

Sigma Algebra

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1 Basics

Remark. I will directly start with *Borel σ -algebra* because that is where things get complicated

Definition 1.1. The σ -algebra $\sigma(\mathcal{O})$ generated by the open sets \mathcal{O} of \mathbb{R}^n is called *Borel σ -algebra*

Remark. For every system of sets $\mathcal{G} \subset \mathcal{P}(X)$ there exists a smallest σ -algebra containing \mathcal{G} . This is the *σ -algebra generated by \mathcal{G}* which is just

$$\mathcal{A} = \bigcap_{\substack{\mathcal{F} \text{ } \sigma\text{-alg} \\ \mathcal{F} \supset \mathcal{G}}} \mathcal{F}$$

Since Borel sets are fundamental for both measure theory and topology, we consider the following amazing theorem

Theorem 1.1. Let $\mathcal{O}, \mathcal{C}, \mathcal{K}$ be families of open, closed, compact sets in \mathbb{R}^n . Then

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}) = \sigma(\mathcal{C}) = \sigma(\mathcal{K})$$

Proof. Since compact sets are closed, we have $\mathcal{K} \subset \mathcal{C}$, hence $\sigma(\mathcal{K}) \subset \sigma(\mathcal{C})$. On the other hand, if $C \in \mathcal{C}$, then $C_k = C \cap \overline{B_k(0)}$ is closed and bounded, hence $C_k \in \mathcal{K}$. Notice $C = \bigcup_{k \in \mathbb{N}} C_k$, thus $\mathcal{C} \subset \sigma(\mathcal{K})$, hence $\sigma(\mathcal{C}) \subset \sigma(\mathcal{K})$

Since $(\mathcal{O})^c = \mathcal{C}$ we have $\mathcal{C} \subset \sigma(\mathcal{O})$. And the converse is similar \square

Notice from our above theorem, a lot of unexpected (at least for me) sets can be a Borel σ -algebra. Consider:

$$\mathcal{J}^o(\mathbb{R})^n = \{(a_i, b_i) : a_i, b_i \in \mathbb{R}, i \in \mathbb{N}\} \quad \mathcal{J}(\mathbb{R})^n = \{[a_i, b_i] : a_i, b_i \in \mathbb{R}, i \in \mathbb{N}\}$$

For notation, we write $\mathcal{J}_{rat}, \mathcal{J}_{rat}^o$ as (half-)open interval with rational endpoints. For which we have the following theorem

Theorem 1.2. $\mathcal{B}(\mathbb{R})^n = \sigma(\mathcal{J}_{rat}^n) = \sigma(\mathcal{J}_{rat}^{o,n}) = \sigma(\mathcal{J}^n) = \sigma(\mathcal{J}^{n,o})$

Proof. Consider an obvious fact: $\sigma(\mathcal{O}) \supset \sigma(\mathcal{J}^o) \subset \sigma(\mathcal{J}_{rat}^o)$. For converse direction, if $U \in \mathcal{O}$, we have

$$U = \bigcup_{\substack{I \in \mathcal{J}_{rat}^o \\ I \subset U}} I$$

The \supset direction is obvious, for the other direction we fix some $x \in U$. Since U is open, there is some ball $B_\epsilon(x) \subset U$ and we can inscribe a square into the ball $B_\epsilon(x)$ and shrink this square to get a rectangle $I' = I' \in \mathcal{J}_{rat}^o$ containing x . Hence

$$U \in \mathcal{O} \subset \sigma(\mathcal{J}_{rat}^o)$$

Every half-open rectangle can be written as

$$[a_i, b_i) \times \cdots \times [a_n, b_n) = \bigcap_{i \in \mathbb{N}} (a_1 - \frac{1}{i}, b_i) \times \cdots \times (a_n - \frac{1}{i}, b_n)$$

while every open rectangle can be represented as

$$(c_1, d_1) \times \cdots \times (c_n, d_n) = \bigcup_{i \in \mathbb{N}} [c_1 + \frac{1}{i}d_1) \times \cdots \times [c_n + \frac{1}{i}d_n)$$

Hence $\sigma(\mathcal{J}) = \sigma(\mathcal{J}^o)$ and similarly $\sigma(\mathcal{J}_{hat}) = \sigma(\mathcal{J}_{hat}^o)$ which completes the proof \square

Lemma 1. Every singleton set $\{x\}$ is an element of Borel σ -algebra

Proof. Consider $\{(b - \frac{1}{n}, b + \frac{1}{n}), n \geq 1\}$. By definition,

$$(b - \frac{1}{n}, b + \frac{1}{n}) \in \sigma(\mathcal{O})$$

Using property of σ -algebra

$$\begin{aligned} (b - \frac{1}{n}, b + \frac{1}{n})^c &\in \sigma(\mathcal{B}) \\ \bigcup_{n=1}^{\infty} (b - \frac{1}{n}, b + \frac{1}{n})^c &\in \sigma(\mathcal{B}) \\ (\bigcap_{n=1}^{\infty} (b - \frac{1}{n}, b + \frac{1}{n}))^c &\in \sigma(\mathcal{B}) \\ \bigcap_{n=1}^{\infty} (b - \frac{1}{n}, b + \frac{1}{n}) &\in \sigma(\mathcal{B}) \end{aligned}$$

and

$$\{b\} = \bigcap_{n=1}^{\infty} (b - \frac{1}{n}, b + \frac{1}{n})$$

\square