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Problem

We consider the transient heat equation in a two-dimensional (2D) unit square $\Omega := (0,1) \times (0,1)$. The boundary of the domain Γ can be decomposed into

the bottom face $\Gamma_b := \{(x,y) : x \in (0,1), y = 0\}$;

the top face $\Gamma_t := \{(x,y) : x \in (0,1), y = 1\}$;

the left face $\Gamma_l := \{(x,y) : x = 0, y \in (0,1)\}$;

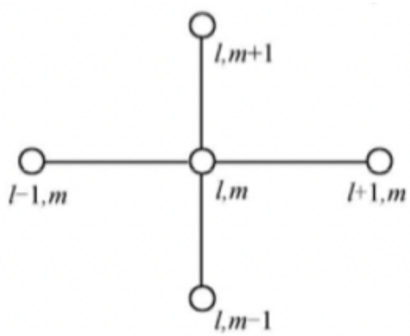
the right face $\Gamma_r := \{(x,y) : x = 1, y \in (0,1)\}$.

Let f be the heat supply per unit volume, u be the temperature, ρ be the density, c be the heat capacity, u_0 be the initial temperature, κ be the conductivity, n_x and n_y be the Cartesian components of the unit outward normal vector . The boundary data involves the prescribed temperature g on Γ_g and heat flux h on Γ_h. The boundary Γ admits a non-overlapping decomposition: $\Gamma = \overline{\Gamma_g \cup \Gamma_h}$ and $\emptyset = \Gamma_g \cap \Gamma_h$. The transient heat equation may be stated as follows.

$$\begin{aligned}
\rho c \frac{\partial u}{\partial t} - \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= f \quad \text{on } \Omega \times (0,T) \\
u &= g \quad \text{on } \Gamma_g \times (0,T) \\
\kappa \frac{\partial u}{\partial x} n_x + \kappa \frac{\partial u}{\partial y} n_y &= h \quad \text{on } \Gamma_h \times (0,T) \\
u|_{t=0} &= u_0 \quad \text{in } \Omega.
\end{aligned}$$

Solutions

We learned the solution methods from the videos on bilibili and the books from class MAE5032-HPC by Ju-LIU



$u_{l,m}^t$ is denoted as temperature u in node l,m at time t.

(1) For each internal node, the $u_{l,m}^{t+1}$ should be depended on five data at time t (i.e., $u_{l,m}^{t}$, $u_{l+1,m}^{t}$, $u_{l-,m}^{t}$, $u_{l,m+1}^{t}$, $u_{l,m-1}^{t}$) spatially and temporally.

The basic equation should be

$$\rho c\Delta x\Delta y(u_{l,m}^{t+1}-u_{l,m}^{t})=\kappa((\frac{\partial u}{\partial x})_{l+1,m}^{t\to t+1}-(\frac{\partial u}{\partial x})_{l-1,m}^{t\to t+1})\Delta y\Delta t+\kappa((\frac{\partial u}{\partial y})_{l,m+1}^{t\to t+1}-(\frac{\partial u}{\partial y})_{l,m-1}^{t\to t+1})\Delta x\Delta t$$

In explicit format, the relationship can be decribed by the following.

$$\begin{aligned}\frac{\rho c\Delta x\Delta y}{\Delta t}(u_{l,m}^{t+1}-u_{l,m}^{t}) &= \kappa(\frac{u_{l+1,m}^{t}-u_{l,m}^{t}}{\delta x_{l+1}}-\frac{u_{l,m}^{t}-u_{l-1,m}^{t}}{\delta x_{l-1}})\Delta y+\kappa(\frac{u_{l,m+1}^{t}-u_{l,m}^{t}}{\delta y_{m+1}}-\frac{u_{l,m}^{t}-u_{l,m-1}^{t}}{\delta y_{m+1}})\Delta x \\ a_{l,m}^{t+1}u_{l,m}^{t+1} &= a_{l+1,m}^{t}u_{l+1,m}^{t}+a_{l-1,m}^{t}u_{l-1,m}^{t}+a_{l,m+1}^{t}u_{l,m+1}^{t}+a_{l,m-1}^{t}u_{l,m-1}^{t}+a_{l,m}^{t}u_{l,m}^{t}\end{aligned}$$

where

$$\begin{aligned}a_{l+1,m}^{t} &= \frac{\kappa\Delta y}{\delta x_{l+1}}=\frac{\kappa\Delta y}{\Delta x}=a_{l-1,m}^{t}=\frac{\kappa\Delta y}{\delta x_{l-1}} \\ a_{l,m+1}^{t} &= \frac{\kappa\Delta x}{\delta y_{m+1}}=\frac{\kappa\Delta x}{\Delta y}=a_{l,m-1}^{t}=\frac{\kappa\Delta x}{\delta y_{m-1}} \\ a_{l,m}^{t} &= \frac{\rho c\Delta x\Delta y}{\Delta t}-\frac{\kappa\Delta y}{\delta x_{l+1}}-\frac{\kappa\Delta y}{\delta x_{l-1}}-\frac{\kappa\Delta x}{\delta y_{m+1}}-\frac{\kappa\Delta x}{\delta y_{m-1}} \\ &= \frac{\rho c\Delta x\Delta y}{\Delta t}-2\frac{\kappa\Delta y}{\Delta x}-2\frac{\kappa\Delta x}{\Delta y} \\ a_{l,m}^{t+1} &= \frac{\rho c\Delta x\Delta y}{\Delta t}=2a_{l+1,m}^{t}+2a_{l,m+1}^{t}+a_{l,m}^{t}\end{aligned}$$

Note that δ is actually the same as Δ for internal nodes.

In implicit format, the relationship can be decribed by the following.

$$\begin{aligned}\frac{\rho c\Delta x\Delta y}{\Delta t}(u_{l,m}^{t+1}-u_{l,m}^{t}) &= \kappa(\frac{u_{l+1,m}^{t+1}-u_{l,m}^{t+1}}{\delta x_{l+1}}-\frac{u_{l,m}^{t+1}-u_{l-1,m}^{t+1}}{\delta x_{l-1}})\Delta y+\kappa(\frac{u_{l,m+1}^{t+1}-u_{l,m}^{t+1}}{\delta y_{m+1}}-\frac{u_{l,m}^{t+1}-u_{l,m-1}^{t+1}}{\delta x_{m-1}})\Delta x \\ a_{l,m}^{t+1}u_{l,m}^{t+1}-a_{l+1,m}^{t+1}u_{l+1,m}^{t+1}-a_{l-1,m}^{t+1}u_{l-1,m}^{t+1}-a_{l,m+1}^{t+1}u_{l,m+1}^{t+1}-a_{l,m-1}^{t+1}u_{l,m-1}^{t+1} &= a_{l,m}^{t}u_{l,m}^{t}\end{aligned}$$

where

$$\begin{aligned}a_{l+1,m}^{t+1} &= \frac{\kappa\Delta y}{\delta x_{l+1}}=\frac{\kappa\Delta y}{\Delta x} \\ a_{l-1,m}^{t+1} &= \frac{\kappa\Delta y}{\delta x_{l-1}}=\frac{\kappa\Delta y}{\Delta x} \\ a_{l,m+1}^{t+1} &= \frac{\kappa\Delta x}{\delta y_{m+1}}=\frac{\kappa\Delta x}{\Delta y} \\ a_{l,m-1}^{t+1} &= \frac{\kappa\Delta x}{\delta y_{m-1}}=\frac{\kappa\Delta x}{\Delta y} \\ a_{l,m}^{t} &= \frac{\rho c\Delta x\Delta y}{\Delta t} \\ a_{l,m}^{t+1} &= a_{l+1,m}^{t+1}+a_{l-1,m}^{t+1}+a_{l,m+1}^{t+1}+a_{l,m-1}^{t+1}+a_{l,m}^{t}\end{aligned}$$

(2) Considering the boundary condition

When $l, m \in \Gamma$, data are given that $u = g$ or $\kappa\frac{\partial u}{\partial x}n_x + \kappa\frac{\partial u}{\partial y}n_y = h$.

Starting from only considering h from the left and down sides, the basic equation can be decribed as the following.

$$\rho c\Delta x\Delta y(u^{t+1}_{l,m} - u^{t}_{l,m}) = (\kappa(\frac{\partial u}{\partial x})^{t\to t+1}_{l+1,m} + h^{t\to t+1}_{l-1,m})\Delta y\Delta t + (\kappa(\frac{\partial u}{\partial y})^{t\to t+1}_{l,m+1} + h^{t\to t+1}_{l,m-1})\Delta x\Delta t$$

In explicit format,

$$\begin{aligned}
\rho c\Delta x\Delta y(u^{t+1}_{l,m} - u^{t}_{l,m}) &= \kappa(\frac{u^t_{l+1,m} - u^t_{l,m}}{\delta x_{l+1}} + h^t_{l-1,m})\Delta y\Delta t + \kappa(\frac{u^t_{l,m+1} - u^t_{l,m}}{\delta y_{m+1}} + h^t_{l,m-1})\Delta x\Delta t \\
\frac{\rho c\Delta x\Delta y(u^{t+1}_{l,m} - u^{t}_{l,m})}{\Delta t} &= \kappa(\frac{u^t_{l+1,m} - u^t_{l,m}}{\delta x_{l+1}} + h^t_{l-1,m})\Delta y + \kappa(\frac{u^t_{l,m+1} - u^t_{l,m}}{\delta y_{m+1}} + h^t_{l,m-1})\Delta x \\
a^{t+1}_{l,m}u^{t+1}_{l,m} &= a^t_{l+1,m}u^t_{l+1,m} + \kappa h^t_{l-1,m}\Delta y + a^t_{l,m+1}u^t_{l,m+1} + \kappa h^t_{l,m-1}\Delta x + a^t_{l,m}u^t_{l,m}
\end{aligned}$$

We described this in general format.

$$\begin{aligned}
a^{t+1}_{l,m}u^{t+1}_{l,m} &= a^t_{l+1,m}u^t_{l+1,m} + a^t_{l-1,m}u^t_{l-1,m} + a^t_{l,m+1}u^t_{l,m+1} + a^t_{l,m-1}u^t_{l,m-1} \\
&\quad + \kappa h^t_{l+1,m}\Delta y + \kappa h^t_{l-1,m}\Delta y + \kappa h^t_{l,m+1}\Delta x + \kappa h^t_{l,m-1}\Delta x + a^t_{l,m}u^t_{l,m}
\end{aligned}$$

In implicit format, similarly we get that

$$\begin{aligned}
a^{t+1}_{l,m}u^{t+1}_{l,m} &= a^{t+1}_{l+1,m}u^{t+1}_{l+1,m} + a^{t+1}_{l-1,m}u^{t+1}_{l-1,m} + a^{t+1}_{l,m+1}u^{t+1}_{l,m+1} + a^{t+1}_{l,m-1}u^{t+1}_{l,m-1} \\
&\quad + \kappa h^{t+1}_{l+1,m}\Delta y + \kappa h^{t+1}_{l-1,m}\Delta y + \kappa h^{t+1}_{l,m+1}\Delta x + \kappa h^{t+1}_{l,m-1}\Delta x + a^t_{l,m}u^t_{l,m}
\end{aligned}$$

(a) If prescribed $u = g$ for the boundary nodes in Ω

$h_{l+1,m} = h_{l-1,m} = h_{l,m+1} = h_{l,m-1} = 0$;

when $l + \Delta x = 1$, $u_{l+1,m} = g$;

when $l - \Delta x = 0$, $u_{l-1,m} = g$;

when $m + \Delta y = 0$, $u_{l,m+1} = g$;

when $m - \Delta y = 0$, $u_{l,m-1} = g$.

(b) If prescribed $\kappa\frac{\partial u}{\partial x}n_x + \kappa\frac{\partial u}{\partial y}n_y = h$ for the boundary nodes in Ω

when $l + 1 = 1$, $h_{l+1,m} = -h$ and $a_{l+1,m} = h_{l-1,m} = h_{l,m+1} = h_{l,m-1} = 0$;

when $l - 1 = 0$, $h_{l-1,m} = h$ and $a_{l-1,m} = h_{l+1,m} = h_{l,m+1} = h_{l,m-1} = 0$;

when $m + 1 = 1$, $h_{l,m+1} = -h$ and $a_{l,m+1} = h_{l+1,m} = h_{l-1,m} = h_{l,m-1} = 0$;

when $m - 1 = 0$, $h_{l,m-1} = h$ and $a_{l,m-1} = h_{l+1,m} = h_{l-1,m} = h_{l,m+1} = 0$.

Targets (Course Video 15)

编写程序

- [ ] 程序可以重启（需要hfd5来帮我们）
- [ ] 防御性写法，能处理异常，有断言，有详细的注释
- [ ] 编译的时候不能有任何warning
- [ ] 有makfile或者cmake，可以在太乙上直接编译
- [ ] 有profiling，程序的哪里耗时最多
- [ ] 利用github进行版本控制和团队合作
- [ ] 用开源软件可视化，二维的话需要用VTK或者Paraview
- [ ] 技术性的报告，最好包括LaTex的使用

测试

- [ ] 方法的稳定性
- [ ] 误差分析
- [ ] 并行效率（强可扩展性、弱可扩展性、Petsc不同求解器的影响）