Analyzing On-Line Portfolio Selection for Randomized Asset Price

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Portfolio Selection

- n choices of investments
- Relative price at time t: $\boldsymbol{x}^t = (x_1^t, \cdots, x_n^t) \in \mathbb{R}_+^n$
- Portfolio choice at time t: $\boldsymbol{w}^t = (w_1^t, \cdots, w_n^t) \in \mathbb{R}_+^n, \sum_{i=1}^n w_i^t = 1$
- Total wealth at time T:

$$S_T(\{\boldsymbol{w}^t\}, \{\boldsymbol{x}^t\}) = \prod_{t=1}^{T} \langle \boldsymbol{w}^t, \boldsymbol{x}^t \rangle$$

- Log total wealth at time T:

$$LS_T(\{\boldsymbol{w}^t\}, \{\boldsymbol{x}^t\}) = \sum_{t=1}^{T} \log \langle \boldsymbol{w}^t, \boldsymbol{x}^t \rangle$$

- Objective: maximize $LS_T(\{\boldsymbol{w}^t\}, \{\boldsymbol{x}^t\})$ such that $\boldsymbol{w}^t = \boldsymbol{w}^t(\{\boldsymbol{w}^s, \boldsymbol{x}^s\}_{s=1}^{t-1})$

Randomized Asset Price

- We consider the case where the relative price follows a distribution, i.e.

$$(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^t) \sim \Pr(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^t) \tag{1}$$

- Example: independent log-normal distribution.
- Multiplicative updates [Helmbold et al., 1998]:

Algorithm On-Line Portfolio Selection with Multiplicative Update EG(η)

Input: $\boldsymbol{w}^1 \in \Delta_{n-1}$, $\eta > 0$

1: **for** $t = 1$ to $T - 1$ **do**

2: $\quad w_i^{t+1} \propto w_i^t \exp\left(\eta \frac{x_i^t}{\langle \boldsymbol{w}^t, \boldsymbol{x}^t \rangle}\right), \quad i = 1, 2, \cdots, n.$

3: **end for**

Theorems

Assumption 1 (Finite inverse expectation)

The inverse square of price relative x_i^t *has a finite expectation for any* $t = 1, 2, \cdots, T$ *and* $i = 1, 2, \cdots, n$. *Furthermore,* $\mathbb{E}\frac{1}{(x_i^t)^2} \leq \sigma_i^t$.

Assumption 2 (Uniform boundness)

The price relative x_i^t *is uniformly bounded by* $M > 0$ *for all* $t = 1, 2, \cdots, T$ *and* $i = 1, 2, \cdots, n$.

Remark 1

Under Assumption 1 and 2, no dependent structures are assumed.

Theorems

Theorem 1

Suppose Assumption 1 and 2 holds. For any fixed portfolio vector $\boldsymbol{u} \in \Delta_{n-1}$, the on-line portfolio selection algorithm EG(η) has the expected regret for the logrithmic of the total wealth until time T:

$$\mathbb{E}\sum_{t=1}^{T}\log(\langle \boldsymbol{w}^t, \boldsymbol{x}^t\rangle) \geq \mathbb{E}\sum_{t=1}^{T}\log(\langle \boldsymbol{u}, \boldsymbol{x}^t\rangle) - \frac{KL(\boldsymbol{u}||\boldsymbol{w}^1)}{\eta} - \frac{M^2\eta}{8}\sum_{i,t}\sigma_i^t, \qquad (2)$$

where $KL(\boldsymbol{u}||\boldsymbol{w}^1)$ is the KL divergence between $\boldsymbol{u}$ and $\boldsymbol{w}^1$.

Remark 2

In practice we may assume the i.i.d. price relatives across all time t for each investment each investment n. In this case, the last term in equation (4) becomes $\frac{M^2\eta T}{8}\sum_{i=1}^{n}\sigma_i$, which can further be bounded by $\frac{M^2\eta Tn}{8}\max\{\sigma_i\}$.

Theorems

Assumption 3 (Uniformly bounded expectation)

The expectation of price relative $\{x_i^t\}$ *and their reciprocal* $\{\frac{1}{x_i^t}\}$ *are uniformly bounded across all* $i = 1, 2, \cdots, n$ *and* $t = 1, 2, \cdots, T$.

Assumption 4 (Independence and sub-gaussian tails)

The price relative $\{x_i^t\}$ *is independent. Furthermore, for each for all* $i = 1, 2, \cdots, n$ *and* $t = 1, 2, \cdots, T$, *the price relative* $x_i^t \sim$ *sub-*$G(\sigma^2)$ *and* $\frac{1}{x_i^t} \sim$ *sub-*$G(s^2)$, *where sub-*$G(\cdot)$ *is the sub-Gaussian distribution.*

Theorems

Theorem 2

Suppose Assumption 3 and 4 holds. For any fixed portfolio vector $\boldsymbol{u} \in \Delta_{n-1}$, the on-line portfolio selection algorithm EG(η) has the expected regret for the logrithmic of the total wealth until time T:

$$\mathbb{E}\sum_{t=1}^{T}\log(\langle \boldsymbol{w}^t, \boldsymbol{x}^t\rangle) \geq \mathbb{E}\sum_{t=1}^{T}\log(\langle \boldsymbol{u}, \boldsymbol{x}^t\rangle) - \frac{KL(\boldsymbol{u}||\boldsymbol{w}^1)}{\eta} - \frac{CT\sigma^2 s^2(\log n)^2\eta}{8}, \quad (3)$$

where C is a constant not related to n, T, σ and s.

Remark 3

In practice we can regard η as a tuning parameter. If we initialize $\boldsymbol{w}^1$ as the uniform distribution and $\{\sigma_i^t\}$ has a uniform bound K, the regret of Theorem 1 has order $O(M\sqrt{nTK\log n})$, and Theorem 2 gives an $O(\sigma s\sqrt{T\log^3 n})$ regret.

Theorems

Assumption 5 (Finite expectation)

The square of price relative x_i^t *has a finite expectation for any* $t = 1, 2, \cdots, T$ *and* $i = 1, 2, \cdots, n$. *Furthermore,* $\mathbb{E}(x_i^t)^2 \leq s_i^t$.

Theorem 3

Suppose Assumption 1 and 5 holds. For any fixed portfolio vector $\boldsymbol{u} \in \Delta_{n-1}$*, the on-line portfolio selection algorithm EG(*η*) has the expected regret for the logrithmic of the total wealth until time* T*:*

$$\mathbb{E}\sum_{t=1}^{T}\log(\langle \boldsymbol{w}^t, \boldsymbol{x}^t\rangle) \geq \mathbb{E}\sum_{t=1}^{T}\log(\langle \boldsymbol{u}, \boldsymbol{x}^t\rangle) - \frac{KL(\boldsymbol{u}||\boldsymbol{w}^1)}{\eta} - \sum_{t=1}^{T}\sqrt{\sum_{i=1}^{n}\sigma_i^t\sum_{i=1}^{n}s_i^t} + T. \tag{4}$$

Experiments

- We test performance of multiplicative update rules on historical stock data
- 28 stocks from the New York Stock Exchange over the period between 01/11/2003 and 01/11/2023
- Experiment I: Portfolio consisted of two stocks with low/high correlations
- Experiment II: Dow Jones Industrial Average

References

David P Helmbold, Robert E Schapire, Yoram Singer, and Manfred K Warmuth. On-line portfolio selection using multiplicative updates. *Mathematical Finance*, 8(4):325–347, 1998.