機率領域的和積演算法
Probability-Domain Sum-Product Algorithm

對數領域和積演算法
Log-Domain Sum-Product Algorithm

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Probability-Domain Sum-Product Algorithm

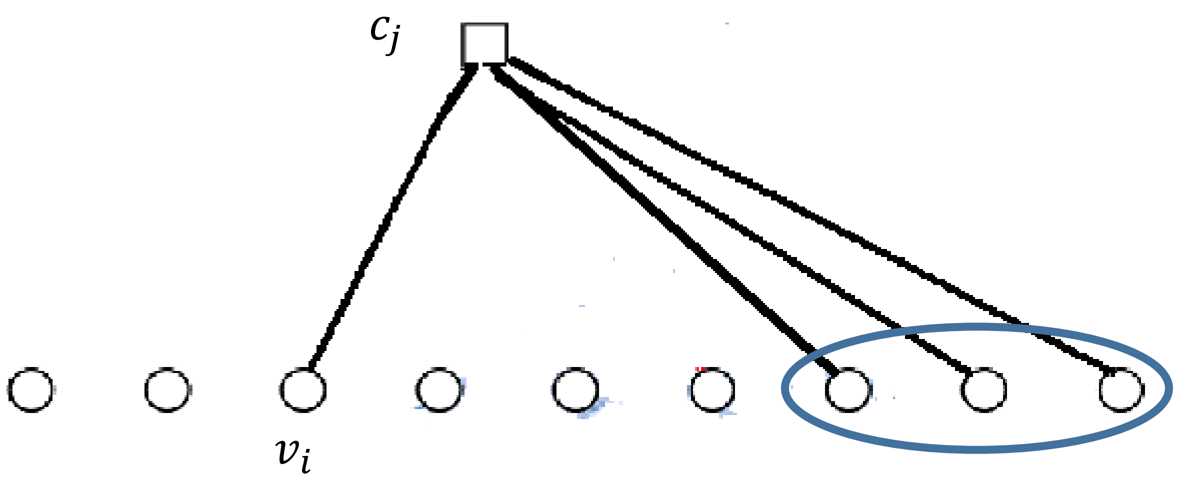
- 算APP 後驗機率

Decoding overview

- Initialize：q值
- v - c nodes：傳q值算r值
- c - v nodes：傳r值算q值
- 用r值判定 $\hat{x}$

Definitions

- C_j
- V_j
- $C_j \setminus \mathrm{i}$
- $V_j \setminus \mathrm{i}$
- $P_j = P_r(x_j = 1 \mid y_j)$
- $1 - P_j = P_r(x_j = 0 \mid y_j)$



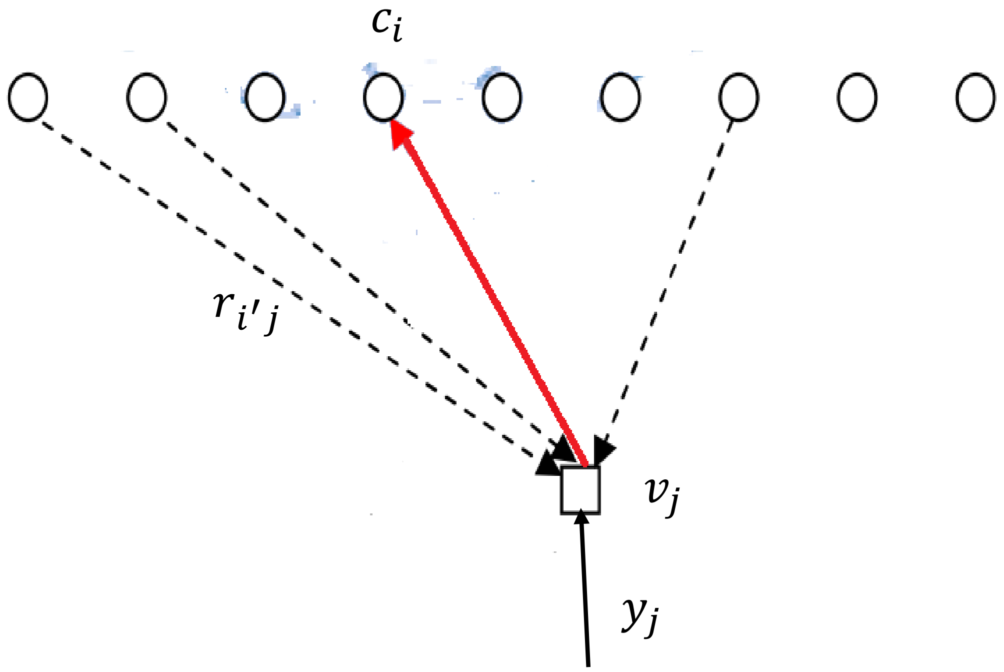
q值

- $q_{ij}(b) = m_{\uparrow ij} = P_r\left(x_j = b \,\middle|\, y_j\right.$，滿足有$x_j$的檢查方程式$c_{i'}$, $i' \neq i$)

$$q_{ij}(0) = \mathrm{K}_{ij}(1 - \mathrm{P}_j) \prod_{i' \in \mathrm{C}_j \setminus i} r_{i'j}(0)$$

$$q_{ij}(1) = \mathrm{K}_{ij}\mathrm{P}_j \prod_{i' \in \mathrm{C}_j \setminus i} r_{i'j}(1)$$

$$q_{ij}(0) + q_{ij}(1) = 1$$



Lemma

數列$\{\, x_j \,\}_{j=1}^{M}$，$P_r\left(x_j = 1\right) = P_j$。偶數個一的機率$= \dfrac{1}{2} + \dfrac{1}{2}\displaystyle\prod_{j=1}^{M}(1 - 2P_j)$

Proof.

令F(t) $= \prod_{j=1}^{M}\left[\left(1 - P_j\right) + P_j t\right]$ => t^j的係數為含j個1的機率

$$\frac{\mathrm{F}(1) - F(-1)}{2} = \frac{\prod_{j=1}^{M}\left[\left(1 - P_j\right) + P_j\right] + \prod_{j=1}^{M}\left[\left(1 - P_j\right) - P_j\right]}{2}$$

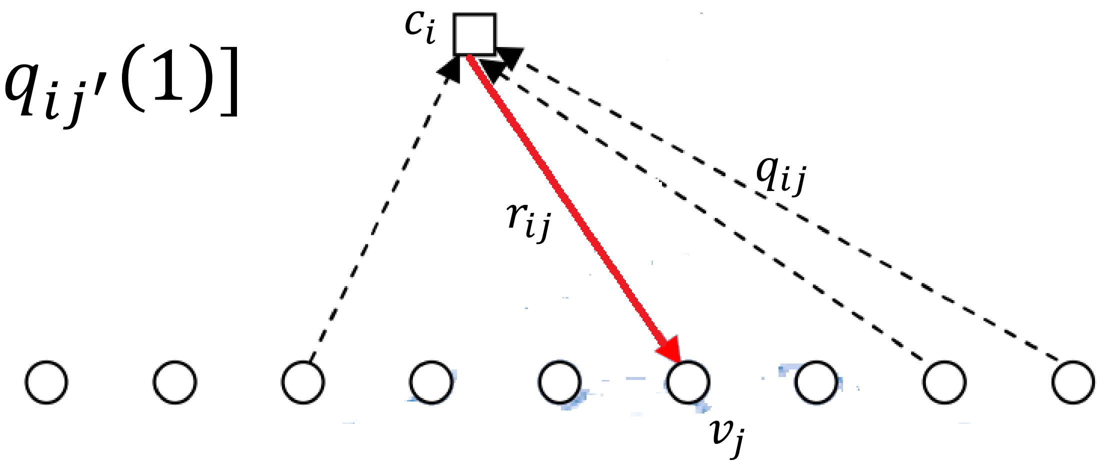
$$= \frac{1}{2} + \frac{1}{2}\prod_{j=1}^{M}(1 - 2P_j)$$

r值

- Lemma中的P_j=>$q_{ij'}(1)$
- 外來位元(接收到的codeword) $\{x_{j'}\}_{j'\neq j}$ 有偶數個1 且 $x_j = 0$
  外來位元(接收到的codeword) $\{x_{j'}\}_{j'\neq j}$ 有奇數個1 且 $x_j = 1$
  =>滿足c_i
- $r_{ij}(0) = P_r\{滿足c_i \mid x_j = 0, c_i含有\{x_{j'}\}_{j'\neq j}\}$

$$= \frac{1}{2} + \frac{1}{2}\prod_{j'\in v_i\backslash j}[1 - 2q_{ij'}(1)]$$

- $r_{ij}(1) = 1 - r_{ij}(0)$



初始值Initialize

- BI-AWGNC :binary input additive white Gaussian noise channel
- 設定P_j=>$q_{ij}(1)$
  - 設$t_j = 1 - 2x_j$
  - 設接收的訊號$y_j = x_j + n_j$，n_j是N(0 , σ^2)的常態分佈

$$\Pr(t_j = t \mid y_j) = \frac{\Pr(t_j = t, y_j)}{\Pr(y_j)}$$

$$= \frac{\Pr(t_j = t, y_j)}{\Pr(t_j = t, y_j) + \Pr(t_j = -t, y_j)} \quad 其中 t \in \{1,-1\}$$

$$= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y_j - t)^2}{2\sigma^2}\right)}{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y_j - t)^2}{2\sigma^2}\right) + \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y_j + t)^2}{2\sigma^2}\right)}$$

$$= \frac{1}{1 + \exp(-\frac{(y_j + t)^2}{2\sigma^2}) / \exp(-\frac{(y_j - t)^2}{2\sigma^2})} \boxed{= \frac{1}{1 + \exp(-\frac{2y_j t}{\sigma^2})}}$$

Q值

- 判定原來傳送的codeword $\hat{x}$

$$Q_j(0) = K_j(1 - P_j)\prod_{i \in C_j} r_{ij}(0)$$

$$Q_j(1) = K_j \prod_{i \in C_j} r_{ij}(1)$$

- $Q_j(0) > Q_j(1)$: $\hat{x}_j = 0$
- $Q_j(1) > Q_j(0)$: $\hat{x}_j = 1$

Example

- consider an (8, 4) product code ($d_{min}=4$) comprised of a (3, 2) single parity check code ($d_{min}=2$) along rows and columns:

c_0	c_1	c_2
c_3	c_4	c_5
c_6	c_7	

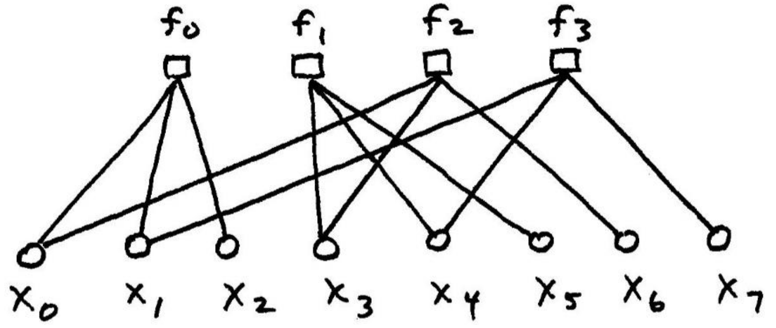
- thus,

$$c_2 = c_0 + c_1$$
$$c_5 = c_3 + c_4$$
$$c_6 = c_0 + c_3$$
$$c_7 = c_1 + c_4$$

from which

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- the graph corresponding to H is



- note that the code is neither low-density nor regular, but it will suffice to demonstrate the decoding algorithm

- the codeword we choose for this example is

$$\bar{c} = \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & \end{array} \xrightarrow{x_i = (-1)^{c_i}} \bar{x} = \begin{array}{ccc} -1 & +1 & -1 \\ +1 & -1 & -1 \\ -1 & -1 & \end{array}$$

- the received word $\bar{y}$ is

$$\bar{y} = \begin{matrix} y_0 & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & \end{matrix} = \begin{matrix} +.2 & +.2 & -.9 \\ +.6 & +.5 & -1.1 \\ -.4 & -1.2 & \end{matrix}$$

so that there are sign errors in y_0 and y_4

- initialization (assume $\sigma^2 = 0.5$):

$$\{q_{ij}(1)\}_{i=0}^{7} = \{P_i\}_{i=0}^{7}$$

for each valid j ($h_{ij}=1$)

$$= \left\{ \frac{1}{1+e^{2y_i/\sigma^2}} \right\}$$

$$= (.31, .31, .97, .083, .12, .99, .83, .99)$$

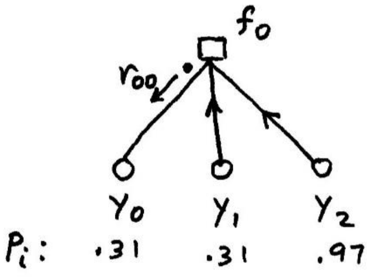
$$q_{ij}(0) = 1 - q_{ij}(1) \quad \forall\, i,j \text{ s.t. } h_{ij} = 1$$

- computation of $\{r_{ji}\}$

$$r_{00}(0) = \frac{1}{2} + \frac{1}{2} \prod_{i' \in \{1,2\}} (1 - 2q_{i'0}(1))$$

$$= \frac{1}{2} + \frac{1}{2}(1-2(.31))(1-2(.97))$$

$$= .32$$



$$r_{01}(0) = \frac{1}{2} + \frac{1}{2}(1-2(.31))(1-2(.97))$$

$$= .32$$

$$r_{02}(0) = \frac{1}{2} + \frac{1}{2}(1-2(.31))(1-2(.31))$$

$$= .57$$

... and so on

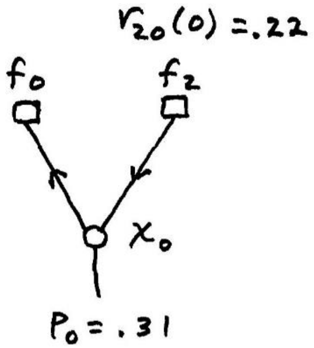
$r_{ji}(1)$ computed via $r_{ji}(1) = 1 - r_{ji}(0)$

- computation of $\{q_{ij}\}$

$$\tilde{q}_{00}(0) = (1-P_0) \prod_{j' \in \{2\}} r_{j'0}(0)$$

$$= (1-.31)(.22)$$

$$= .15$$



$$\tilde{q}_{00}(1) = P_0 \prod_{j' \in \{2\}} r_{j'0}(1)$$

$$= (.31)(.78)$$

$$= .24$$

$$\Rightarrow \quad q_{00}(0) = \frac{.15}{.15+.24} = .38$$

$$q_{00}(1) = \frac{.24}{.15+.24} = .62$$

- computation of $\{Q_i\}$: can be obtained from $\{q_{ij}\}$ by multiplying by additional factor

• Results [Q1 $\triangleq (Q_0(1), \ldots, Q_7(1))$]

iteration = 1
Q1 =0.7686 0.8694 0.9647 0.5076 0.7426 0.9479 0.7199 0.9853

c = 1 1 1 1 1 1 1 1

iteration = 2
Q1 = 0.1751 0.1836 0.9668 0.2330 0.4637 0.9722 0.8305 0.9919

c = 0 0 1 0 0 1 1 1

iteration = 3
Q1 = 0.7232 0.4609 0.9667 0.6308 0.8091 0.9498 0.6802 0.9909

c = 1 0 1 1 1 1 1 1

iteration = 4
Q1 = 0.5307 0.2683 0.9731 0.1477 0.4095 0.9811 0.8392 0.9922

c = 1 0 1 0 0 1 1 1

iteration = 5
Q1 = 0.7564 0.5799 0.9672 0.3425 0.7573 0.9558 0.8102 0.9896

c = 1 1 1 0 1 1 1 1

iteration = 6
Q1 = 0.4544 0.2089 0.9736 0.2136 0.6243 0.9686 0.8412 0.9924

c = 0 0 1 0 1 1 1 1

iteration = 7
Q1 = 0.7399 0.3381 0.9692 0.4086 0.7869 0.9567 0.7754 0.9923

c = 1 0 1 0 1 1 1 1

• note converges to the correct codeword after 7 iterations

對數領域和積演算法
Log-Domain Sum-Product Algorithm

- 算LLR 後驗機率比對數
- 因為很多乘法運算讓機率領域SPA不穩定，所以對數領域SPA比較受歡迎

Definitions

- $LLR\left(x_j\right) = \log(\frac{P_r\left(x_j=0 \mid y_j\right)}{P_r\left(x_j=1 \mid y_j\right)})$
- $LLR\left(r_{ij}\right) = \log(\frac{r_{ij}(0)}{r_{ij}(1)})$
- $LLR\left(q_{ij}\right) = \log(\frac{q_{ij}(0)}{q_{ij}(1)})$
- $LLR\left(Q_j\right) = \log(\frac{Q_j(0)}{Q_j(1)})$

初始值Initialize

$$LLR(q_{ij}) = LLR(x_j) = \log\left(\frac{\Pr(x_j = 0 \mid y_j)}{\Pr(x_j = 1 \mid y_j)}\right)$$

$$= \log\left(\frac{\Pr(t_j = 1 \mid y_j)}{\Pr(t_j = -1 \mid y_j)}\right)$$

$$= \log\frac{(1 + \mathrm{e}^{-\frac{2y_j}{\sigma^2}})^{-1}}{(1 + \mathrm{e}^{\frac{2y_j}{\sigma^2}})^{-1}} = \log\frac{1 + \mathrm{e}^{\frac{2y_j}{\sigma^2}}}{1 + \mathrm{e}^{-\frac{2y_j}{\sigma^2}}} = \log(\mathrm{e}^{\frac{2y_j}{\sigma^2}}) = \boxed{\frac{2y_j}{\sigma^2}}$$

Lemma2

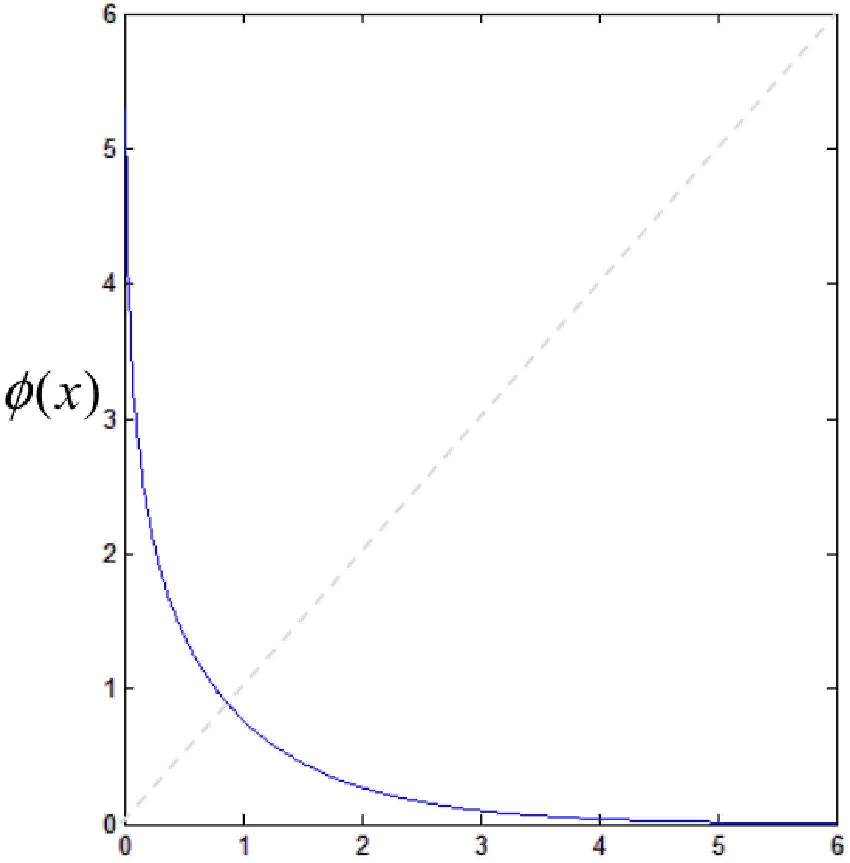
- 若$p_0 + p_1 = 1$，$\tanh\left(\frac{1}{2}\log\frac{p_0}{p_1}\right) = 1 - 2p_1$
- Proof.

$$\because \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\therefore \tanh\left(\frac{1}{2}\log\frac{p_0}{p_1}\right) = \frac{e^{2(\frac{1}{2}\log\frac{p_0}{p_1})} - 1}{e^{2(\frac{1}{2}\log\frac{p_0}{p_1})} + 1} = \frac{\frac{p_0}{p_1} - 1}{\frac{p_0}{p_1} + 1} = \frac{p_0 - p_1}{p_0 + p_1} = p_0 - p_1 = 1 - 2p_1$$

Lemma3

如圖 4-8，若 $\phi(\mathrm{x}) = -\log\left(\tanh\dfrac{x}{2}\right)$，且 $x>0$，則 $\phi(x)=\phi^{-1}(x)$



- Proof

$$\phi(\mathrm{x}) = -\log\left(\tanh\frac{x}{2}\right) = -\log\left(\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}\right) = -\log\left(\frac{e^{x}-1}{e^{x}+1}\right) = \log\left(\frac{e^{x}+1}{e^{x}-1}\right)$$

$$\Rightarrow \phi(\phi(\mathrm{x})) = \log\left(\frac{e^{\phi(\mathrm{x})}+1}{e^{\phi(\mathrm{x})}-1}\right) = \log\left(\frac{\frac{e^{x}+1}{e^{x}-1}+1}{\frac{e^{x}+1}{e^{x}-1}-1}\right)$$

$$= \log\left(\frac{(e^{x}+1)+(e^{x}-1)}{(e^{x}+1)-(e^{x}-1)}\right)$$

$$= \log e^{x} = x$$

$$\Rightarrow \phi(\mathrm{x}) = \phi^{-1}(\mathrm{x}) \text{ 。}$$

$LLR(r_{ij})$

$$\because \ r_{ij}(0) = \frac{1}{2} + \frac{1}{2} \prod_{j' \in V_i \setminus j} (1 - 2q_{ij'}(1))$$ Lemma1

$$\Rightarrow 1 - r_{ij}(1) = \frac{1}{2} + \frac{1}{2} \prod_{j' \in V_i \setminus j} (1 - 2q_{ij'}(1))$$

$$\Rightarrow \frac{1}{2} - r_{ij}(1) = \frac{1}{2} \prod_{j' \in V_i \setminus j} (1 - 2q_{ij'}(1))$$

$$\Rightarrow 1 - 2r_{ij}(1) = \prod_{j' \in V_i \setminus j} (1 - 2q_{ij'}(1))$$

$$\Rightarrow \tanh\left(\frac{1}{2} \log \frac{r_{ij}(0)}{r_{ij}(1)}\right) = \prod_{j' \in V_i \setminus j} \tanh\left(\frac{1}{2} \log \frac{q_{ij'}(0)}{q_{ij'}(1)}\right)$$ Lemma2

$$\therefore \tanh\left(\frac{1}{2} LLR(r_{ij})\right) = \prod_{j' \in V_i \setminus j} \tanh\left(\frac{1}{2} LLR(q_{ij'})\right)$$

$$\Rightarrow \frac{1}{2}LLR(r_{ij}) = \tanh^{-1}\Big(\prod_{j'\in V_i\setminus j}\tanh\big(\frac{1}{2}LLR(q_{ij'})\big)\Big)$$

$$\Rightarrow LLR(r_{ij}) = 2\tanh^{-1}\Big(\prod_{j'\in V_i\setminus j}\tanh\big(\frac{1}{2}LLR(q_{ij'})\big)\Big) \quad 令\ LLR(q_{ij'}) = \alpha_{ij'}\beta_{ij'}$$

$$= 2\tanh^{-1}\Big(\prod_{j'\in V_i\setminus j}\tanh\big(\frac{1}{2}\alpha_{ij'}\beta_{ij'}\big)\Big) \quad 其中\ \alpha_{ij'} = sign(LLR(q_{ij'}));\ \beta_{ij'} = \left|LLR(q_{ij'})\right|$$

$$= \prod_{j'\in V_i\setminus j}\alpha_{ij'} \cdot 2\tanh^{-1}\Big(\prod_{j'\in V_i\setminus j}\tanh\big(\frac{1}{2}\beta_{ij'}\big)\Big)$$

$$= \prod_{j'\in V_i\setminus j}\alpha_{ij'} \cdot 2\tanh^{-1}\log^{-1}\log\Big(\prod_{j'\in V_i\setminus j}\tanh\big(\frac{1}{2}\beta_{ij'}\big)\Big)$$

$$= \prod_{j'\in V_i\setminus j}\alpha_{ij'} \cdot 2\tanh^{-1}\log^{-1}\sum_{j'\in V_i\setminus j} log\Big(\tanh\big(\frac{1}{2}\beta_{ij'}\big)\Big) \quad 令\ \phi(\mathrm{x}) = -\log\left(\tanh\frac{x}{2}\right)$$

$$= \prod_{j'\in V_i\setminus j}\alpha_{ij'} \cdot \phi^{-1}\left(\sum_{j'\in V_i\setminus j}\phi(\beta_{ij'})\right)$$

$$LLR(r_{ij}) = \prod_{j' \in V_i \setminus j} \alpha_{ij'} \cdot \phi\left(\sum_{j' \in V_i \setminus j} \phi(\beta_{ij'}) \right) = \boxed{\prod_{j' \in V_i \setminus j} sign(LLR(q_{ij'})) \cdot \phi\left(\sum_{j' \in V_i \setminus j} \phi(\left| LLR(q_{ij'}) \right|) \right)}$$

Lemma3

$LLR(q_{ij})$

$$\Rightarrow \frac{q_{ij}(0)}{q_{ij}(1)} = \frac{\mathrm{K}_{ij}(1-\mathrm{P}_j)\prod_{i'\in \mathrm{C}_j \setminus i} r_{i'j}(0)}{\mathrm{K}_{ij}\mathrm{P}_j \prod_{i'\in \mathrm{C}_j \setminus i} r_{i'j}(1)} = \frac{(1-P_j)\prod_{i'\in \mathrm{C}_j \setminus i} r_{i'j}(0)}{P_j \prod_{i'\in \mathrm{C}_j \setminus i} r_{i'j}(1)}$$

$$\Rightarrow LLR(q_{ij}) = \log\left(\frac{q_{ij}(0)}{q_{ij}(1)}\right) = \log\left(\frac{(1-P_j)\prod_{i'\in \mathrm{C}_j \setminus i} r_{i'j}(0)}{P_j \prod_{i'\in \mathrm{C}_j \setminus i} r_{i'j}(1)}\right) = \log\left(\frac{1-P_j}{P_j}\right) + \log\left(\prod_{i'\in \mathrm{C}_j \setminus i} \frac{r_{i'j}(0)}{r_{i'j}(1)}\right)$$

$$= \log\left(\frac{\Pr(x_j = 0 \mid y_j)}{\Pr(x_j = 1 \mid y_j)}\right) + \sum_{i'\in \mathrm{C}_j \setminus i} \log\left(\frac{r_{i'j}(0)}{r_{i'j}(1)}\right)$$

$$= \boxed{LLR(x_j) + \sum_{i'\in \mathrm{C}_j \setminus i} LLR(r_{i'j})}$$

$$LLR(Q_j)$$

$$LLR(Q_j) = \boxed{LLR(x_j) + \sum_{i \in C_j} LLR(r_{ij})}$$