Sample Solution

Solution 1. i) let $A_1, \dots, A_n \in \mathscr{A}$, and then by definition $\bigcup_{1\le i\le n} A_n \in \mathscr{A}$, since $A_i^c \in \mathscr{A}$ for all i. Hence,

$$\bigcup A_i^c = (\bigcap A_i)^c \in \mathscr{A}$$

implies

$$\bigcap A_i \in \mathscr{A}$$

where $i \in \{1, \dots, n\}$

ii) Just use the complement axiom twice

iii) Notice that $A\backslash B = A \cap B^c$ and $A \triangle B = (A\backslash B) \cup (B\backslash A)$ then it is obvious

Solution 2. It would be crazy to write to write down exactly what is contained in it, which is the case for all this kind of objects that has the well-known property of smallest and intersection. But we can at least make the following observation

Since every singleton is a Borel σ-algebra, $\sigma(A) \subset \sigma(\mathcal{O})$. Same observation can be made from the fact that $A \subset \mathcal{O}$ hence $\sigma(A) \subset \sigma(\mathcal{O})$(exercise).

However $\sigma(A) \not\supset \sigma(\mathcal{O})$. Since countable unions (or intersections) of countable sets are countable, so any $X \in \sigma(A)$ is either countable or compliment of a countable set. Then consider $(0, 1) \notin \sigma(A)$ but is a Borel set.