Theoretical Physics Group Project

Chaos in the Double Pendulum

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Abstract

In this study, Python is used in the JupyterHub environment to simulate and examine the dynamics and behavior of both the simple and double pendulum experiments. The equations of motion were used to model the basic pendulum analytically, and the results were compared against various initial conditions. A numerical integration approach was employed to model the double pendulum, and the outcomes were analysed in terms of time-steps and phase space.

The simulation results demonstrate that the basic pendulum moves in a predictable harmonic manner, while the double pendulum displays a more complex and chaotic behavior. The report concludes by discussing the implications of these results for our understanding of pendulum dynamics and the role of chaos in physics. The full simulation code is also included at the end of the report for further analysis and reference.

Contents

Introduction

The study of simple pendulum motion has been a subject of fascination for generations of physicists. This simple yet elegant system embodies the essence of classical mechanics, and its periodic swinging motion has captured the attention of physicists for generations. The simple pendulum consists of a mass (known as the bob) hanging from a fixed position (the pivot) by a string or rod of a given length.

On the other hand, the double pendulum system is an intricate example of the connection between mechanics and chaos. This system exhibits a wide range of dynamic behaviors, from predictable swinging to chaotic oscillations and even predetermined chaotic activity. The double pendulum system is composed of two simple pendulums, each with a mass (the bob) suspended from an immovable pivot by a rod of fixed length. The pivot of the first pendulum serves as the attachment point for the second pendulum, creating a complex and interrelated system. The behavior of the double pendulum can be described mathematically using a series of nonlinear differential equations, but it is also subject to the principles of classical physics.

In this report, we will use mathematical models and simulations to investigate the behavior of both the simple and double pendulum under various conditions. Our goal is to gain insights into the factors affecting pendulum motion and to demonstrate how these factors can be controlled in the case of the simple pendulum, or result in significant differences in the behavior of the double pendulum over time.

Simple Pendulum

Dissecting and Understanding the Problem

Before deriving the equation of motion for a simple pendulum, it is suitable to first plot a properly labeled diagram (as with any other kinematic problems).

Consider a simple pendulum of length L and mass m with its bob located at a distance θ from the vertical as seen in Graph 1.

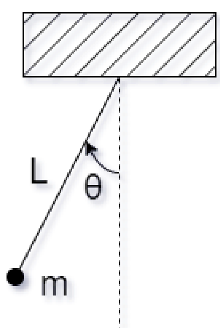

Figure 1: Simple Pendulum Model

The system can be treated as a point mass that is acted upon by a torque τ, with a moment of inertia of the entire system I and an angular acceleration α. The equation of motion can then be derived from Newton's second law of angular motion, which states the net torque acting on an object is the sum of products of the object's moment of inertia and angular acceleration. With the corresponding substitutions:

$$\tau = I\alpha \tag{1}$$

$$-mg\sin\theta L = mL^2\frac{d^2\theta}{dt^2} \tag{2}$$

where t is the time and $\frac{d^2\theta}{dt^2}$ is the acceleration of the pendulum. The tension in the string is equal to the weight component vertical to the pendulum. The equation can be simplified by rearranging and dividing both sides of the equation by the mass m:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta) \tag{3}$$

This equation describes the motion of the pendulum in terms of its angular position and velocity and can be used to simulate its behavior over time.

The equation is a second-order differential equation and can be solved analytically or numerically to determine the position and velocity of the pendulum at any given time. The solution of this equation provides information about the pendulum's period, amplitude, and phase, which can be used to analyse its motion under different conditions.

Evaluating the Analytical Solution and some Numerical Examples

To solve the simple pendulum analytically, an assumption must be brought in to simplify the differential equation. This assumption is the small angle approximation which assumes that for small angles (less than 5°s) $sin\theta$ is approximately equal to θ.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0$$
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

This equation can then be solved giving the following characteristic equation:

$$\theta = A[\cos(\sqrt{\frac{g}{L}}t) + \sin(\sqrt{\frac{g}{L}}t)] + B[\cos(-\sqrt{\frac{g}{L}}t) + \sin(-\sqrt{\frac{g}{L}}t)] \tag{4}$$

To find θ_0, solve equation at $t = 0$

$$\theta = A[\cos 0 + i\sin 0] + B[\cos 0 + i\sin 0]$$
$$\theta_0 = A + B \tag{5}$$

The derivative of the characteristic equation is:

$$\theta \doteq -(A+B)\sqrt{\frac{g}{L}}\sin\sqrt{\frac{g}{L}}t + (A-B)\sqrt{\frac{g}{L}}\cos\sqrt{\frac{g}{L}}t \tag{6}$$

At $t = 0$ set the velocity to $\theta \doteq 0$

$$0 = -(A+B)\sqrt{\frac{g}{L}}\sin 0 + (A-B)\sqrt{\frac{g}{L}}\cos 0$$

$$
\begin{aligned}
0 &= \sqrt{\frac{g}{L}}(A - B) \\
A &= B
\end{aligned}
\tag{7}
$$

Using $A = B$ it can be seen that the initial equation simplifies to:

$$\theta = 2A \cos \sqrt{\frac{g}{L}} t$$

and hence,

$$\theta = \theta_0 \cos \sqrt{\frac{g}{L}} t \tag{8}$$

Double Pendulum

Dissecting and Understanding the Problem

The double pendulum system presents a more complex derivation for its equation of motion compared to the simple pendulum case. The system consists of an upper and lower part, as illustrated in Figure 2.

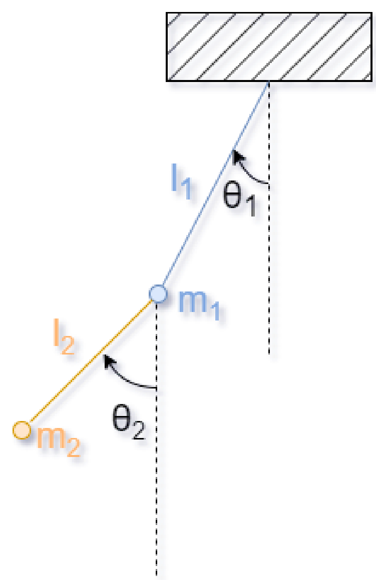

Figure 2: Double Pendulum Model

To simplify the analysis, we can treat each part separately, with a clear distinction between them shown in the figure using contrasting colors.

To begin the analysis, we must perform a force analysis for both the upper and lower parts, as with any kinematic problem. This is done by examining the free body diagram (FBD) of each part as shown in figure 3.

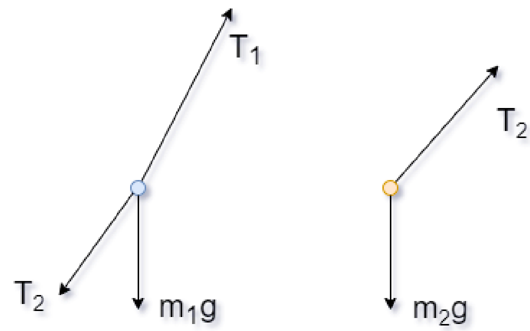

Figure 3: Upper and Lower Pendulum FBD

Starting with the upper pendulum, the bob is subjected to two sources of tension: one from the pivot and another from the lower pendulum, in addition to its own weight. To analyze the forces, we can break them down into their horizontal and vertical components using Pythagoras's theorem. An example of force depiction is in Figure 4 for the lower pendulum.

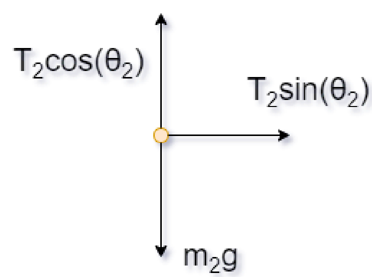

Figure 4: Lower Pendulum Force Depiction

The net horizontal force acting on the upper pendulum is the sum of the horizontal components of the tensions, while the net vertical force is the sum of the tensions' vertical components and the weight of the bob. By applying the equation $F = ma$ to the upper pendulum, we can determine its net force 9.

$$\begin{aligned} m_1\ddot{x}_1 &= -T_1 \sin\theta_1 + T_2 \sin\theta 2 \\ m_1\ddot{y}_1 &= T_1 \cos\theta_1 - T_2 \sin\theta 2 - m_1 g \end{aligned} \tag{9}$$

where $x_1^{"}$ and $y_1^{"}$ represent the horizontal and vertical accelerations (of the upper pendulum) respectively, and g the gravitational acceleration. Using the similar approach, forces of the lower pendulum can also be deduced:

$$
\begin{aligned}
m_2 x_2^{''} &= -T_2 \sin\theta_2 \\
m_2 y_2^{''} &= T_2 \cos\theta_2 - m_2 g
\end{aligned}
\tag{10}
$$

While the set of four simultaneous equations 9 and 10 appears to be a reasonable description of the pendulum's motion, the inclusion of tensions and Cartesian accelerations makes the equations difficult to work with. A proper equation of motion should only consider the pendulum's fundamental kinematic characteristics, including its angular variations, length, and mass.

To simplify the equations, we can eliminate the tensions by rearranging the equations. However, we must still address the presence of Cartesian accelerations. To do so, we can choose the pivot point of the upper pendulum as the origin for our Cartesian coordinate system. Using Pythagoras's theorem, we can easily determine the corresponding x and y coordinates, as shown in Equation 11 based on Figure 2.

$$
\begin{aligned}
x_1 &= l_1 \sin\theta_1 \\
y_1 &= -l_1 \cos\theta_1
\end{aligned}
\tag{11}
$$

Despite appearing complicated, the lower part of the double pendulum system is connected to the end of the upper pendulum at its pivot point. Therefore, we can determine the coordinate of the lower bob by adding the coordinate of the upper bob to the coordinate of the lower bob relative to the upper bob. This relationship is expressed in Equation 12. This approach is similar to how vectors and relative coordinates work.

$$
\begin{aligned}
x_2 &= x_1 + l_2 \sin\theta_2 \\
y_2 &= y_1 - l_2 \cos\theta_2
\end{aligned}
\tag{12}
$$

With all of the Cartesian coordinates determined, we can now take the second derivative to obtain the accelerations in terms of the physical characteristics of the double pendulum system. This process allows us to express the accelerations solely in terms of the system's fundamental kinematic characteristics. The resulting equations are shown in Equation 13:

$$
\begin{aligned}
\ddot{x}_1 &= -\dot{\theta}_1^2 l_1 \sin\theta_1 + \dot{\theta}_1 l_1 \cos\theta_1 \\
\ddot{y}_1 &= \dot{\theta}_1^2 l_1 \sin\theta_1 + \ddot{\theta}_1 l_1 \cos\theta_1 \\
\ddot{x}_2 &= \ddot{x}_1 - \dot{\theta}_2^2 l_2 \sin\theta_2 + \ddot{\theta}_2 l_2 \cos\theta_2 \\
\ddot{y}_2 &= \ddot{y}_1 + \dot{\theta}_1^2 l_2 \sin\theta_2 + \dot{\theta}_2 l_2 \cos\theta_2
\end{aligned}
\tag{13}
$$

Once we have all of the Cartesian acceleration equations, we can substitute them into the force equations expressed in Equations 9 and 10. Through some rearrangement and elimination of the tensions, we can arrive at the final equation of motion for the double pendulum system.

$$
\begin{aligned}
l_1[(m_1+m_2)(g\sin\theta_1+\ddot{\theta}_1 l_1)+\ddot{\theta}_2 l_2 m_2\cos\theta_1-\theta_2+\ddot{\theta}_2^2 l_2 m_2\sin\theta_1-\theta_2]=0\\
l_2 m_2[g\sin\theta_2+\ddot{\theta}_1 l_1\cos\theta_1-\theta_2+\dot{\theta}_1^2 l_1(-\sin\theta_1-\theta_2)+\ddot{\theta}_2 l_2]=0
\end{aligned}
\tag{14}
$$

The resulting equation of motion for the double pendulum system consists of two coupled second-order ordinary differential equations, which are relatively complex and chaotic when compared to the equation of motion for the simple pendulum system.

Numerical Solution Plan

The mathematical equation that governs the motion of a double pendulum is highly complex and does not lend itself to analytical solutions as seen 15. The initial step in simplifying the problem involves decoupling the equations of motion, which can be achieved through substitutions and the use of trigonometric identities. For the sake of brevity, we will not delve into the derivation here.

$$
\begin{aligned}
\ddot{\theta_1}&=\frac{-g(2m_1+m_2)\sin(\theta_1)-m_2 g\sin(\theta_1-2\theta_2)-2\sin(\theta_1-\theta_2)m_2(\dot{\theta_2^2}l_2+\dot{\theta_1^2}l_1\cos(\theta_1-\theta_2))}{l_1(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}\\
\ddot{\theta_2}&=\frac{2\sin(\theta_1-\theta_2)(\dot{\theta_1^2}l_1(m_1+m_2)+g(m_1+m_2)\cos(\theta_1)+\dot{\theta_2^2}l_2 m_2 cos(\theta_1-\theta_2))}{l_2(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}
\end{aligned}
\tag{15}
$$

One frequently employed numerical approach for solving such problems is the Runge-Kutta Method (RK4). The basic principle of this method involves transforming the equations into a set of several first-order Ordinary Differential Equations (ODEs), which can be numerically integrated through the use of time-steps and phase space.

To apply this method to the double pendulum, we can begin by defining the angular velocity of each pendulum, which can be represented by ω_1 and ω_2, respectively. We can further define these velocities as the first derivatives of the corresponding angles, $\dot{\theta_1}$ and $\dot{\theta_2}$, respectively.

$$
\begin{aligned}
\dot{\theta_1}&=\omega_1\\
\dot{\theta_2}&=\omega_2
\end{aligned}
\tag{16}
$$

To avoid dealing with any second derivatives, we can substitute the expressions for the angular velocities that we previously defined into the decoupled equations of motion, as represented by Equation 15. Specifically, we can represent the second derivative of the angles, $\ddot{\theta}$, as the first derivative of the corresponding angular velocities, $\dot{\omega}$.

$$\dot{\omega_1} = \frac{-g(2m_1+m_2)\sin(\theta_1) - m_2 g\sin(\theta_1-2\theta_2) - 2\sin(\theta_1-\theta_2)m_2(\omega_2^2 l_2 + \omega_1^2 l_1\cos(\theta_1-\theta_2))}{l_1(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))}$$
$$\dot{\omega_2} = \frac{2\sin(\theta_1-\theta_2)(\omega_1^2 l_1(m_1+m_2)+g(m_1+m_2)\cos(\theta_1)+\omega_2^2 l_2 m_2 cos(\theta_1-\theta_2))}{l_2(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))} \tag{17}$$

After following the aforementioned steps, we will end up with four Ordinary Differential Equations (ODEs) that need to be solved, as represented by Equations 16 and 17. These equations can be numerically analyzed by specifying initial conditions, such as random starting angular positions and a random step size, h.

In order to obtain the solutions for θ_1, θ_2, ω_1, and ω_2, their numerical values at each time step must be calculated, along with their corresponding k values.

We can initiate the program by specifying an arbitrary initial condition (as determined by user input), and then use the Runge-Kutta formulas to approximate the values of θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$ at the next time step, as depicted by Equation 18.

$$\begin{aligned}
\theta_1(tn+1) &= \theta_1(tn) + h\omega_1(tn) \\
\theta 2(tn+1) &= \theta_2(tn) + h\omega_2(tn) \\
\omega_1(tn+1) &= \omega_1(tn) + \frac{k_{1,\omega_1} + 2k_{2,\omega_1} + 2k_{3,\omega_1} + k_{4,\omega_1}}{6} \\
\omega_2(tn+1) &= \omega_2(tn) + \frac{k_{1,\omega_2} + 2k_{2,\omega_2} + 2k_{3,\omega_2} + k_{4,\omega_2}}{6}
\end{aligned} \tag{18}$$

The variables $k1$, $k2$, $k3$, and $k4$ correspond to the slopes at the start, two midpoints, and end of the function. These values can be calculated using the equations presented in Equation 19.

$$\begin{aligned}
k_1(\omega_1) &= h\dot{\omega}_1(t_n) \\
k_2(\omega_1) &= h\dot{\omega}_1(t_n + \frac{h}{2}) \\
k_3(\omega_1) &= h\dot{\omega}_1(t_n + \frac{h}{2}) \\
k_4(\omega_1) &= h\dot{\omega}_1(t_n + h)
\end{aligned} \tag{19}$$

A similar approach can be used to calculate the k values for ω_2. Once we have determined these values, we can then incorporate them into the program's code and use it to simulate the motion of the double pendulum, allowing us to observe its chaotic behavior.

Numerical Results and Simulation Evaluation

Summary and Error Analysing

Conclusion

Bibliography