

Problem

We consider the transient heat equation in a two-dimensional (2D) unit square $\Omega := (0, 1) \times (0, 1)$. The boundary of the domain Γ can be decomposed into

the bottom face $\Gamma_b := \{(x, y) : x \in (0, 1), y = 0\}$;

the top face $\Gamma_t := \{(x, y) : x \in (0, 1), y = 1\}$;

the left face $\Gamma_l := \{(x, y) : x = 0, y \in (0, 1)\}$;

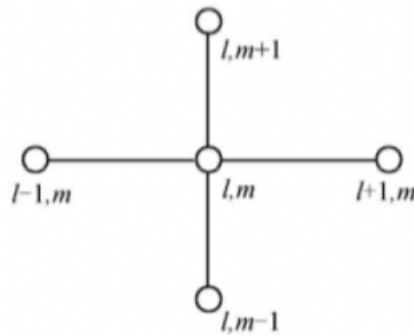
the right face $\Gamma_r := \{(x, y) : x = 1, y \in (0, 1)\}$.

Let f be the heat supply per unit volume, u be the temperature, ρ be the density, c be the heat capacity, u_0 be the initial temperature, κ be the conductivity, n_x and n_y be the Cartesian components of the unit outward normal vector. The boundary data involves the prescribed temperature g on Γ_g and heat flux h on Γ_h . The boundary Γ admits a non-overlapping decomposition: $\Gamma = \overline{\Gamma_g} \cup \overline{\Gamma_h}$ and $\emptyset = \Gamma_g \cap \Gamma_h$. The transient heat equation may be stated as follows.

$$\begin{aligned} \rho c \frac{\partial u}{\partial t} - \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= f \quad \text{on } \Omega \times (0, T) \\ u &= g \quad \text{on } \Gamma_g \times (0, T) \\ \kappa \frac{\partial u}{\partial x} n_x + \kappa \frac{\partial u}{\partial y} n_y &= h \quad \text{on } \Gamma_h \times (0, T) \\ u|_{t=0} &= u_0 \quad \text{in } \Omega. \end{aligned}$$

Solutions

We learned the solution methods from the [videos](#) on bilibili and the books from class MAE5032-HPC by [Ju-LIU](#)



$u_{l,m}^t$ is denoted as temperature u in node l, m at time t .

(1) For each internal node, the $u_{l,m}^{t+1}$ should be depended on five data at time t (i.e., $u_{l,m}^t, u_{l+1,m}^t, u_{l-1,m}^t, u_{l,m+1}^t, u_{l,m-1}^t$) spatially and temporally.

The basic equation should be

$$\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t) = \kappa \left(\left(\frac{\partial u}{\partial x} \right)_{l+1,m}^{t \rightarrow t+1} - \left(\frac{\partial u}{\partial x} \right)_{l-1,m}^{t \rightarrow t+1} \right) \Delta y \Delta t + \kappa \left(\left(\frac{\partial u}{\partial y} \right)_{l,m+1}^{t \rightarrow t+1} - \left(\frac{\partial u}{\partial y} \right)_{l,m-1}^{t \rightarrow t+1} \right) \Delta x \Delta t$$

In explicit format, the relationship can be described by the following.

$$\begin{aligned} \frac{\rho c \Delta x \Delta y}{\Delta t} (u_{l,m}^{t+1} - u_{l,m}^t) &= \kappa \left(\frac{u_{l+1,m}^t - u_{l,m}^t}{\delta x_{l+1}} + \frac{u_{l,m}^t - u_{l-1,m}^t}{\delta x_{l-1}} \right) \Delta y + \kappa \left(\frac{u_{l,m+1}^t - u_{l,m}^t}{\delta y_{m+1}} + \frac{u_{l,m}^t - u_{l,m-1}^t}{\delta y_{m-1}} \right) \Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^t u_{l+1,m}^t + a_{l-1,m}^t u_{l-1,m}^t + a_{l,m+1}^t u_{l,m+1}^t + a_{l,m-1}^t u_{l,m-1}^t + a_{l,m}^t u_{l,m}^t \end{aligned}$$

where

$$\begin{aligned} a_{l+1,m}^t &= \frac{\kappa \Delta y}{\delta x_{l+1}} = \frac{\kappa \Delta y}{\Delta x} = a_{l-1,m}^t = \frac{\kappa \Delta y}{\delta x_{l-1}} \\ a_{l,m+1}^t &= \frac{\kappa \Delta x}{\delta y_{m+1}} = \frac{\kappa \Delta x}{\Delta y} = a_{l,m-1}^t = \frac{\kappa \Delta x}{\delta y_{m-1}} \\ a_{l,m}^t &= \frac{\rho c \Delta x \Delta y}{\Delta t} - \frac{\kappa \Delta y}{\delta x_{l+1}} - \frac{\kappa \Delta y}{\delta x_{l-1}} - \frac{\kappa \Delta x}{\delta y_{m+1}} - \frac{\kappa \Delta x}{\delta y_{m-1}} \\ &= \frac{\rho c \Delta x \Delta y}{\Delta t} - 2 \frac{\kappa \Delta y}{\Delta x} - 2 \frac{\kappa \Delta x}{\Delta y} \\ a_{l,m}^{t+1} &= \frac{\rho c \Delta x \Delta y}{\Delta t} = 2a_{l+1,m}^t + 2a_{l,m+1}^t + a_{l,m}^t \end{aligned}$$

Note that δ is actually the same as Δ for internal nodes.

In implicit format, the relationship can be described by the following.

$$\begin{aligned} \frac{\rho c \Delta x \Delta y}{\Delta t} (u_{l,m}^{t+1} - u_{l,m}^t) &= \kappa \left(\frac{u_{l+1,m}^{t+1} - u_{l,m}^{t+1}}{\delta x_{l+1}} - \frac{u_{l,m}^{t+1} - u_{l-1,m}^{t+1}}{\delta x_{l-1}} \right) \Delta y + \kappa \left(\frac{u_{l,m+1}^{t+1} - u_{l,m}^{t+1}}{\delta y_{m+1}} - \frac{u_{l,m}^{t+1} - u_{l,m-1}^{t+1}}{\delta y_{m-1}} \right) \Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} - a_{l+1,m}^{t+1} u_{l+1,m}^{t+1} - a_{l-1,m}^{t+1} u_{l-1,m}^{t+1} - a_{l,m+1}^{t+1} u_{l,m+1}^{t+1} - a_{l,m-1}^{t+1} u_{l,m-1}^{t+1} &= a_{l,m}^t u_{l,m}^t \end{aligned}$$

where

$$\begin{aligned} a_{l+1,m}^{t+1} &= \frac{\kappa \Delta y}{\delta x_{l+1}} \\ a_{l-1,m}^{t+1} &= \frac{\kappa \Delta y}{\delta x_{l-1}} \\ a_{l,m+1}^{t+1} &= \frac{\kappa \Delta x}{\delta y_{m+1}} \\ a_{l,m-1}^{t+1} &= \frac{\kappa \Delta x}{\delta y_{m-1}} \\ a_{l,m}^t &= \frac{\rho c \Delta x \Delta y}{\Delta t} \\ a_{l,m}^{t+1} &= a_{l+1,m}^{t+1} + a_{l-1,m}^{t+1} + a_{l,m+1}^{t+1} + a_{l,m-1}^{t+1} + a_{l,m}^t \end{aligned}$$

(2) Considering the boundary condition

As there is $\kappa \frac{\partial u}{\partial x} n_x + \kappa \frac{\partial u}{\partial y} n_y = h$, we considered this is a Neumann problem.

From the basic equation, we know that

$$\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t) = (\kappa (\frac{\partial u}{\partial x})_{l+1,m}^{t \rightarrow t+1} + h_{l-1,m}^{t \rightarrow t+1}) \Delta y \Delta t + (\kappa (\frac{\partial u}{\partial y})_{l,m+1}^{t \rightarrow t+1} + h_{l,m-1}^{t \rightarrow t+1}) \Delta x \Delta t$$

In explicit format,

$$\begin{aligned} \rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t) &= \kappa (\frac{u_{l+1,m}^t - u_{l,m}^t}{\delta x_{l+1}} + h_{l-1,m}^t) \Delta y \Delta t + \kappa (\frac{u_{l,m+1}^t - u_{l,m}^t}{\delta y_{m+1}} + h_{l,m-1}^t) \Delta x \Delta t \\ \frac{\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t)}{\Delta t} &= \kappa (\frac{u_{l+1,m}^t - u_{l,m}^t}{\delta x_{l+1}} + h_{l-1,m}^t) \Delta y + \kappa (\frac{u_{l,m+1}^t - u_{l,m}^t}{\delta y_{m+1}} + h_{l,m-1}^t) \Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^t u_{l+1,m}^t + \kappa h_{l-1,m}^t \Delta y + a_{l,m+1}^t u_{l,m+1}^t + \kappa h_{l,m-1}^t \Delta x + a_{l,m}^t u_{l,m}^t \end{aligned}$$

where

$$\begin{aligned} \kappa h_{l-1,m}^t \Delta y + \kappa h_{l,m-1}^t \Delta x &= h \Delta x \Delta y \\ a_{l+1,m}^t &= \frac{\kappa \Delta y}{\delta x_{l+1}} \\ a_{l,m+1}^t &= \frac{\kappa \Delta x}{\delta y_{m+1}} \\ a_{l,m}^t &= \frac{\rho c \Delta x \Delta y}{\Delta t} - \frac{\kappa \Delta y}{\delta x_{l+1}} - \frac{\kappa \Delta x}{\delta y_{m+1}} \\ a_{l,m}^{t+1} &= \frac{\rho c \Delta x \Delta y}{\Delta t} = a_{l+1,m}^t + a_{l,m+1}^t \end{aligned}$$

when $l = 0$, $h_{l-1,m}^t = 0$; when $m = 0$, $h_{l,m-1}^t = 0$;

when $l = 1$, $\delta x_{l+1} = \Delta x/2$; when $m = 1$, $\delta y_{m+1} = \Delta y/2$;

In implicit format,

$$\begin{aligned} \frac{\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t)}{\Delta t} &= \kappa (\frac{u_{l+1,m}^{t+1} - u_{l,m}^{t+1}}{\delta x_{l+1}} + h_{l-1,m}^{t+1}) \Delta y + \kappa (\frac{u_{l,m+1}^{t+1} - u_{l,m}^{t+1}}{\delta y_{m+1}} + h_{l,m-1}^{t+1}) \Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^{t+1} u_{l+1,m}^{t+1} + \kappa h_{l-1,m}^{t+1} \Delta y + a_{l,m+1}^{t+1} u_{l,m+1}^{t+1} + \kappa h_{l,m-1}^{t+1} \Delta x + a_{l,m}^t u_{l,m}^t \end{aligned}$$

where

$$\begin{aligned} \kappa h_{l-1,m}^{t+1} \Delta y + \kappa h_{l,m-1}^{t+1} \Delta x &= h \Delta x \Delta y \\ a_{l+1,m}^{t+1} &= \frac{\kappa \Delta y}{\delta x_{l+1}} \\ a_{l,m+1}^{t+1} &= \frac{\kappa \Delta x}{\delta y_{m+1}} \\ a_{l,m}^t &= \frac{\rho c \Delta x \Delta y}{\Delta t} \\ a_{l,m}^{t+1} &= \frac{\rho c \Delta x \Delta y}{\Delta t} + a_{l+1,m}^{t+1} + a_{l,m+1}^{t+1} \end{aligned}$$

when $l = 0$, $h_{l-1,m}^t = 0$; when $m = 0$, $h_{l,m-1}^t = 0$;

when $l = 1$, $\delta x_{l+1} = \Delta x/2$; when $m = 1$, $\delta y_{m+1} = \Delta y/2$;

Targets (Course Video 15)

编写程序

- ☐ 程序可以重启（需要hfd5来帮我们）
- ☐ 防御性写法，能处理异常，有断言，有详细的注释
- ☐ 编译的时候不能有任何warning
- ☐ 有makfile或者cmake，可以在太乙上直接编译
- ☐ 有profiling，程序的哪里耗时最多
- ☐ 利用github进行版本控制和团队合作
- ☐ 用开源软件可视化，二维的话需要用VTK或者Paraview
- ☐ 技术性的报告，最好包括LaTeX的使用

测试

- ☐ 方法的稳定性
- ☐ 误差分析
- ☐ 并行效率（强可扩展性、弱可扩展性、Petsc不同求解器的影响）