Log SPA 2

definition

$p_{10}=\Pr(a_1=0)$，$p_{20}=\Pr(a_2=0)$，$p_{11}=\Pr(a_1=1)$，$p_{21}=\Pr(a_2=1)$。

$L_i=LLR(a_i)=\log\left(\frac{\Pr(a_i=0)}{\Pr(a_i=1)}\right)=\log\left(\frac{p_{i0}}{p_{i1}}\right)$表示變數$a_i$之概似比的對數($LLR$)。

$a_1\oplus a_2$表示a_1與a_2的二進位的和，即a_1+a_2的和除以2之餘數。

$L_1 \boxplus L_2=LLR(a_1\oplus a_2)=\log\left(\frac{\Pr(a_1\oplus a_2=0)}{\Pr(a_1\oplus a_2=1)}\right)$。

Lemma 4

設 a_1 與 a_2 是兩個獨立的二進位變數

L_1 與 L_2 分別是 a_1 與 a_2 的LLR

若 $L_1 \boxplus L_2$ 是 $a_1 \oplus a_2$ 的 LLR，則

$$L_1 \boxplus L_2 = \log\left(\frac{1 + e^{L_1 + L_2}}{e^{L_1} + e^{L_2}}\right)$$

Proof.

$$L_1 \boxplus L_2 = LLR(a_1 \oplus a_2)$$

$$= \log\left(\frac{\Pr(a_1 \oplus a_2 = 0)}{\Pr(a_1 \oplus a_2 = 1)}\right) \quad \text{其中} \begin{array}{l} a_1 \oplus a_2 = 0 \Rightarrow (a_1 = 0 \text{且} a_2 = 0) \text{或} (a_1 = 1 \text{且} a_2 = 1) \\ a_1 \oplus a_2 = 1 \Rightarrow (a_1 = 0 \text{且} a_2 = 1) \text{或} (a_1 = 1 \text{且} a_2 = 0) \end{array}$$

$$= \log\left(\frac{p_{10}p_{20} + p_{11}p_{21}}{p_{10}p_{21} + p_{11}p_{20}}\right) = \log\left(\frac{\frac{p_{10}}{p_{11}}\frac{p_{20}}{p_{21}} + 1}{\frac{p_{10}}{p_{11}} + \frac{p_{20}}{p_{21}}}\right)$$

$$= \log\left(\frac{e^{L_1}e^{L_2} + 1}{e^{L_1} + e^{L_2}}\right) = \log\left(\frac{1 + e^{L_1 + L_2}}{e^{L_1} + e^{L_2}}\right) \text{。}$$

Lemma 5

對任意兩個實數 x, y ，定義 $\max^*(x, y) = \log(e^x + e^y)$

$$\max^*(x, y) = \max(x, y) + \log(1 + e^{-|x-y|})$$

Proof.

若 $x \geq y$ ， $\max^*(x,y) = \log(e^x + e^y) = \log e^x(1+e^{-(x-y)})$

$$= \log e^x + \log(1+e^{-(x-y)})$$

$$= x + \log(1+e^{-|x-y|})$$

$$= \max(x,y) + \log(1+e^{-|x-y|})$$

若 $x < y$ ，同理可證。

By Lemma 4, 5

$$L_1 \boxplus L_2 = \log\left(\frac{1+e^{L_1+L_2}}{e^{L_1}+e^{L_2}}\right) = \log\left(\frac{e^{0}+e^{L_1+L_2}}{e^{L_1}+e^{L_2}}\right)$$

$$= \log(e^{0}+e^{L_1+L_2}) - \log(e^{L_1}+e^{L_2})$$

$$= \max{}^{*}(0, L_1+L_2) - \max{}^{*}(L_1, L_2)$$

$$= [\max(0, L_1+L_2) + \log(1+e^{-|0-(L_1+L_2)|})] - [\max(L_1, L_2) + \log(1+e^{-|L_1-L_2|})]$$

$$= [\max(0, L_1+L_2) - \max(L_1, L_2)] + [\log(1+e^{-|L_1+L_2|}) - \log(1+e^{-|L_1-L_2|})]$$

$$L_1 \boxplus L_2 = [\max(0, L_1 + L_2) - \max(L_1, L_2)] + [\log(1 + e^{-|L_1 + L_2|}) - \log(1 + e^{-|L_1 - L_2|})]$$

$$= [\max(0, L_1 + L_2) - \max(L_1, L_2)] + s(L_1, L_2)$$

$$s(x, y) = \log(1 + e^{-|x+y|}) - \log(1 + e^{-|x-y|})$$

Lemma 6

x, y 為實數，則 $\max(0, x+y) - \max(x, y) = sign(x) sign(y) \min(|x|, |y|)$

Proof.

若$x+y>0$且$x>y \Rightarrow x>y>-x \Rightarrow |x|>|y|$且$x>0$

$$\Rightarrow 左式=(x+y)-x=y=+sign(y)|y|=sign(x)sign(y)\min(|x|,|y|)=右式$$

若$x+y>0$且$y>x$,同理可證

若$x+y<0$且$x>y \Rightarrow -y>x>y>\Rightarrow |x|<|y|$且$y<0$

$$\Rightarrow 左式=0-x=-x=-sign(x)|x|=sign(x)sign(y)\min(|x|,|y|)=右式$$

若$x+y<0$且$y>x$,同理可證。

定義 $A_2 = a_1 \oplus a_2$

$$LLR(A_2) = LLR(a_1 \oplus a_2) = L_1 \boxplus L_2$$

$$L_1 \boxplus L_2 \boxplus L_3 = LLR(a_1 \oplus a_2 \oplus a_3) = LLR(A_2 \oplus a_3)$$

$$= \log\left(\frac{1+e^{LLR(A_2)+L_3}}{e^{LLR(A_2)} + e^{L_3}}\right) = \log\left(\frac{1+e^{LLR(A_2)}e^{L_3}}{e^{LLR(A_2)} + e^{L_3}}\right) = \log\left(\frac{e^{L_1} + e^{L_2} + e^{L_3} + e^{L_1+L_2+L_3}}{1+e^{L_1+L_2} + e^{L_1+L_3} + e^{L_2+L_3}}\right)$$

因為$LLR(r_{i\,j})$的計算相當於求二進位的隨機變數之和的 LLR 隨機變數為獨立的假設下

$$LLR(r_{i\,j}) = LLR(q_{i,j'_1}) \boxplus LLR(q_{i,j'_2}) \boxplus LLR(q_{i,j'_3}) \cdots \boxplus LLR(q_{i,j'_\omega}) \quad \text{其中 } j'_t \in V_{\mathrm{i}} \setminus j$$

Approximation

$$L_1 \boxplus L_2 = [\max(0, L_1 + L_2) - \max(L_1, L_2)] + s(L_1, L_2)$$

$$|s(x, y)| \leq 0.693$$

$$L_1 \boxplus L_2 \approx sign(L_1) sign(L_2) \min(|L_1|, |L_2|)]$$

$$LLR(r_{i\,j}) = \prod_{j' \in V_i \setminus j} sign(LLR(q_{i\,j'})) \cdot \min_{j' \in V_i \setminus j} |LLR(q_{i\,j'})|$$

$$s(x,y)=\log(1+e^{-|x+y|})-\log(1+e^{-|x-y|})$$ ，則 $|s(x,y)|\le 0.693$

Proof. 若 $|x+y|\ge|x-y|$ ，則 $0\le\log(1+\frac{1}{e^{|x+y|}})\le\log(1+\frac{1}{e^{|x-y|}})$

$$|s(x,y)|=\log(1+\frac{1}{e^{|x-y|}})-\log(1+\frac{1}{e^{|x+y|}})$$

$$\le\log(1+\frac{1}{e^{|x-y|}})\le\log(1+\frac{1}{e^{0}})=\log 2\approx 0.693$$

若 $|x+y|<|x-y|$ ，則 $\log(1+\frac{1}{e^{|x+y|}})>\log(1+\frac{1}{e^{|x-y|}})\ge 0$

$$|s(x,y)|=\log(1+\frac{1}{e^{|x+y|}})-\log(1+\frac{1}{e^{|x-y|}})$$

$$\le\log(1+\frac{1}{e^{|x+y|}})\le\log(1+\frac{1}{e^{0}})=\log 2\approx 0.693$$