Measure Theory Notes

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1 Basics

Remark. I will directly start with *Borel σ-algebra* because that is where things get compilcated

Definition 1.1. The σ-algebra $\sigma(\mathscr{O})$ generated by the open sets $\mathscr{O}$ of $\mathbb{R}^n$ is called *Borel σ-algebra*

Remark. For every system of sets $\mathscr{G} \subset \mathscr{P}(X)$ there exists a smallest σ-algebra containing $\mathscr{G}$. This is the *σ-algebra generated by $\mathscr{G}$* which is just

$$\mathscr{A} = \bigcap_{\substack{\mathscr{F} \ \sigma\text{-alg} \\ \mathscr{F} \supset \mathscr{G}}} \mathscr{F}$$

Since Borel sets are fundamental for both measure theory and topology, we consider the following amaizng theorem

Theorem 1.1. *Let $\mathscr{O}, \mathscr{C}, \mathscr{K}$ be families of open, closed, comapct sets in $\mathbb{R}^n$.Then*

$$\mathscr{B}(\mathbb{R}^n) = \sigma(\mathscr{O}) = \sigma(\mathscr{C}) = \sigma(\mathscr{K})$$

Proof. Since compact sets are closed, we have $\mathscr{K} \subset \mathscr{C}$, hence $\sigma(\mathscr{K}) \subset \sigma(\mathscr{C})$. On the other hand, if $C \in \mathscr{C}$, then $C_k = C \cap \overline{B_k(0)}$ is closed and bounded, hence $C_k \in \mathscr{K}$. Notice $C = \bigcup_{k \in \mathbb{N} C_k}$,thus $\mathscr{C} \subset \sigma(\mathscr{K})$, hence $\sigma(\mathscr{C}) \subset \sigma(\mathscr{K})$

Since $(\mathscr{O})^c = \mathscr{C}$ we have $\mathscr{C} \subset \sigma(\mathscr{O})$. And the converse is similar □

Notice from our above theorem, a lot of unexpected(at least for me) sets can be a Borel σ-algebra. Consider:

$$\mathscr{J}^o(\mathbb{R})^n = \{(a_i, b_i) : a_i, b_i \in \mathbb{R}, i \in \mathbb{N}\} \ \mathscr{J}(\mathbb{R})^n = \{[a_i, b_i) : a_i, b_i \in \mathbb{R}, i \in \mathbb{N}\}$$

For for notation, we write $\mathscr{J}_{rat}$, $\mathscr{J}^o_{rat}$ as (half-)open interval with rational endpoints. For which we have the following theorem

Theorem 1.2. $\mathscr{B}(\mathbb{R})^n = \sigma(\mathscr{J}^n_{rat}) = \sigma(\mathscr{J}^{o,n}_{rat}) = \sigma(\mathscr{J}^n) = \sigma(\mathscr{J}^{n,o})$

Proof. Consider an obvious fact: $\sigma(\mathscr{O}) \supset \sigma(\mathscr{J}^o) \subset \sigma(\mathscr{J}^o_{rat})$. For converse direction, if $U \in \mathscr{O}$, we have

$$U = \bigcup_{\substack{I \in \mathscr{J}^o_{rat} \\ I \subset U}} I$$

The $\supset$ direction is obvious, for the other direction we fix some $x \in U$. Since U is open, there is some ball $B_\epsilon(x) \subset U$ and we can inscribe a square into the ball $B_\epsilon(x)$ and shrink this square to get a rectangle $I' = I' \in \mathscr{J}^o_{rat}$ containing x. Hence

$$U \in \mathscr{O} \subset \sigma(\mathscr{J}^o_{rat})$$

Every half-open rectangle can be written as

$$[a_i, b_i) \times \cdots \times [a_n, b_n) = \bigcap_{i \in \mathbb{N}} (a_1 - \frac{1}{i}, b_i) \times \cdots \times (a_n - \frac{1}{i}, b_n)$$

whilw every open rectangle can be represented as

$$(c_1, d_1) \times \cdots \times (c_n, d_n) = \bigcup_{i \in \mathbb{N}} [c_1 + \frac{1}{i} d_1) \times \cdots \times [c_n + \frac{1}{i} d_n)$$

Hence $\sigma(\mathscr{J}) = \sigma(\mathscr{J}^o)$ and similarly $\sigma(\mathscr{J}_{hat}) = \sigma(\mathscr{J}_{hat}^o)$ which completes the proof $\square$