Theoretical Physics Group Project

Chaos in the Double Pendulum

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Contents

Introduction 2

Simple pendulum 3

Work Process and Contributions 3

Results and Findings 4

Double pendulum 5

Work Process and Contributions 5

Results and Findings 7

Conclusion 9

Bibliography 10

Introduction

Our project was a challenging and exciting journey that allowed us to explore the complex dynamics of simple and double pendulum experiments through simulation. As a team, we recognized the magnitude of the task ahead and were determined to come up with innovative solutions to help us achieve our goals.

Throughout the project, I played a significant role, conducting extensive research and providing crucial support for the simple pendulum experiments. Moreover, I took on the challenging task of analyzing and deriving the motion equations for the double pendulum system. It required applying advanced mathematical concepts to develop a robust solution, and we had to overcome obstacles and find alternative approaches, such as implementing numerical methods like the Runge-Kutta method.

In addition to contributing to the mathematical analysis, I created the simulation code and automated graphical plotter for the double pendulum. It was a significant achievement for our team, and the simulation and plotter leveraged the simulation model to generate a range of visualizations showcasing the displacements, velocity, and time variation of the double pendulum.

Overall, I am proud of the active role I played in the project's success and the meaningful contributions I made towards achieving our goals. The project provided us with valuable insights into the intricacies of pendulum systems.

Simple Pendulum

Work Process and Contributions

At the start of the project, I faced some personal issues that limited my involvement in the simple pendulum aspect. Nevertheless, I provided my team with research and support throughout the entire project.

Although I wasn't able to contribute as much to the simple pendulum part, I did play a significant role in deriving the equation of motion for the system 1.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta) \tag{1}$$

To achieve this, I analyzed the system's torque in a comprehensive manner, which helped me gain a deeper understanding of the underlying principles. Furthermore, my contribution provided valuable insights for my colleagues, and I was thrilled to be a part of such an exciting project.

My colleagues were exceptional in handling the simple pendulum with precision and skill. We managed to solve the equation of motion for the system using auxiliary equations and a simple ansatz as shown in group report [**1**], resulting in the analytical solution we were excited to incorporate into our simulation.

$$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{L}}t\right) \tag{2}$$

As we delved deeper into our exploration of the simple pendulum, I made sure to keep our team motivated and focused. I contributed significantly to the project by conducting extensive research and analysis, which helped to inform and shape our approach. Additionally, I was always available to offer guidance and answer questions, ensuring that my colleagues had the support they needed to succeed.

With my support and guidance, my colleagues were able to successfully code the simulation using the analytical solution we derived. To ensure the accuracy of our results, we conducted numerical examples using both the Euler and Runge-Kutta methods. All our hard work paid off when we saw the range plot we created, which clearly demonstrated the results of our simulation.

Results and Findings

The simplicity of the simple pendulum is a testament to the beauty of its physical laws. The angular displacement of the pendulum is solely dependent on its length and gravity, resulting in a predictable, oscillatory motion. As shown in Figure , the pendulum maintains a consistent shape regardless of its initial angular displacement, making it an excellent example of a system with a regular pattern.

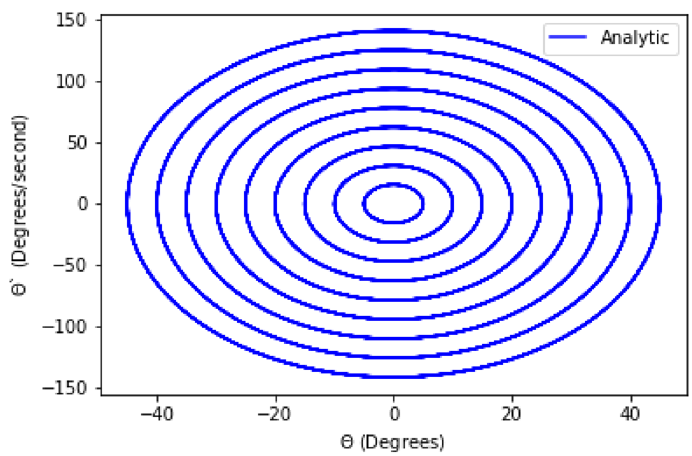

Figure 1: A graph of the analytic solution for displacement against velocity for a variety of θ_0

This makes it an ideal model for understanding the basics of oscillation and periodic motion. However, as we explore the limits of the simple pendulum's motion, we begin to see the limitations of its simplicity.

For instance, at higher angles, the small angle approximation on which the analytical solution is based fails, making the numerical methods more reliable for modelling the motion.

However, even with varying starting angles, the numerical method still follows a steady and uncomplicated pattern. In any case, the basic pendulum is still an important aid in comprehending the essential concepts of mechanics.

Double Pendulum

Work Process and Contributions

Throughout the project, I dedicated my efforts to the double pendulum system, and my contributions spanned from start to finish across analysis, planning, coding, and plotting.

I started by taking on the challenging task of deriving and solving the equation of motion for the double pendulum system. Although I won't go into the full derivation here, the basic idea is to calculate the pendulums' accelerations through kinematic analysis 3. This involves describing the movement of objects in terms of position, velocity, and acceleration, without considering the forces that cause those changes.

$$\begin{aligned}
\ddot{x}_1 &= -\dot{\theta}_1^2 l_1 \sin\theta_1 + \dot{\theta}_1 l_1 \cos\theta_1 \\
\ddot{y}_1 &= \dot{\theta}_1^2 l_1 \sin\theta_1 + \ddot{\theta}_1 l_1 \cos\theta_1 \\
\ddot{x}_2 &= \ddot{x}_1 - \dot{\theta}_2^2 l_2 \sin\theta_2 + \ddot{\theta}_2 l_2 \cos\theta_2 \\
\ddot{y}_2 &= \ddot{y}_1 + \dot{\theta}_1^2 l_2 \sin\theta_2 + \dot{\theta}_2 l_2 \cos\theta_2
\end{aligned} \tag{3}$$

Next, I rearranged the force analysis equations 4, which were obtained through free body diagrams, by eliminating the tensions T_1 and T_2. Then, I substituted the accelerations 3 into these equations.

$$\begin{aligned}
m_1\ddot{x}_1 &= -T_1 \sin\theta_1 + T_2 \sin\theta 2 \\
m_1\ddot{y}_1 &= T_1 \cos\theta_1 - T_2 \sin\theta 2 - m_1 g \\
m_2\ddot{x}_2 &= -T_2 \sin\theta_2 \\
m_2\ddot{y}_2 &= T_2 \cos\theta_2 - m_2 g
\end{aligned} \tag{4}$$

Following these steps, we derived two coupled second-order ordinary differential equations. To simplify the analysis, I'll present the decoupled version. However, it's important to note that we achieved the decoupling by rearranging the equations and using trigonometric identities. I should also mention that I received some assistance from one of my colleagues during the process.

$$\ddot{\theta_1} = \frac{-g(2m_1+m_2)\sin(\theta_1) - m_2 g\sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1-\theta_2)m_2(\dot{\theta_2^2}l_2 + \dot{\theta_1^2}l_1\cos(\theta_1-\theta_2))}{l_1(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}$$
$$\ddot{\theta_2} = \frac{2\sin(\theta_1-\theta_2)(\dot{\theta_1^2}l_1(m_1+m_2) + g(m_1+m_2)\cos(\theta_1) + \dot{\theta_2^2}l_2 m_2 cos(\theta_1-\theta_2))}{l_2(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))} \tag{5}$$

To solve the complex equations of motion, we needed a plan. Unfortunately, these equations were so intricate that we couldn't find a simple solution using analytical methods alone.

Therefore, I turned to numerical methods to find a solution. After careful consideration, I decided to use the popular Runge-Kutta method. I put a lot of effort into implementing it in our simulations, and it proved to be a valuable tool.

The solutions obtained in equation 6 came directly from the Runge-Kutta method. However, to get there, I had to use mathematical techniques to transform the original equation 5 into several 1st order ODEs. The process is explained in detail in our group report [**1**].

$$\begin{aligned}
\theta_1(tn+1) &= \theta_1(tn) + h\omega_1(tn) \\
\theta 2(tn+1) &= \theta_2(tn) + h\omega_2(tn) \\
\omega_1(tn+1) &= \omega_1(tn) + \frac{k_{1,\omega_1} + 2k_{2,\omega_1} + 2k_{3,\omega_1} + k_{4,\omega_1}}{6} \\
\omega_2(tn+1) &= \omega_2(tn) + \frac{k_{1,\omega_2} + 2k_{2,\omega_2} + 2k_{3,\omega_2} + k_{4,\omega_2}}{6}
\end{aligned} \tag{6}$$

$$\begin{aligned}
k_1(\omega_1) &= h\dot{\omega}_1(t_n) \\
k_2(\omega_1) &= h\dot{\omega}_1(t_n + \frac{h}{2}) \\
k_3(\omega_1) &= h\dot{\omega}_1(t_n + \frac{h}{2}) \\
k_4(\omega_1) &= h\dot{\omega}_1(t_n + h)
\end{aligned} \tag{7}$$

After finalizing the solution plan, I was excited to implement the simulation in a Python notebook. Initially, I created some basic functions for the 1st order ODEs and k-values. However, I encountered some overflow errors when attempting to run the simulation.

Thankfully, my colleague was able to help me out by not only fixing the errors, but also reorganizing and refining the functions using classes.

Once the simulation code was functioning correctly, I took it to the next level by creating a Python library to house the model. Using this library, I developed an automated plotter notebook that generated a range of visualizations, highlighting the displacements, velocity, and time variation of the double pendulum.

It was incredibly satisfying to see the plots come to life and add another layer of depth to the simulation. However, my colleague took it a step further by creating even more detailed plots that illustrated the trajectory of the pendulums.

Results and Findings

During the simulations, we observed that the behavior of the double pendulum simulations is quite different from that of a simple pendulum, as it exhibits chaotic movement.

The figure illustrates the various paths taken by the lower pendulum with varying values of θ and $\dot{\theta}$.

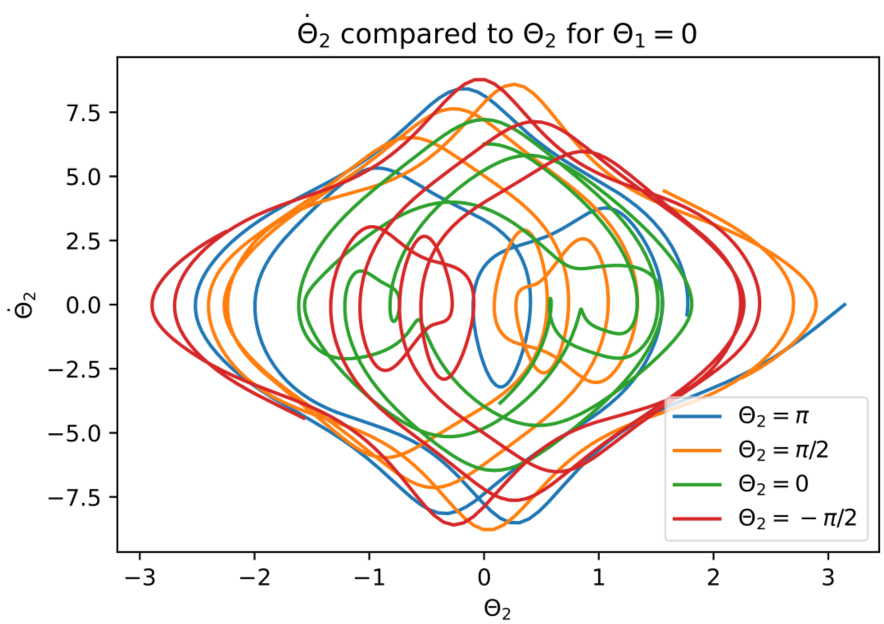

Figure 2: Poincare Section with varying initial conditions for $\dot{\theta}_2$ and θ_2

It is quite evident that there are no discernible shapes or patterns in any of the curves. In fact, even the slightest variation in the pendulum's initial conditions causes the bob to follow a completely different trajectory.

It is worth noting that the total energy of the system must remain conserved at all times. To test this, we came up with an idea to use the equations for the potential and kinetic energy of the double pendulum, as shown in 8 and 9.

$$P = -m_1 g L_1 \cos\theta_1 - m_2 g(L_1 \cos\theta_1 + L_2 \cos\theta_2) \tag{8}$$

$$K = \frac{1}{2} m_1 \dot{\theta}_1 L_1^2 + \frac{1}{2} m_2 (\dot{\theta}_1 L_1^2 + \dot{\theta}_2 L_2^2 + 2\dot{\theta}_1 L_1 \dot{\theta}_2 L_2 \cos(\theta_1 - \theta_2)) \tag{9}$$

Although we expected the total energy to fluctuate over time, as the numerical solution is only an estimation, it is still interesting to observe the wobbling of the total energy. This also highlights the importance of controlling errors by limiting the step-size of the model.

Overall, these findings emphasize that the double pendulum is heavily influenced by its current conditions, which can have a profound impact on its future behavior.

Conclusion

As we delved into the simple pendulum, we discovered that applying the small angle approximation can assist in finding an analytical solution. It's worth noting that this method may not be effective for all starting conditions. We also observed that when dealing with larger angles, numerical methods yield more precise results. Despite the fact that the motion of the pendulum can be varied, we were able to identify a pattern to it.

Moving on to the double pendulum, it's intriguing to note that its motion is chaotic and unpredictable. There isn't an analytical solution available to estimate its movement; instead, we must rely on numerical methods to make predictions. Although the simulation may contain some errors due to its intrinsic properties, it functions well and produces various plots.

Overall, I am incredibly proud of the part I played in our team's success in exploring the complex dynamics of the pendulum experiment. The project provided us with useful insights into the complexities of pendulum systems, and I'm eager to continue studying this field in the future.

Bibliography

[1] Theoretical Physics Group Project, Chaos in the Double Pendulum. 2547473m, 2557113s, 2565037w. 2023-01-31