Penghan CHEN (陈鹏翰) Hao HUANG (黄灏)

Problem

We consider the transient heat equation in a two-dimensional (2D) unit square $\Omega := (0,1) \times (0,1)$. The boundary of the domain Γ can be decomposed into

the bottom face $\Gamma_b := \{(x,y) : x \in (0,1), y = 0\}$;

the top face $\Gamma_t := \{(x,y) : x \in (0,1), y = 1\}$;

the left face $\Gamma_l := \{(x,y) : x = 0, y \in (0,1)\}$;

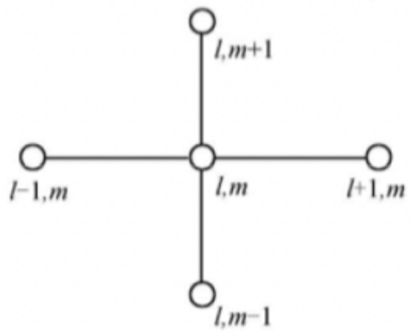
the right face $\Gamma_r := \{(x,y) : x = 1, y \in (0,1)\}$.

Let f be the heat supply per unit volume, u be the temperature, ρ be the density, c be the heat capacity, u_0 be the initial temperature, κ be the conductivity, n_x and n_y be the Cartesian components of the unit outward normal vector . The boundary data involves the prescribed temperature g on Γ_g and heat flux h on Γ_h. The boundary Γ admits a non-overlapping decomposition: $\Gamma = \overline{\Gamma_g \cup \Gamma_h}$ and $\emptyset = \Gamma_g \cap \Gamma_h$. The transient heat equation may be stated as follows.

$$\begin{aligned} \rho c \frac{\partial u}{\partial t} - \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= f \quad \text{on } \Omega \times (0,T) \\ u &= g \quad \text{on } \Gamma_g \times (0,T) \\ \kappa \frac{\partial u}{\partial x} n_x + \kappa \frac{\partial u}{\partial y} n_y &= h \quad \text{on } \Gamma_h \times (0,T) \\ u|_{t=0} &= u_0 \quad \text{in } \Omega. \end{aligned}$$

Solutions

We learned the solution methods from the videos on bilibili and the books from class MAE5032-HPC by Ju-LIU



$u^t_{l,m}$ is denoted as temperature u in node l,m at time t.

(1) For each internal node, the $u_{l,m}^{t+1}$ should be depended on five data at time t (i.e., $u_{l,m}^{t}$, $u_{l+1,m}^{t}$, $u_{l-,m}^{t}$, $u_{l,m+1}^{t}$, $u_{l,m-1}^{t}$) spatially and temporally.

The basic equation should be

$$\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^{t}) = \kappa((\frac{\partial u}{\partial x})_{l+1,m}^{t \to t+1} - (\frac{\partial u}{\partial x})_{l-1,m}^{t \to t+1})\Delta y \Delta t + \kappa((\frac{\partial u}{\partial y})_{l,m+1}^{t \to t+1} - (\frac{\partial u}{\partial y})_{l,m-1}^{t \to t+1})\Delta x \Delta t$$

In explicit format, the relationship can be decribed by the following.

$$\begin{aligned} \frac{\rho c \Delta x \Delta y}{\Delta t}(u_{l,m}^{t+1} - u_{l,m}^{t}) &= \kappa(\frac{u_{l+1,m}^{t} - u_{l,m}^{t}}{\delta x_{l+1}} - \frac{u_{l,m}^{t} - u_{l-1,m}^{t}}{\delta x_{l-1}})\Delta y + \kappa(\frac{u_{l,m+1}^{t} - u_{l,m}^{t}}{\delta y_{m+1}} - \frac{u_{l,m}^{t} - u_{l,m-1}^{t}}{\delta y_{m+1}})\Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} = a_{l+1,m}^{t} u_{l+1,m}^{t} &+ a_{l-1,m}^{t} u_{l-1,m}^{t} + a_{l,m+1}^{t} u_{l,m+1}^{t} + a_{l,m-1}^{t} u_{l,m-1}^{t} + a_{l,m}^{t} u_{l,m}^{t} \end{aligned}$$

where

$$\begin{aligned} a_{l+1,m}^{t} &= \frac{\kappa \Delta y}{\delta x_{l+1}} = \frac{\kappa \Delta y}{\Delta x} = a_{l-1,m}^{t} = \frac{\kappa \Delta y}{\delta x_{l-1}} \\ a_{l,m+1}^{t} &= \frac{\kappa \Delta x}{\delta y_{m+1}} = \frac{\kappa \Delta x}{\Delta y} = a_{l,m-1}^{t} = \frac{\kappa \Delta x}{\delta y_{m-1}} \\ a_{l,m}^{t} &= \frac{\rho c \Delta x \Delta y}{\Delta t} - \frac{\kappa \Delta y}{\delta x_{l+1}} - \frac{\kappa \Delta y}{\delta x_{l-1}} - \frac{\kappa \Delta x}{\delta y_{m+1}} - \frac{\kappa \Delta x}{\delta y_{m-1}} \\ &= \frac{\rho c \Delta x \Delta y}{\Delta t} - 2\frac{\kappa \Delta y}{\Delta x} - 2\frac{\kappa \Delta x}{\Delta y} \\ a_{l,m}^{t+1} &= \frac{\rho c \Delta x \Delta y}{\Delta t} = 2a_{l+1,m}^{t} + 2a_{l,m+1}^{t} + a_{l,m}^{t} \end{aligned}$$

Note that δ is actually the same as Δ for internal nodes.

In implicit format, the relationship can be decribed by the following.

$$\begin{aligned} \frac{\rho c \Delta x \Delta y}{\Delta t}(u_{l,m}^{t+1} - u_{l,m}^{t}) &= \kappa(\frac{u_{l+1,m}^{t+1} - u_{l,m}^{t+1}}{\delta x_{l+1}} - \frac{u_{l,m}^{t+1} - u_{l-1,m}^{t+1}}{\delta x_{l-1}})\Delta y + \kappa(\frac{u_{l,m+1}^{t+1} - u_{l,m}^{t+1}}{\delta y_{m+1}} - \frac{u_{l,m}^{t+1} - u_{l,m-1}^{t+1}}{\delta x_{m-1}})\Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} - a_{l+1,m}^{t+1} u_{l+1,m}^{t+1} &- a_{l-1,m}^{t+1} u_{l-1,m}^{t+1} - a_{l,m+1}^{t+1} u_{l,m+1}^{t+1} - a_{l,m-1}^{t+1} u_{l,m-1}^{t+1} = a_{l,m}^{t} u_{l,m}^{t} \end{aligned}$$

where

$$\begin{aligned} a_{l+1,m}^{t+1} &= \frac{\kappa \Delta y}{\delta x_{l+1}} \\ a_{l-1,m}^{t+1} &= \frac{\kappa \Delta y}{\delta x_{l-1}} \\ a_{l,m+1}^{t+1} &= \frac{\kappa \Delta x}{\delta y_{m+1}} \\ a_{l,m-1}^{t+1} &= \frac{\kappa \Delta x}{\delta y_{m-1}} \\ a_{l,m}^{t} &= \frac{\rho c \Delta x \Delta y}{\Delta t} \\ a_{l,m}^{t+1} &= a_{l+1,m}^{t+1} + a_{l-1,m}^{t+1} + a_{l,m+1}^{t+1} + a_{l,m-1}^{t+1} + a_{l,m}^{t} \end{aligned}$$

(2) Considering the boundary condition

When $l, m \in \Gamma$, data are given that $u = g$ or $\kappa \dfrac{\partial u}{\partial x} n_x + \kappa \dfrac{\partial u}{\partial y} n_y = h$.

Starting from only considering h from the left and down sides, the basic equation can be decribed as the following.

$$\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^{t}) = (\kappa (\frac{\partial u}{\partial x})_{l+1,m}^{t \to t+1} + h_{l-1,m}^{t \to t+1}) \Delta y \Delta t + (\kappa (\frac{\partial u}{\partial y})_{l,m+1}^{t \to t+1} + h_{l,m-1}^{t \to t+1}) \Delta x \Delta t$$

In explicit format,

$$\begin{aligned}
\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^{t}) &= \kappa (\frac{u_{l+1,m}^{t} - u_{l,m}^{t}}{\delta x_{l+1}} + h_{l-1,m}^{t}) \Delta y \Delta t + \kappa (\frac{u_{l,m+1}^{t} - u_{l,m}^{t}}{\delta y_{m+1}} + h_{l,m-1}^{t}) \Delta x \Delta t \\
\frac{\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^{t})}{\Delta t} &= \kappa (\frac{u_{l+1,m}^{t} - u_{l,m}^{t}}{\delta x_{l+1}} + h_{l-1,m}^{t}) \Delta y + \kappa (\frac{u_{l,m+1}^{t} - u_{l,m}^{t}}{\delta y_{m+1}} + h_{l,m-1}^{t}) \Delta x \\
a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^{t} u_{l+1,m}^{t} + \kappa h_{l-1,m}^{t} \Delta y + a_{l,m+1}^{t} u_{l,m+1}^{t} + \kappa h_{l,m-1}^{t} \Delta x + a_{l,m}^{t} u_{l,m}^{t}
\end{aligned}$$

We described this in general format.

$$\begin{aligned}
a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^{t} u_{l+1,m}^{t} + a_{l-1,m}^{t} u_{l-1,m}^{t} + a_{l,m+1}^{t} u_{l,m+1}^{t} + a_{l,m-1}^{t} u_{l,m-1}^{t} \\
&+ \kappa h_{l+1,m}^{t} \Delta y + \kappa h_{l-1,m}^{t} \Delta y + \kappa h_{l,m+1}^{t} \Delta x + \kappa h_{l,m-1}^{t} \Delta x + a_{l,m}^{t} u_{l,m}^{t}
\end{aligned}$$

In implicit format, similarly we get that

$$\begin{aligned}
a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^{t+1} u_{l+1,m}^{t+1} + a_{l-1,m}^{t+1} u_{l-1,m}^{t+1} + a_{l,m+1}^{t+1} u_{l,m+1}^{t+1} + a_{l,m-1}^{t+1} u_{l,m-1}^{t+1} \\
&+ \kappa h_{l+1,m}^{t+1} \Delta y + \kappa h_{l-1,m}^{t+1} \Delta y + \kappa h_{l,m+1}^{t+1} \Delta x + \kappa h_{l,m-1}^{t+1} \Delta x + a_{l,m}^{t} u_{l,m}^{t}
\end{aligned}$$

(a) If prescribed $u = g$ for a boundary node

$h_{l+1,m} = h_{l-1,m} = h_{l,m+1} = h_{l,m-1} = 0$;

when $l+1 = 1$, $u_{l+1,m} = g$ and $a_{l+1,m} = \dfrac{\kappa \Delta y}{\delta x_{l+1}} = \dfrac{\kappa \Delta y}{\Delta x / 2}$;

when $l-1 = 0$, $u_{l-1,m} = g$ and $a_{l-1,m} = \dfrac{\kappa \Delta y}{\delta x_{l-1}} = \dfrac{\kappa \Delta y}{\Delta x / 2}$;

when $m+1 = 0$, $u_{l,m+1} = g$ and $a_{l,m+1} = \dfrac{\kappa \Delta x}{\delta y_{m+1}} = \dfrac{\kappa \Delta x}{\Delta y / 2}$;

when $m-1 = 0$, $u_{l,m-1} = g$ and $a_{l,m-1} = \dfrac{\kappa \Delta x}{\delta y_{m-1}} = \dfrac{\kappa \Delta x}{\Delta y / 2}$.

(b) If prescribed $\kappa \dfrac{\partial u}{\partial x} n_x + \kappa \dfrac{\partial u}{\partial y} n_y = h$ for a boundary node

when $l+1 = 1$, $h_{l+1,m} = -h$ and $a_{l+1,m} = h_{l-1,m} = h_{l,m+1} = h_{l,m-1} = 0$;

when $l-1 = 0$, $h_{l-1,m} = h$ and $a_{l-1,m} = h_{l+1,m} = h_{l,m+1} = h_{l,m-1} = 0$;

when $m+1 = 1$, $h_{l,m+1} = -h$ and $a_{l,m+1} = h_{l+1,m} = h_{l-1,m} = h_{l,m-1} = 0$;

when $m-1 = 0$, $h_{l,m-1} = h$ and $a_{l,m-1} = h_{l+1,m} = h_{l-1,m} = h_{l,m+1} = 0$;

Targets (Course Video 15)

编写程序

- [ ] 程序可以重启（需要hfd5来帮我们）
- [ ] 防御性写法，能处理异常，有断言，有详细的注释
- [ ] 编译的时候不能有任何warning
- [ ] 有makfile或者cmake，可以在太乙上直接编译
- [ ] 有profiling，程序的哪里耗时最多
- [ ] 利用github进行版本控制和团队合作
- [ ] 用开源软件可视化，二维的话需要用VTK或者Paraview
- [ ] 技术性的报告，最好包括LaTex的使用

测试

- [ ] 方法的稳定性
- [ ] 误差分析
- [ ] 并行效率（强可扩展性、弱可扩展性、Petsc不同求解器的影响）