

## Problem

We consider the transient heat equation in a two-dimensional (2D) unit square  $\Omega := (0, 1) \times (0, 1)$ . The boundary of the domain  $\Gamma$  can be decomposed into

the bottom face  $\Gamma_b := \{(x, y) : x \in (0, 1), y = 0\}$ ;

the top face  $\Gamma_t := \{(x, y) : x \in (0, 1), y = 1\}$ ;

the left face  $\Gamma_l := \{(x, y) : x = 0, y \in (0, 1)\}$ ;

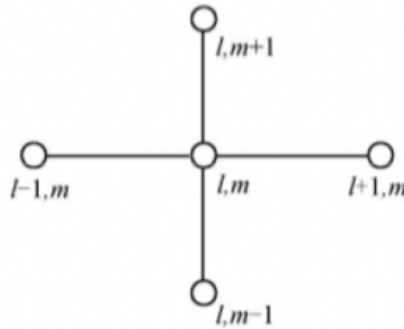
the right face  $\Gamma_r := \{(x, y) : x = 1, y \in (0, 1)\}$ .

Let  $f$  be the heat supply per unit volume,  $u$  be the temperature,  $\rho$  be the density,  $c$  be the heat capacity,  $u_0$  be the initial temperature,  $\kappa$  be the conductivity,  $n_x$  and  $n_y$  be the Cartesian components of the unit outward normal vector. The boundary data involves the prescribed temperature  $g$  on  $\Gamma_g$  and heat flux  $h$  on  $\Gamma_h$ . The boundary  $\Gamma$  admits a non-overlapping decomposition:  $\Gamma = \overline{\Gamma_g} \cup \overline{\Gamma_h}$  and  $\emptyset = \Gamma_g \cap \Gamma_h$ . The transient heat equation may be stated as follows.

$$\begin{aligned} \rho c \frac{\partial u}{\partial t} - \kappa \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &= f \quad \text{on } \Omega \times (0, T) \\ u &= g \quad \text{on } \Gamma_g \times (0, T) \\ \kappa \frac{\partial u}{\partial x} n_x + \kappa \frac{\partial u}{\partial y} n_y &= h \quad \text{on } \Gamma_h \times (0, T) \\ u|_{t=0} &= u_0 \quad \text{in } \Omega. \end{aligned}$$

## Solutions

We learned the solution methods from the [videos](#) on bilibili and the books from class MAE5032-HPC by [Ju-LIU](#)



$u_{l,m}^t$  is denoted as temperature  $u$  in node  $l, m$  at time  $t$ .

**(1) For each internal node, the  $u_{l,m}^{t+1}$  should be depended on five data at time  $t$  (i.e.,  $u_{l,m}^t$ ,  $u_{l+1,m}^t$ ,  $u_{l-1,m}^t$ ,  $u_{l,m+1}^t$ ,  $u_{l,m-1}^t$ ) spatially and temporally.**

The basic equation should be

$$\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t) = \kappa \left( \left( \frac{\partial u}{\partial x} \right)_{l+1,m}^{t \rightarrow t+1} - \left( \frac{\partial u}{\partial x} \right)_{l-1,m}^{t \rightarrow t+1} \right) \Delta y \Delta t + \kappa \left( \left( \frac{\partial u}{\partial y} \right)_{l,m+1}^{t \rightarrow t+1} - \left( \frac{\partial u}{\partial y} \right)_{l,m-1}^{t \rightarrow t+1} \right) \Delta x \Delta t$$

In explicit format, the relationship can be decribed by the following.

$$\begin{aligned} \frac{\rho c \Delta x \Delta y}{\Delta t} (u_{l,m}^{t+1} - u_{l,m}^t) &= \kappa \left( \frac{u_{l+1,m}^t - u_{l,m}^t}{\delta x_{l+1}} + \frac{u_{l,m}^t - u_{l-1,m}^t}{\delta x_{l-1}} \right) \Delta y + \kappa \left( \frac{u_{l,m+1}^t - u_{l,m}^t}{\delta y_{m+1}} + \frac{u_{l,m}^t - u_{l,m-1}^t}{\delta y_{m-1}} \right) \Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^t u_{l+1,m}^t + a_{l-1,m}^t u_{l-1,m}^t + a_{l,m+1}^t u_{l,m+1}^t + a_{l,m-1}^t u_{l,m-1}^t + a_{l,m}^t u_{l,m}^t \end{aligned}$$

where

$$\begin{aligned} a_{l+1,m}^t &= \frac{\kappa \Delta y}{\delta x_{l+1}} = \frac{\kappa \Delta y}{\Delta x} = a_{l-1,m}^t = \frac{\kappa \Delta y}{\delta x_{l-1}} \\ a_{l,m+1}^t &= \frac{\kappa \Delta x}{\delta y_{m+1}} = \frac{\kappa \Delta x}{\Delta y} = a_{l,m-1}^t = \frac{\kappa \Delta x}{\delta y_{m-1}} \\ a_{l,m}^t &= \frac{\rho c \Delta x \Delta y}{\Delta t} - \frac{\kappa \Delta y}{\delta x_{l+1}} - \frac{\kappa \Delta y}{\delta x_{l-1}} - \frac{\kappa \Delta x}{\delta y_{m+1}} - \frac{\kappa \Delta x}{\delta y_{m-1}} \\ &= \frac{\rho c \Delta x \Delta y}{\Delta t} - 2 \frac{\kappa \Delta y}{\Delta x} - 2 \frac{\kappa \Delta x}{\Delta y} \\ a_{l,m}^{t+1} &= \frac{\rho c \Delta x \Delta y}{\Delta t} = 2a_{l+1,m}^t + 2a_{l,m+1}^t + a_{l,m}^t \end{aligned}$$

Note that  $\delta$  is actually the same as  $\Delta$  for internal nodes.

In implicit format, the relationship can be decribed by the following.

$$\begin{aligned} \frac{\rho c \Delta x \Delta y}{\Delta t} (u_{l,m}^{t+1} - u_{l,m}^t) &= \kappa \left( \frac{u_{l+1,m}^{t+1} - u_{l,m}^{t+1}}{\delta x_{l+1}} - \frac{u_{l,m}^{t+1} - u_{l-1,m}^{t+1}}{\delta x_{l-1}} \right) \Delta y + \kappa \left( \frac{u_{l,m+1}^{t+1} - u_{l,m}^{t+1}}{\delta y_{m+1}} - \frac{u_{l,m}^{t+1} - u_{l,m-1}^{t+1}}{\delta y_{m-1}} \right) \Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} - a_{l+1,m}^{t+1} u_{l+1,m}^{t+1} - a_{l-1,m}^{t+1} u_{l-1,m}^{t+1} - a_{l,m+1}^{t+1} u_{l,m+1}^{t+1} - a_{l,m-1}^{t+1} u_{l,m-1}^{t+1} &= a_{l,m}^t u_{l,m}^t \end{aligned}$$

where

$$\begin{aligned} a_{l+1,m}^{t+1} &= \frac{\kappa \Delta y}{\delta x_{l+1}} \\ a_{l-1,m}^{t+1} &= \frac{\kappa \Delta y}{\delta x_{l-1}} \\ a_{l,m+1}^{t+1} &= \frac{\kappa \Delta x}{\delta y_{m+1}} \\ a_{l,m-1}^{t+1} &= \frac{\kappa \Delta x}{\delta y_{m-1}} \\ a_{l,m}^t &= \frac{\rho c \Delta x \Delta y}{\Delta t} \\ a_{l,m}^{t+1} &= a_{l+1,m}^{t+1} + a_{l-1,m}^{t+1} + a_{l,m+1}^{t+1} + a_{l,m-1}^{t+1} + a_{l,m}^t \end{aligned}$$

## (2) Considering the boundary condition

When  $l, m \in \Gamma$ , data are given that  $u = g$  or  $\kappa \frac{\partial u}{\partial x} n_x + \kappa \frac{\partial u}{\partial y} n_y = h$ .

Starting from only considering  $h$  from the left and down sides, the basic equation can be described as the following.

$$\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t) = (\kappa (\frac{\partial u}{\partial x})_{l+1,m}^{t \rightarrow t+1} + h_{l-1,m}^{t \rightarrow t+1}) \Delta y \Delta t + (\kappa (\frac{\partial u}{\partial y})_{l,m+1}^{t \rightarrow t+1} + h_{l,m-1}^{t \rightarrow t+1}) \Delta x \Delta t$$

In explicit format,

$$\begin{aligned} \rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t) &= \kappa (\frac{u_{l+1,m}^t - u_{l,m}^t}{\delta x_{l+1}} + h_{l-1,m}^t) \Delta y \Delta t + \kappa (\frac{u_{l,m+1}^t - u_{l,m}^t}{\delta y_{m+1}} + h_{l,m-1}^t) \Delta x \Delta t \\ \frac{\rho c \Delta x \Delta y (u_{l,m}^{t+1} - u_{l,m}^t)}{\Delta t} &= \kappa (\frac{u_{l+1,m}^t - u_{l,m}^t}{\delta x_{l+1}} + h_{l-1,m}^t) \Delta y + \kappa (\frac{u_{l,m+1}^t - u_{l,m}^t}{\delta y_{m+1}} + h_{l,m-1}^t) \Delta x \\ a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^t u_{l+1,m}^t + \kappa h_{l-1,m}^t \Delta y + a_{l,m+1}^t u_{l,m+1}^t + \kappa h_{l,m-1}^t \Delta x + a_{l,m}^t u_{l,m}^t \end{aligned}$$

We described this in general format.

$$\begin{aligned} a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^t u_{l+1,m}^t + a_{l-1,m}^t u_{l-1,m}^t + a_{l,m+1}^t u_{l,m+1}^t + a_{l,m-1}^t u_{l,m-1}^t \\ &\quad + \kappa h_{l+1,m}^t \Delta y + \kappa h_{l-1,m}^t \Delta y + \kappa h_{l,m+1}^t \Delta x + \kappa h_{l,m-1}^t \Delta x + a_{l,m}^t u_{l,m}^t \end{aligned}$$

In implicit format, similarly we get that

$$\begin{aligned} a_{l,m}^{t+1} u_{l,m}^{t+1} &= a_{l+1,m}^{t+1} u_{l+1,m}^{t+1} + a_{l-1,m}^{t+1} u_{l-1,m}^{t+1} + a_{l,m+1}^{t+1} u_{l,m+1}^{t+1} + a_{l,m-1}^{t+1} u_{l,m-1}^{t+1} \\ &\quad + \kappa h_{l+1,m}^{t+1} \Delta y + \kappa h_{l-1,m}^{t+1} \Delta y + \kappa h_{l,m+1}^{t+1} \Delta x + \kappa h_{l,m-1}^{t+1} \Delta x + a_{l,m}^t u_{l,m}^t \end{aligned}$$

**(a) If prescribed  $u = g$  for a boundary node**

$$h_{l+1,m} = h_{l-1,m} = h_{l,m+1} = h_{l,m-1} = 0;$$

$$\text{when } l+1=1, u_{l+1,m} = g \text{ and } a_{l+1,m} = \frac{\kappa \Delta y}{\delta x_{l+1}} = \frac{\kappa \Delta y}{\Delta x/2};$$

$$\text{when } l-1=0, u_{l-1,m} = g \text{ and } a_{l-1,m} = \frac{\kappa \Delta y}{\delta x_{l-1}} = \frac{\kappa \Delta y}{\Delta x/2};$$

$$\text{when } m+1=0, u_{l,m+1} = g \text{ and } a_{l,m+1} = \frac{\kappa \Delta x}{\delta y_{m+1}} = \frac{\kappa \Delta x}{\Delta y/2};$$

$$\text{when } m-1=0, u_{l,m-1} = g \text{ and } a_{l,m-1} = \frac{\kappa \Delta x}{\delta y_{m-1}} = \frac{\kappa \Delta x}{\Delta y/2}.$$

**(b) If prescribed  $\kappa \frac{\partial u}{\partial x} n_x + \kappa \frac{\partial u}{\partial y} n_y = h$  for a boundary node**

$$\text{when } l+1=1, h_{l+1,m} = -h \text{ and } a_{l+1,m} = h_{l-1,m} = h_{l,m+1} = h_{l,m-1} = 0;$$

$$\text{when } l-1=0, h_{l-1,m} = h \text{ and } a_{l-1,m} = h_{l+1,m} = h_{l,m+1} = h_{l,m-1} = 0;$$

$$\text{when } m+1=1, h_{l,m+1} = -h \text{ and } a_{l,m+1} = h_{l+1,m} = h_{l-1,m} = h_{l,m-1} = 0;$$

$$\text{when } m-1=0, h_{l,m-1} = h \text{ and } a_{l,m-1} = h_{l+1,m} = h_{l-1,m} = h_{l,m+1} = 0;$$

## Targets (Course Video 15)

### 编写程序

- ☐ 程序可以重启（需要hfd5来帮我们）
- ☐ 防御性写法，能处理异常，有断言，有详细的注释
- ☐ 编译的时候不能有任何warning
- ☐ 有makfile或者cmake，可以在太乙上直接编译
- ☐ 有profiling，程序的哪里耗时最多
- ☐ 利用github进行版本控制和团队合作
- ☐ 用开源软件可视化，二维的话需要用VTK或者Paraview
- ☐ 技术性的报告，最好包括LaTeX的使用

### 测试

- ☐ 方法的稳定性
- ☐ 误差分析
- ☐ 并行效率（强可扩展性、弱可扩展性、Petsc不同求解器的影响）