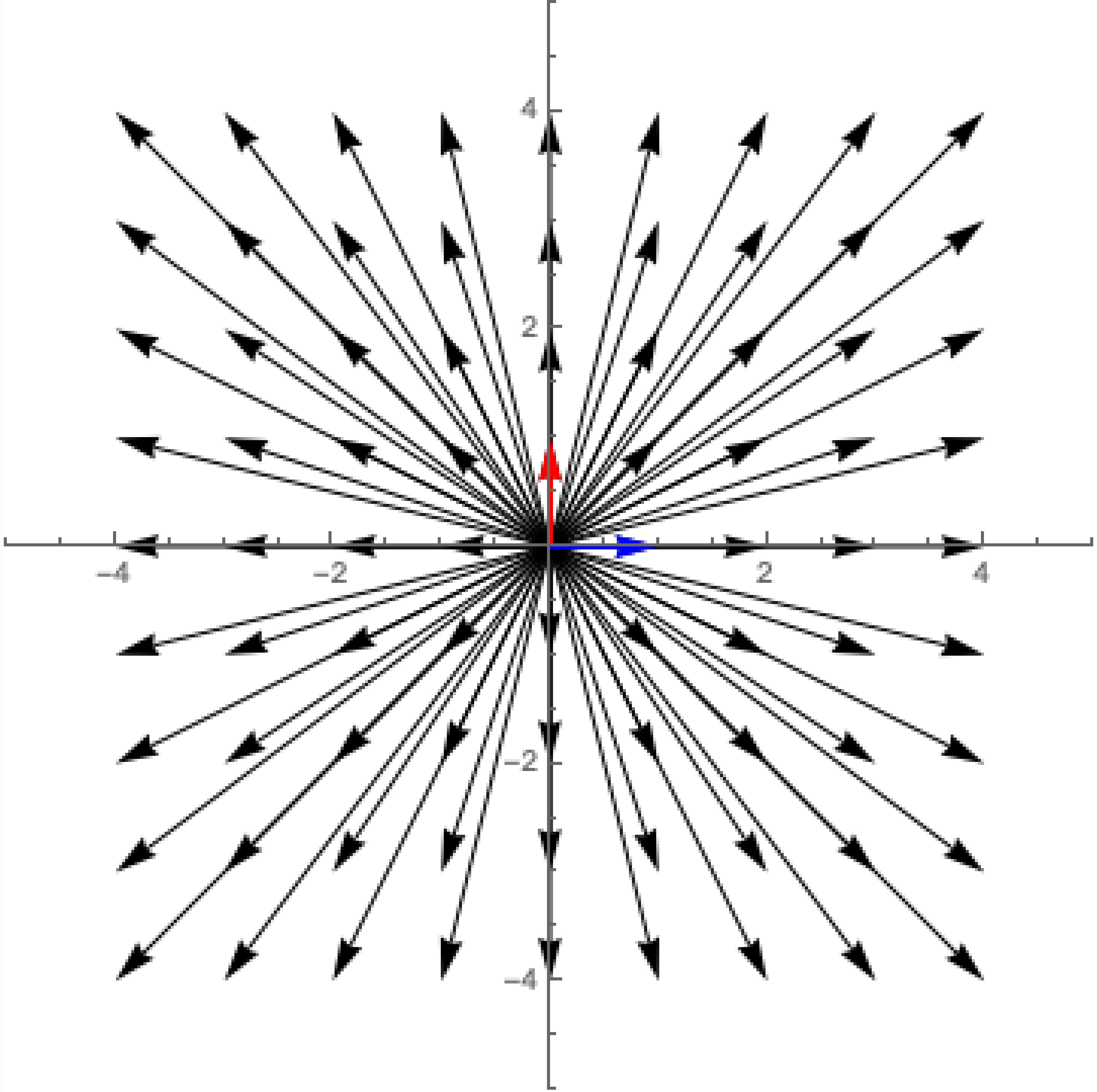
Basis Vectors

Basis Vectors

$$\vec{v} = 3\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$



Basis Vectors



Span

Span

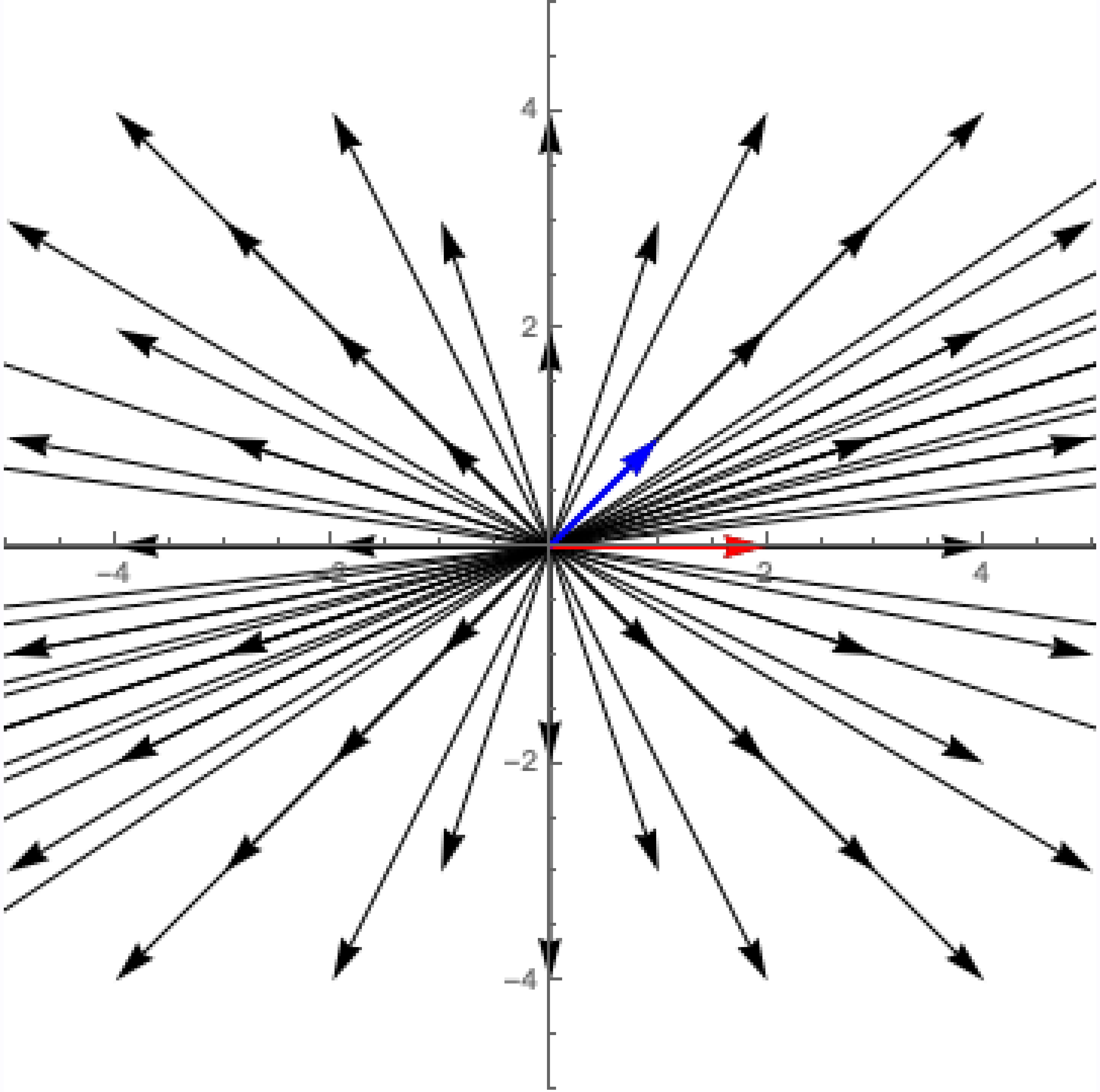
For the set of vectors $\{\hat{\imath}, \hat{\jmath}\}$, you can generate **all** of the possible vectors in the 2 dimensional vector space .

Span

You can also generate **all** of the possible vectors in the 2-dimensional vector space using the basis vectors:

Span

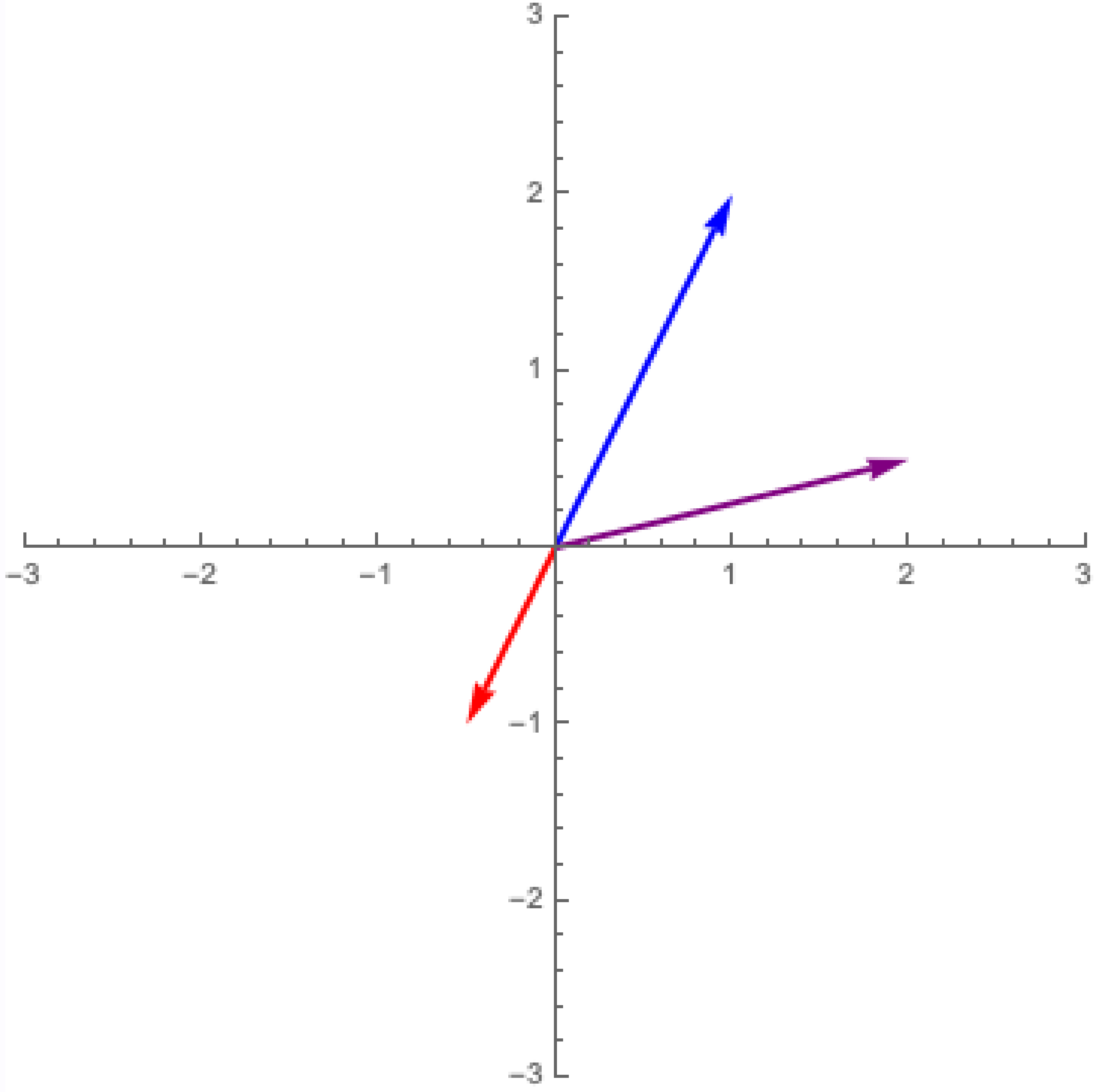
$$\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$



Span

Span

$$\{\vec{r} = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$$



Span

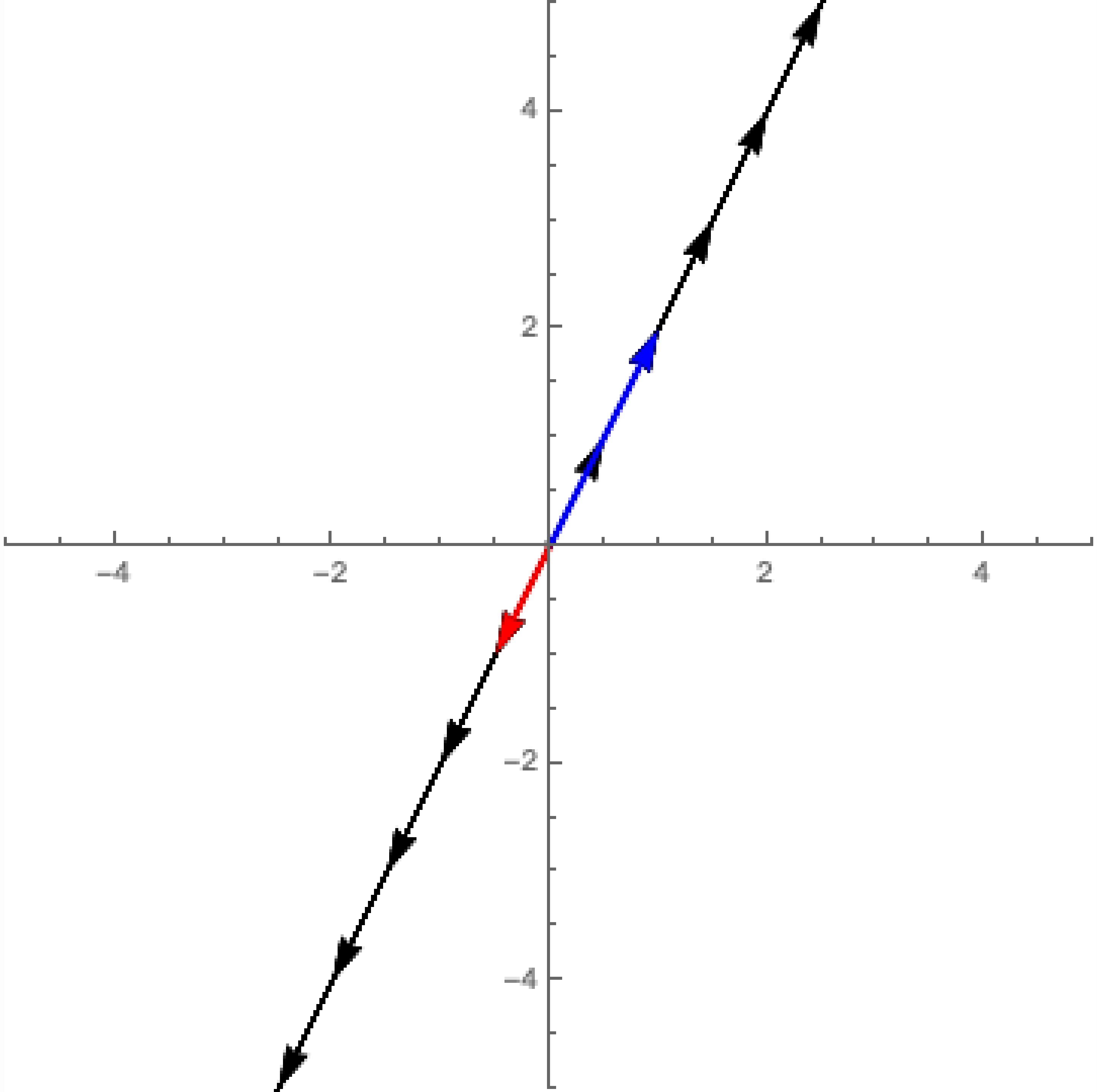
Span

- You **can't** generate the purple vector since one of the basis vectors you are using is **redundant**
- It is redundant in the sense that the $\vec{r}$ is just a **scaled version** on of $\vec{b}$ and vice versa

Span

$$\vec{r} = -0.5\vec{b}$$

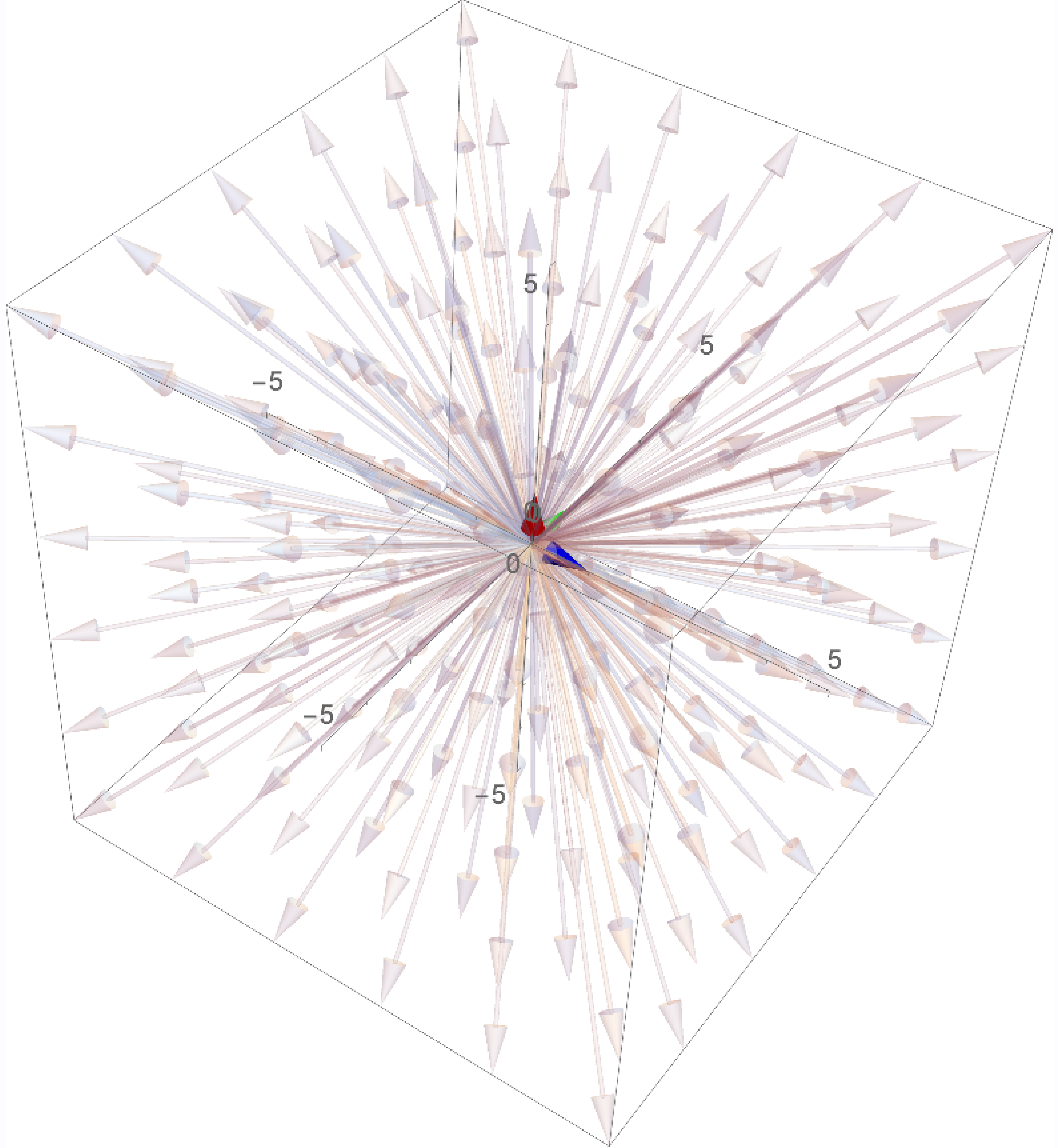
$$\vec{b} = -2\vec{r}$$



Span

Span

- We call $\vec{r}$ and $\vec{b}$ and any set of vectors that has some kind of redundancy as **linearly dependent**
- The basis vectors defined earlier, $\{\hat{\imath}, \hat{\jmath}\}$, and $\{\vec{u}, \vec{w}\}$ as **linearly independent**.

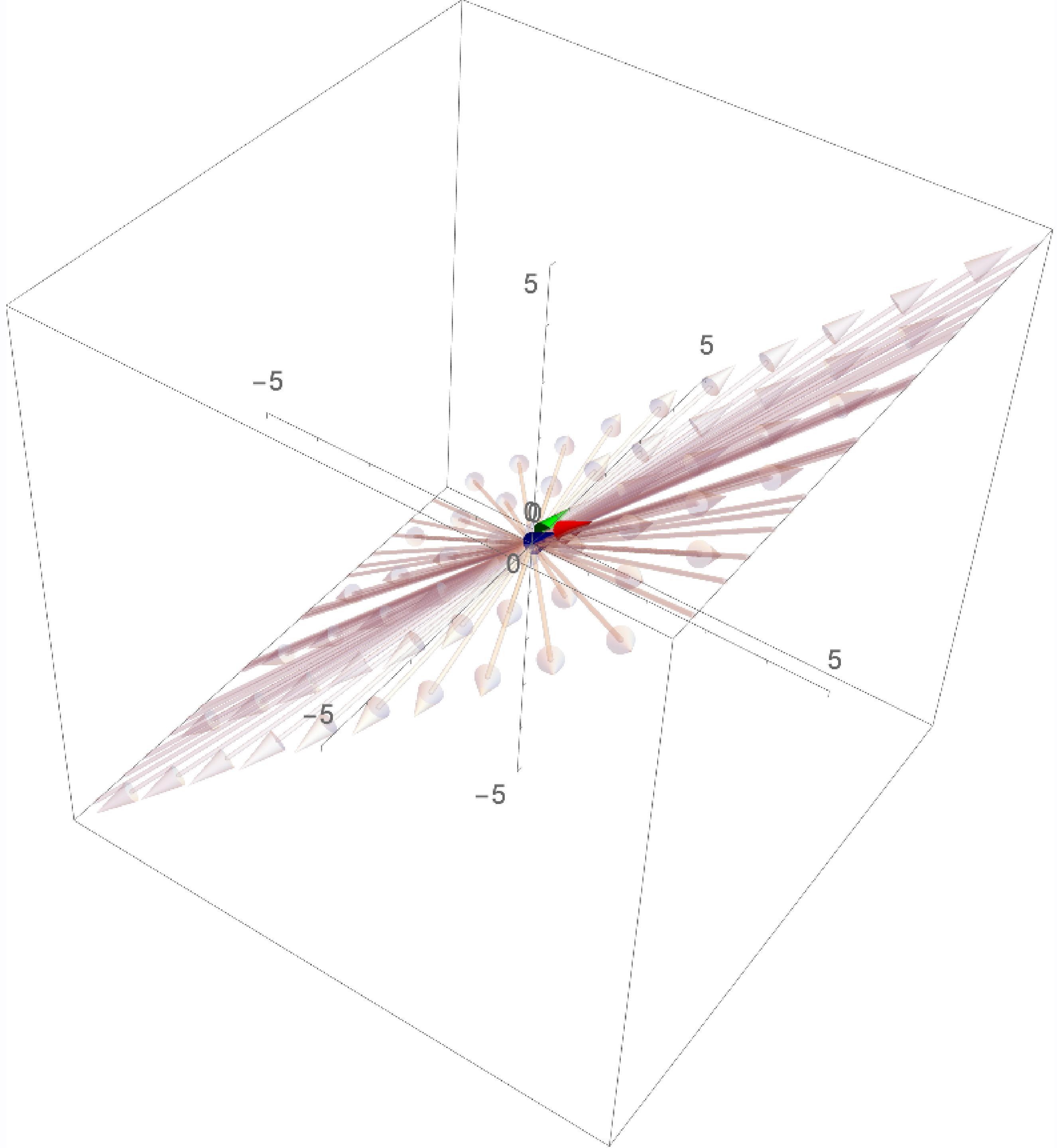


Span

Span

Linear dependence in three dimensions can result to spans that describe a **plane**:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

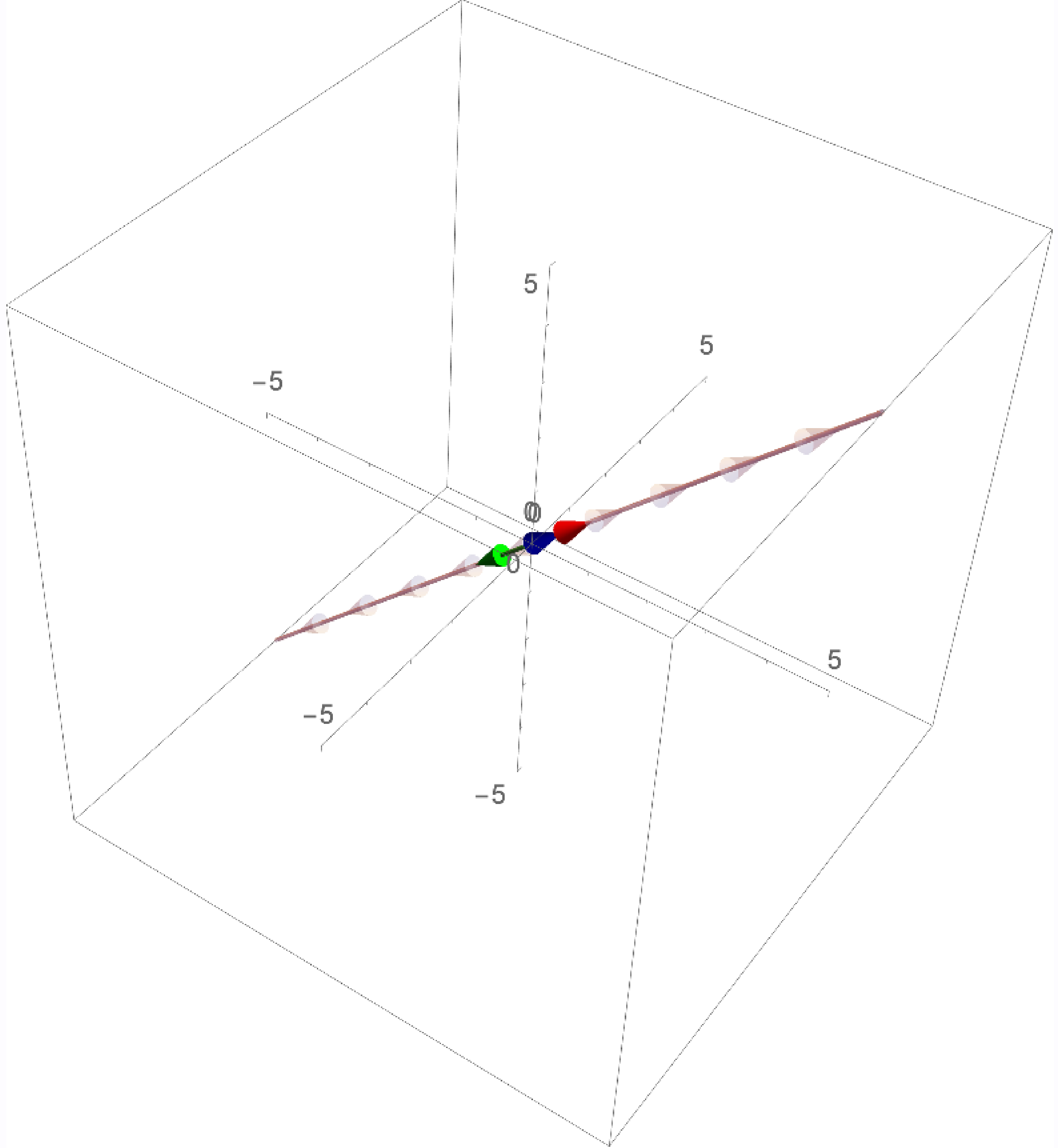


Span

Span

And spans that describe a **line**:

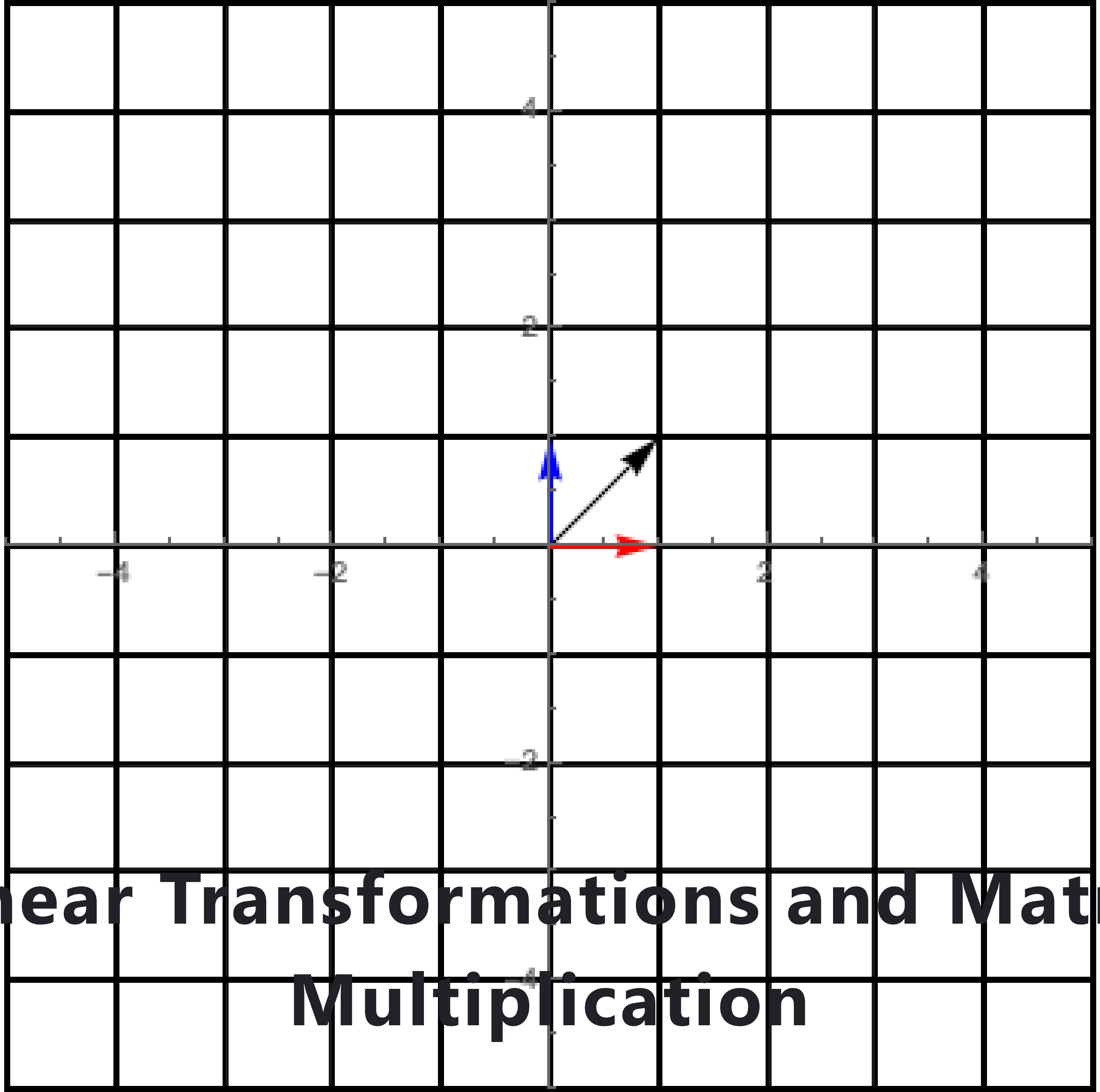
$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$$



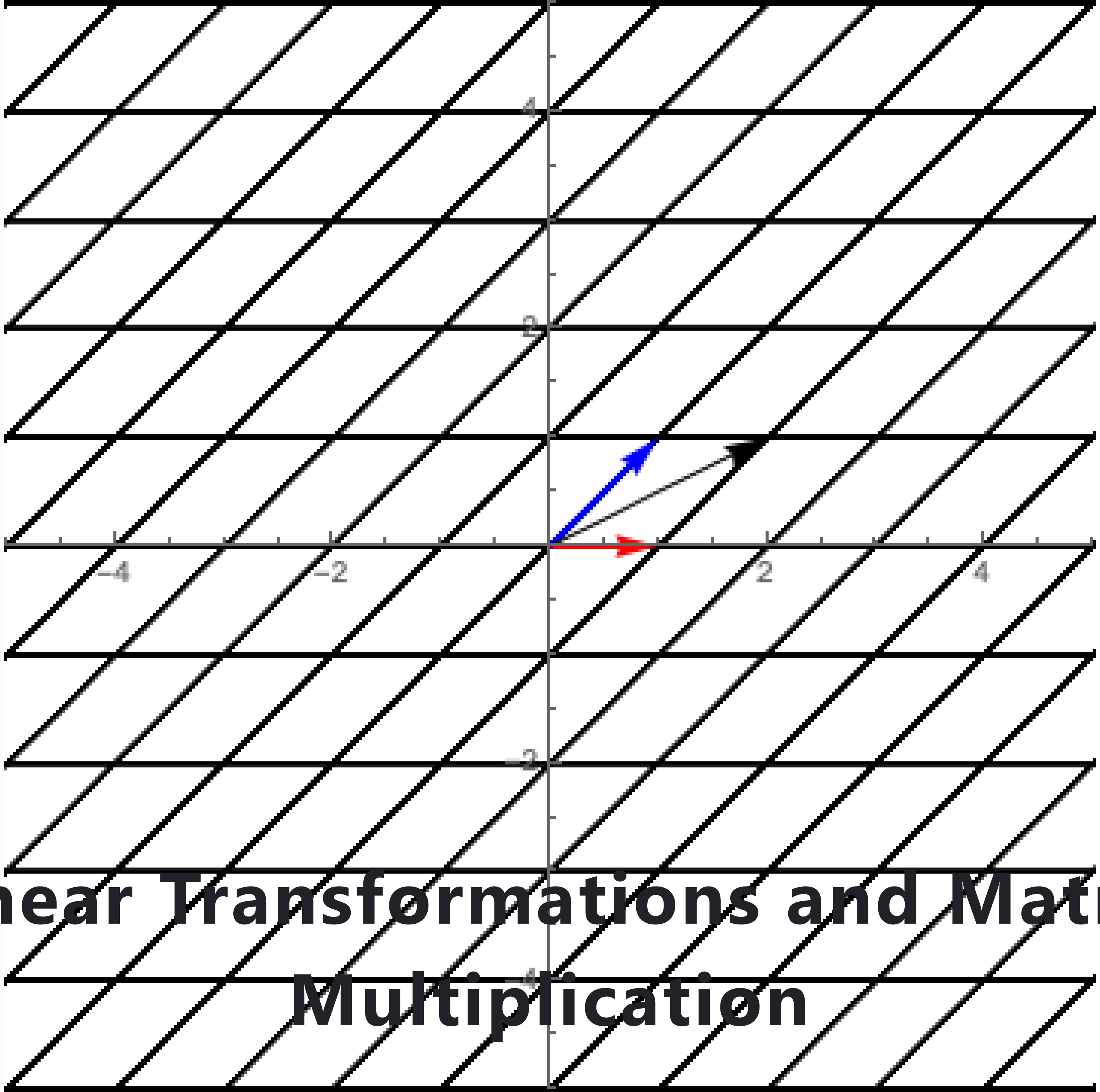
Span

Linear Transformations and Matrix Multiplication

- A **transformation** is basically a function that converts one vector to another vector
- For example the transformation f can be defined as $f(\vec{x}) = 3\vec{x}$
- You can think of a transformation visually as the **distortion** of the entire vector space



Linear Transformations and Matrix Multiplication



Linear Transformations and Matrix Multiplication

Liner Transformations

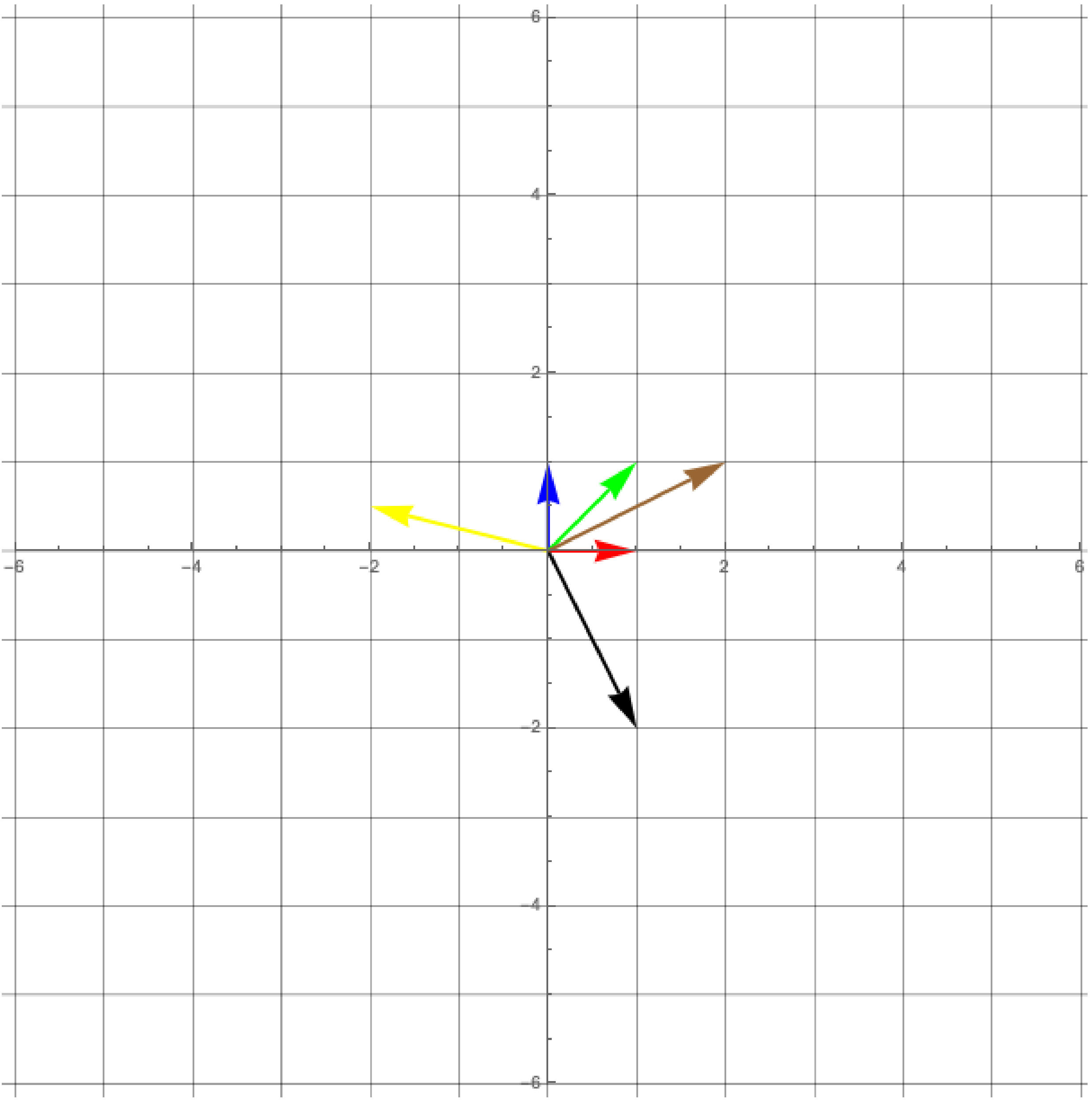
Linear Transformations are special transformations where the distortion of the vector space follows these rules:

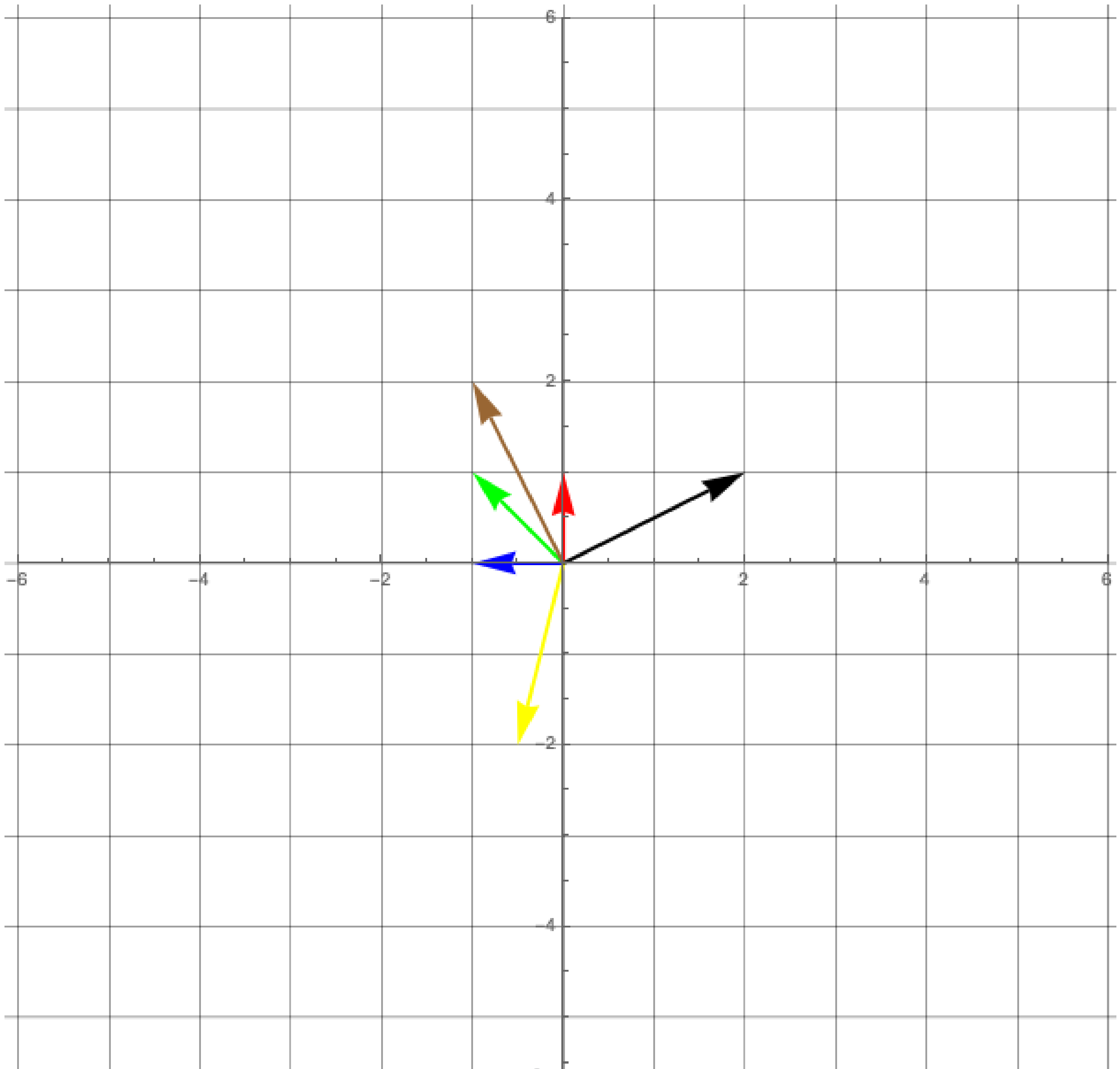
Liner Transformations

1. The origin should not move
2. Parallel lines stay parallel
3. Straight lines stay straight

Liner Transformations

It turns out all transformations that satisfy the above rules can be perfectly described by watching how the **basis vectors** are transformed





Liner Transformations

Liner Transformations

- new $\hat{\imath}$ (red vector) :$\hat{\imath}' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- new $\hat{\jmath}$ (blue vector): $\hat{\jmath}' = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$
- Green vector: $1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Liner Transformations

- Brown vector: $2\begin{bmatrix}0\\1\end{bmatrix}+1\begin{bmatrix}-1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$
- Yellow vector: $-2\begin{bmatrix}0\\1\end{bmatrix}+0.5\begin{bmatrix}-1\\0\end{bmatrix}=\begin{bmatrix}-0.5\\-2\end{bmatrix}$

Liner Transformations

- Black vector: $1\begin{bmatrix}0\\1\end{bmatrix}+-2\begin{bmatrix}-1\\0\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$

Liner Transformations

- All of the other vector values after the transformation is basically **scaled versions** of the new basis vectors in the same way that the pretransformed vector values are combinations of the original basis vectors
- This means that any 2-dimensional linear transformation can be represented by 4 numbers, which we can write as a **matrix**, where each column corresponds to a basis vector

Liner Transformations

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Liner Transformations

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Linear Transformation and Matrix Multiplication

$$x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Linear Transformation and Matrix Multiplication

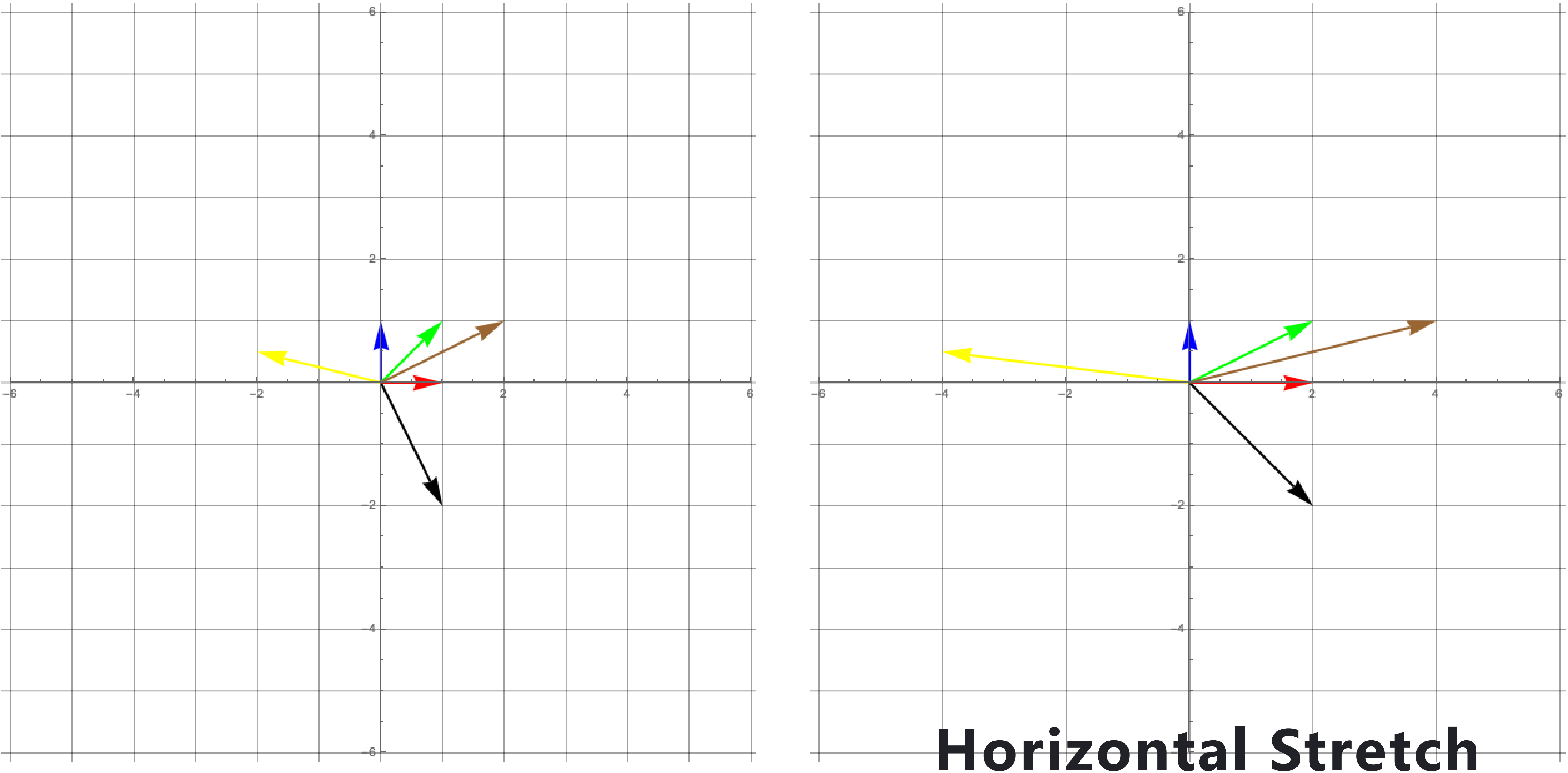
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Linear Transformation and Matrix Multiplication

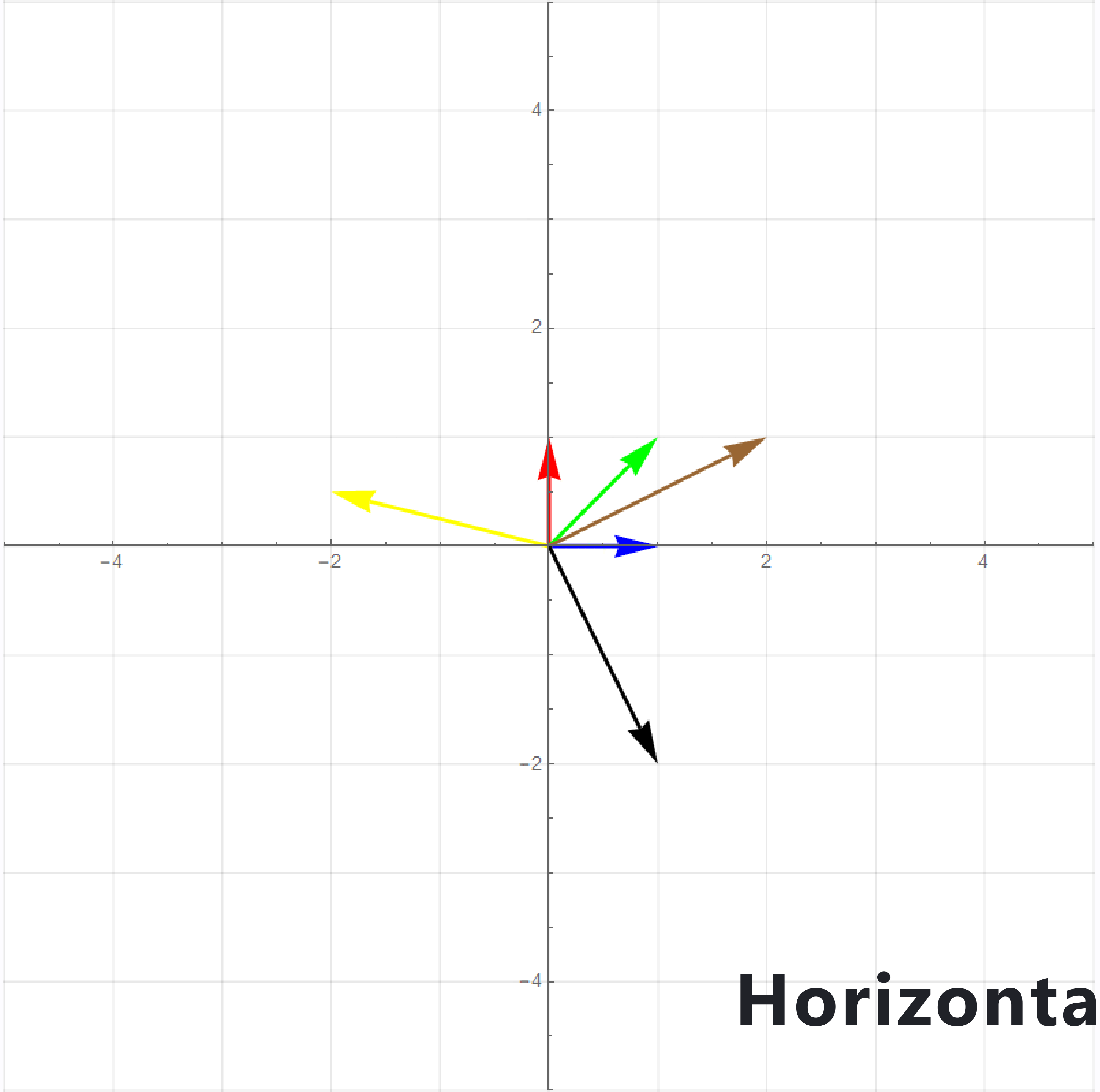
$$f(\vec{v}) = T\vec{v}$$

Horizontal Stretch

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



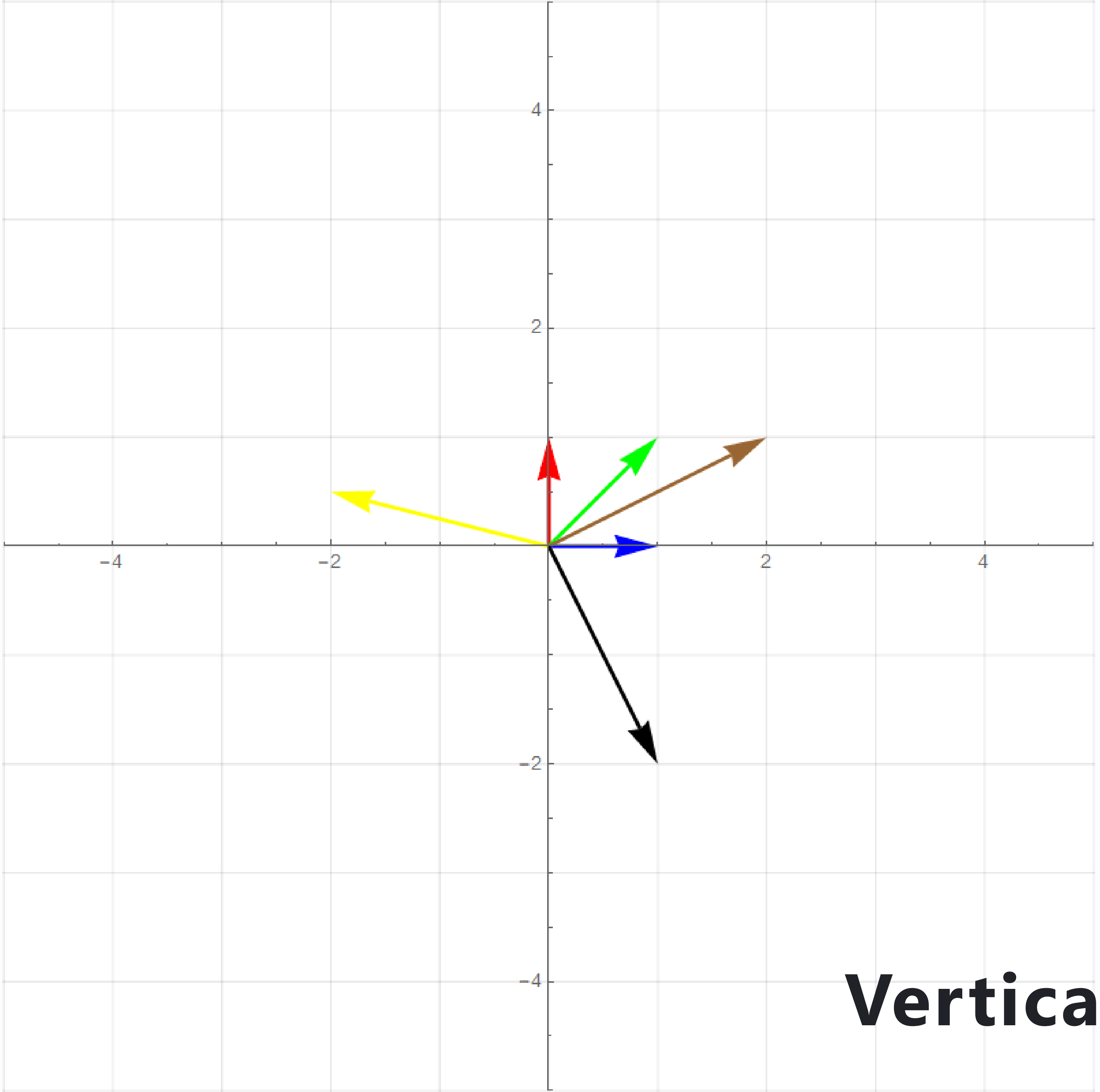
Horizontal Stretch



Horizontal Stretch

Vertical Stretch

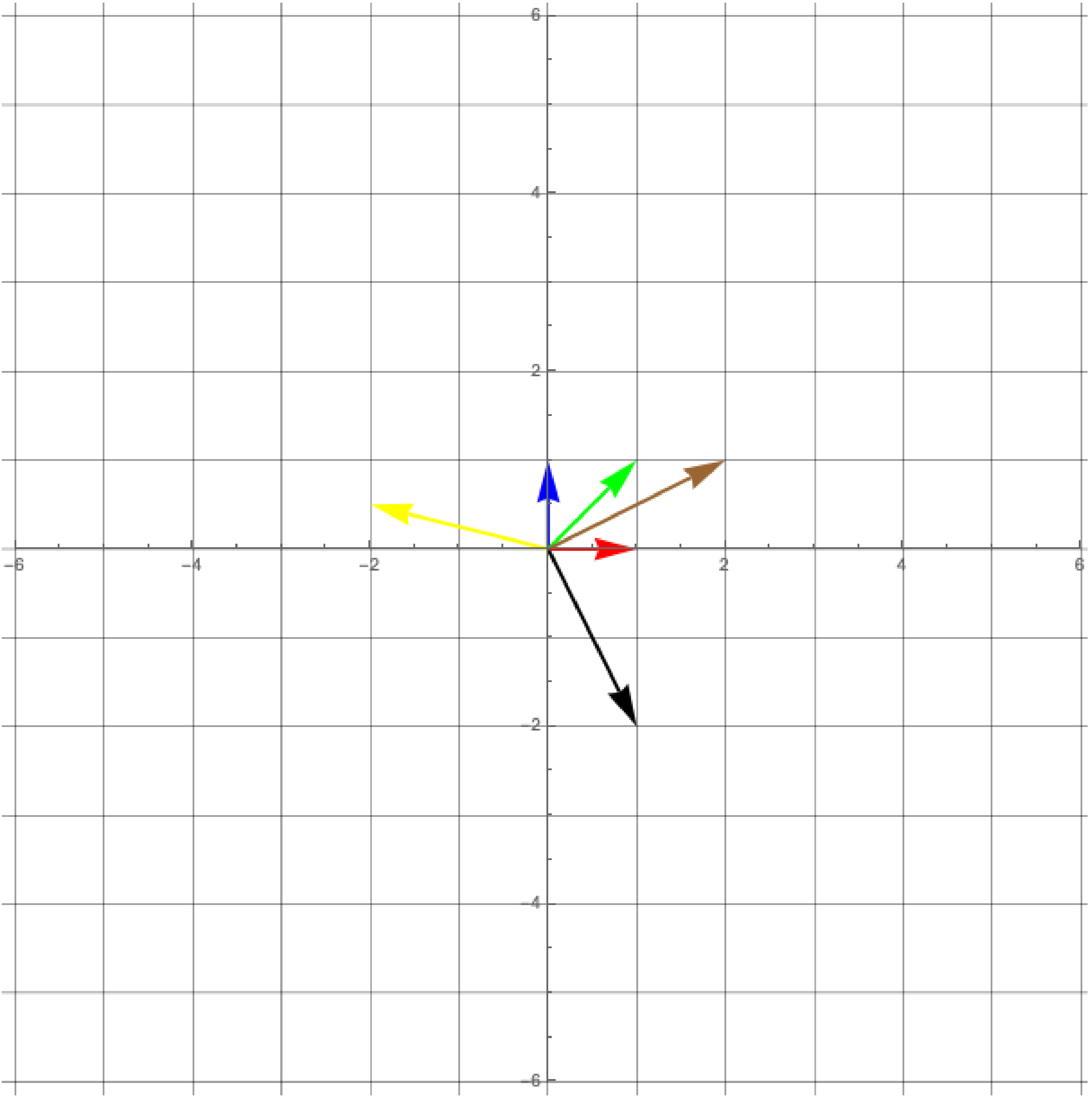
$$T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

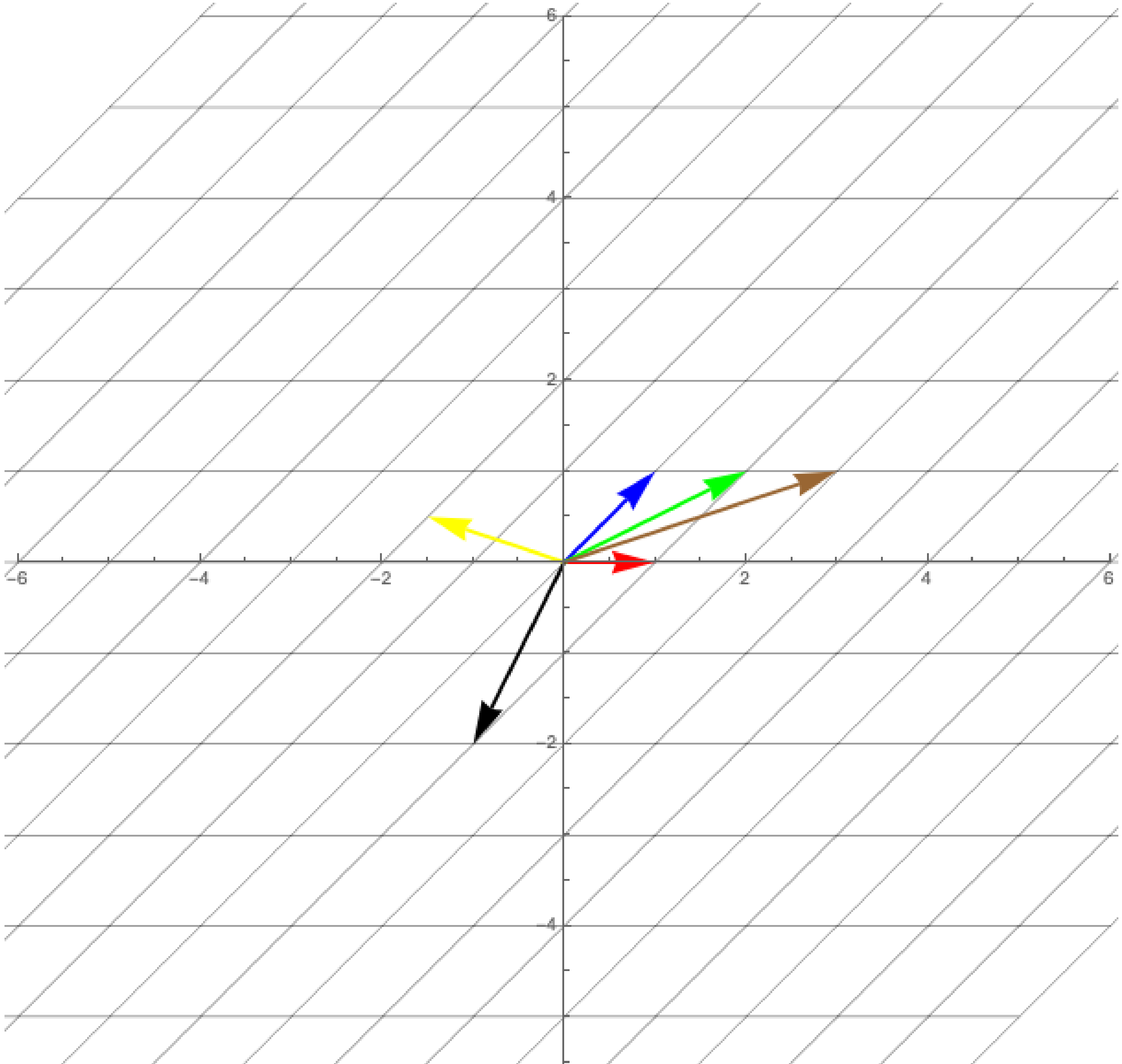


Vertical Stretch

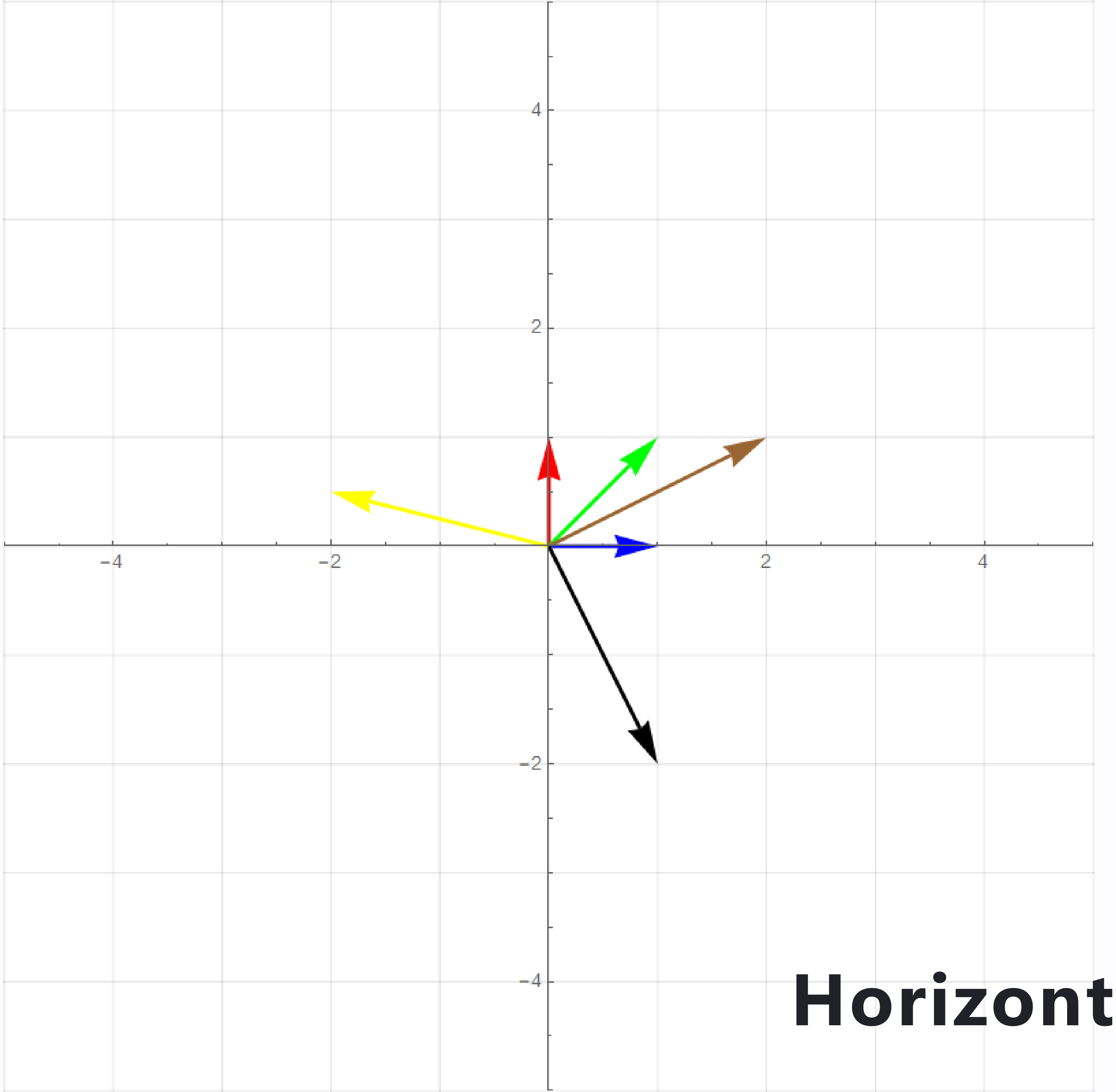
Horizontal Shear

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$





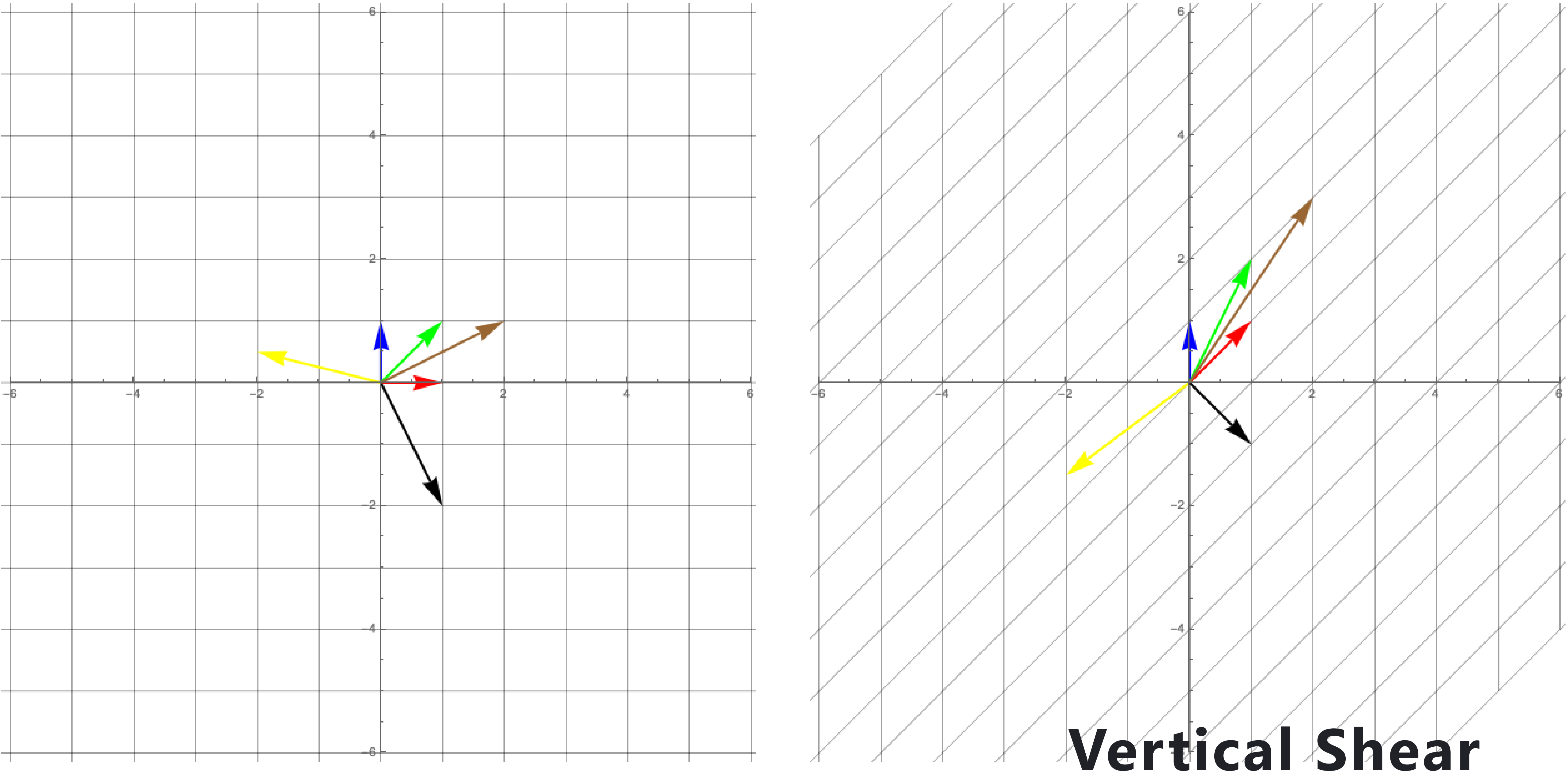
Horizontal Shear



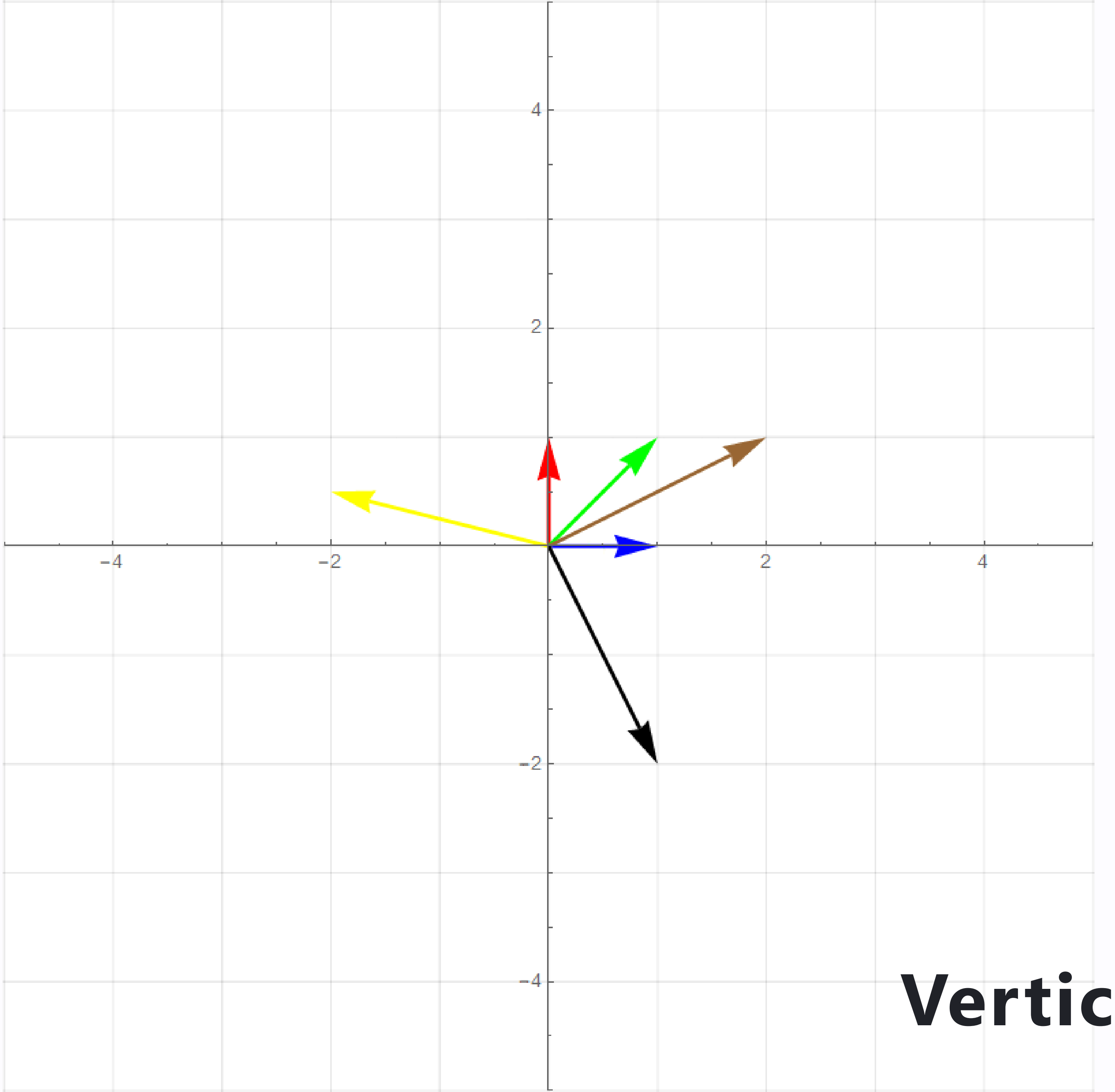
Horizontal Shear

Vertical Shear

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$



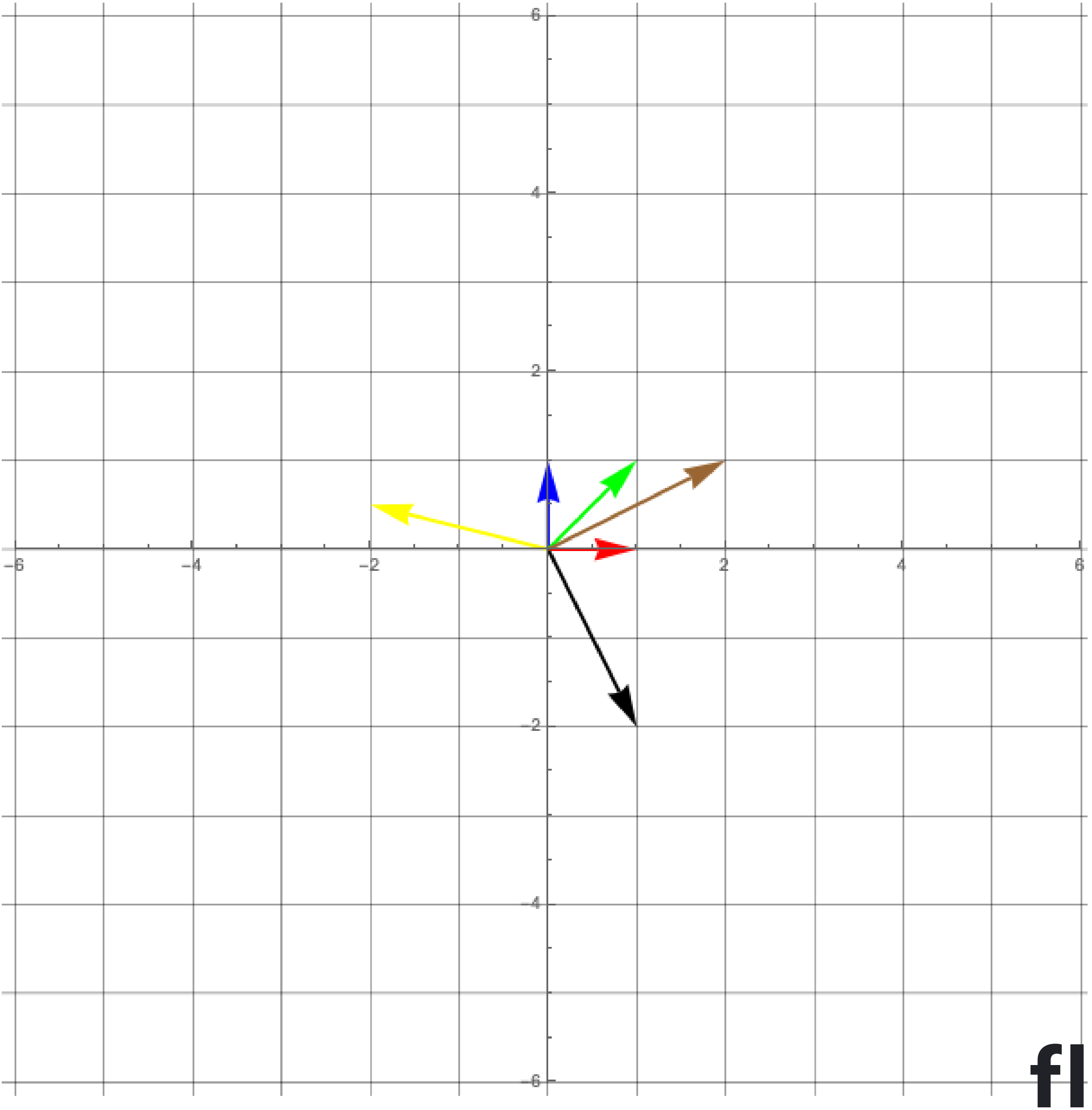
Vertical Shear

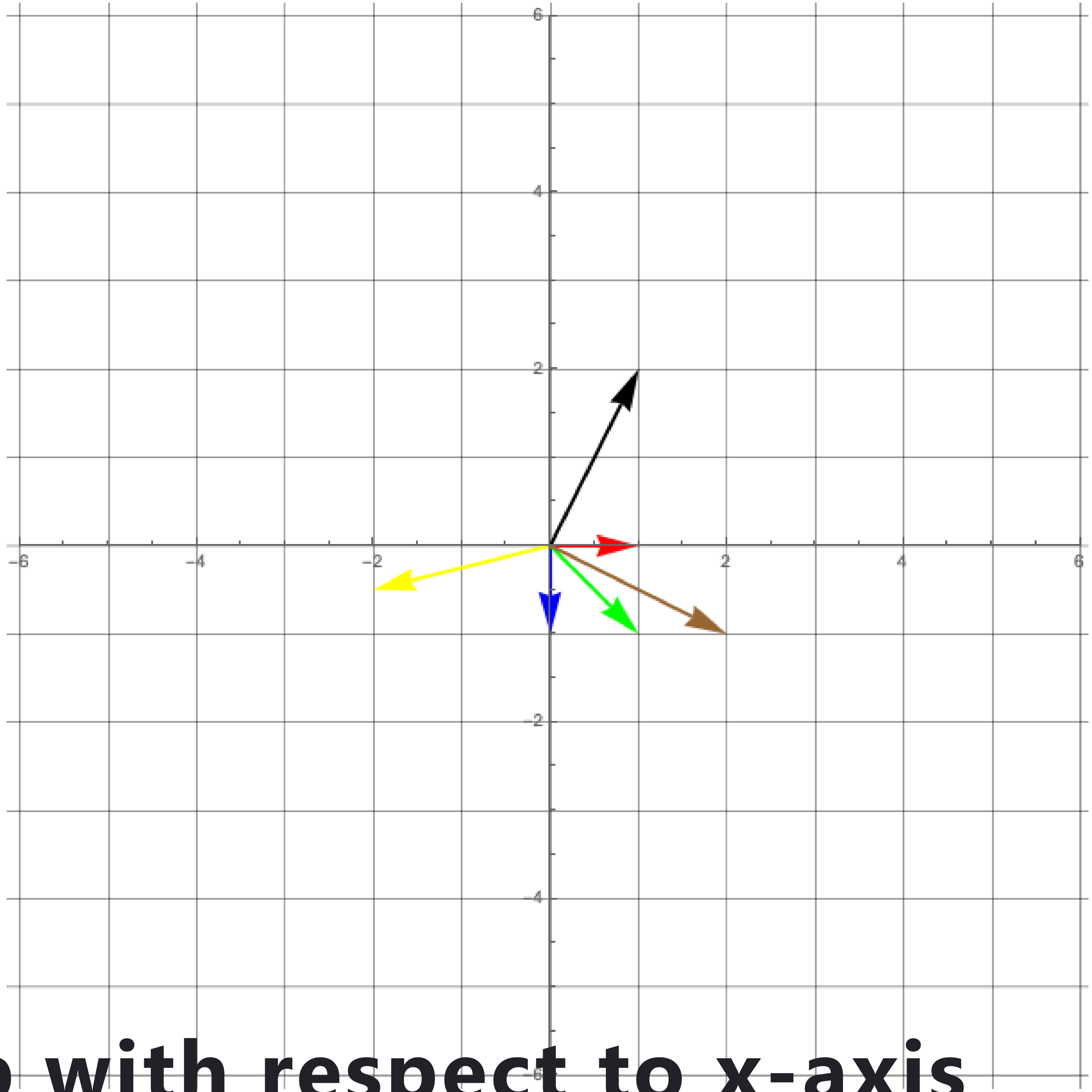


Vertical Shear

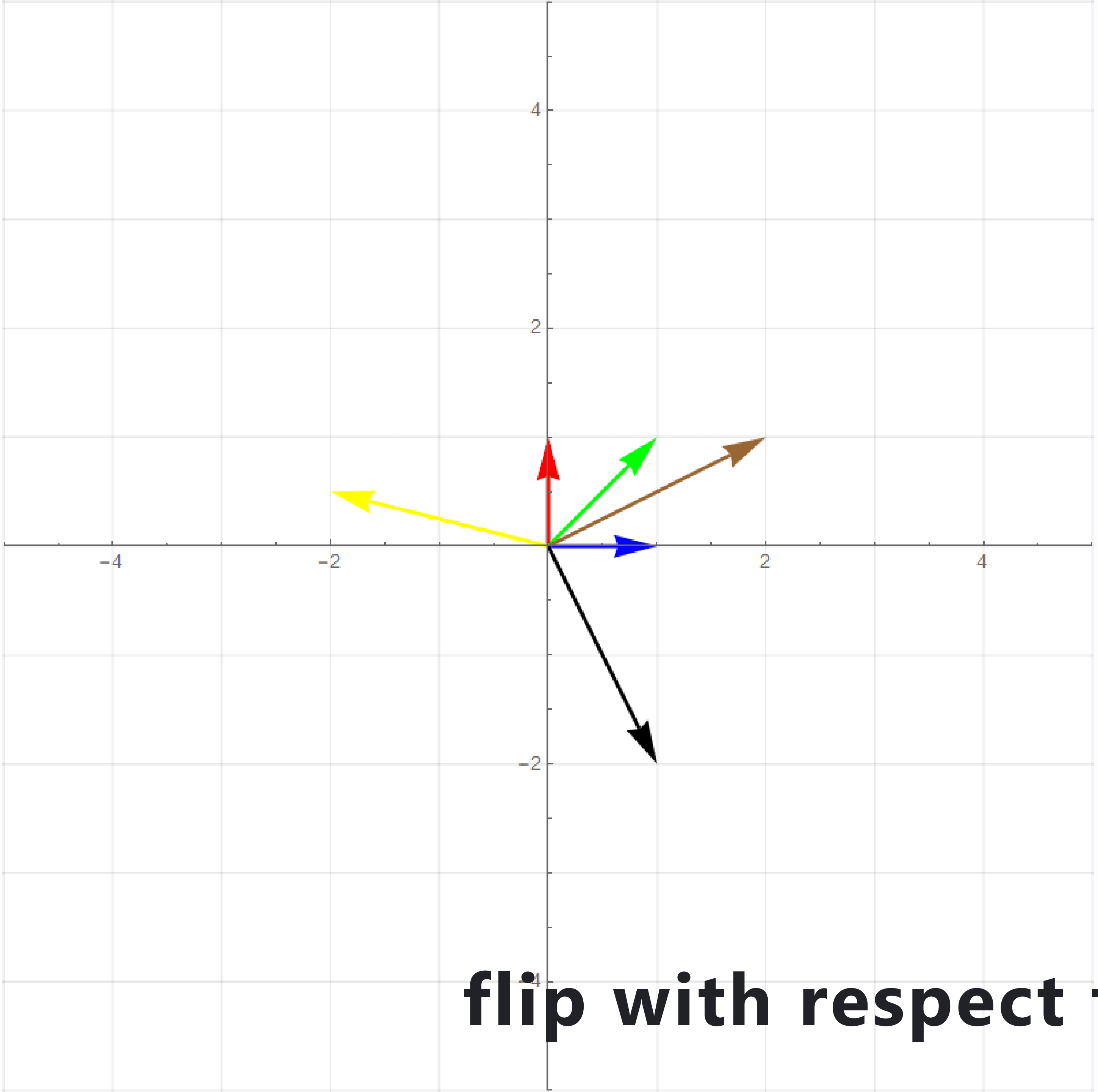
flip with respect to x-axis

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$





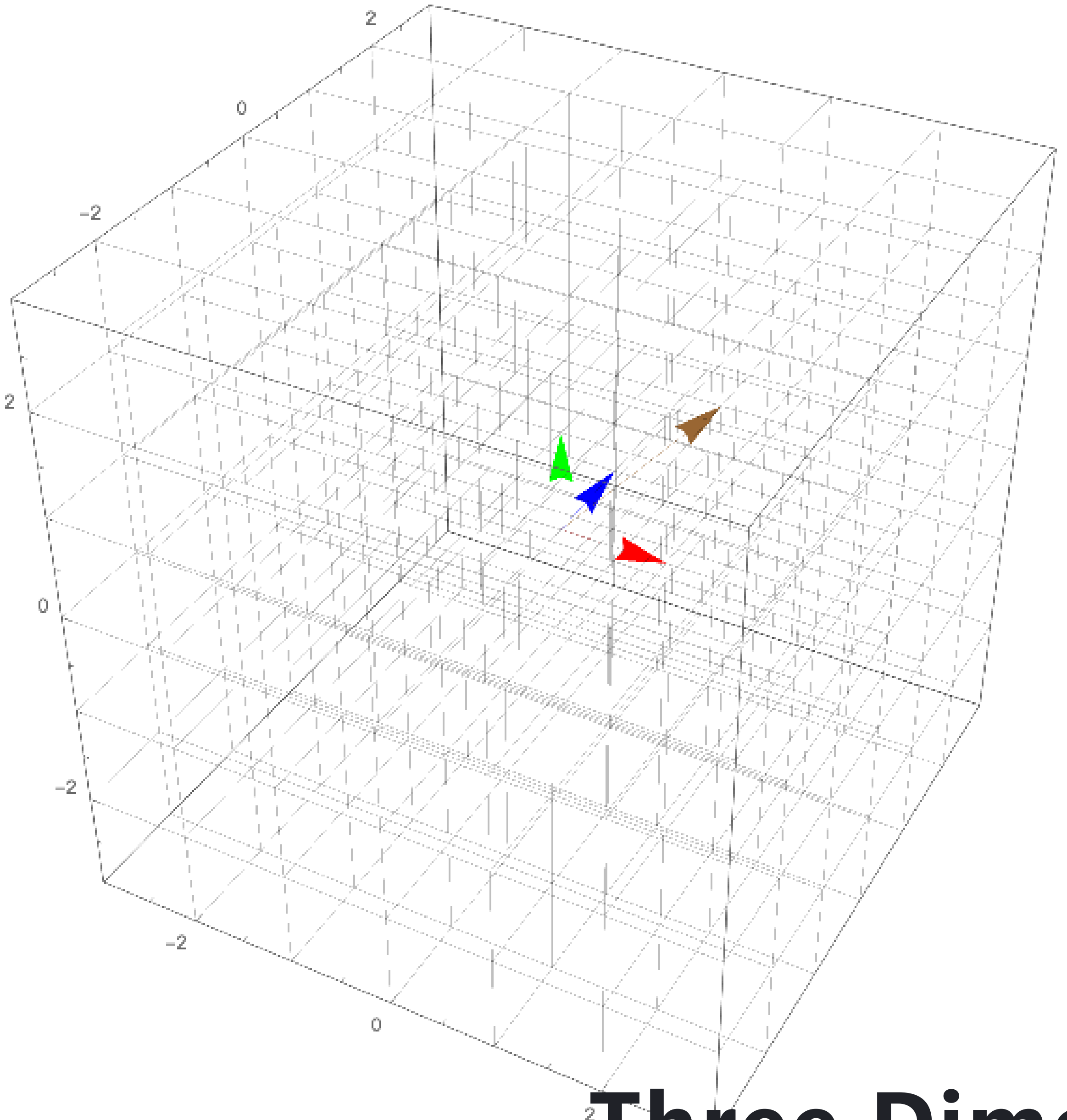
flip with respect to x-axis

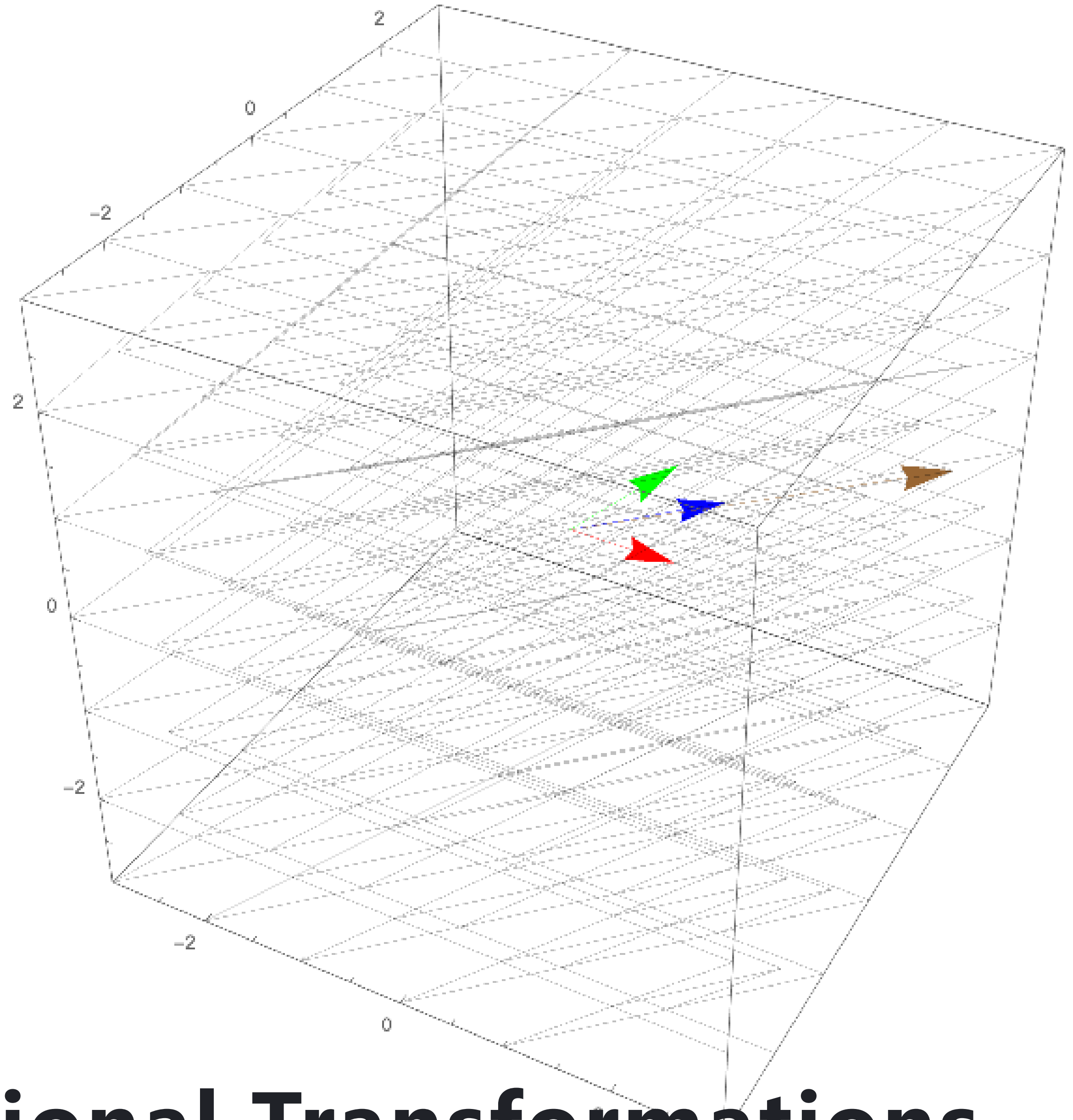


flip with respect to x-axis

Three Dimensional Transformations

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

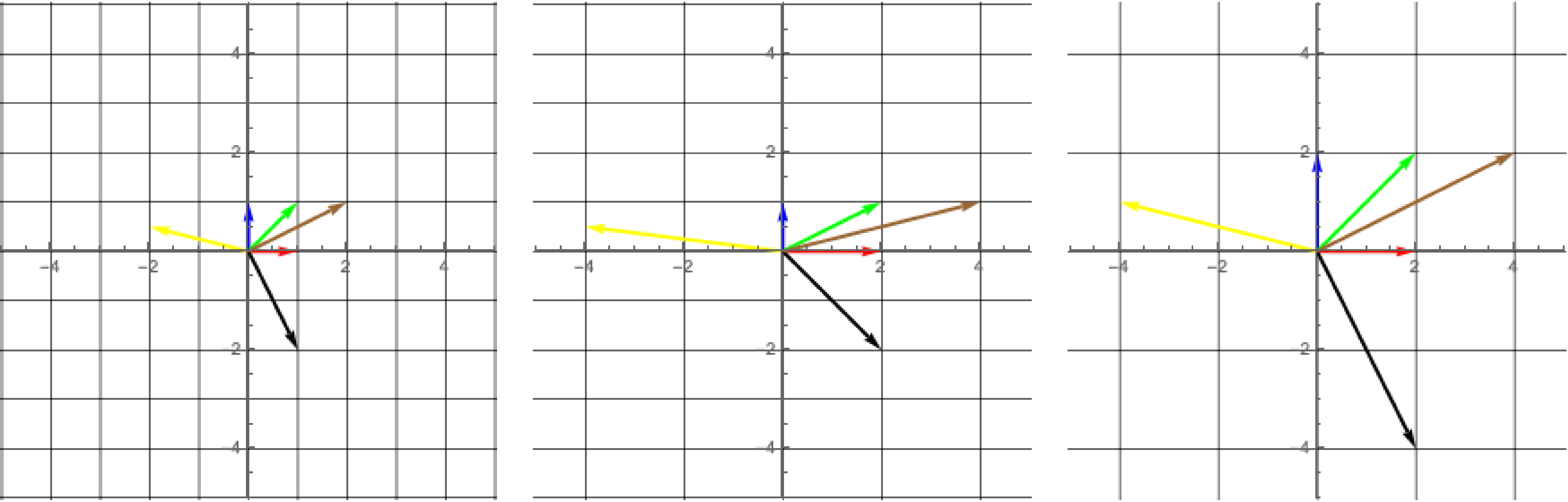




Three Dimensional Transformations

Composing transformations

- You can combine transformations by performing transformations in a **sequence**
- For example, you can **combine** a horizontal stretch with a vertical stretch which can be defined as a new transformation.



Composing transformations

Composing transformations

Since you are applying one transformation after the other, the overall transformation can be defined as a **composition** of the horizontal stretch transformation inside a vertical stretch transformation.

Composing transformations

$$f(\vec{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$$

$$g(\vec{v}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{v}$$

$$g(f(\vec{v})) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{v} \right)$$

Composing transformations

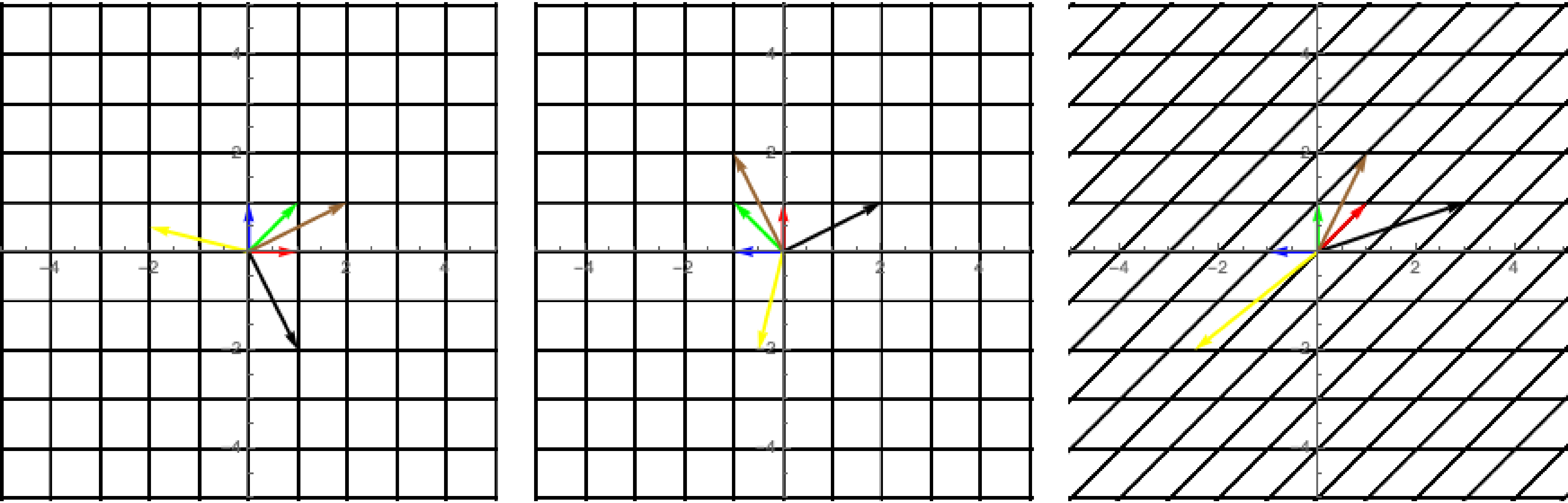
$$g(f(\vec{v})) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{v} \right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{v}$$

Composing transformations

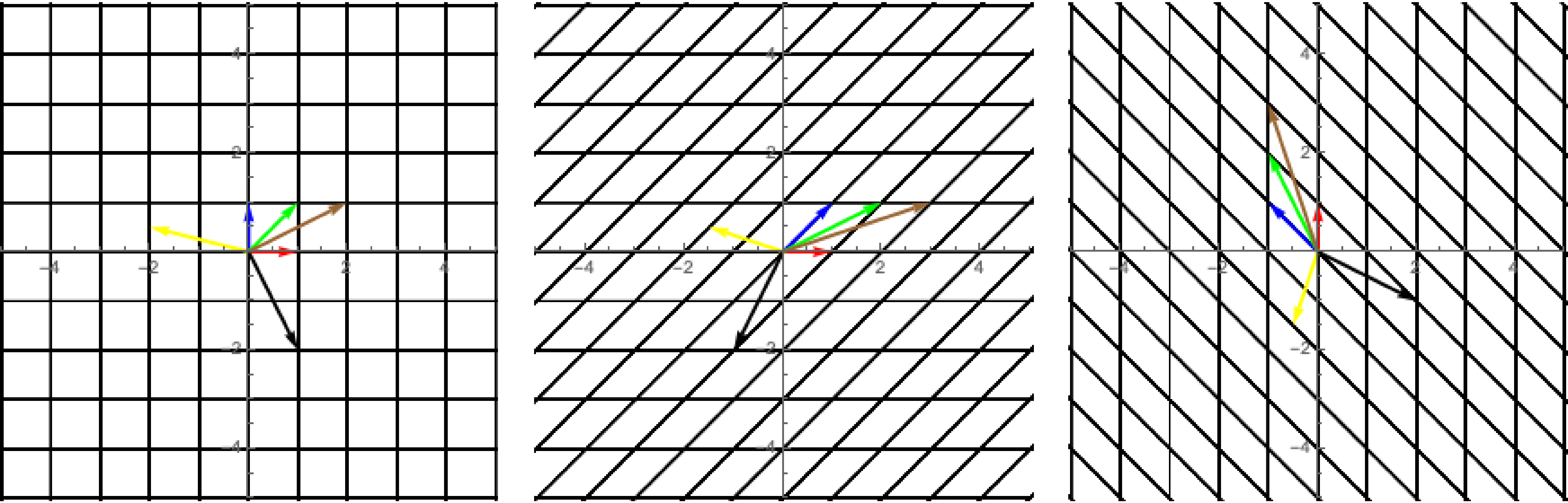
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Composing transformations

$$f(\vec{v}) = T_1\vec{v}$$
$$g(\vec{v}) = T_2\vec{v}$$
$$g(f(\vec{v})) = T_2T_1\vec{v}$$



Composing transformations



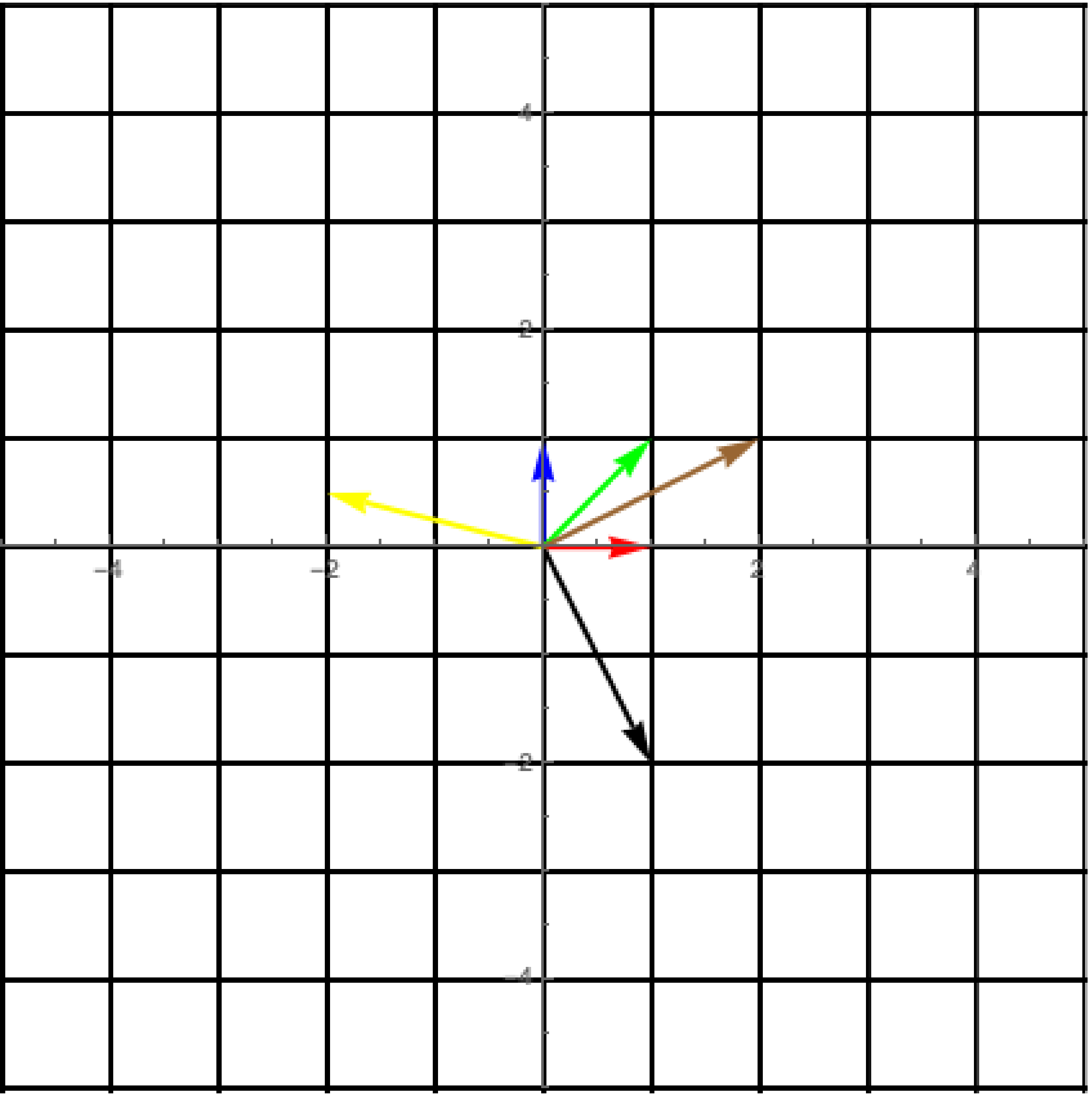
Composing transformations

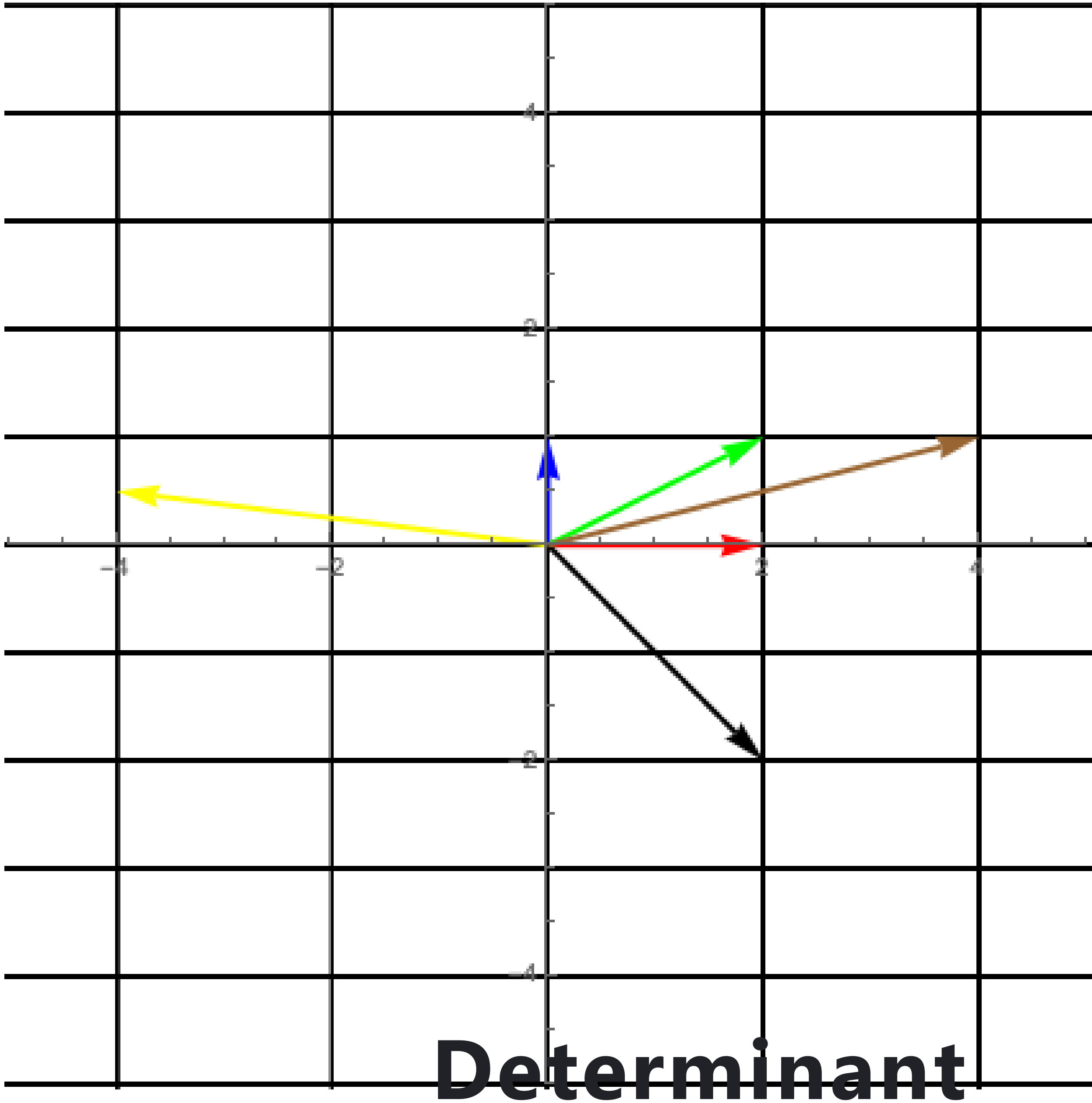
Determinant

One of the important things you can study about a given linear transformation is how it generally **stretches** or **compresses** the space

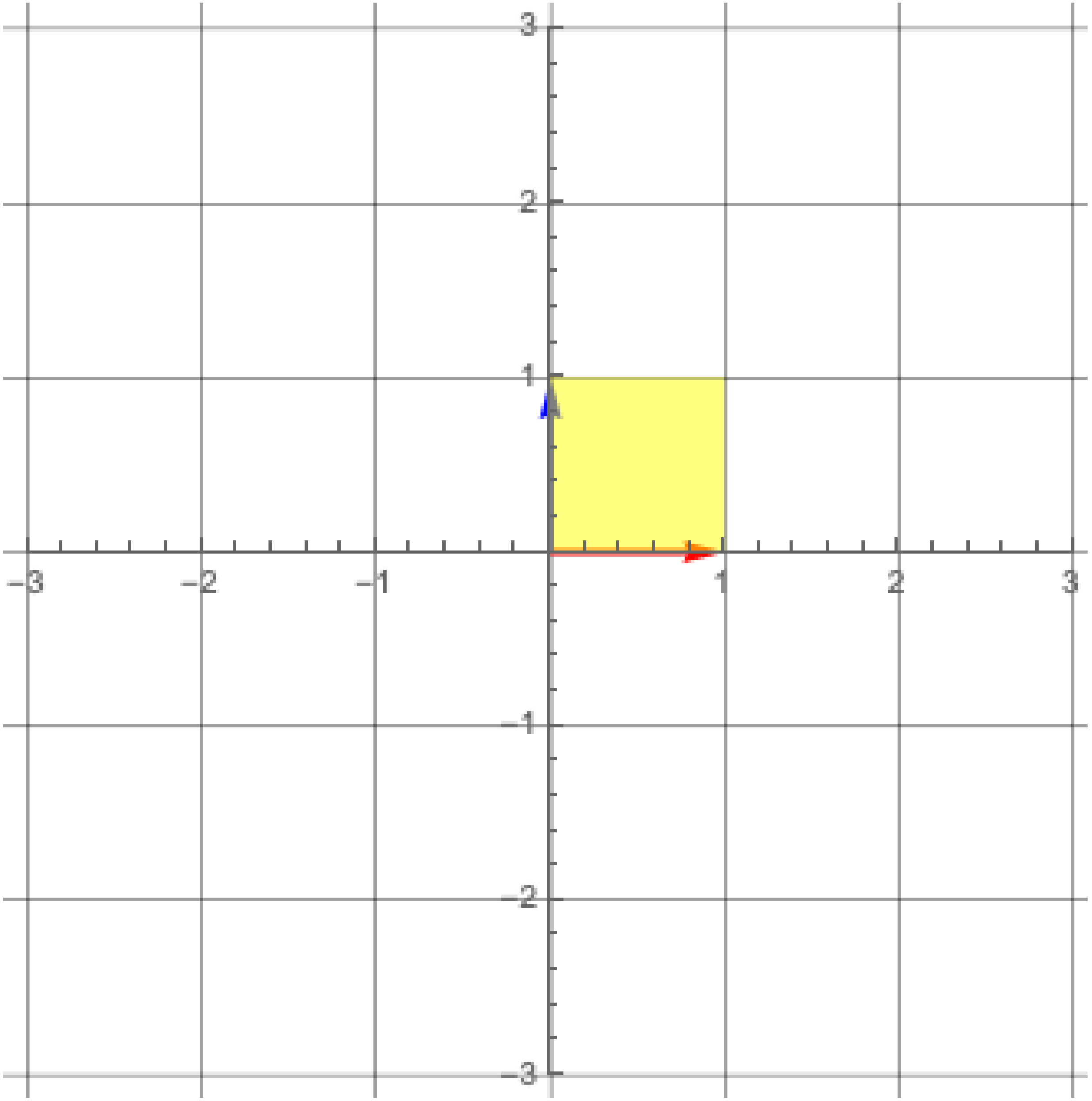
Determinant

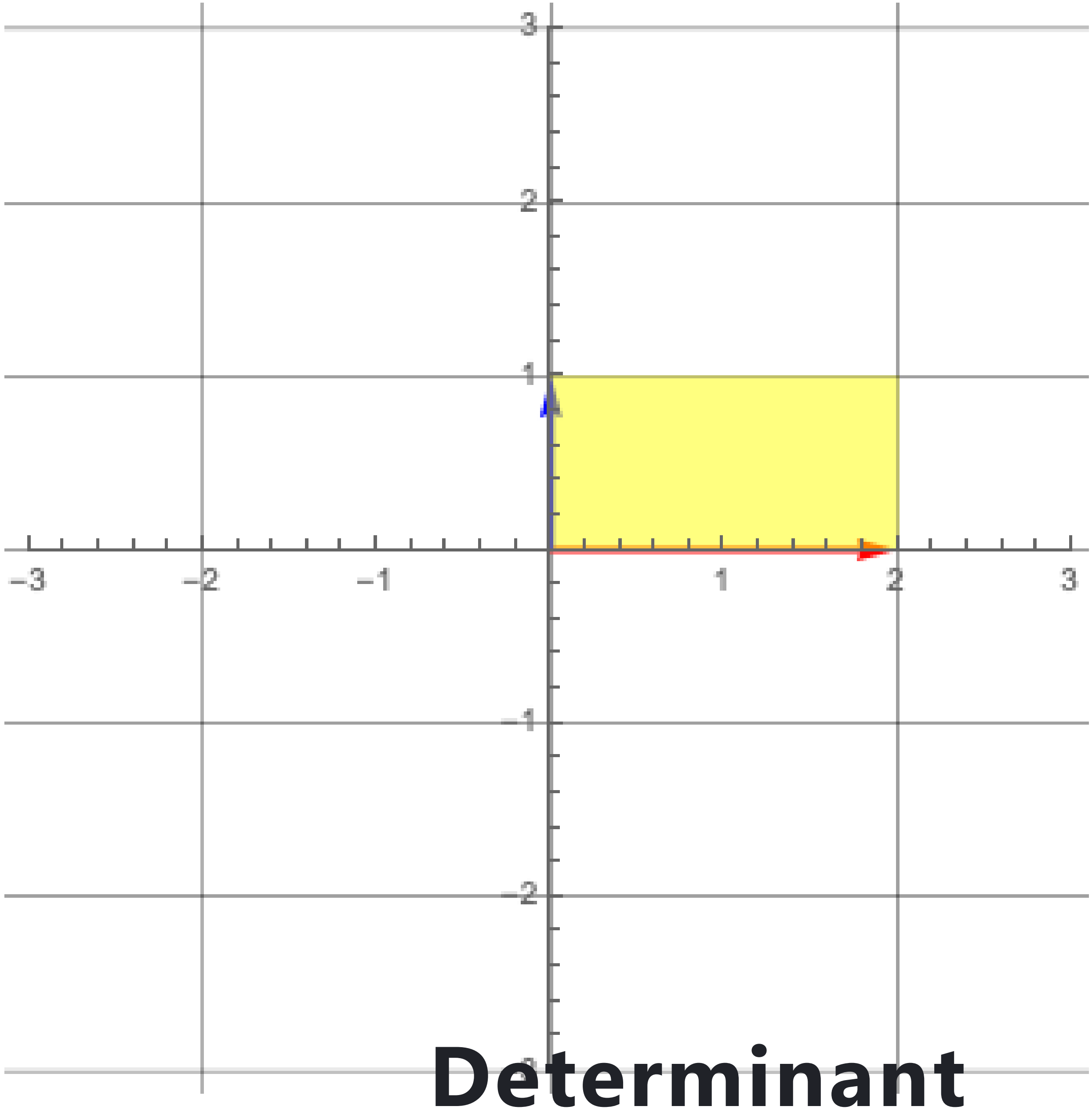
$$T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



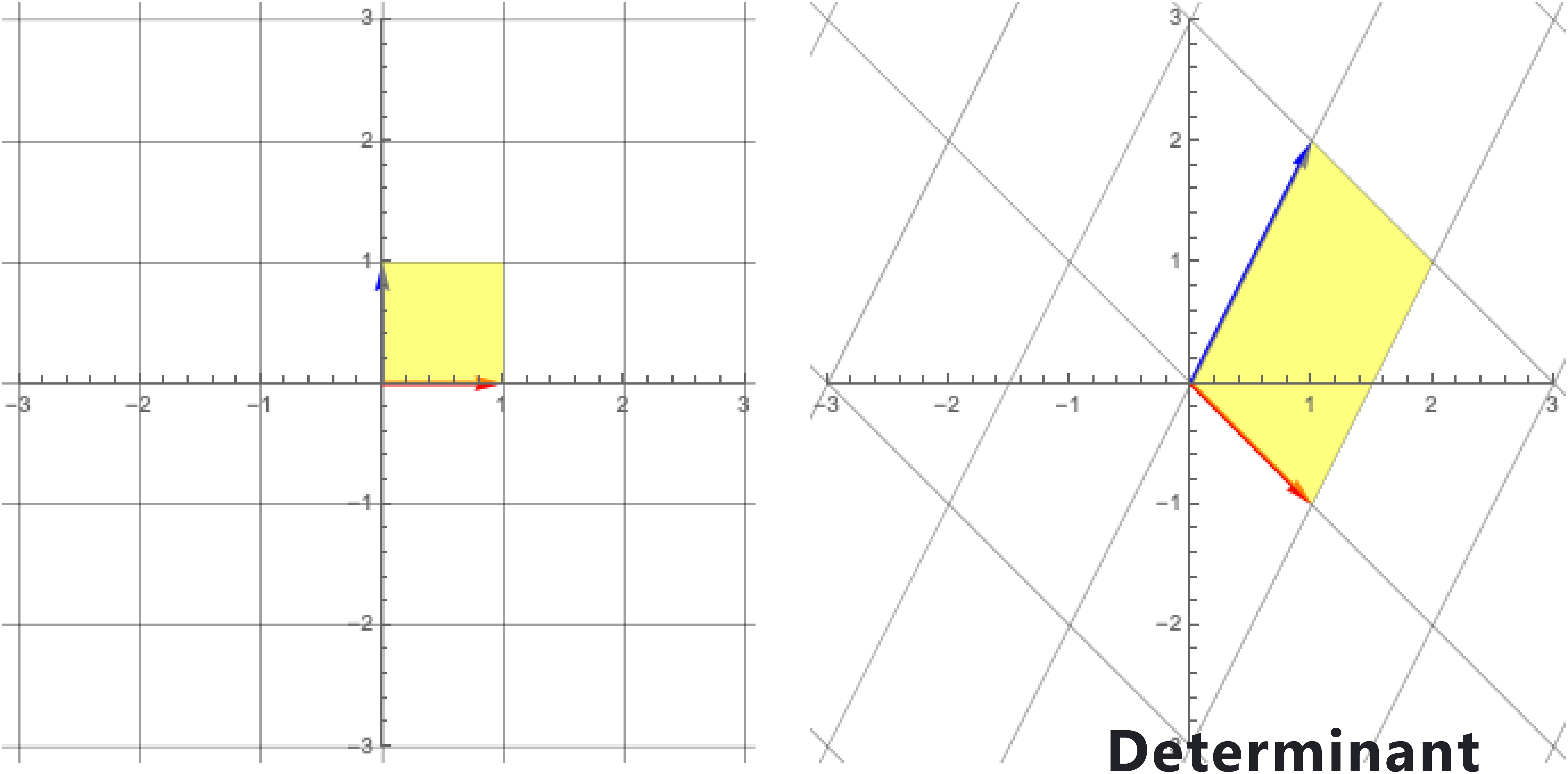


Determinant





Determinant



Determinant

Determinant

The factor stretching/compression of the entire space that occurs during a transformation has a special name called the **determinant** of a transformation

Determinant

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

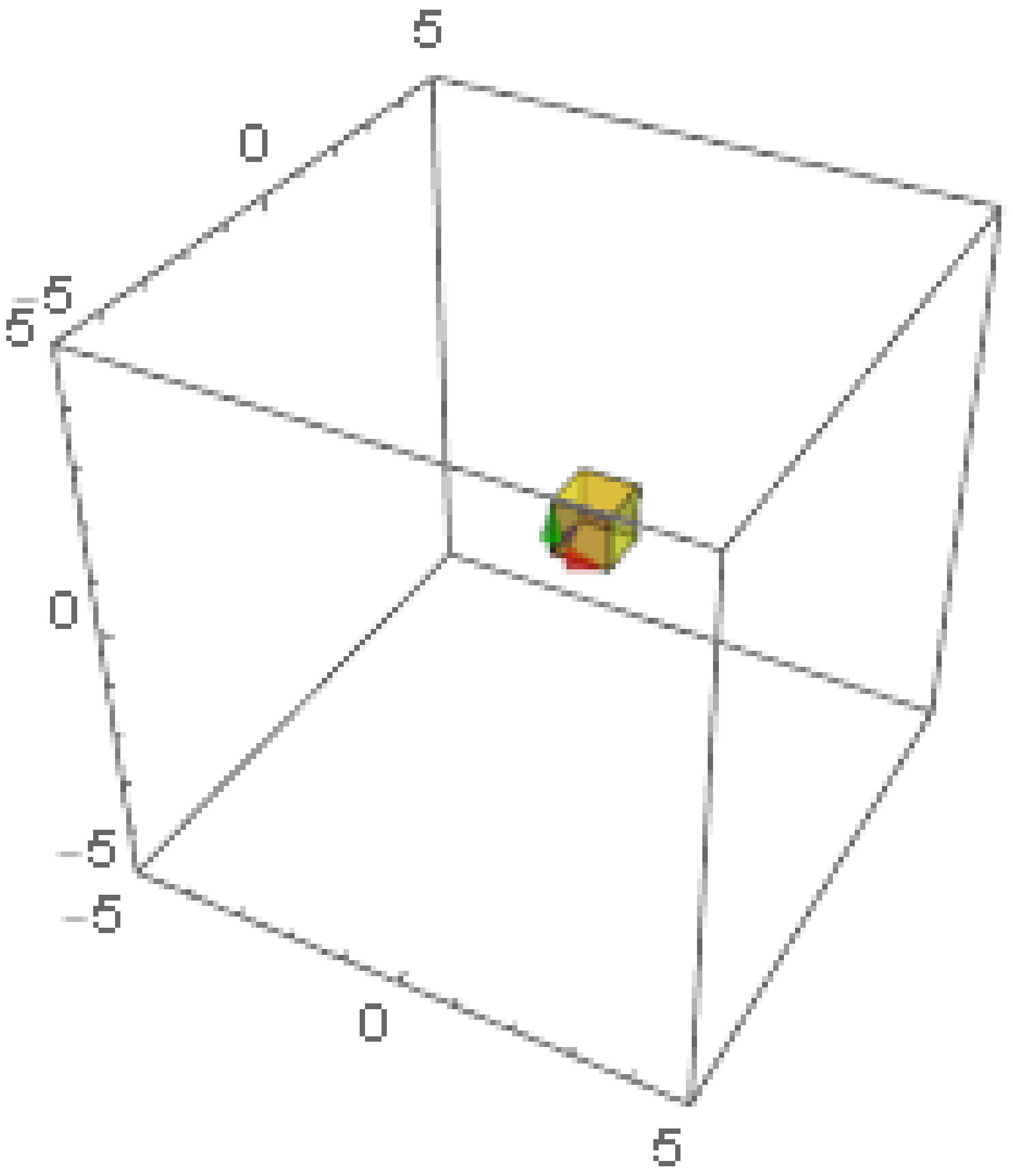
$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

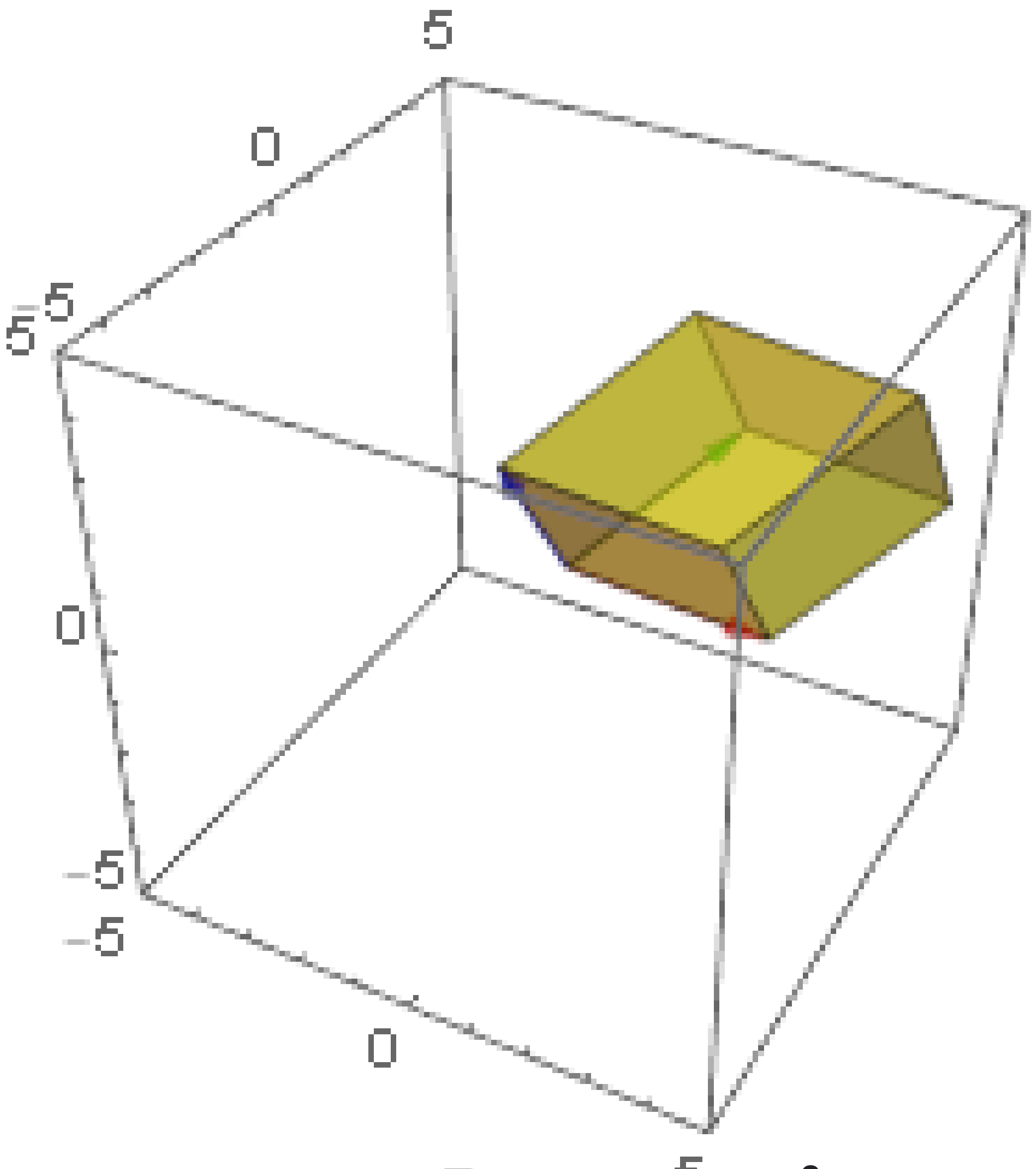
Determinant

This formula can be derived by calculating the **area** of the resulting yellow parallelogram, which is the transformed version of the yellow square

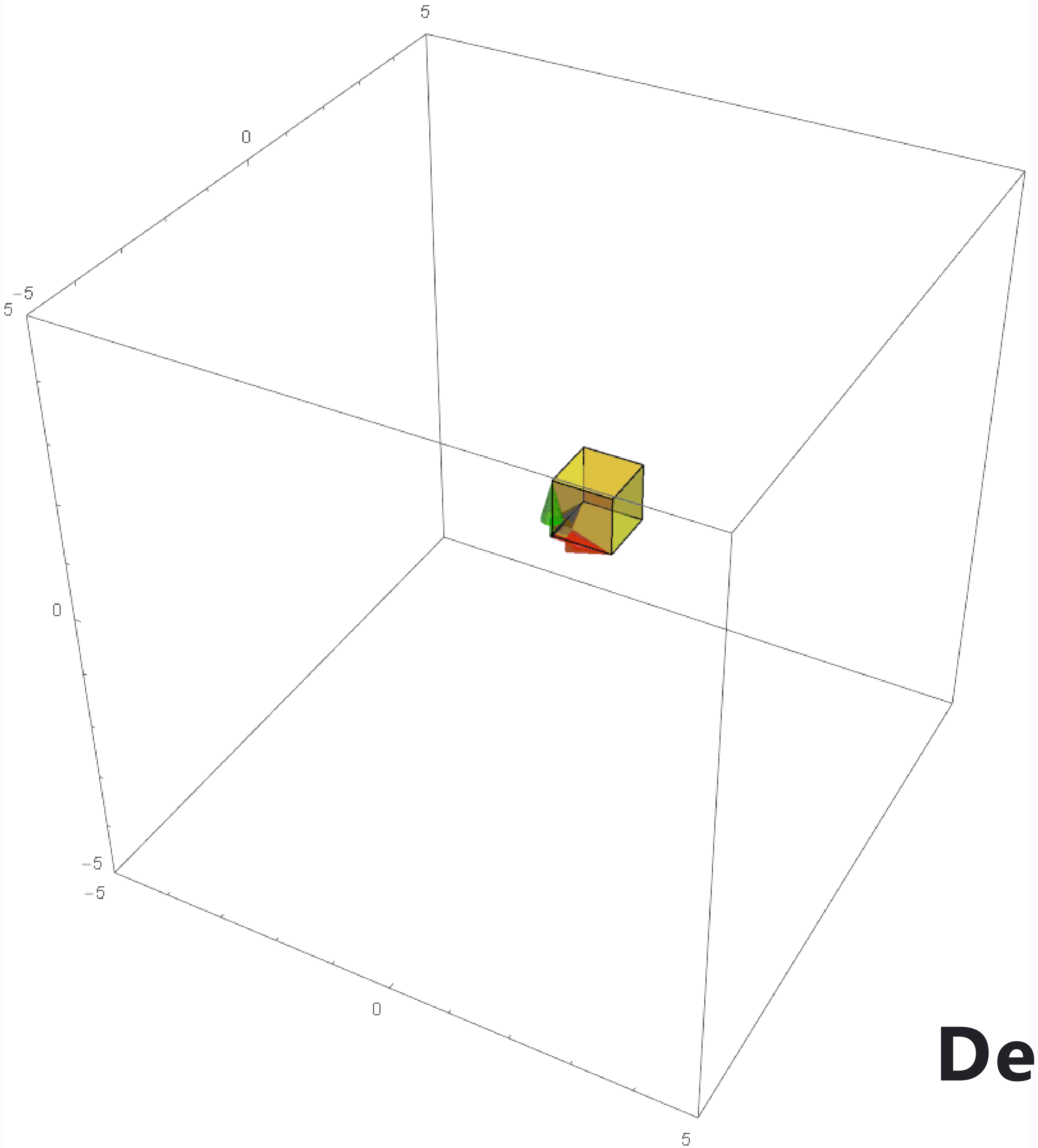
Determinant

The determinant of a 3 dimensional transformation is the factor of stretching/compression of the **volume** of the unit cube to the parallelepiped:

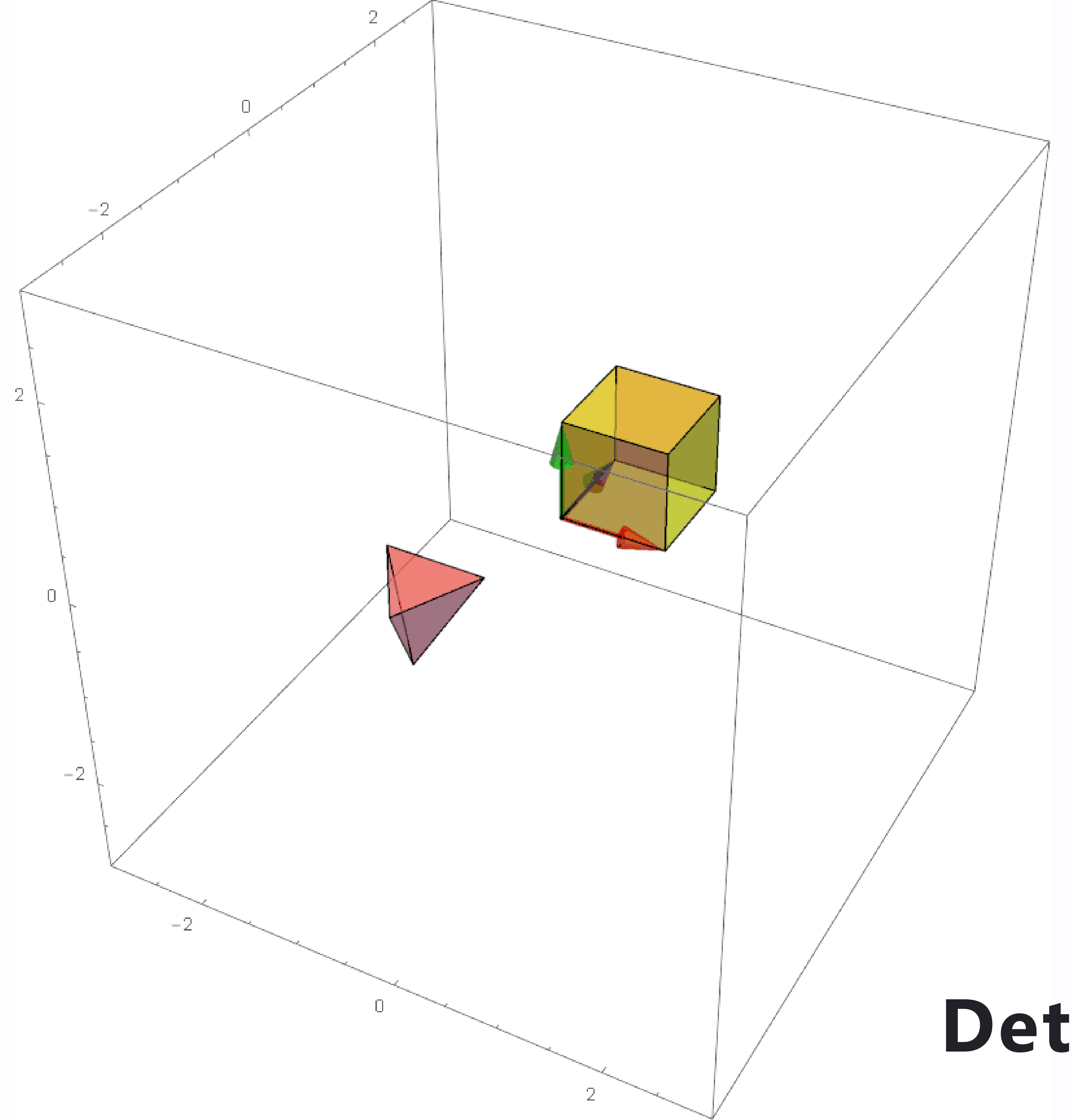




Determinant



Determinant

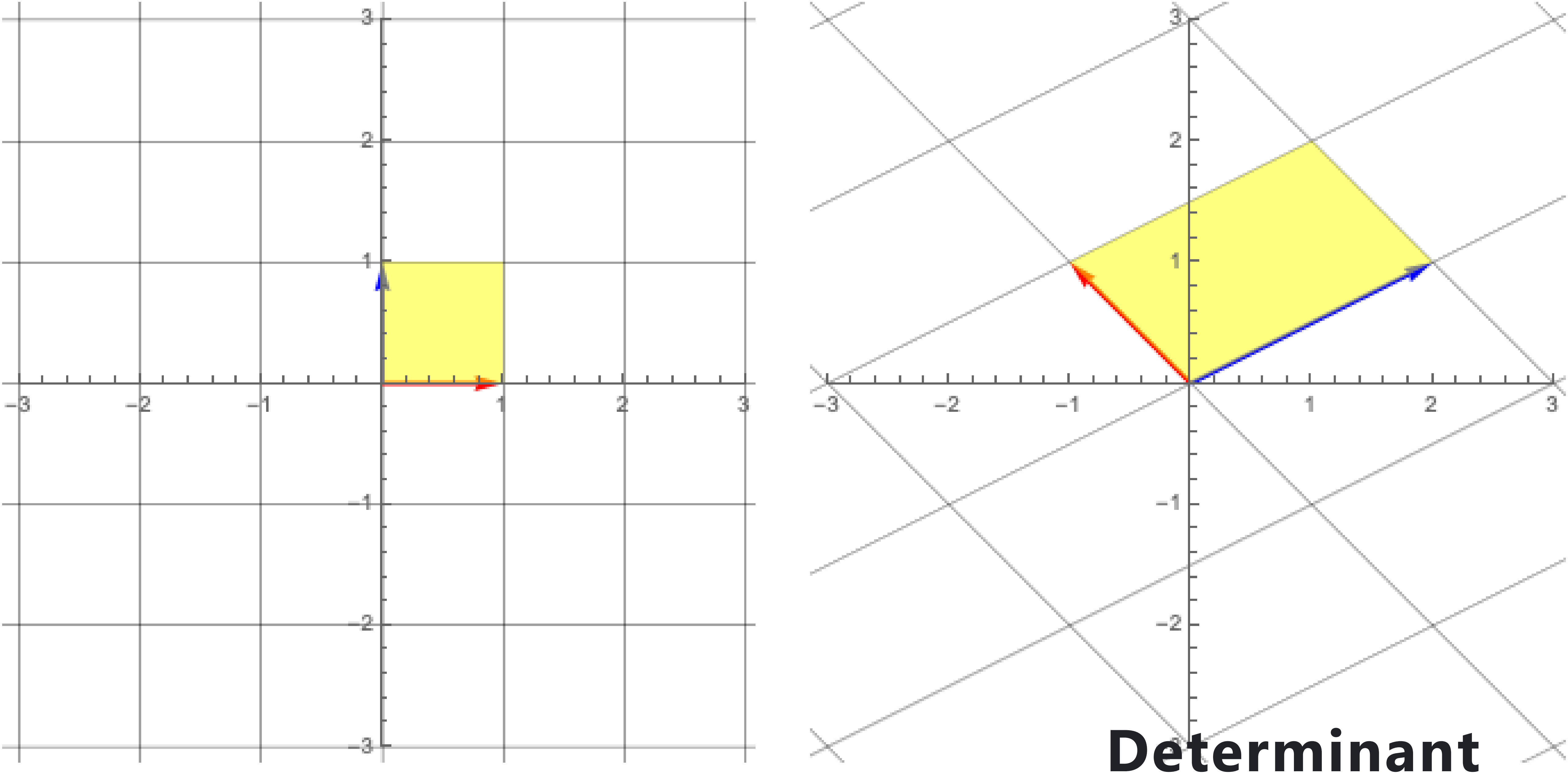


Determinant

Determinant

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right)$$

$$= a\det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$



Determinant

Determinant

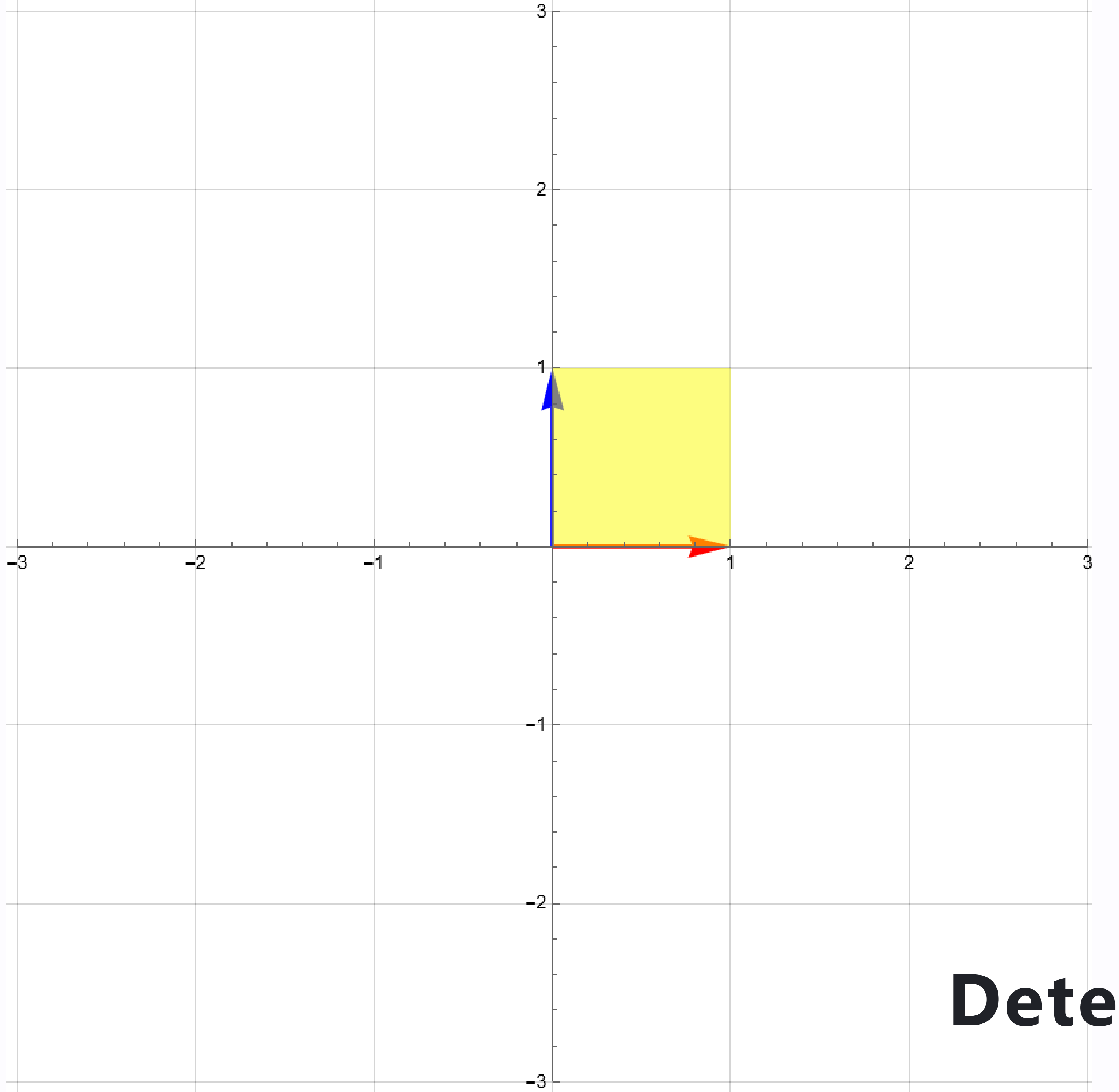
$$\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}$$

Determinant

$$\det\left(\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}\right) = -1(1) - 2(1) = -3$$

Determinant

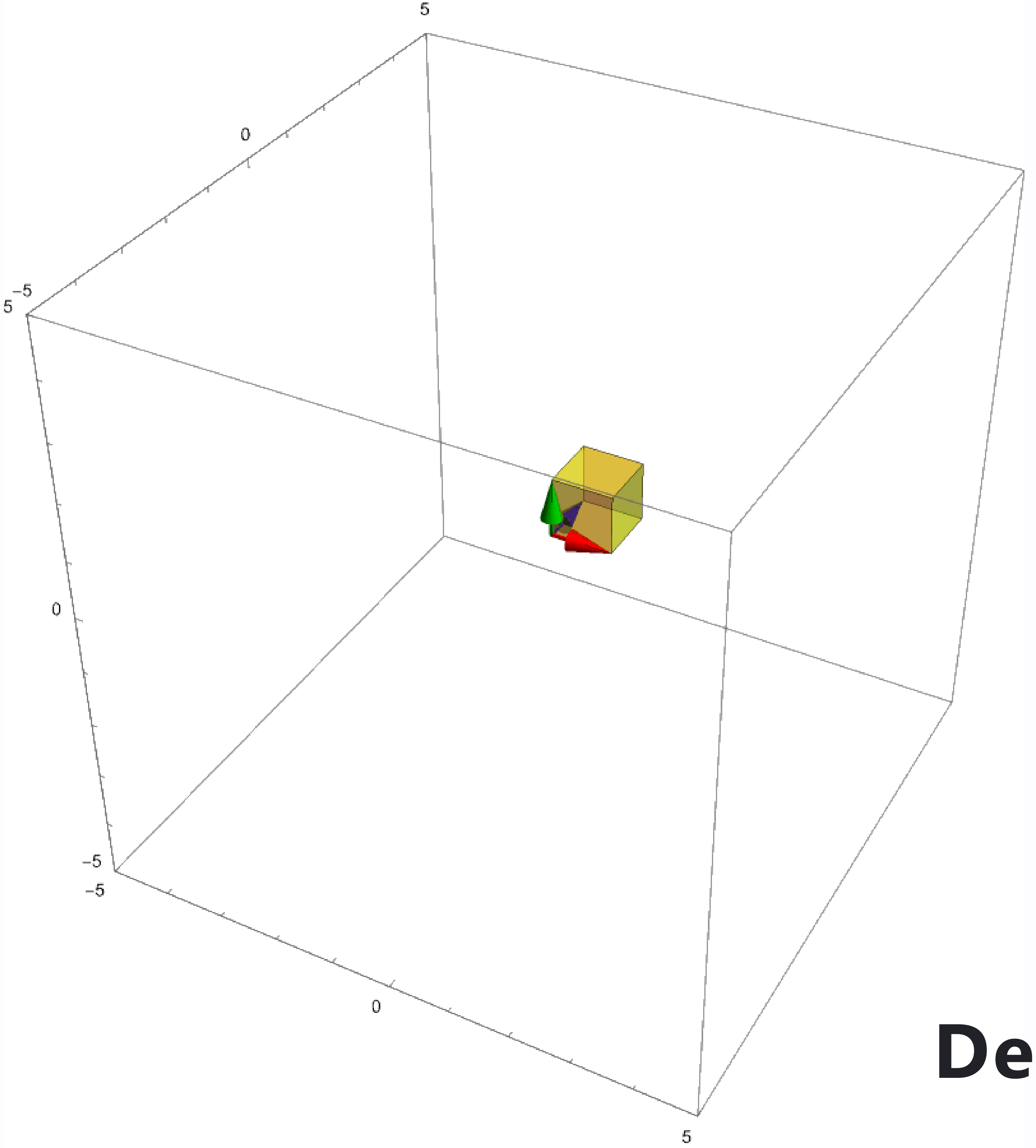
- This means that the factor of stretching/compression for the transformation is -3
- The **sign** of the determinant has meaning
- If the determinant is negative it means that the space is **compressed beyond 0** to the point that the space is flipped.



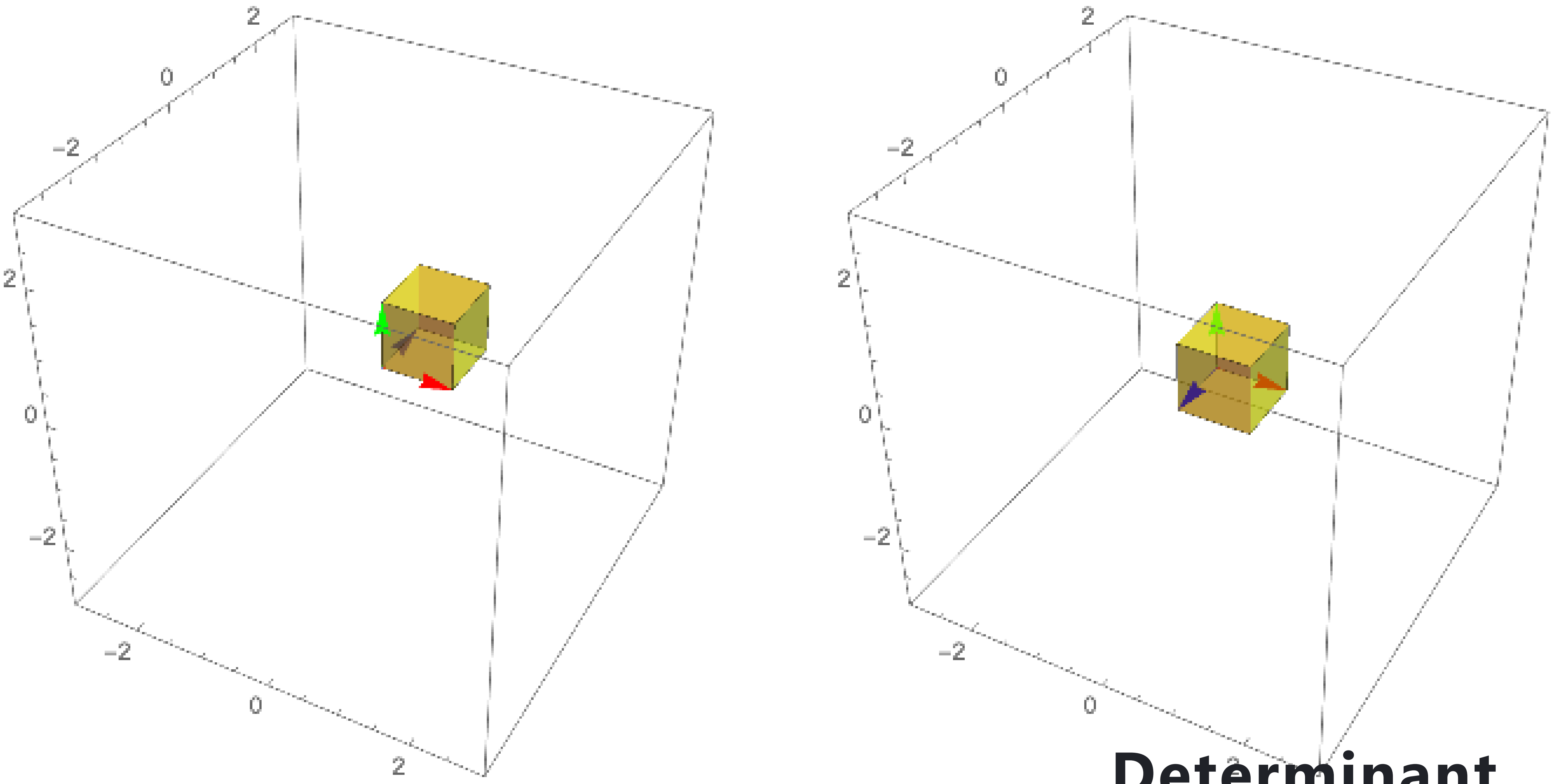
Determinant

Determinant

Flipping space in the 3 dimensions means that the basis vectors cannot follow **right hand rule**.



Determinant



Determinant

Inverse Matrices

$$2x + 0y = 4$$
$$0x + y = 3$$

Inverse Matrices

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Inverse Matrices

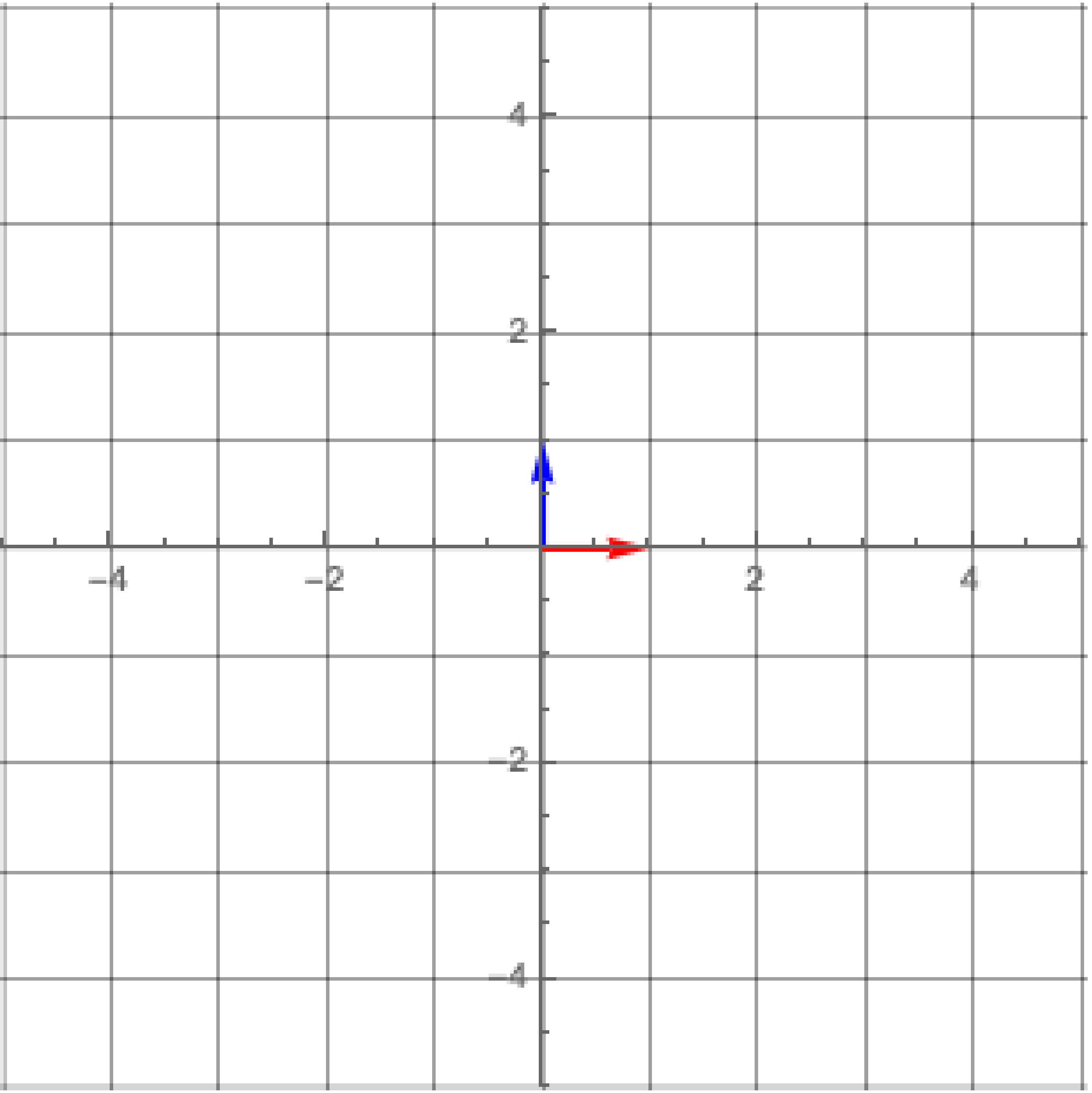
You can think of the **coefficients** for x and y as the numbers that make up the **transformation matrix**, T

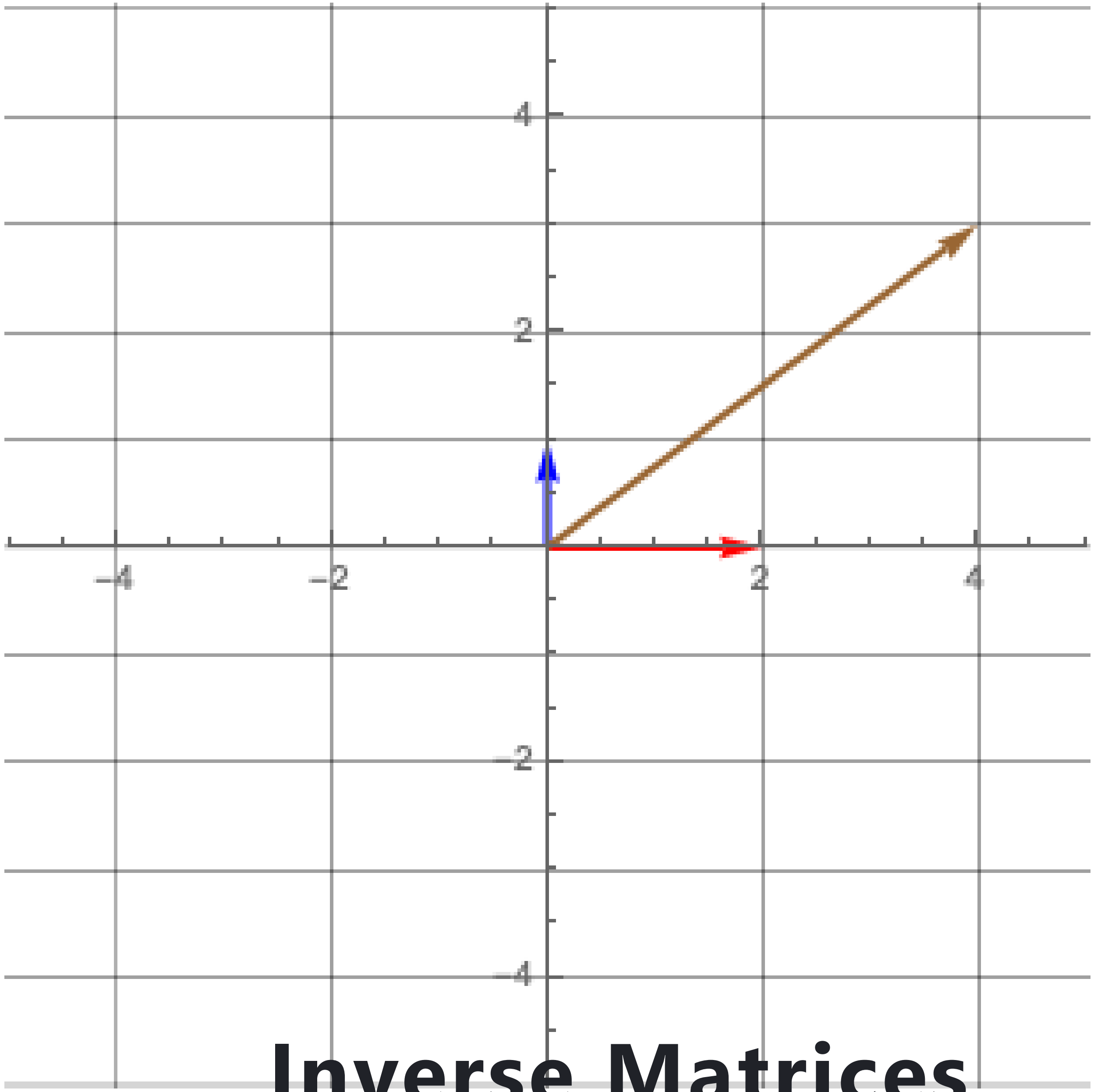
Inverse Matrices

$$T\vec{v} = \vec{v}'$$

Inverse Matrices

To find the solutions of this system of linear equations, you need to look for the **original value** of $\vec{v}'$ **before** the transformation





Inverse Matrices

Inverse Matrices

- To do this you need to make use of a special matrix related to the transformation matrix T, called the **inverse matrix**, T^{-1}
- This matrix serves as the inverse of the transformation T such that, **combining** T **and** T^{-1} results to the original locations of the basis, or the identity matrix

Inverse Matrices

$$T^{-1}T = I$$

Inverse Matrices

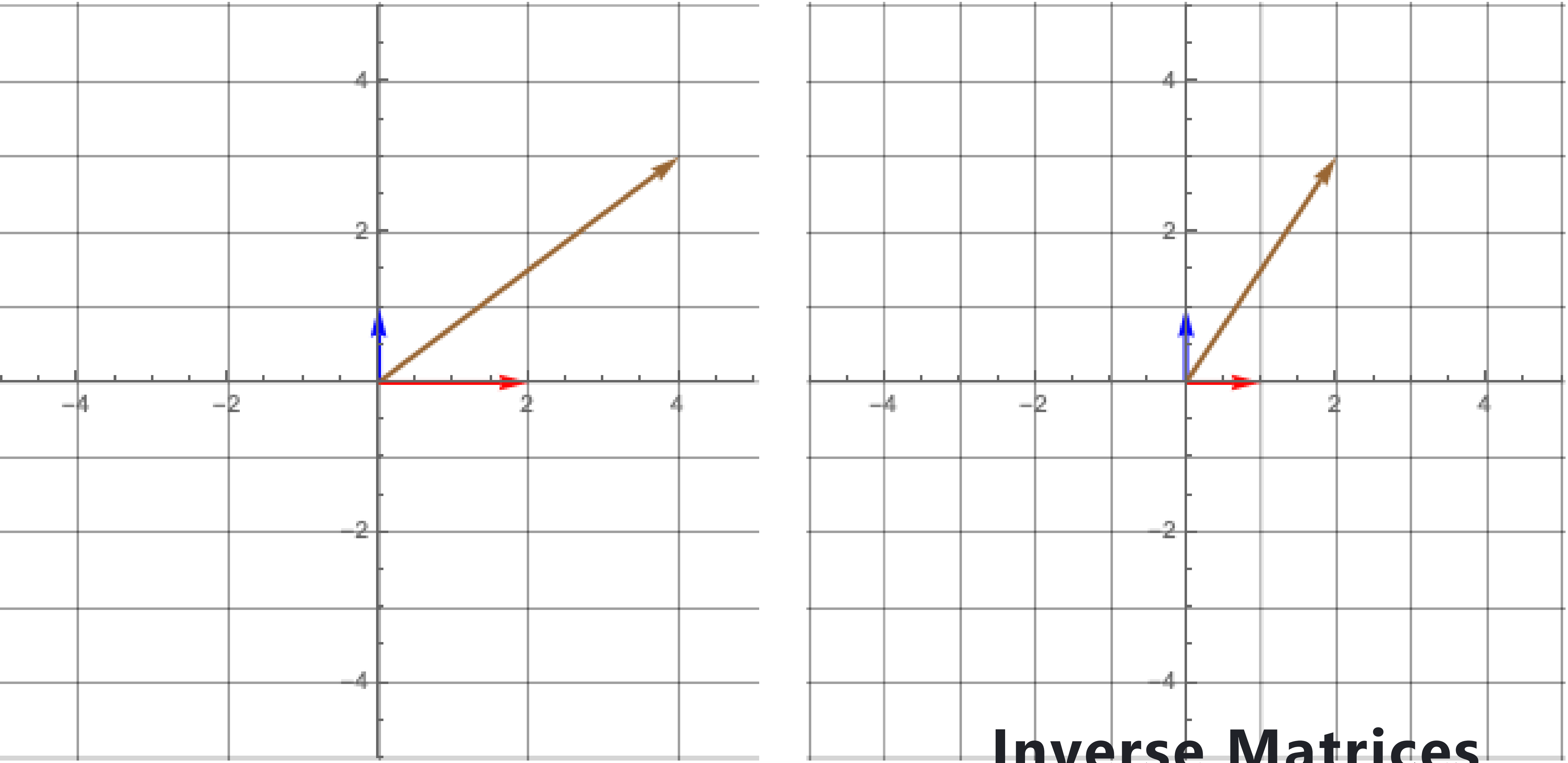
$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse Matrices

$$
\begin{aligned}
T^{-1}T\vec{v} &= T^{-1}\vec{v}' \\
\vec{v} &= T^{-1}\vec{v}'
\end{aligned}
$$

Inverse Matrices

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$



Inverse Matrices

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

Finding the inverse matrix

1. You can change the values of a row by multiplying all of the numbers in the row by a constant
2. You can change rows by adding the elements of other rows to it.

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} (3) & 0 & 2 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \quad (R_1 \to \frac{1}{3} R_1)$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -2 & \frac{4}{3} & -\frac{1}{3} & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \quad (R_2 \to -R_1$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -2 & \frac{4}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 3 & \frac{8}{3} & \frac{1}{3} & 0 & 1 \end{array}\right] \quad (R_3 \to R_1 + R$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 3 & \frac{8}{3} & \frac{1}{3} & 0 & 1 \end{array}\right] \quad (R_2 \to -\frac{1}{2}R_2)$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{14}{3} & -\frac{1}{6} & \frac{3}{2} & 1 \end{array}\right] \quad (R_3 \to -3$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{array}\right] \quad (R_3 \to \frac{3}{14} R$$

Finding the inverse matrix

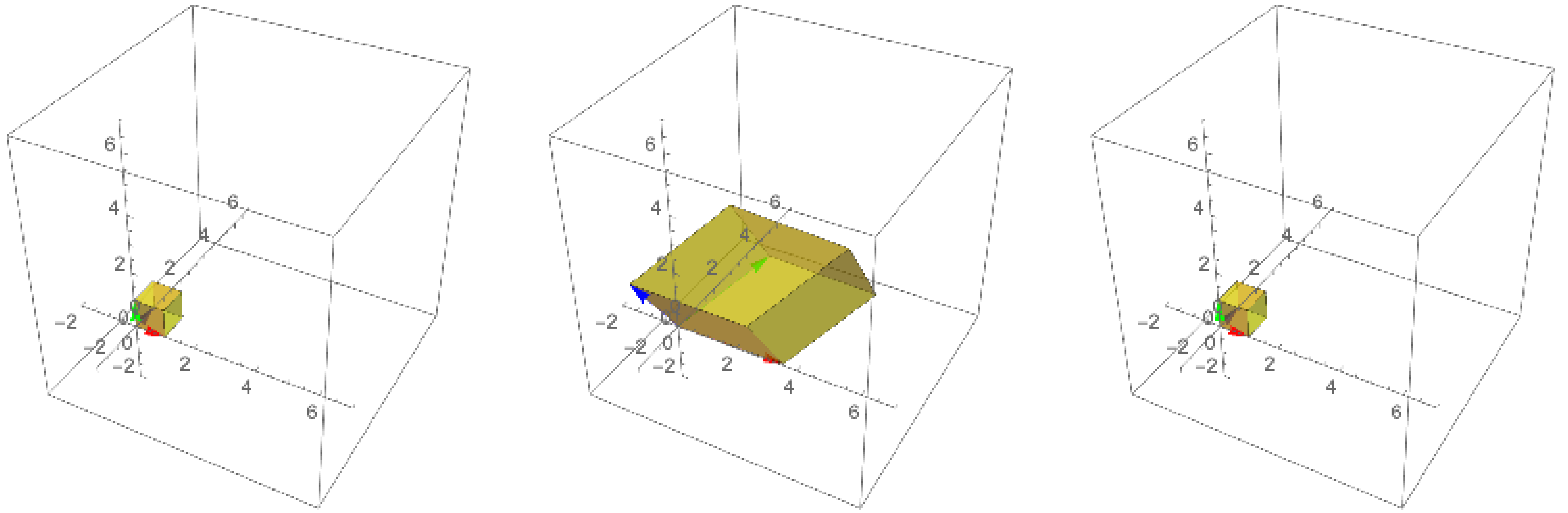
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{14} & -\frac{3}{14} & -\frac{1}{7} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{array}\right] \quad (R_1$$

Finding the inverse matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{14} & -\frac{3}{14} & -\frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & -\frac{2}{7} & \frac{1}{7} \\ 0 & 0 & 1 & -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{array}\right] \quad (R_2 \to \frac{2}{3}$$

Finding the inverse matrix

$$\begin{bmatrix} \frac{5}{14} & -\frac{3}{14} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} & \frac{1}{7} \\ -\frac{1}{28} & \frac{9}{28} & \frac{3}{14} \end{bmatrix}$$



Finding the inverse matrix

Singular Matrices

1. There is one solution
2. There are infinitely many solutions
3. There are no solutions

Singular Matrices

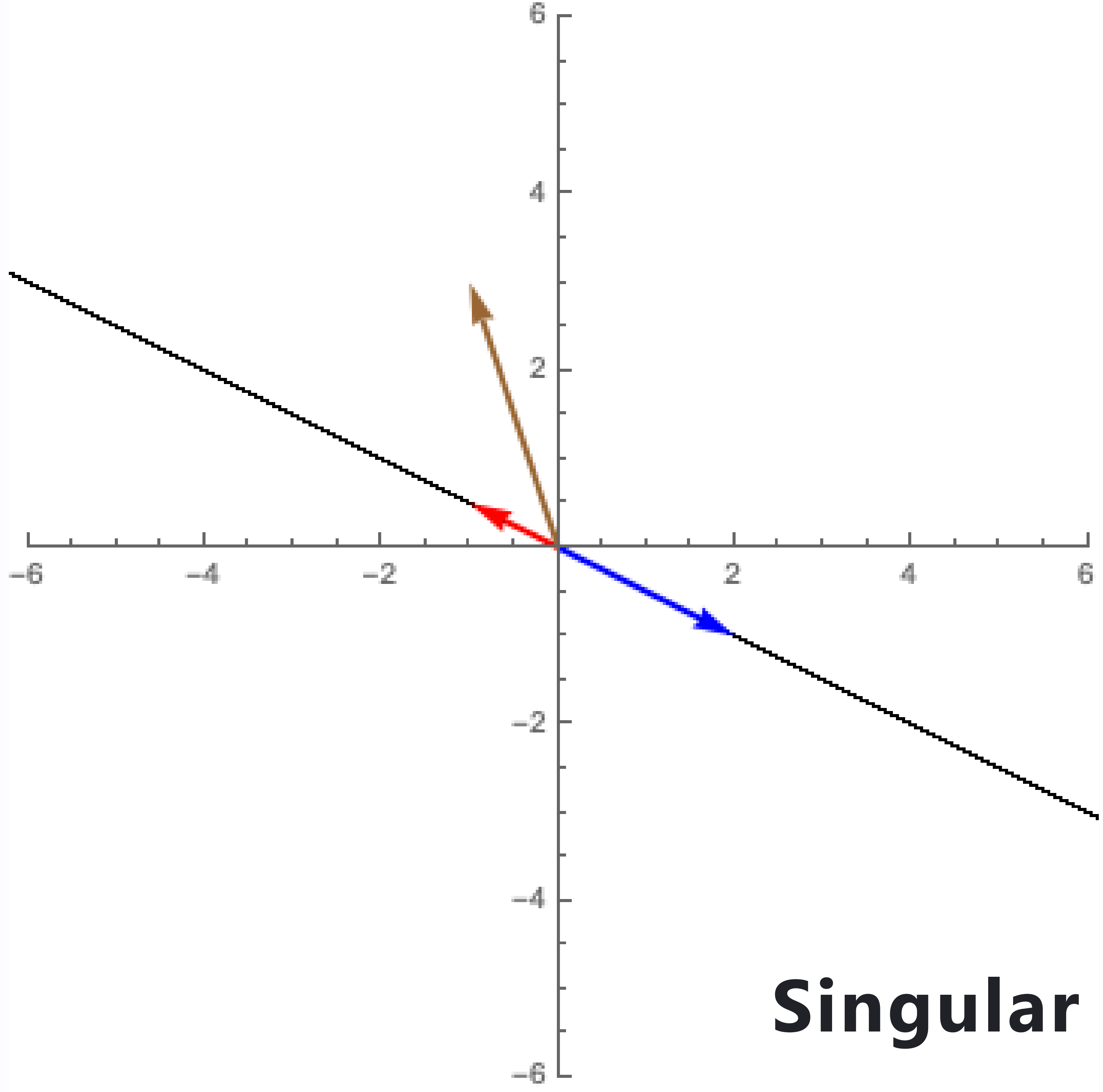
These happen when the span of the transformed basis vectors (also called **column space**) has reduced dimension

Singular Matrices

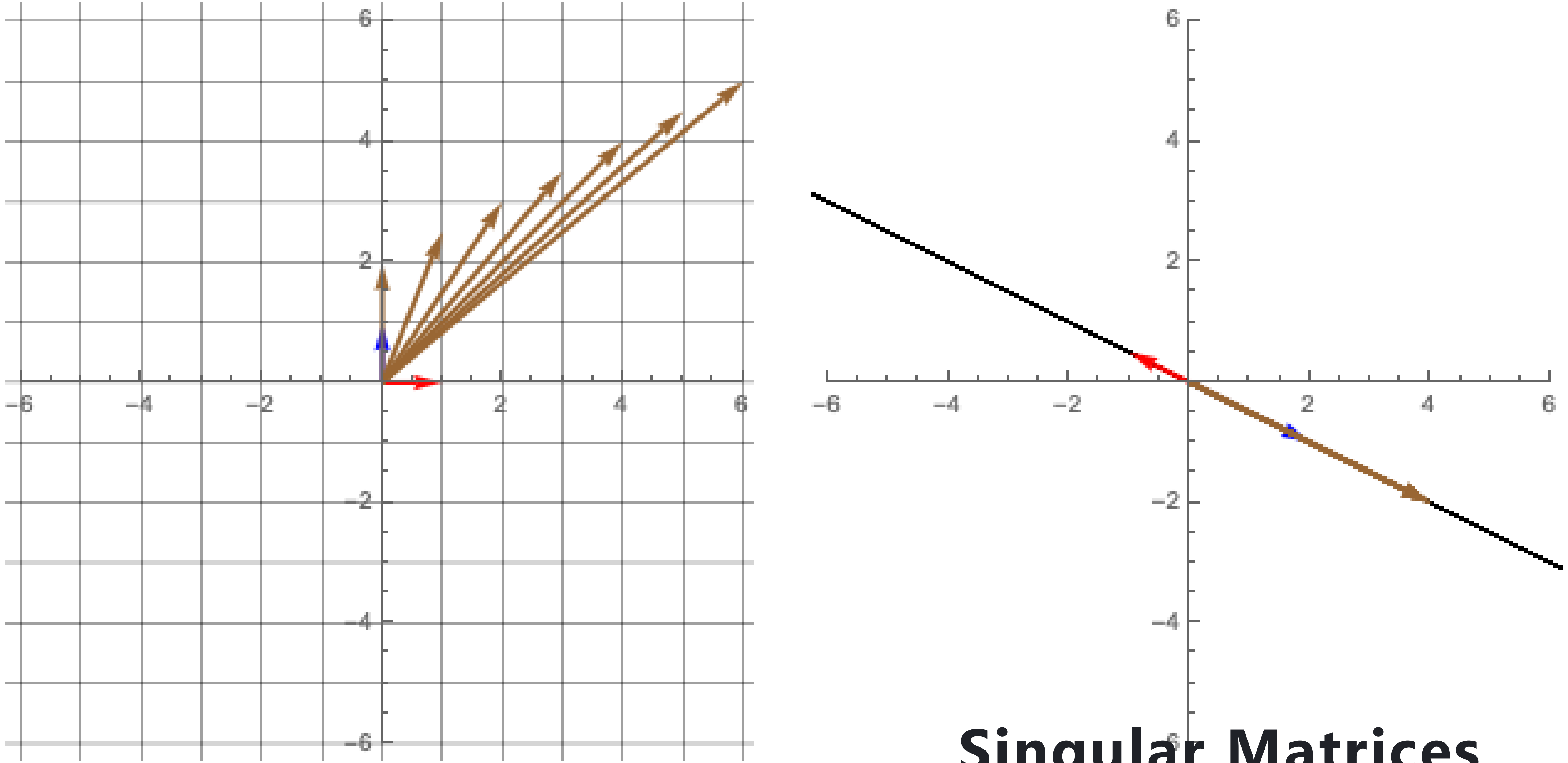
$$T = \begin{bmatrix} -1 & 2 \\ 0.5 & -1 \end{bmatrix}$$

Singular Matrices

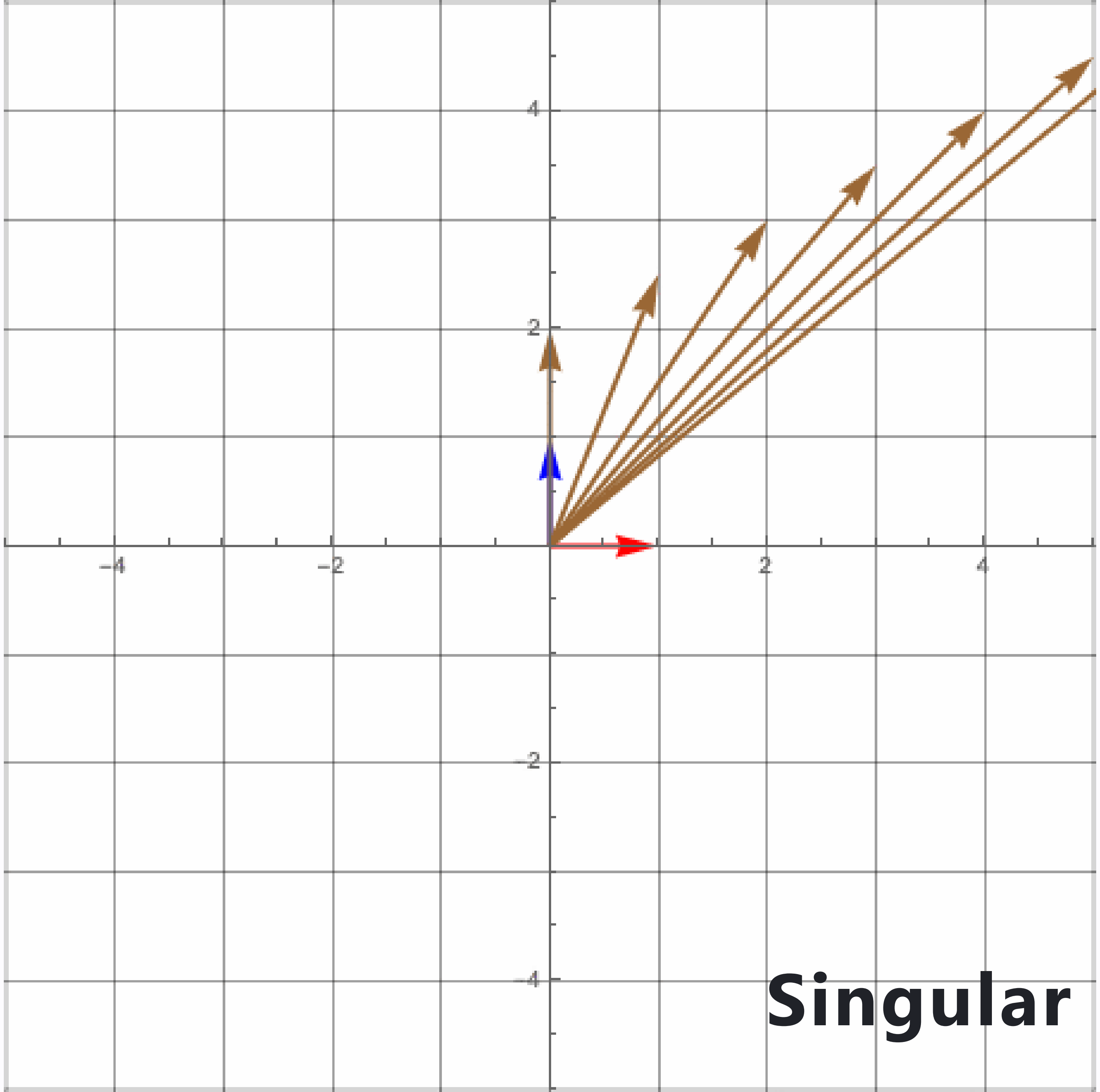
$$\begin{bmatrix} -1 & 2 \\ 0.5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



Singular Matrices



Singular Matrices



Singular Matrices

Singular Matrices

We call non-invertible matrices such as the transformation above, **singular**.

Matrices in the Perspective of Linear Algebra

- The main thesis of this series of lectures should not just be on the definitions of **matrices and their concepts**
- The focus of this lecture is the essence of these concepts both **numerically** and **visually**

Matrices in the Perspective of Linear Algebra

- A **vector** - an arbitrary member of some vector space, presented as an arrow from origin to a corresponding point in some n-dimensional space.
- An n-dimensional **square matrix** - defines some linear transformation, its values correspond to the new location of the basis vectors after the transformation.

Matrices in the Perspective of Linear Algebra

- **matrix multiplication** - composition of linear transformations. The product summarizes the linear transformations into one.
- **determinant** - the factor of scaling of the vector space during a transformation
- **matrix inverse** - the functional inverse of a matrix, reverses the transformation

Non-square Matrices

Let's talk about a class of matrices we haven't talked about before, a **non-square matrix**:

Non-square Matrices

$$\begin{bmatrix} 2 & 2 & -3 \\ -2 & 1 & 0 \end{bmatrix}$$

Non-square Matrices

- But the closest related concept for these matrices are **linear transformations**
- If these are indeed linear transformations, these matrices are meant to be **multiplied** to other vectors to apply some transformation.

Non-square Matrices

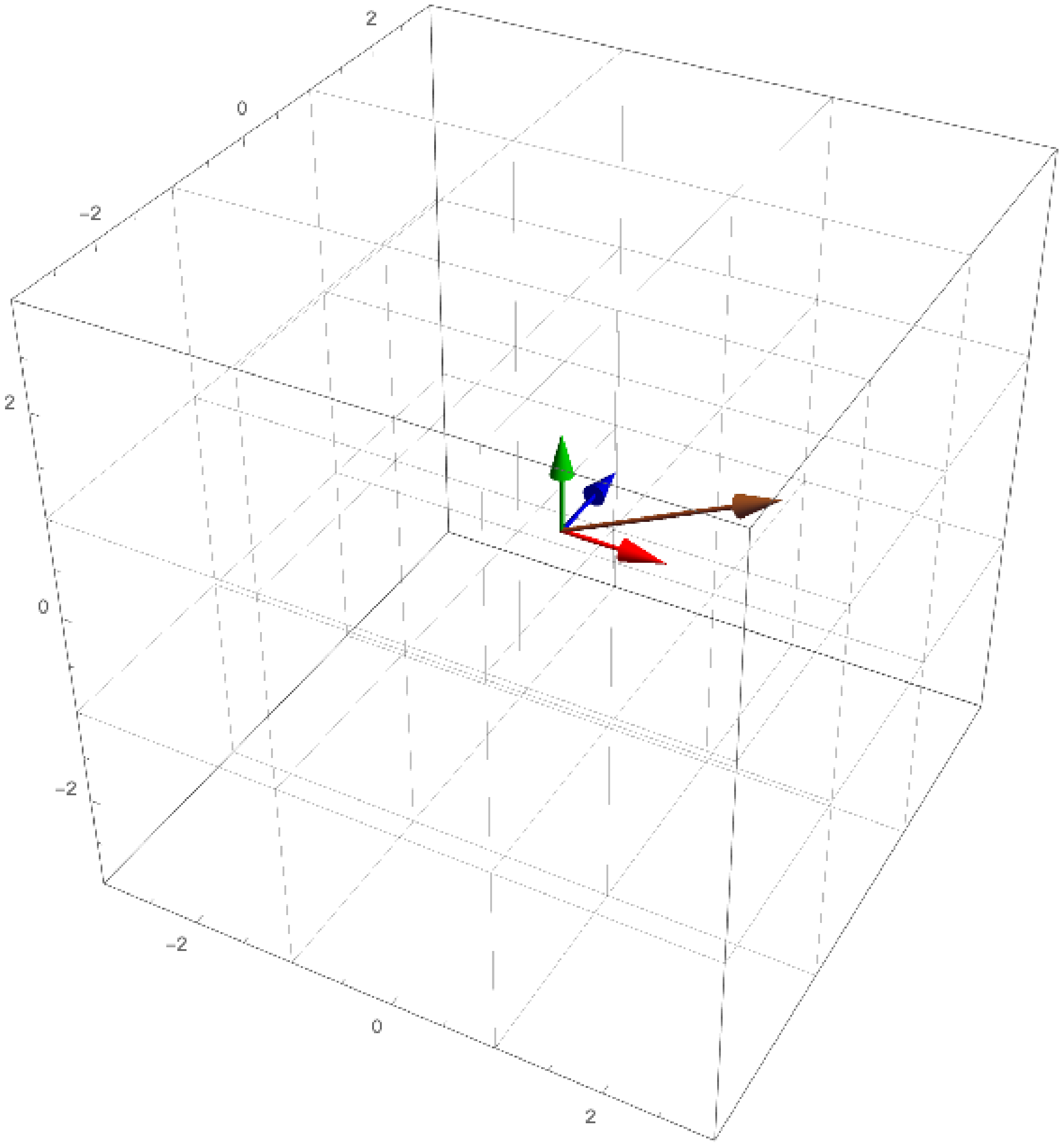
$$\begin{bmatrix} 2 & 2 & -3 \\ -2 & 1 & 0 \end{bmatrix} \vec{v} = \vec{v}'$$

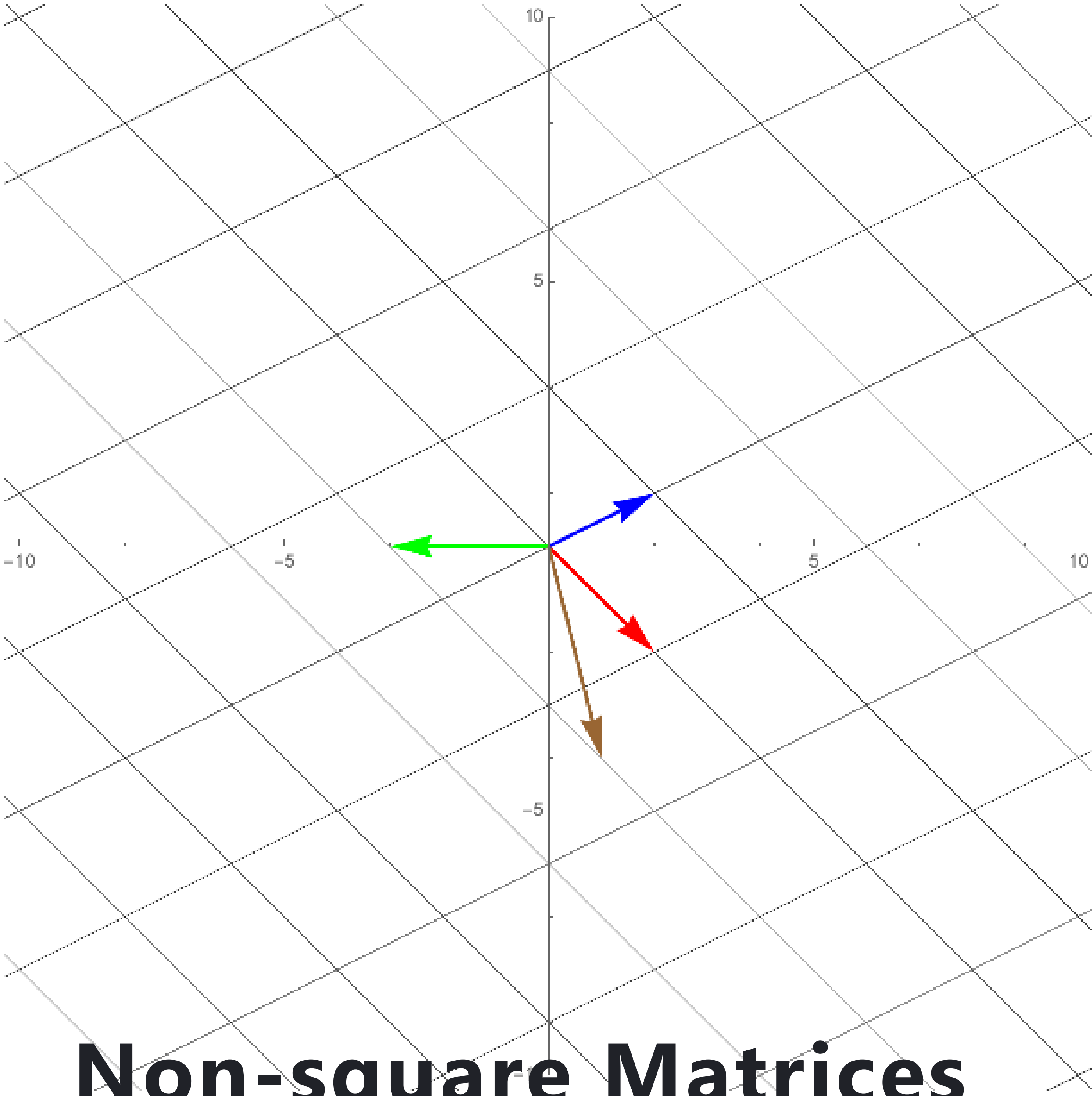
Non-square Matrices

- Since this is a 2×3 matrix, you can only multiply these with vectors of size 3×1
- But the interesting thing about this transformation is that it produces a vector of size 2×1.

Non-square Matrices

$$\begin{bmatrix} 2 & 2 & -3 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$





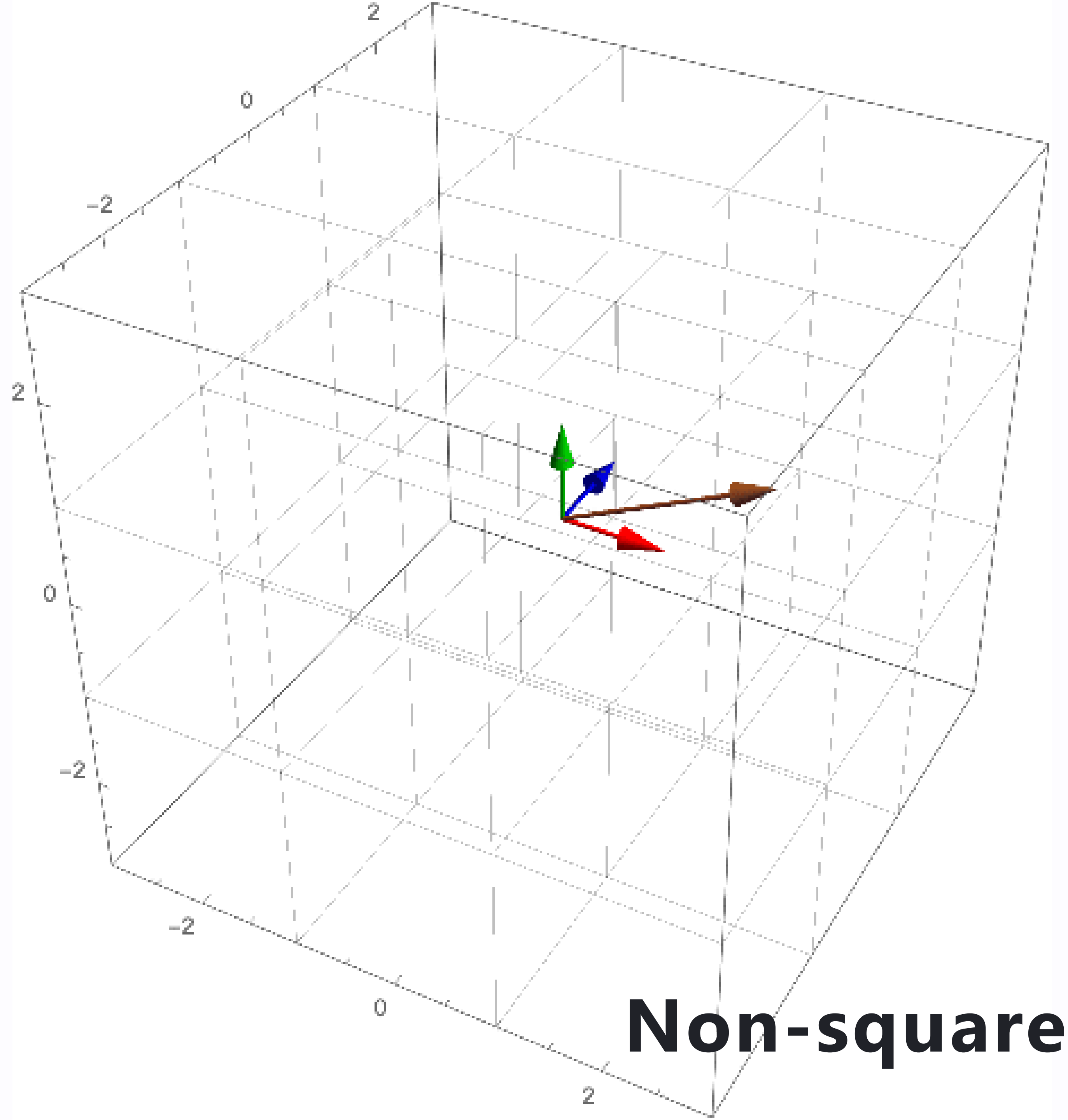
Non-square Matrices

Non-square Matrices

It is also a linear transformation, but it is specifically a transformation that **changes** the number of dimensions of the vector space

Non-square Matrices

$$\begin{bmatrix} 2 & 2 & -3 & \\ -2 & 1 & 0 & \\ 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$$



Non-square Matrices

Non-square Matrices

$$\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$$

Non-square Matrices

$$\begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Non-square Matrices

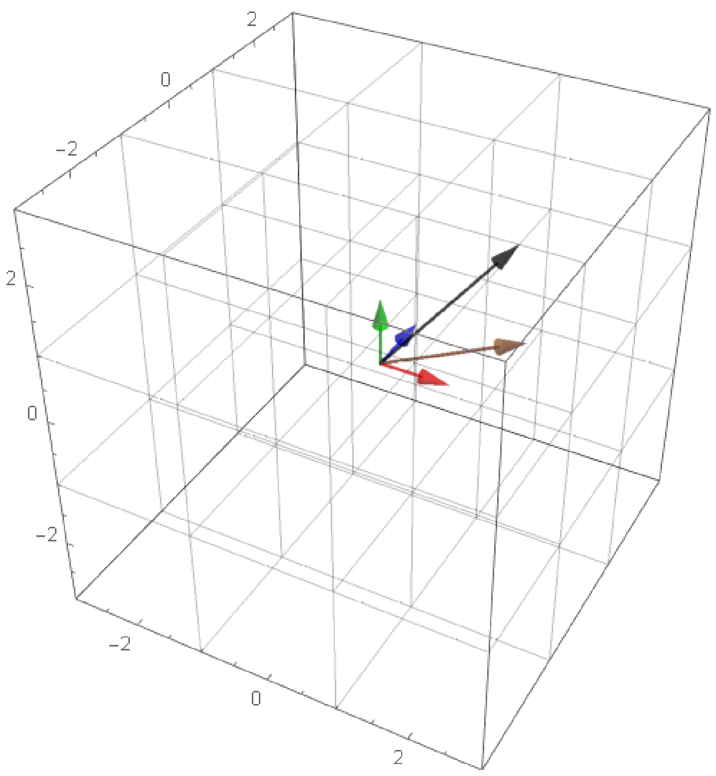
$$\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

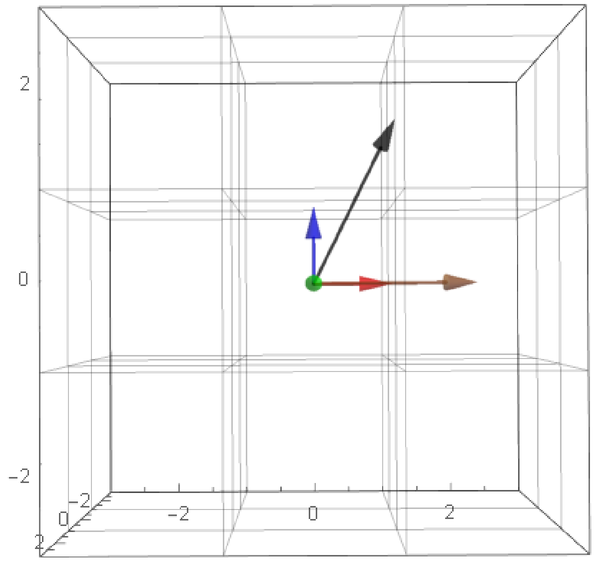
Non-square Matrices

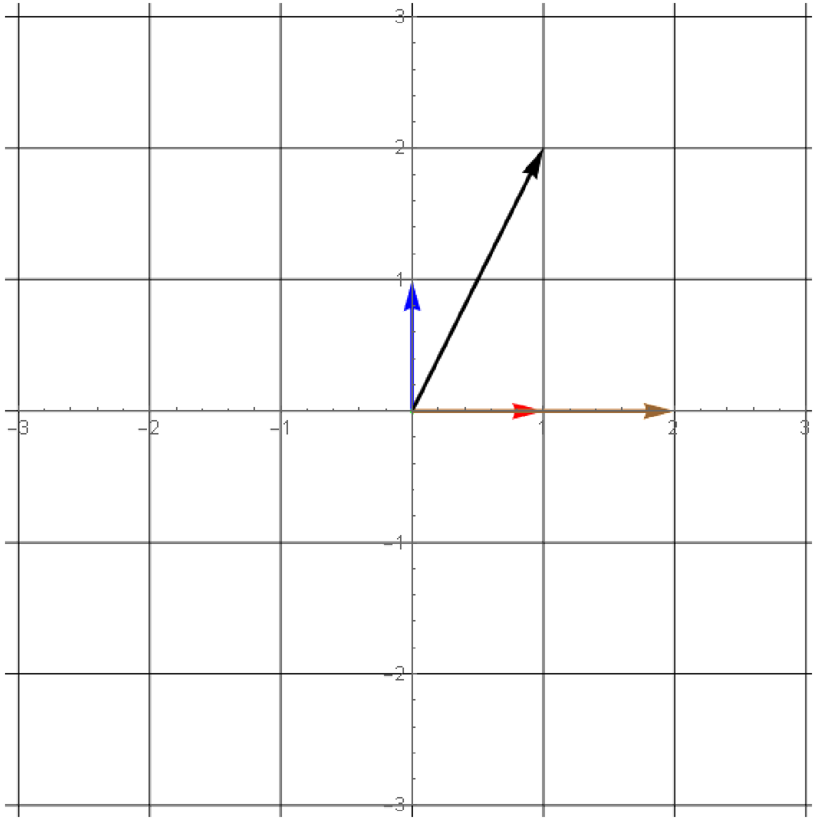
The transformation of any n-dimensional vector to less than n-dimensional space, is similar to producing the **projection** or shadow of the vector

Non-square Matrices

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



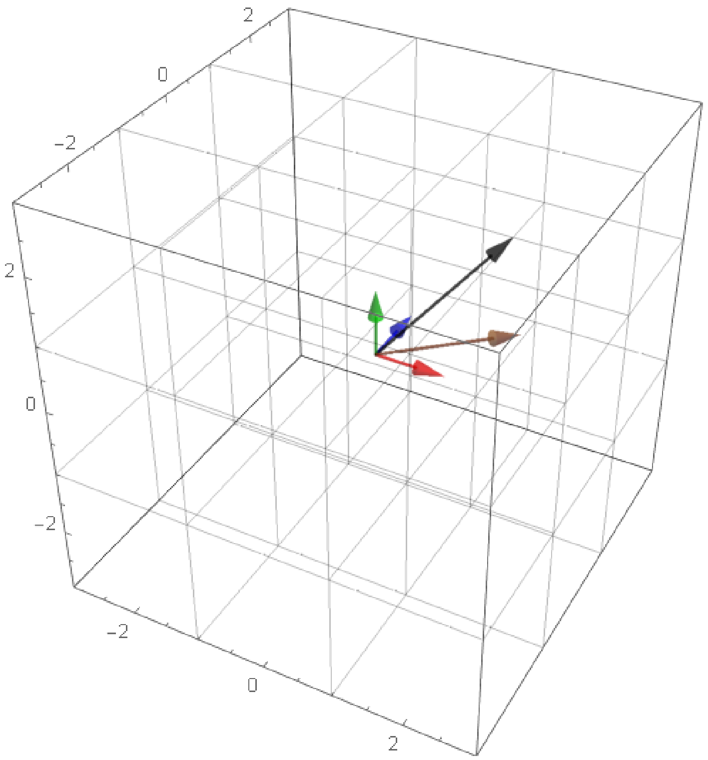


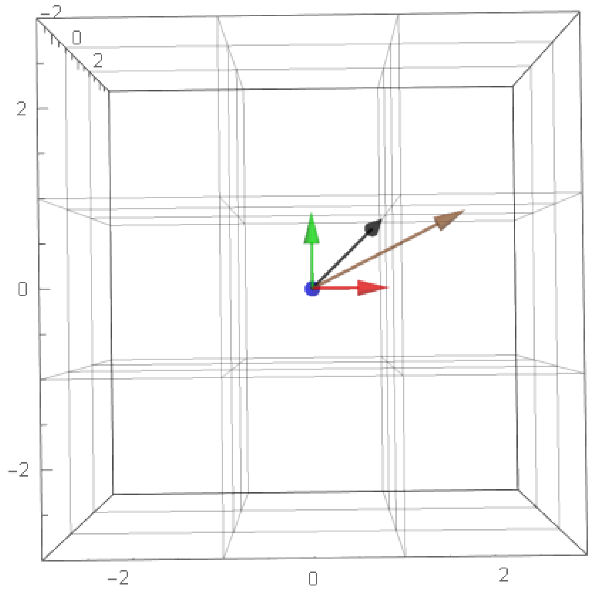


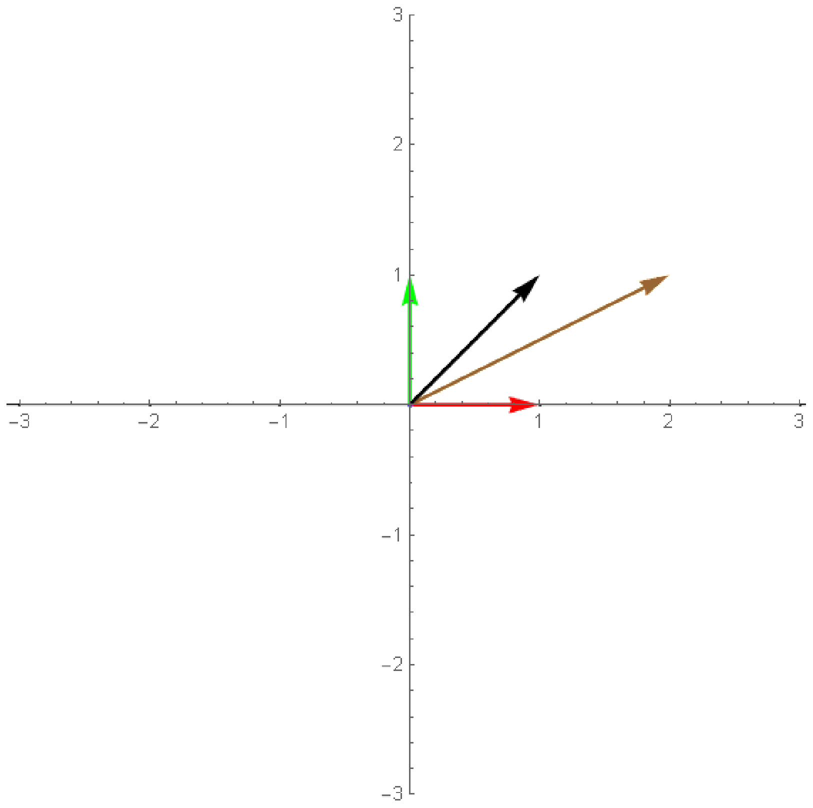
Non-square Matrices

Non-square Matrices

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



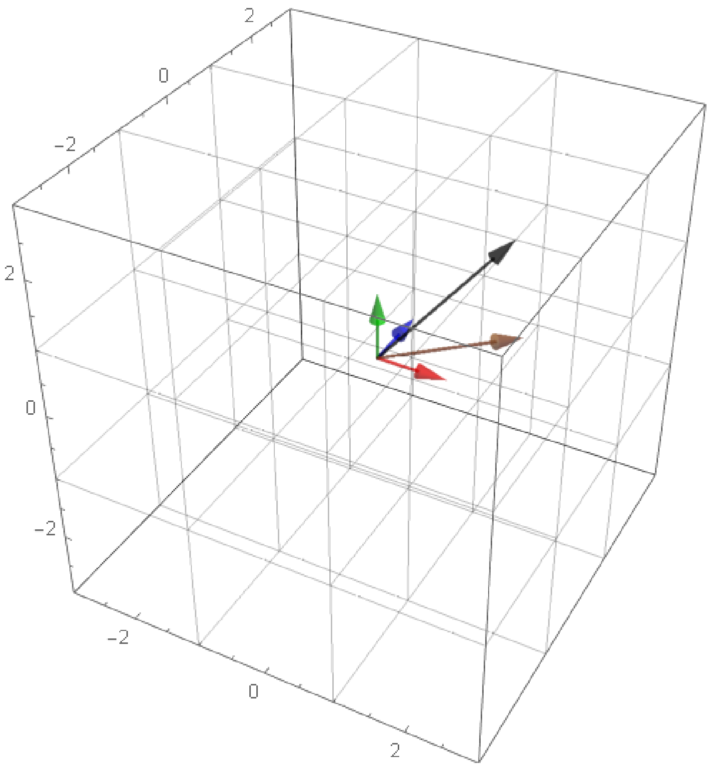


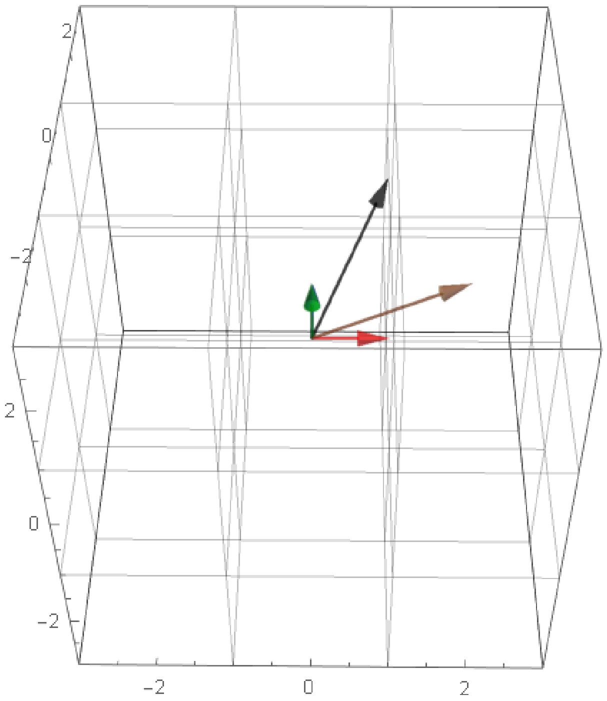


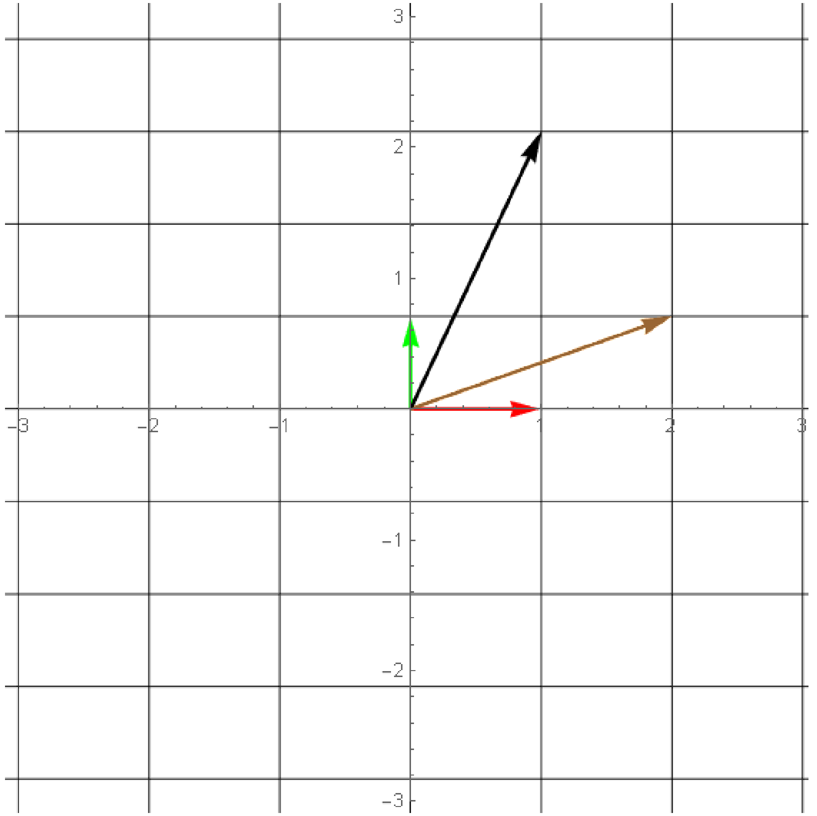
Non-square Matrices

Non-square Matrices

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$







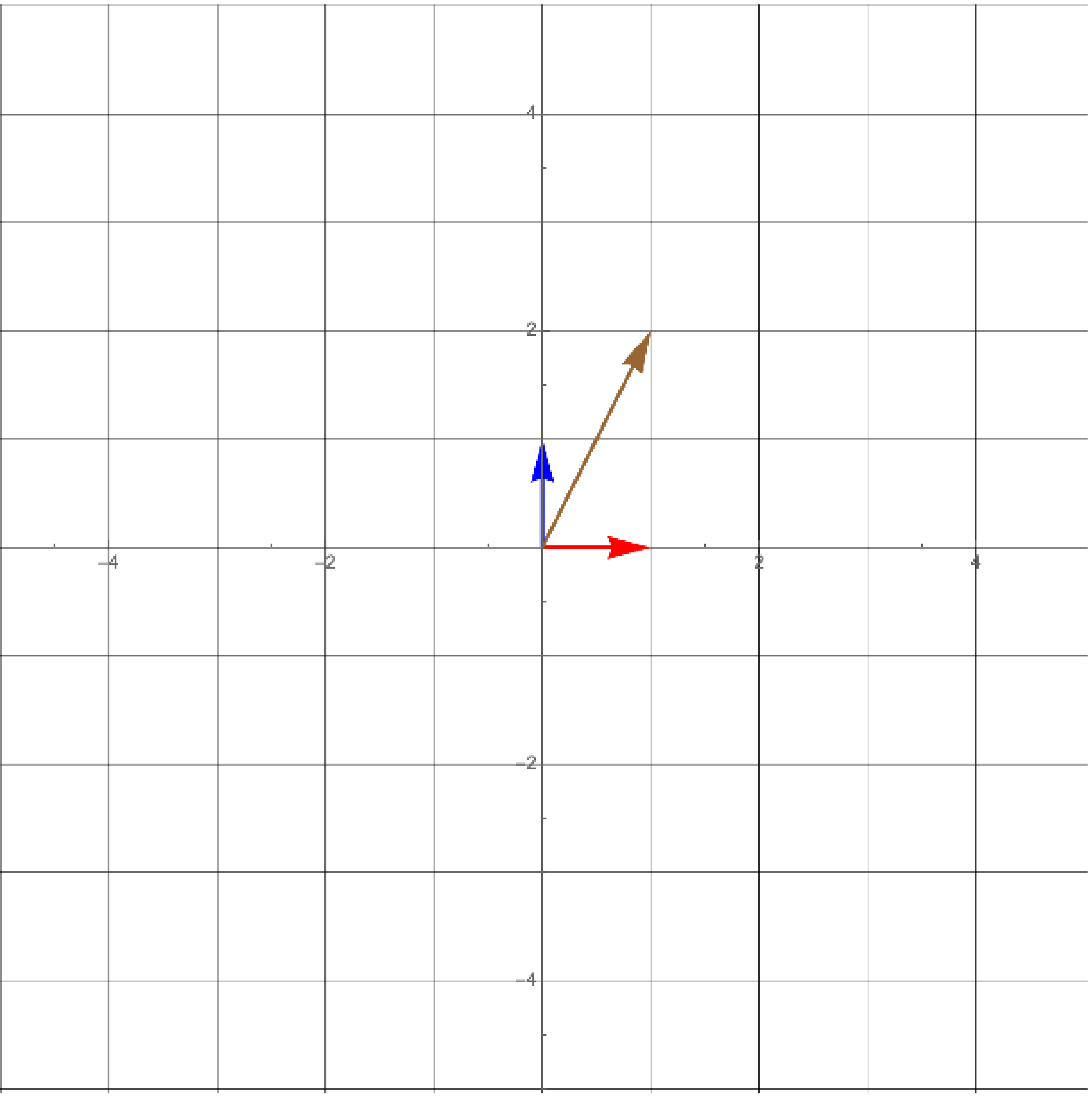
Non-square Matrices

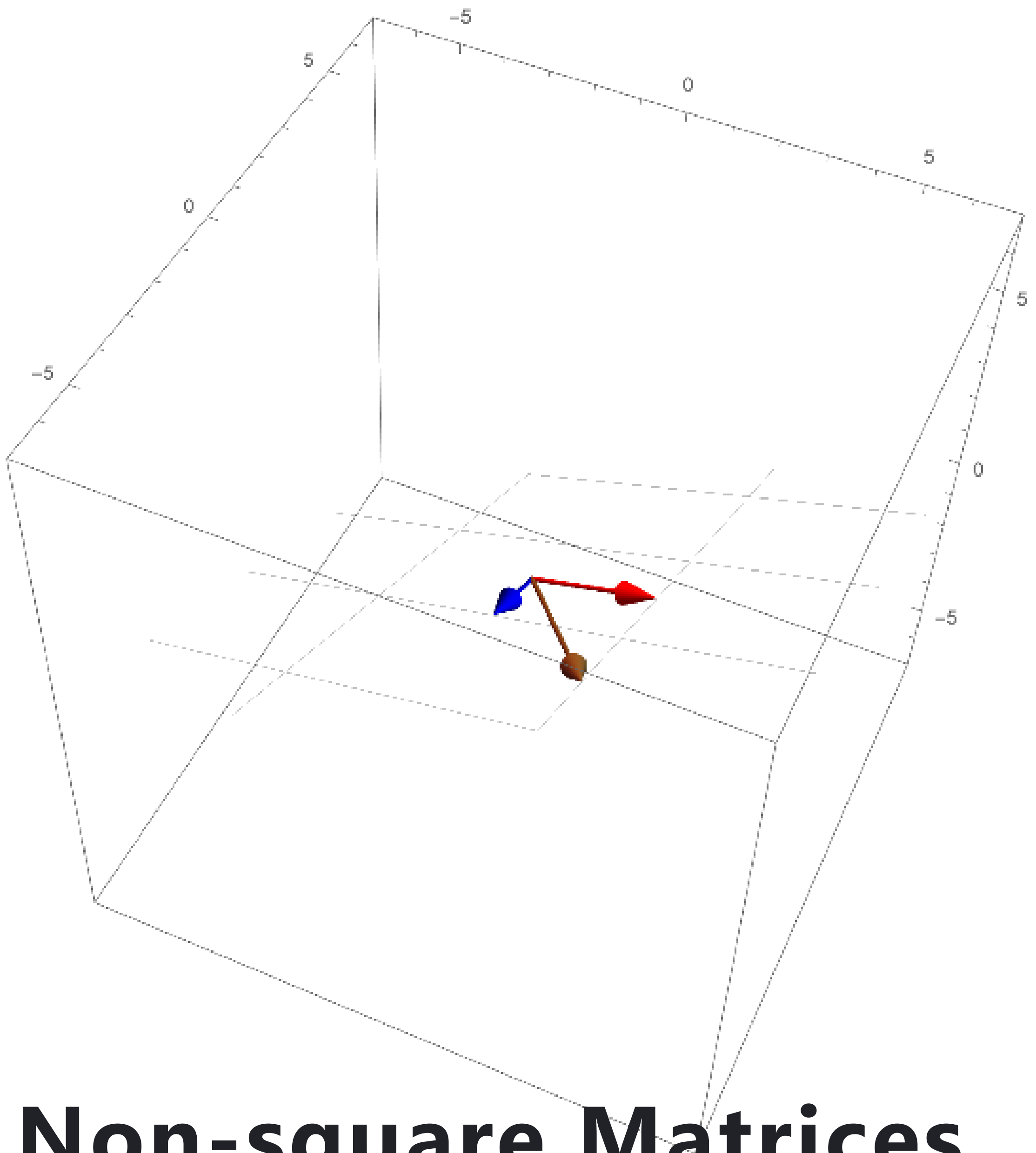
Non-square Matrices

You can also transform a **2-dimensional** vector into a **3-dimensional vector**, to do this you need a 3×2 transformation matrix.

Non-square Matrices

$$\begin{bmatrix} 3 & 0 \\ -1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$$





Non-square Matrices

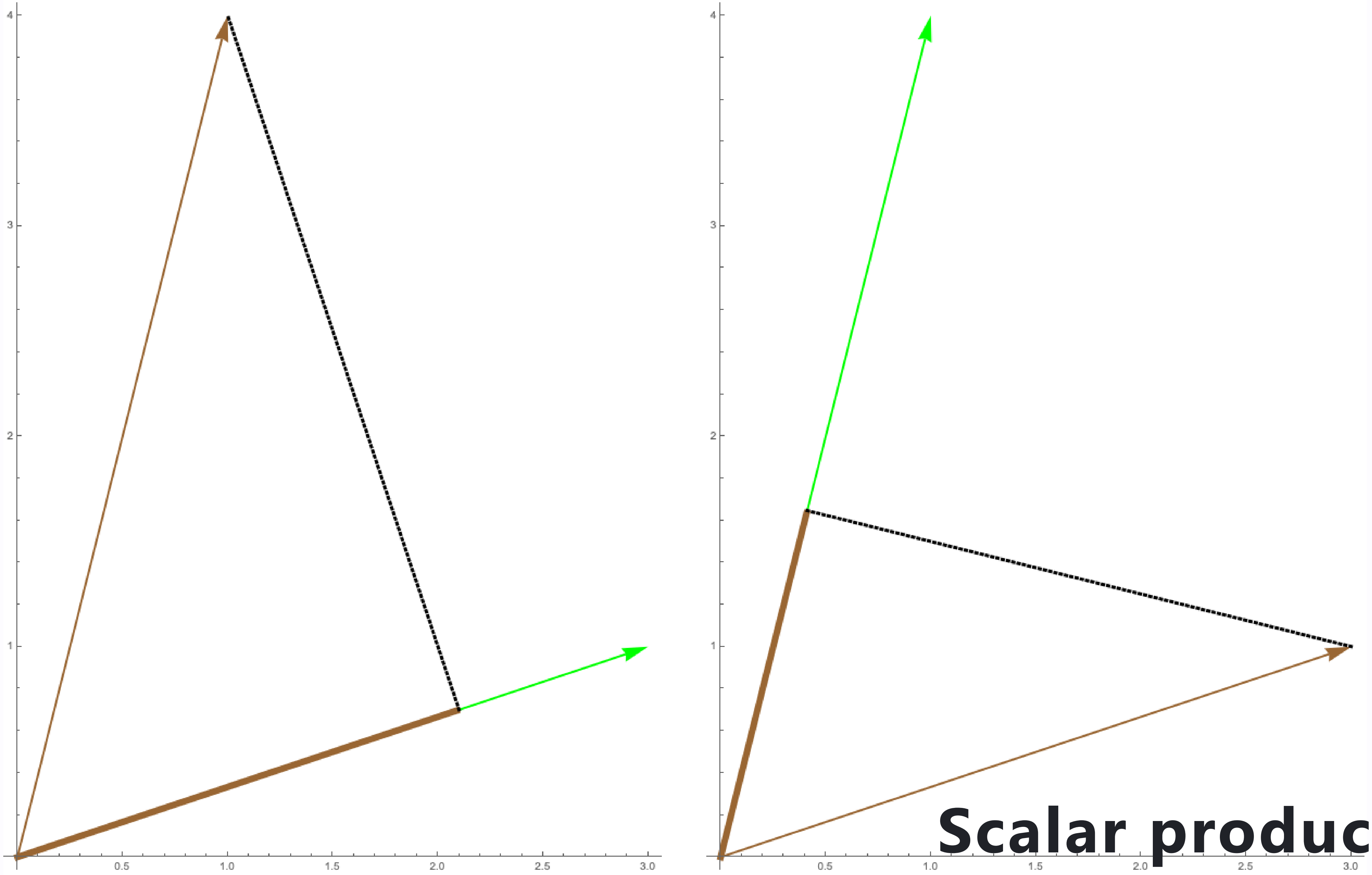
Non-square Matrices

- **the span that the basis vectors produce is still 2 dimensional**
- This is because there is no way for the vectors to be transformed with **extra dimensionality**.

Non-square Matrices

As a general rule, you can think of any $m \times n$ matrix as a transformation that transforms an n-**dimensional** vector into an m-**dimensional vector**.

The scalar product of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$ is the product of the magnitude of $\vec{a}$ and the magnitude of the projected version of $\vec{b}$ onto $\vec{a}$.



Scalar products

Scalar products

$$\begin{bmatrix} a \\ b \\ \vdots \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ \vdots \\ z \end{bmatrix} = ax + by + \cdots + cz$$

Scalar products

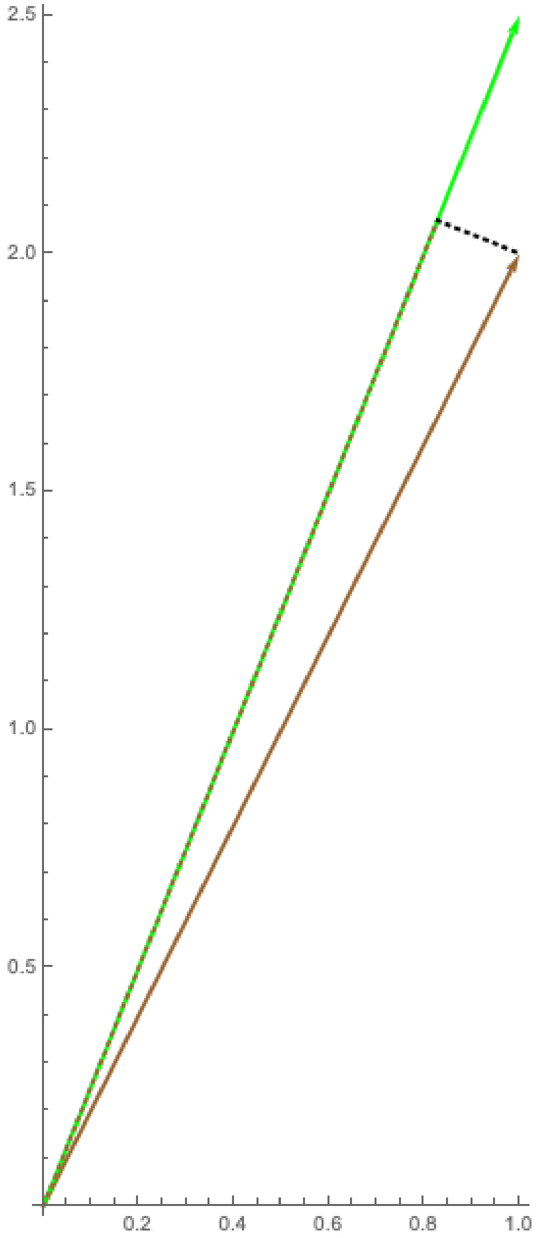
It is the measure of **similarity** between two vectors,
For example, given three vectors,

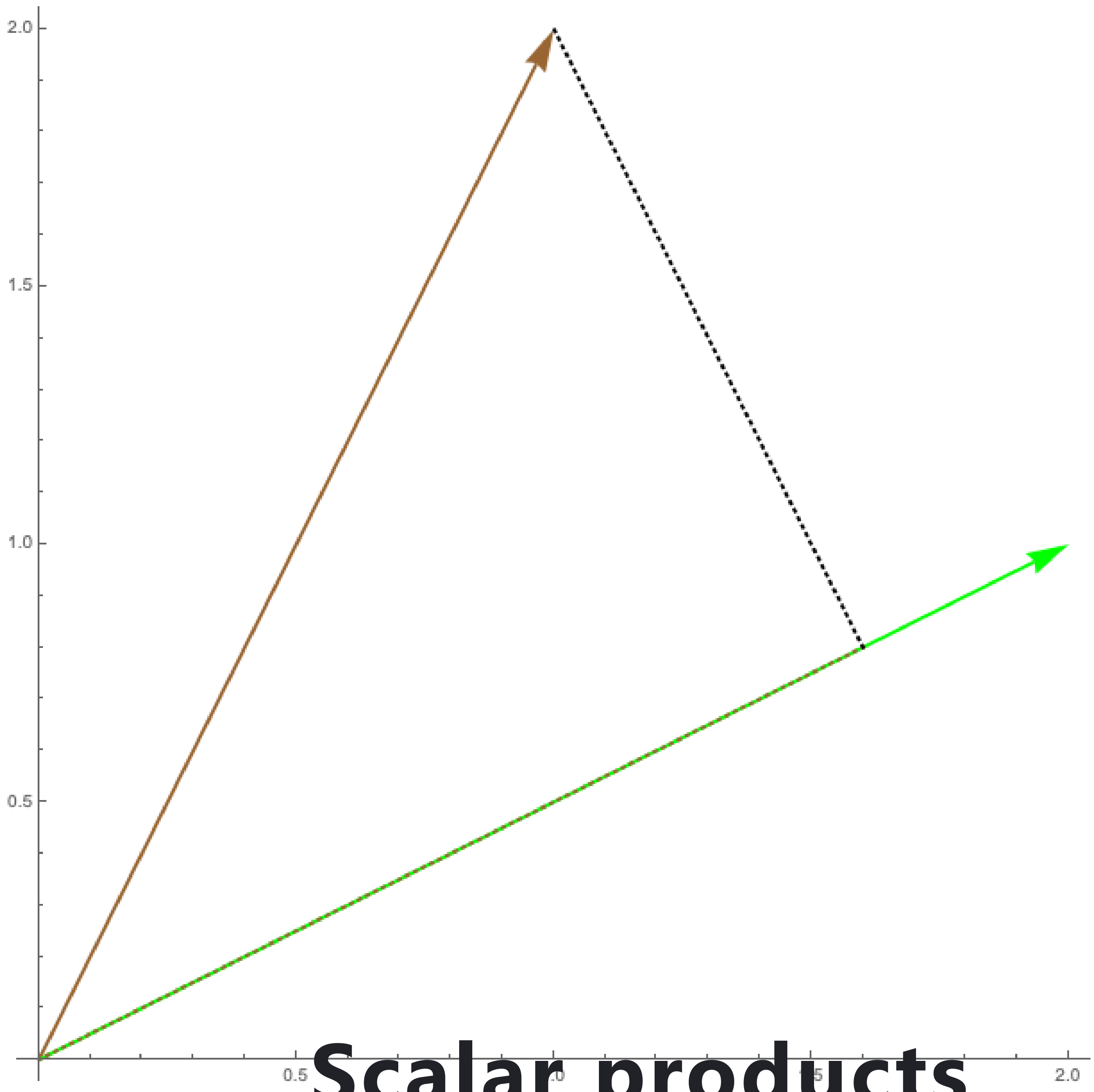
Scalar products

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2.5 \end{bmatrix}, \vec{c} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Scalar products

Is $\vec{a}$ **more similar** to $\vec{b}$ or to $\vec{c}$?





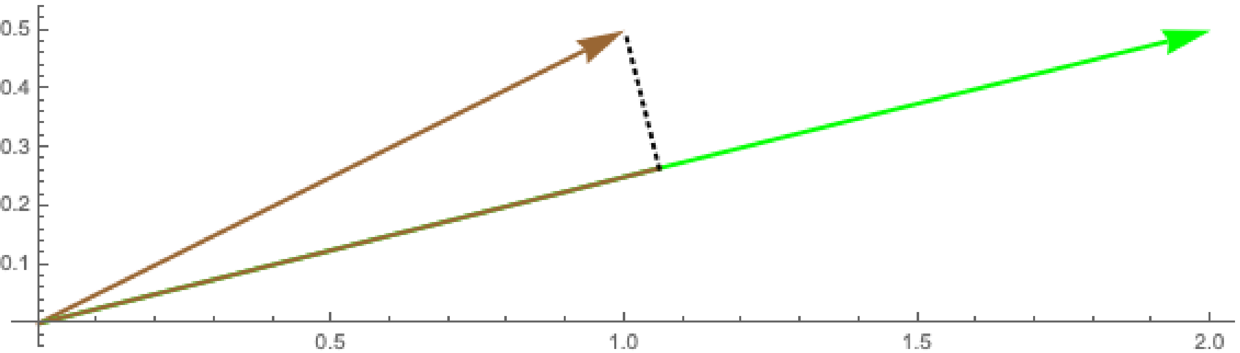
Scalar products

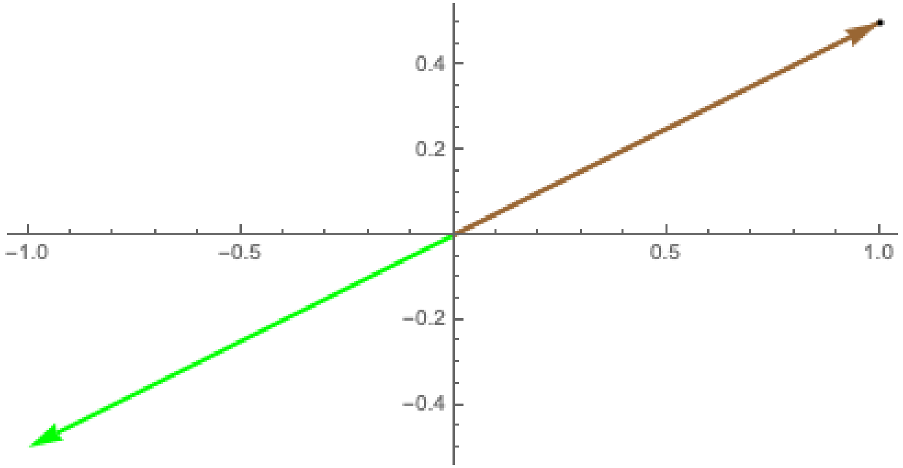
Scalar products

$$\vec{a} \cdot \vec{b} = 6$$
$$\vec{a} \cdot \vec{c} = 4$$

Scalar products

This means that two vectors perpendicular to each other will have a scalar product of **zero** and two vectors pointing in the opposite direction will have a **negative** scalar product:





Scalar products

Scalar products

How does the scalar product relate to **linear transformations**?

Scalar products

And it turns out, a scalar product is merely a transformation of any vector to **one-dimensional space**:

Scalar products

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = u_x a + u_y b$$

Scalar products

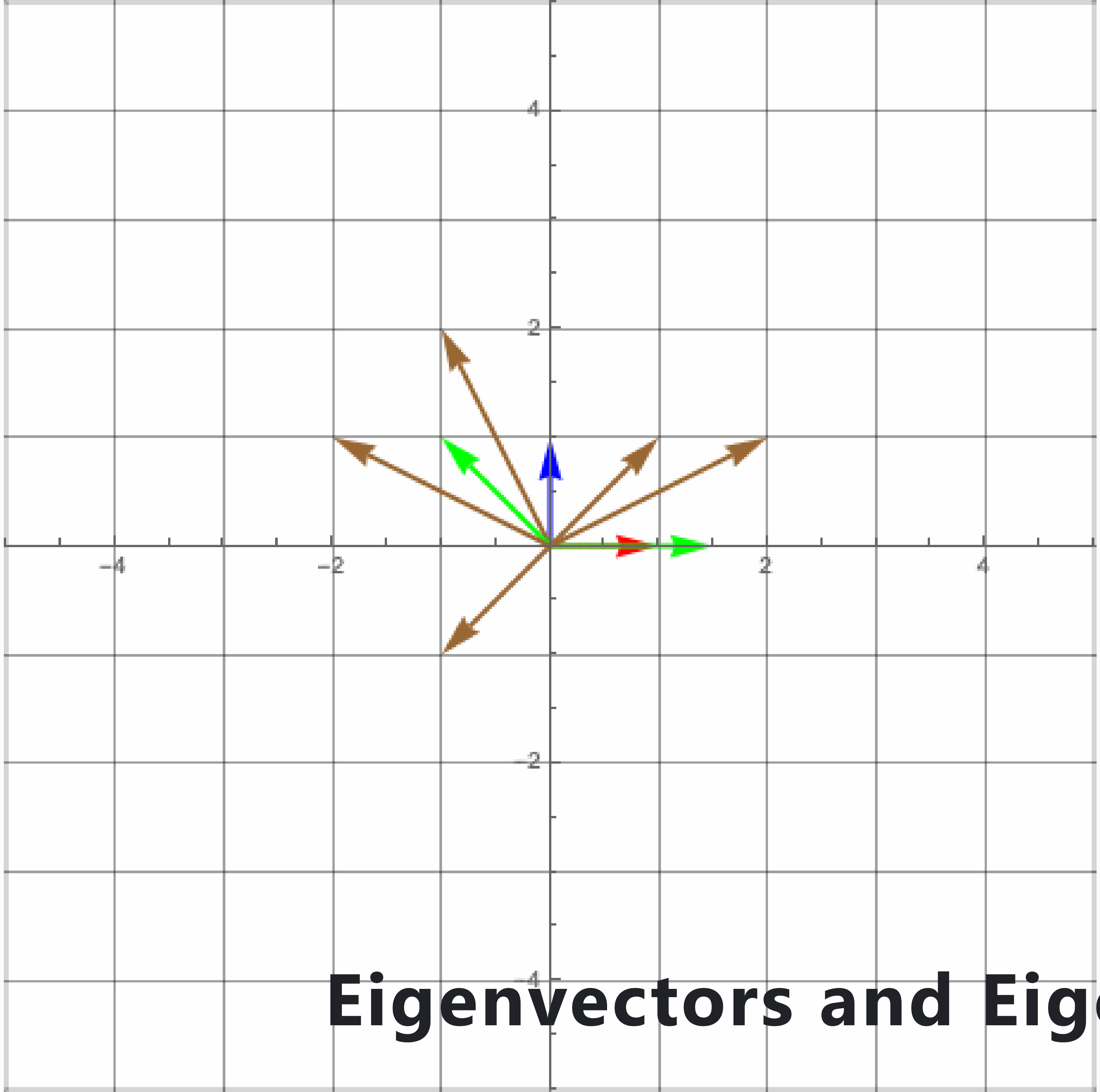
$$\begin{bmatrix} u_x & u_y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = u_x a + u_y b$$

Scalar products

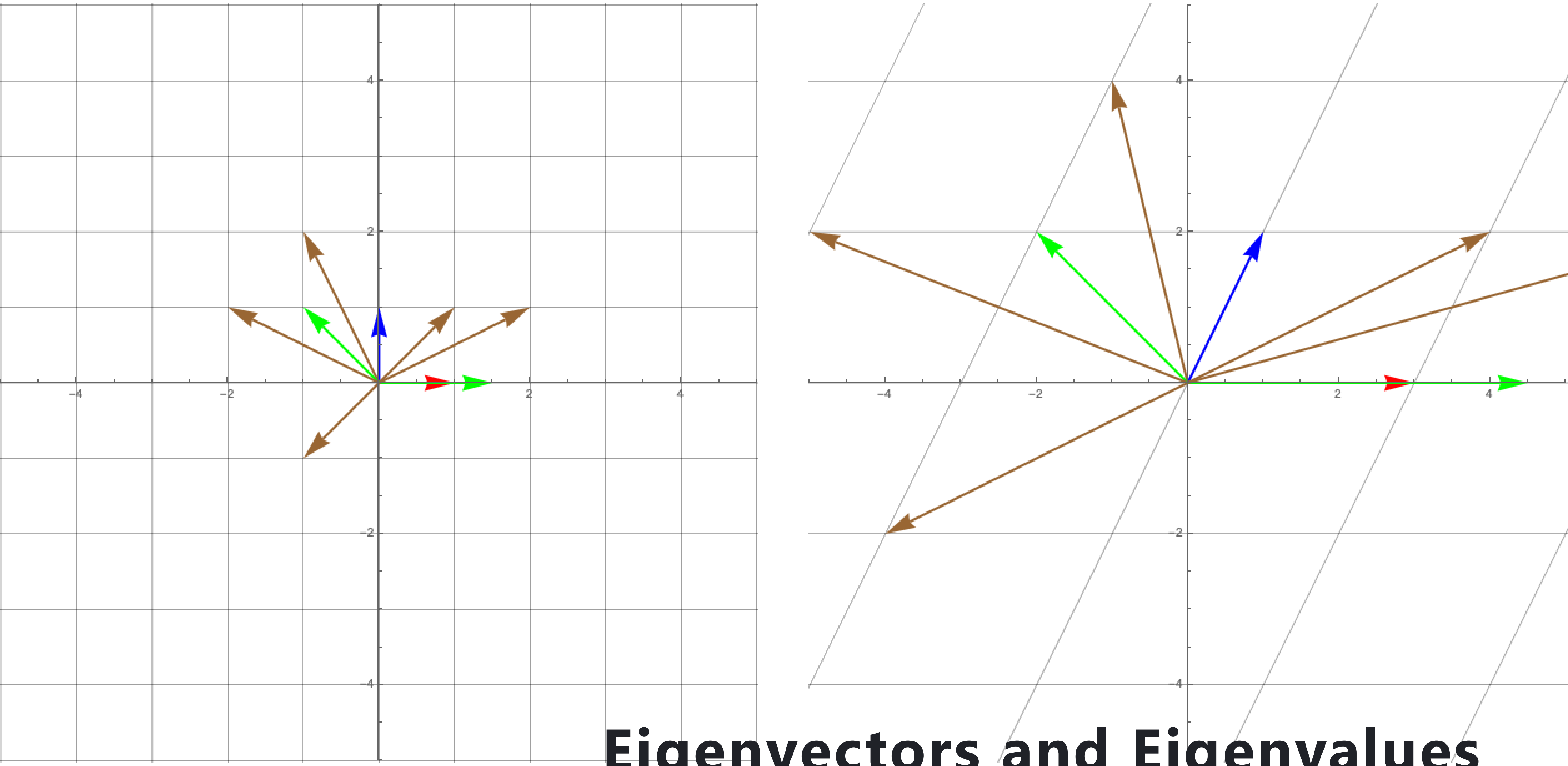
When looking at, the scalar product of two $n \times 1$ vectors, you can imagine that one vector is **reduced** into the **one dimensional vector space**

Eigenvectors and Eigenvalues

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$



Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues

Eigenvectors and Eigenvalues

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

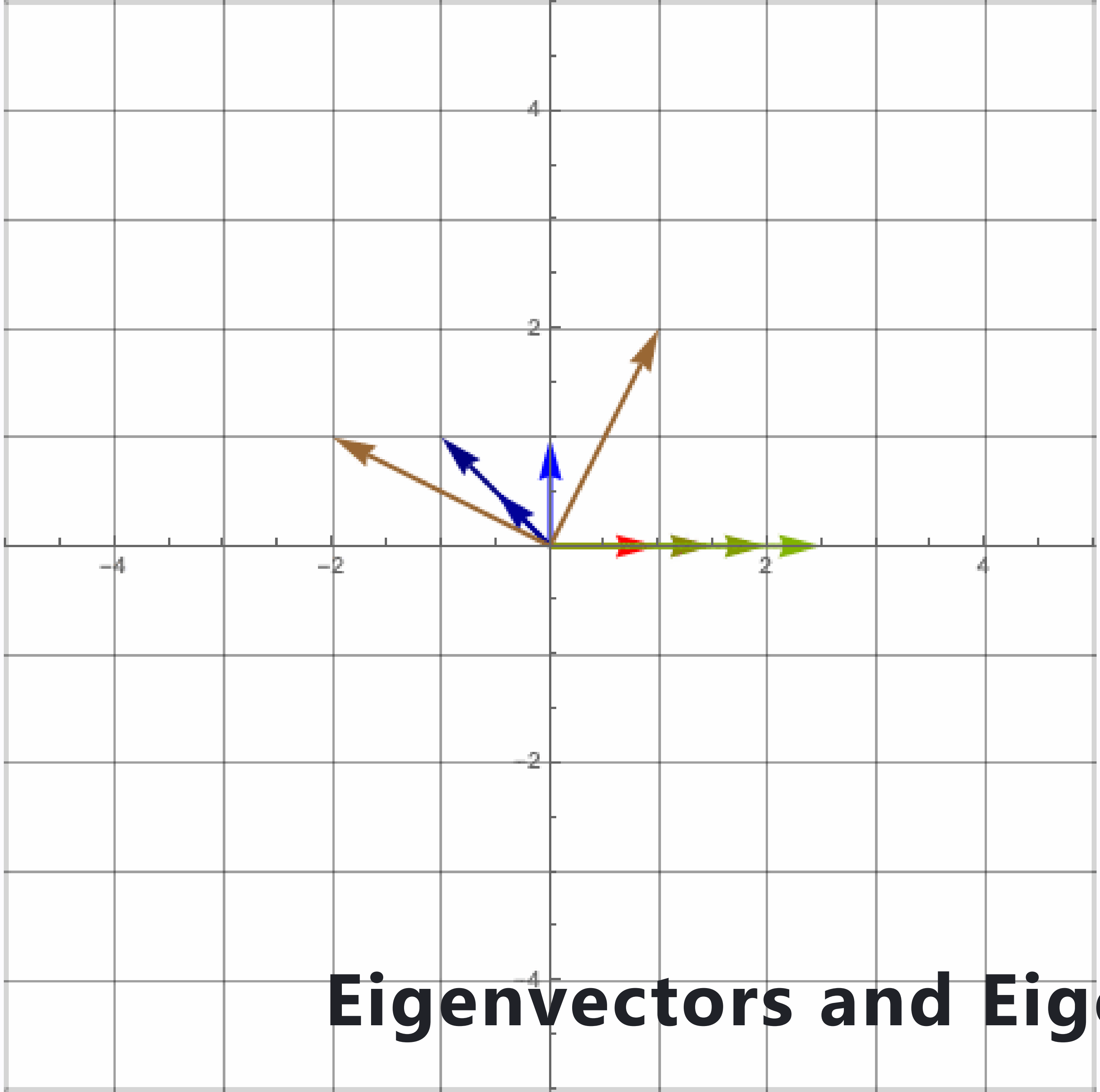
$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

Eigenvectors and Eigenvalues

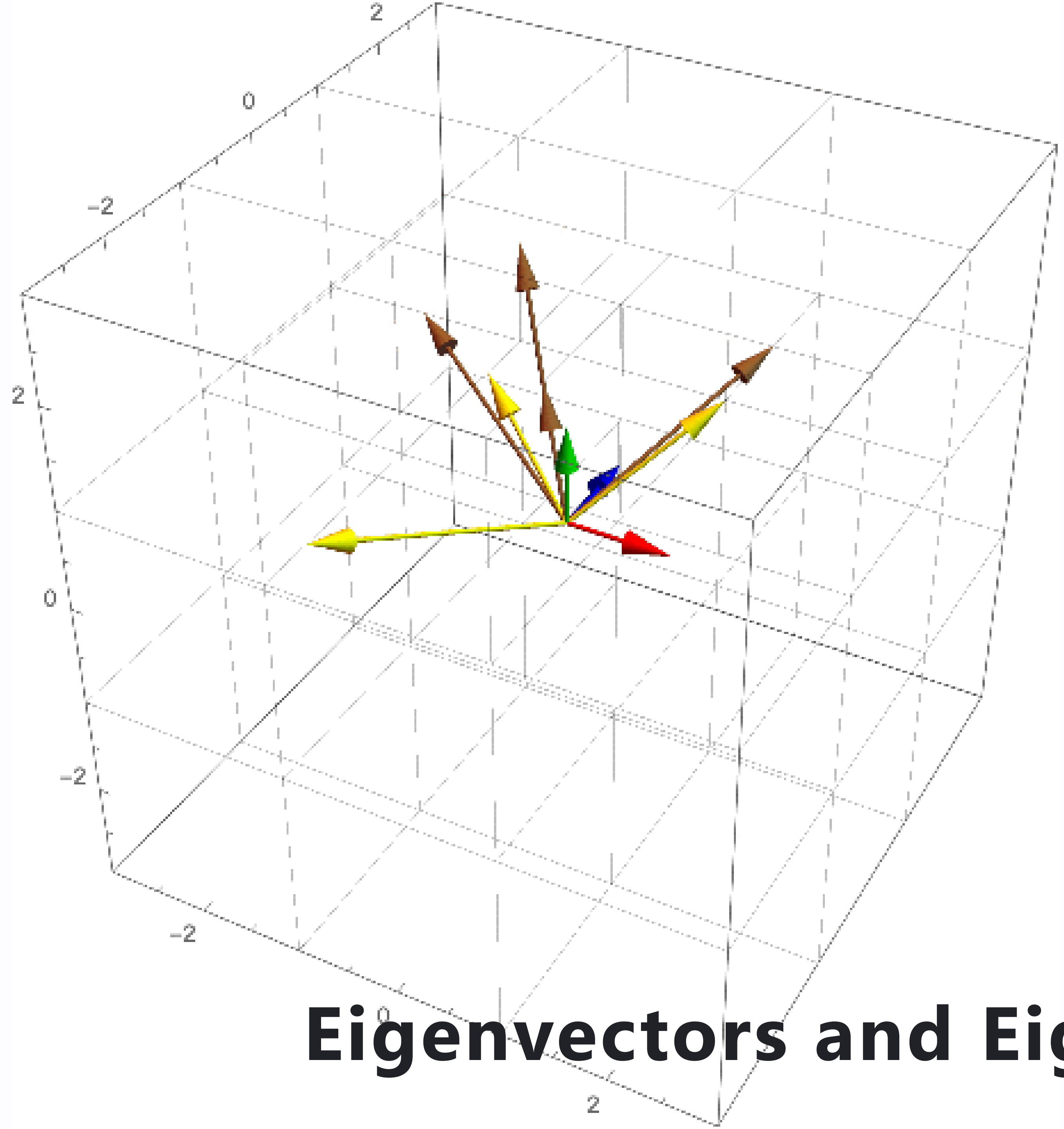
- These special vectors are called **eigenvectors**
- Each eigenvector is always accompanied by special scalar values called **eigenvalues**, which correspond to the factor of scaling for the transformation.

Eigenvectors and Eigenvalues

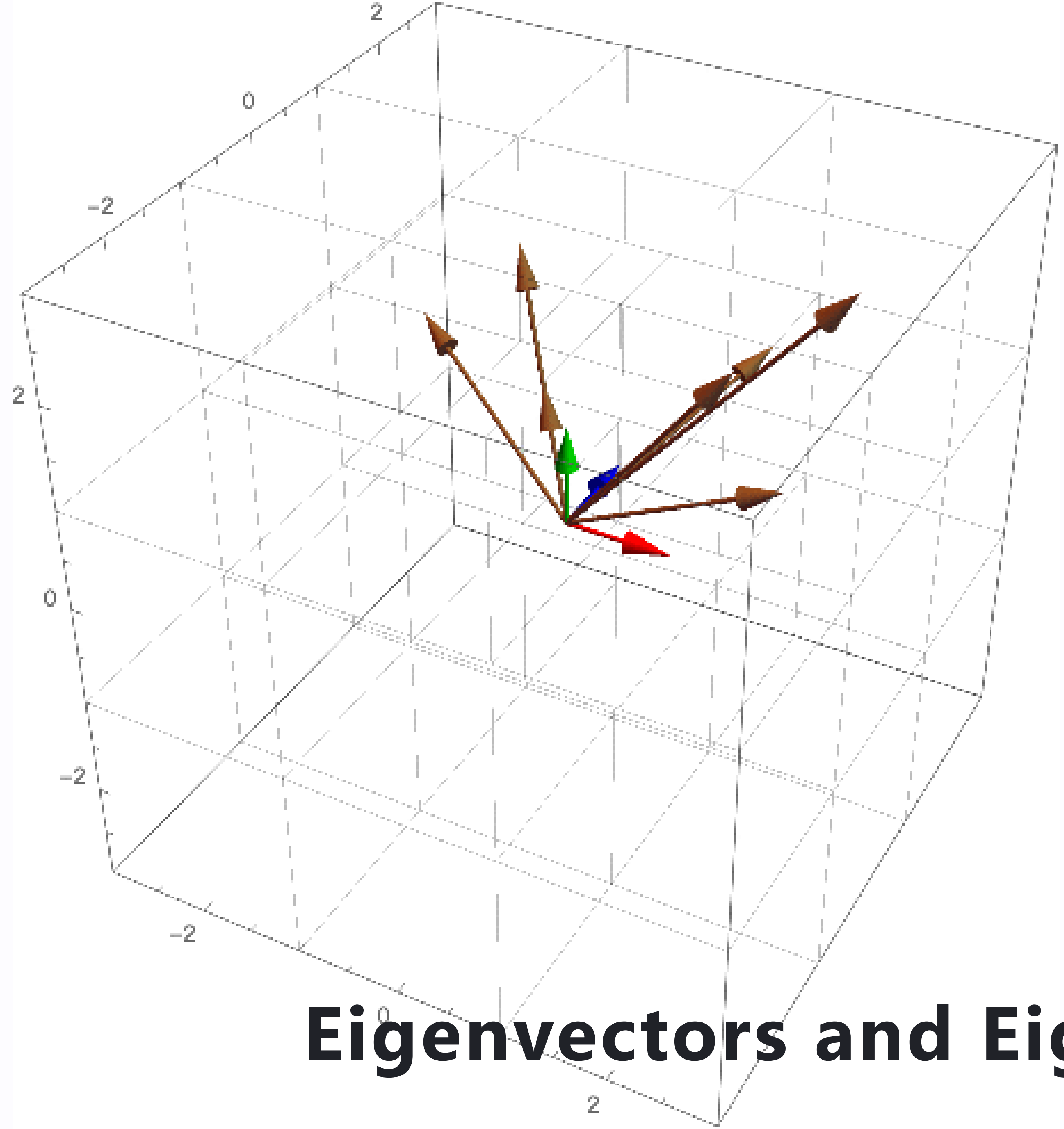
In fact **all** the vectors along the **span of the green vectors** are eigenvectors as well.



Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues

Eigenvectors and Eigenvalues

- Since eigenvectors are vectors that are **only** scaled as a result of the transformation, we can solve for $\vec{e}$ in the following equality
- The scalar value λ refers to the unknown eigenvalue.

Eigenvectors and Eigenvalues

$$T\vec{e} = \lambda\vec{e}$$

Eigenvectors and Eigenvalues

- Scalar times vector multiplication $\lambda\vec{e}$ can be written as a **linear transformation** instead
- **multiplying** λI **to** $\vec{e}$

Eigenvectors and Eigenvalues

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \vec{e}$$

Eigenvectors and Eigenvalues

To solve for the unknowns, we can rewrite $T\vec{e} = \lambda\vec{e}$ into a **solution from zero**:

Eigenvectors and Eigenvalues

$$T\vec{e} = \lambda I\vec{e}$$

$$T\vec{e} - \lambda I\vec{e} = \vec{0}$$

$$(T - \lambda I)\vec{e} = \vec{0}$$

Eigenvectors and Eigenvalues

- If you recall, a non-zero vector can only be transformed to zero if and only if the whole vector space has been **squished to zero itself**
- And this can only happen when the **determinant** of transformation is **zero**.

Eigenvectors and Eigenvalues

$$\det(T - \lambda I) = \det(\begin{bmatrix} t_{11} - \lambda & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} - \lambda & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} - \lambda \end{bmatrix}) = 0$$

Eigenvectors and Eigenvalues

This means that we can find the eigenvalues of any transformation by finding the **lambdas** that reduces the determinant to 0

Eigenvectors and Eigenvalues

$$\det(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}) = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

Eigenvectors and Eigenvalues

$$\begin{aligned}
\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) &= 0 \\
(3-\lambda)(2-\lambda)-(1)(0) &= 0 \\
(3-\lambda)(2-\lambda) &= 0
\end{aligned}$$

$$\begin{aligned}
\lambda &= 2 \\
\lambda &= 3
\end{aligned}$$

Eigenvectors and Eigenvalues

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{e} = 2\vec{e}$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{e} = 3\vec{e}$$

Eigenvectors and Eigenvalues

$$\text{Eigenvectors for } \lambda = 2$$

$$\begin{aligned} 3x + y &= 2x \\ 2y &= 2y \\ x + y &= 0 \\ x &= -y \end{aligned}$$

$$\begin{bmatrix} u \\ -u \end{bmatrix}$$

Eigenvectors and Eigenvalues

$$\text{Eigenvectors for } \lambda = 3$$

$$3x + y = 3x$$

$$2y = 3y$$

$$y = 0$$

$$\begin{bmatrix} v \\ 0 \end{bmatrix}$$

Eigenvectors and Eigenvalues

As you can see, the solutions for $\vec{e}$ is infinitely many, any vector of the form $\begin{bmatrix} u \\ -u \end{bmatrix}$ and any vector of the form $\begin{bmatrix} v \\ 0 \end{bmatrix}$.