CHAOS IN THE DOUBLE PENDULUM

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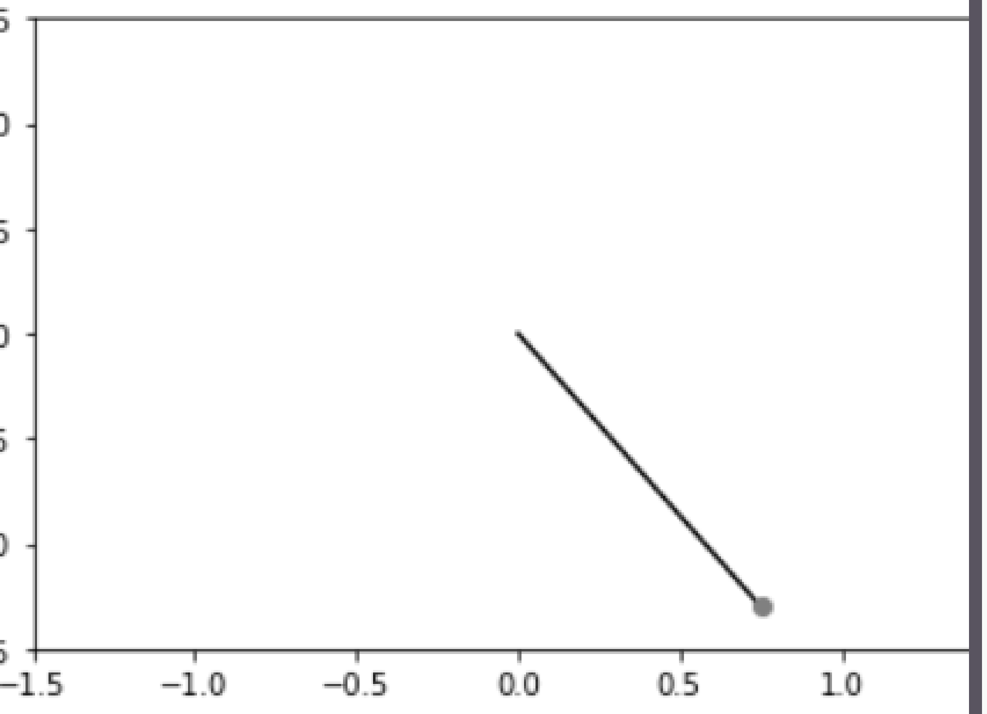
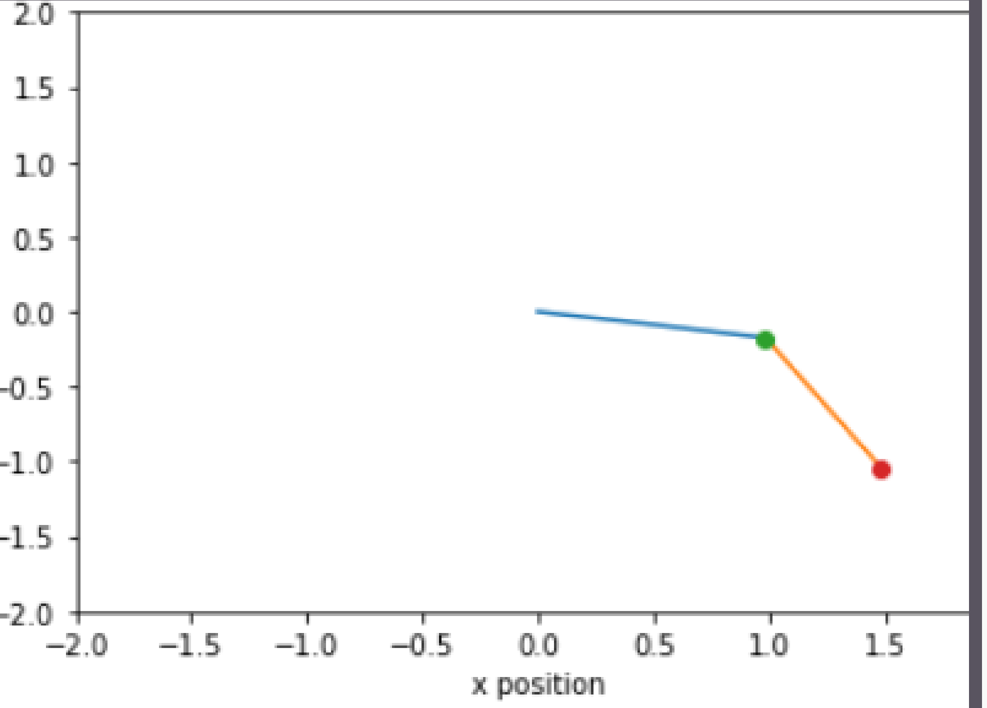
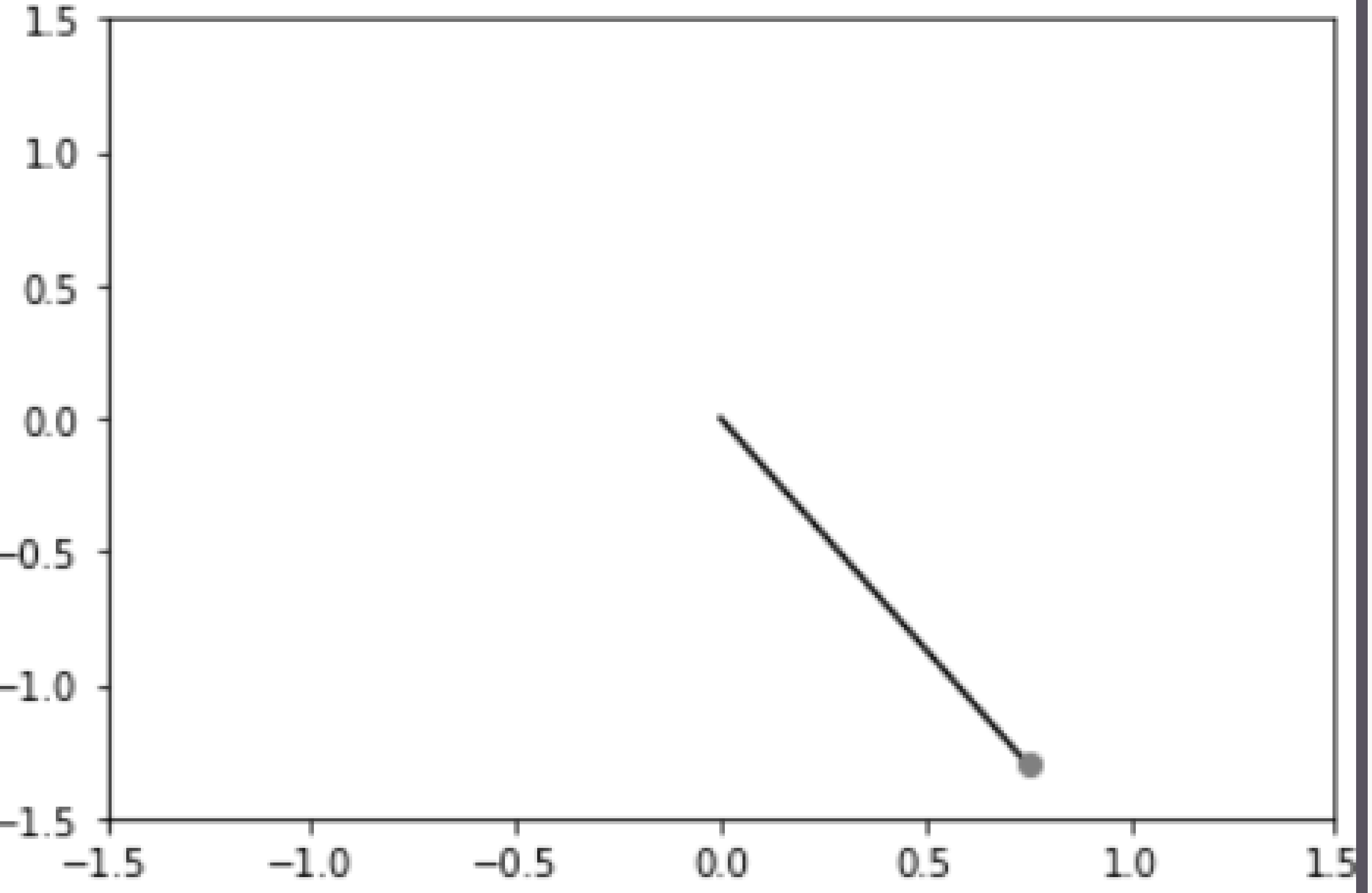




Introduction

During our lab our goal was to investigate and compare the simple pendulum to the chaotic double pendulum.

The double pendulum is a system which exhibits chaos, this means that small changes in the system lead to drastically different outcomes.



Simple Pendulum

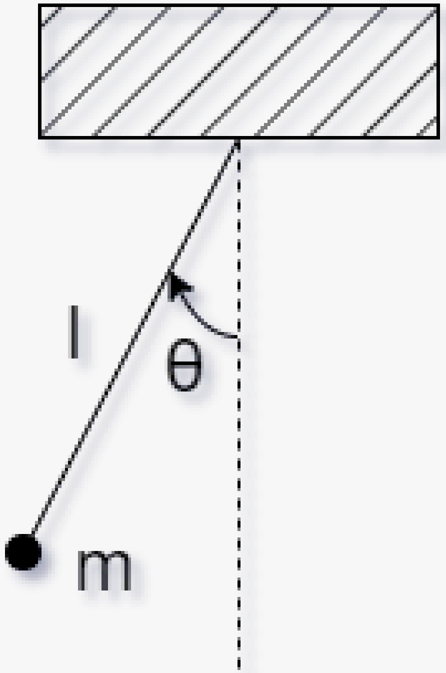
Background

Differential equation

From the diagram on the right it can be seen the force acting on the mass is F=mg. With a component acting directly up the string and the other component making up the restoring force. Solving this force for angular acceleration gives:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Which can be used for solving the simple pendulum.



Analytical

Derivation

Using the small angle approximation the following result was found:

$$\theta = \theta_0 \cos(\sqrt{\frac{g}{l}} t)$$

This is used as an analytical approximation for the simple pendulum at small angles

Limitations

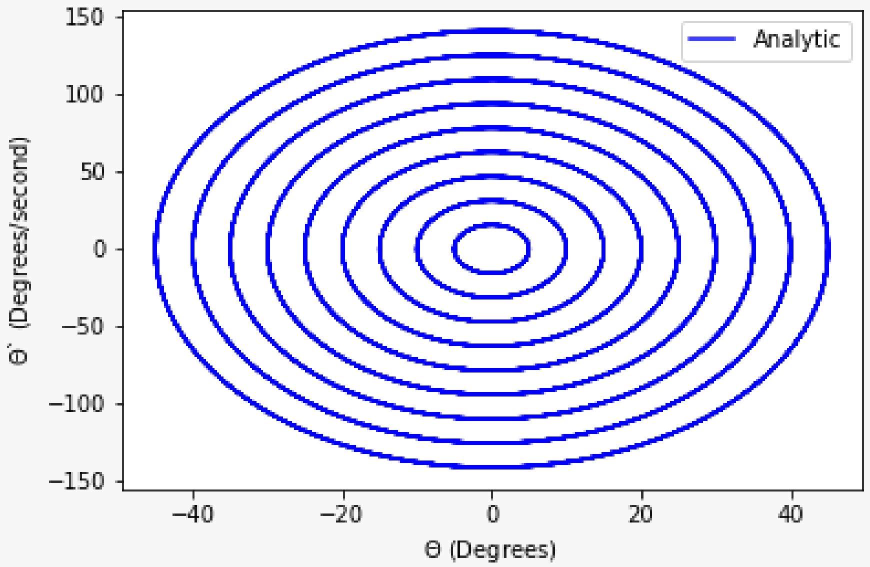
Due to the use of the small angle approximation this solution becomes increasingly inaccurate for higher angles.

Analytical

Results

Here the angular displacement is plotted against the angular velocity. For the analytic solution this creates increasingly large circles.

It can be seen that the maximum velocity is given when the angle is zero



Numerical

Methods

To create a numerical solution a numerical method must be chosen. These vary in accuracy and stability.

We used two methods here (Euler and 4th order Renge-Kutta) to compare how similar they were.

Function

Numerical methods use previous known values to predict values for later times.

It does this mainly by using the derivative of the current point to find the next point using a specified method.

Numerical

Euler’s Method

The most simple method, Euler’s method, uses the derivative at the current point multiplied by the step size to predict the next point.

Renge-Kutta 4th Order

This method is more accurate than Euler’s as it uses a combination of 4 points with differing gradients to find the next point in the simulation.

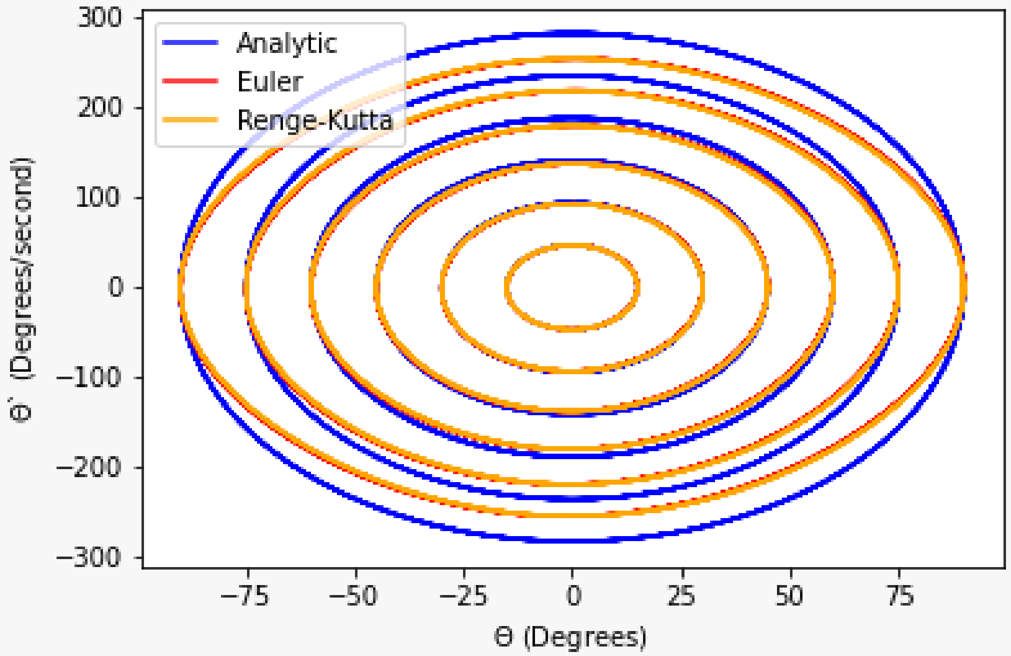
Due to its higher accuracy it is the technique we had the most trust in for producing accurate results.

Overall Findings

Increase in starting angle

Here it can be seen that both numerical methods show that for higher angles the solution becomes more elliptical.

However, this pattern is predictable showing that this system is not chaotic

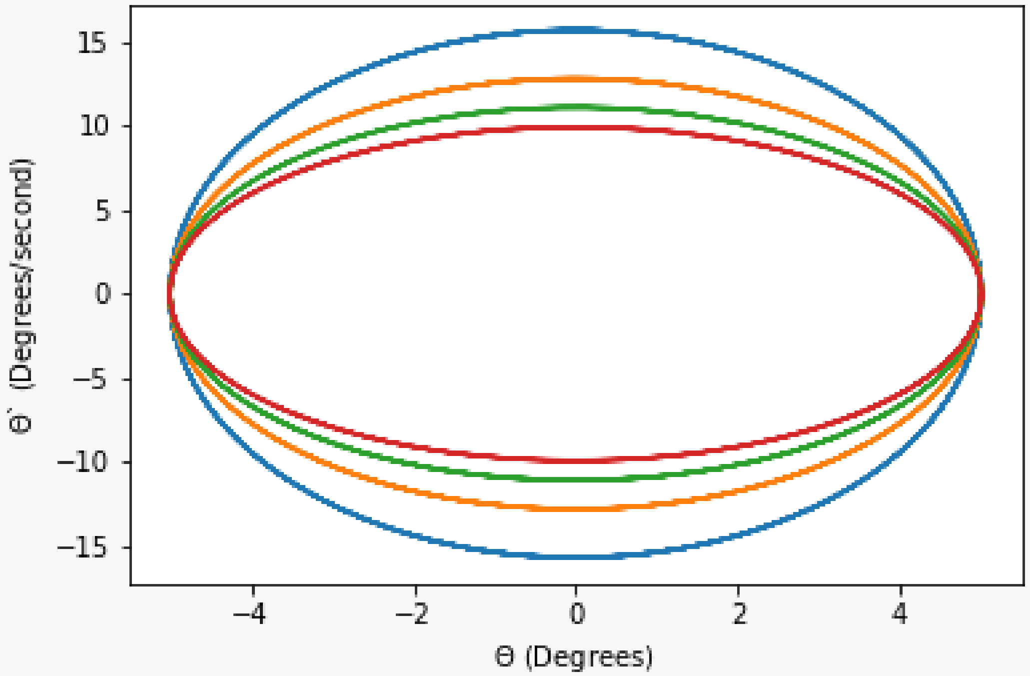


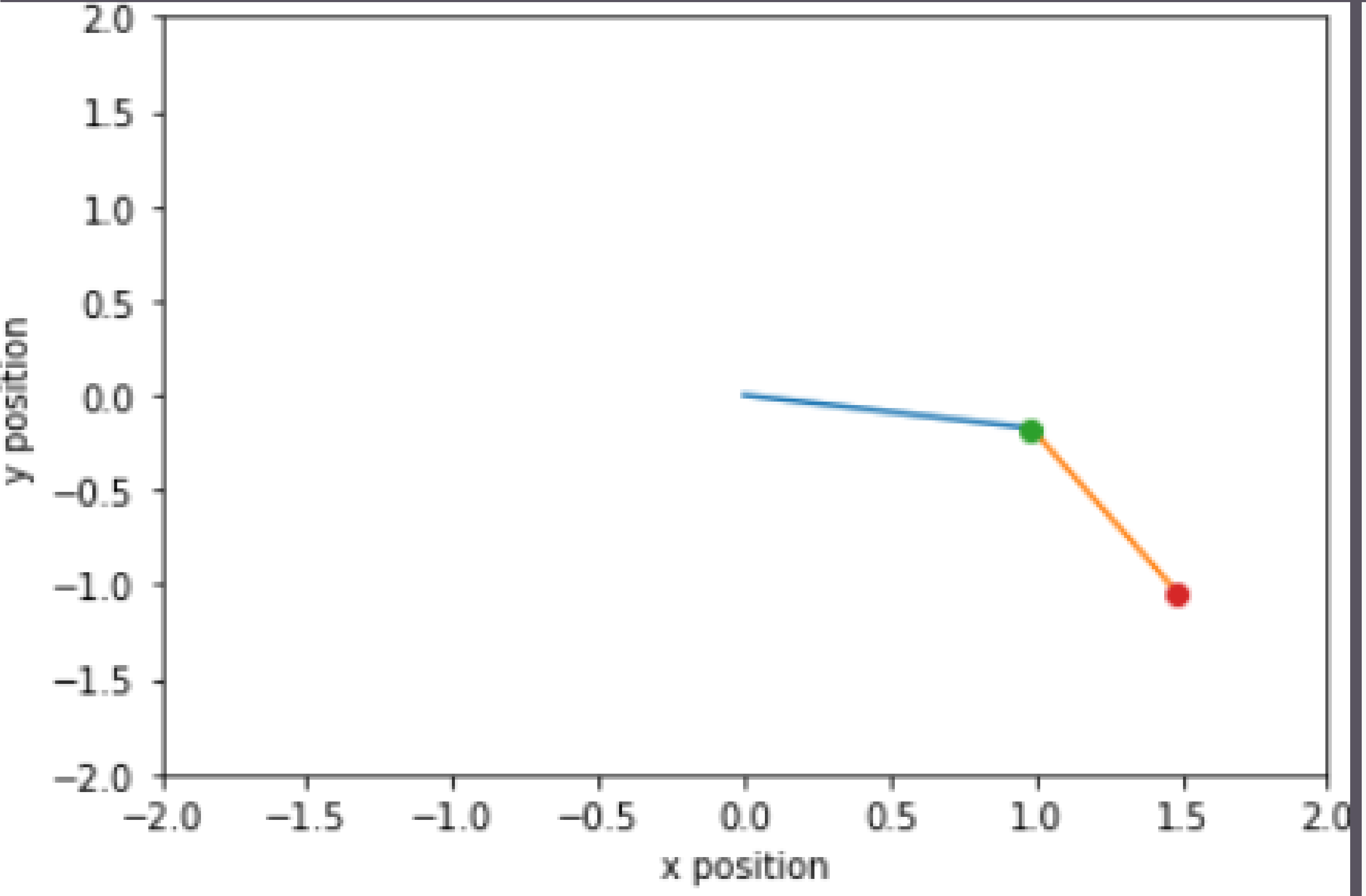
Overall Findings

Increase in string length

Similar to the previous result there is a pattern of an increase in how elliptical the graph appears.

This is further evidence of a non-chaotic system.





Double Pendulum

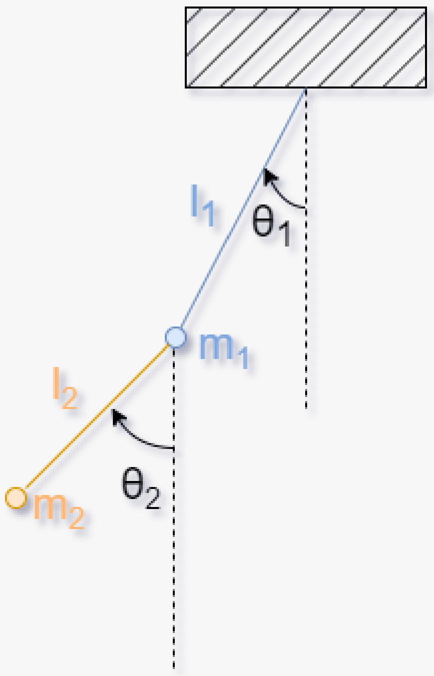
Background

Differential equation

Similar to the simple pendulum a description of the system can be found using force diagrams (although for the double pendulum it’s a little more complex)

$$l_1[(m_1 + m_2)(g \sin\theta_1 + \ddot{\theta}_1 l_1) + \ddot{\theta}_2 l_2 m_2 \cos\theta_1 - \theta_2 + \ddot{\theta}_2^2 l_2 m_2 \sin\theta_1 - \theta_2] = 0$$

$$l_2 m_2[g \sin\theta_2 + \ddot{\theta}_1 l_1 \cos\theta_1 - \theta_2 + \dot{\theta}_1^2 l_1(-\sin\theta_1 - \theta_2) + \ddot{\theta}_2 l_2] = 0$$



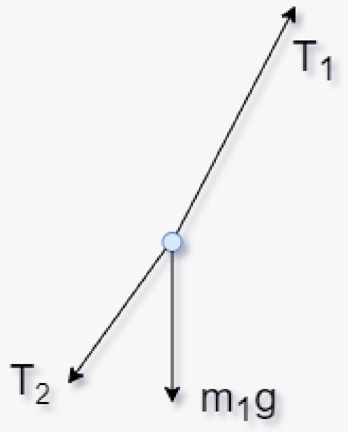
In general, the idea is to separate the pendulum into upper and lower part, then perform individual analysis.

Derivation

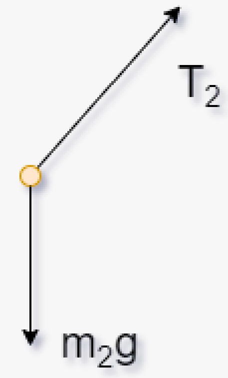
Force analysis

By breaking down the double pendulum system into its upper and lower sections. We were able to perform force analysis using free body diagrams separately.

However, as the tensions and cartesian accelerations are not a direct variable we would be interested in, it is suitable to perform kinematic analysis to eliminate them.



Upper Pendulum



Lower Pendulum

$$m_1\ddot{x}_1 = -T_1 \sin\theta_1 + T_2 \sin\theta 2$$
$$m_1\ddot{y}_1 = T_1 \cos\theta_1 - T_2 \sin\theta 2 - m_1 g$$

$$m_2\ddot{x}_2 = -T_2 \sin\theta_2$$
$$m_2\ddot{y}_2 = T_2 \cos\theta_2 - m_2 g$$

Derivation

Kinematic analysis

Using the same configuration, kinematic analysis is performed to provide us the cartesian coordinates in terms of the pendulum’s characteristics.

They are describing the movement of objects without considering the forces that cause those changes.

To obtain the acceleration for substitution, these equations are then taken into their 2nd derivative, specifically x'' and y‘’.

$$\ddot{x}_1 = -\dot{\theta}_1^2 l_1 \sin\theta_1 + \dot{\theta}_1 l_1 \cos\theta_1$$

$$\ddot{y}_1 = \dot{\theta}_1^2 l_1 \sin\theta_1 + \ddot{\theta}_1 l_1 \cos\theta_1$$

$$\ddot{x}_2 = \ddot{x}_1 - \dot{\theta}_2^2 l_2 \sin\theta_2 + \ddot{\theta}_2 l_2 \cos\theta_2$$

$$\ddot{y}_2 = \ddot{y}_1 + \dot{\theta}_1^2 l_2 \sin\theta_2 + \dot{\theta}_2 l_2 \cos\theta_2$$

Derivation

Equation of Motion

While the full derivation won't be shown here for simplicity, the basic idea is to substitute the derived accelerations, then rearrange to eliminate the tensions and apply some trig-identities during the process for decoupling.

After all these steps, we were left with two complex 2nd order ordinary differential equations.

$$\ddot{\theta}_1 = \frac{-g(2m_1 + m_2)\sin(\theta_1) - m_2 g \sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2)m_2(\dot{\theta}_2^2 l_2 + \dot{\theta}_1^2 l_1 \cos(\theta_1 - \theta_2))}{l_1(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))}$$

$$\ddot{\theta}_2 = \frac{2\sin(\theta_1 - \theta_2)(\dot{\theta}_1^2 l_1(m_1 + m_2) + g(m_1 + m_2)\cos(\theta_1) + \dot{\theta}_2^2 l_2 m_2 cos(\theta_1 - \theta_2))}{l_2(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))}$$

Solution

Limitations

Unfortunately, there's no easy way to come up with an analytical solution due to the lack of a simple ansatz for the two complex 2nd order ODEs.

Numerical Plan

A common approach is to derive the solutions numerically, with the popular Runge-Kutta method being used for this purpose.

To apply this method, it's necessary to define equations that can convert the complex equations into multiple 1st order ODEs.

Numerical Plan

1st order ODEs

If we have a look from previous, our equations of motion are relating to the angular changes and their derivatives.

Therefore, it's helpful to define two 1st order equations for the derivatives of the angular displacements, which is also known as the velocity of both sections.

$$\dot{\theta}_1 = \omega_1$$

$$\dot{\theta}_2 = \omega_2$$

Numerical Plan

1st order ODEs

Using these equations, we can substitute them into the equation of motion to obtain another two 1st order ODEs that represents the double pendulum motion.

$$\dot{\omega_1} = \frac{-g(2m_1 + m_2)\sin(\theta_1) - m_2 g \sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2) m_2 (\omega_2^2 l_2 + \omega_1^2 l_1 \cos(\theta_1 - \theta_2))}{l_1(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))}$$

$$\dot{\omega_2} = \frac{2\sin(\theta_1 - \theta_2)(\omega_1^2 l_1 (m_1 + m_2) + g(m_1 + m_2)\cos(\theta_1) + \omega_2^2 l_2 m_2 cos(\theta_1 - \theta_2))}{l_2(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))}$$

Numerical Plan

Runge-Kutta Method

With the four 1st order ODEs, the Runge-Kutta 4 method provides defined solution estimations.

The solution at a certain time frame can be estimated by an appropriate previous conditions and step-sizes.

$$\theta_1(tn+1) = \theta_1(tn) + h\omega_1(tn)$$

$$\theta_2(tn+1) = \theta_2(tn) + h\omega_2(tn)$$

$$\omega_1(tn+1) = \omega_1(tn) + \frac{k_{1,\omega_1} + 2k_{2,\omega_1} + 2k_{3,\omega_1} + k_{4,\omega_1}}{6}$$

$$\omega_2(tn+1) = \omega_2(tn) + \frac{k_{1,\omega_2} + 2k_{2,\omega_2} + 2k_{3,\omega_2} + k_{4,\omega_2}}{6}$$

Numerical Plan

K-values

The k-values in the solution represent the slopes of the solution at each step.

k1 represents the slope at the beginning, k2 the slope at the first mid-point, k3 the slope at the second mid-point, and k4 the slope at the fourth mid-point.

As the slope of the velocity is the acceleration, we can define the k-values as the angular acceleration instants they are responsible for.

$$k_1(\omega_1) = h\dot{\omega}_1(t_n)$$

$$k_2(\omega_1) = h\dot{\omega}_1(t_n + \frac{h}{2})$$

$$k_3(\omega_1) = h\dot{\omega}_1(t_n + \frac{h}{2})$$

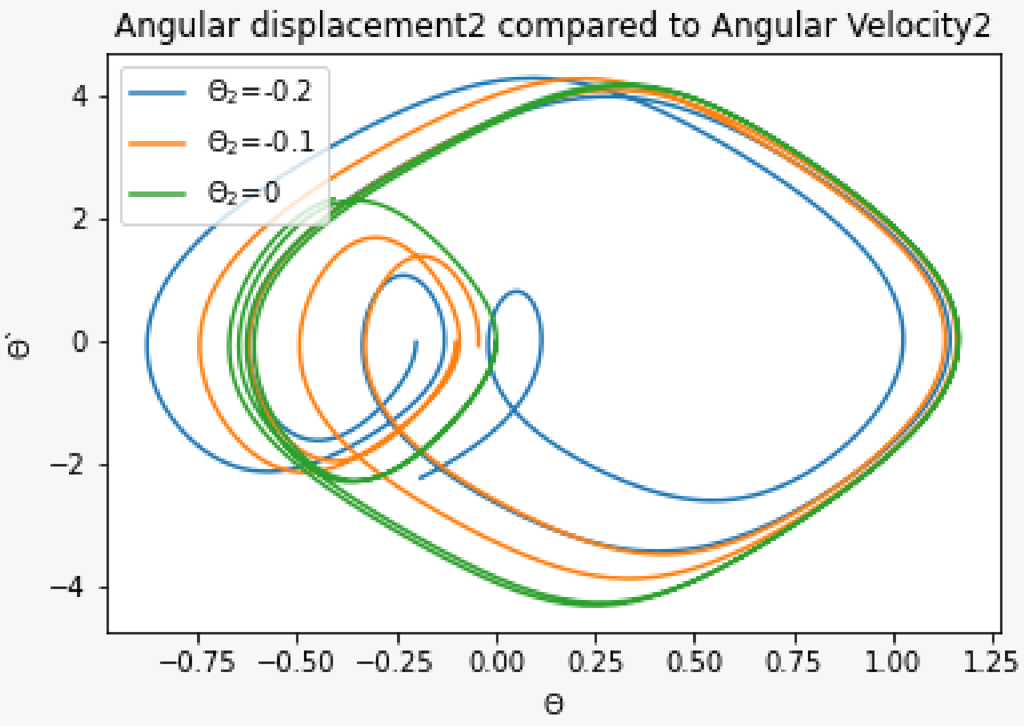
$$k_4(\omega_1) = h\dot{\omega}_1(t_n + h)$$

Findings

Change in Outer Angle

This graph shows that when a change in angle is introduced the change in motion of the pendulum is unpredictable.

This supports the double pendulum being chaotic

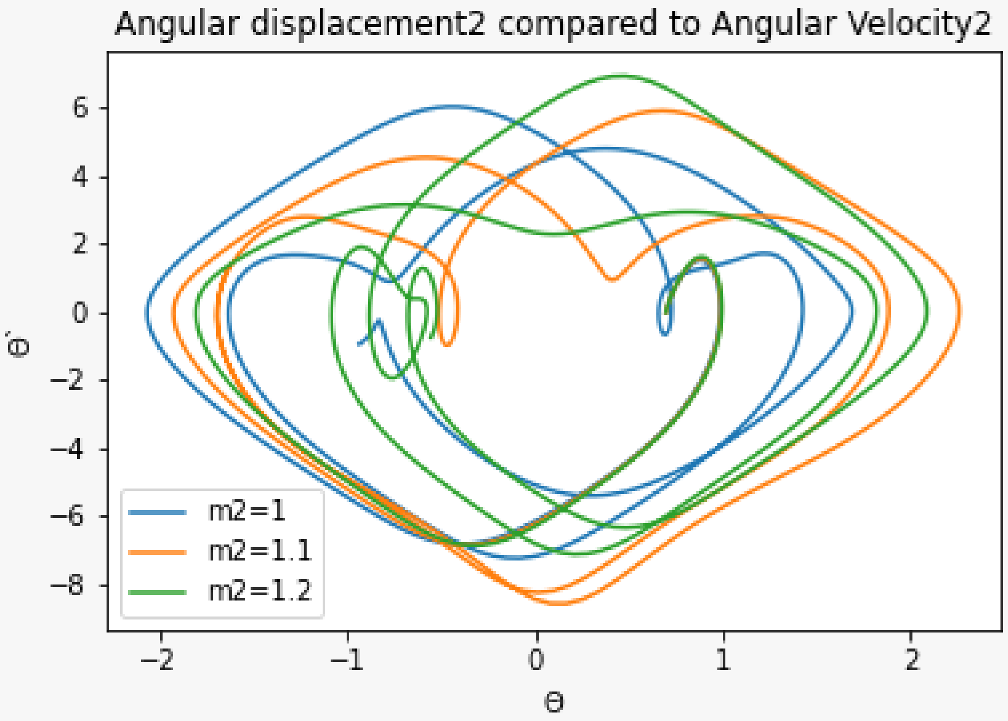


Findings

Change in Mass

Here it is shown that small changes in the outer mass lead to extremely different motions of the double pendulum when all conditions are kept the same.

This adds credence to the hypothesis of chaotic motion

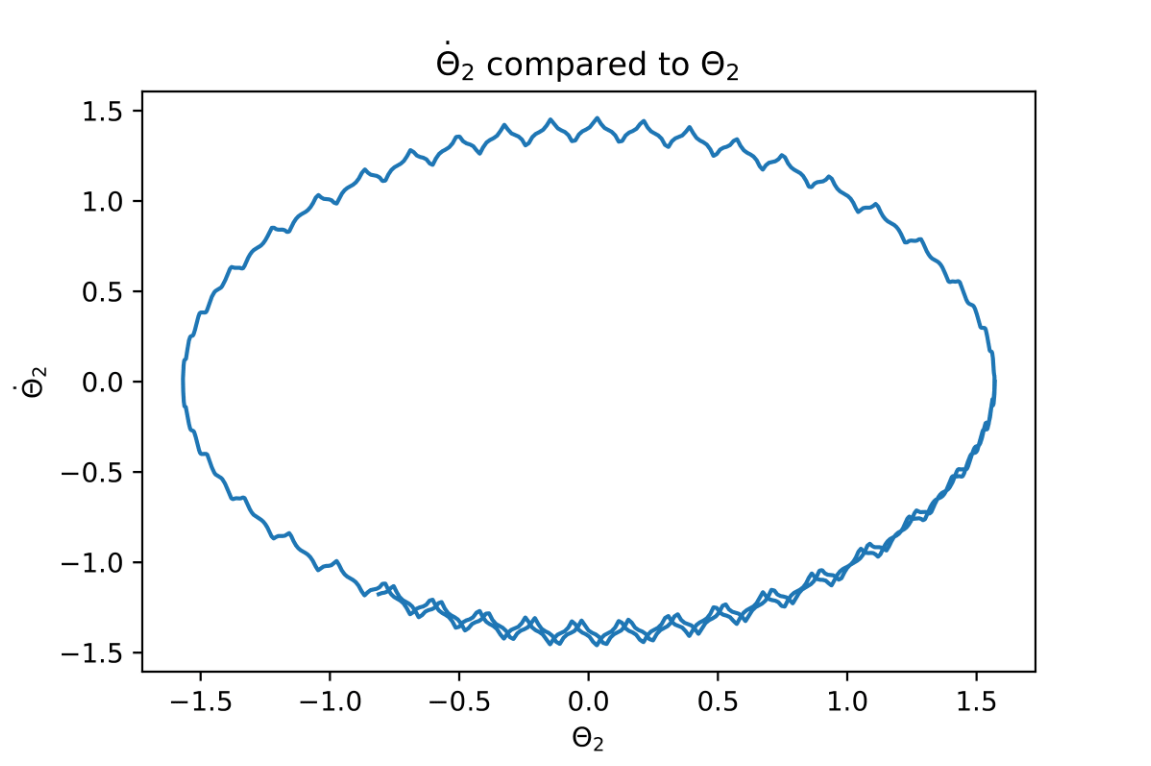


Findings

High L_2

For a large difference between lengths of the rods in the double pendulum it approximates a simple pendulum to a degree.

However, there is a jitter in the graph due to the effect of the small L_1.



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