Part 3
Chapter 8

Linear Algebraic Equations and Matrices

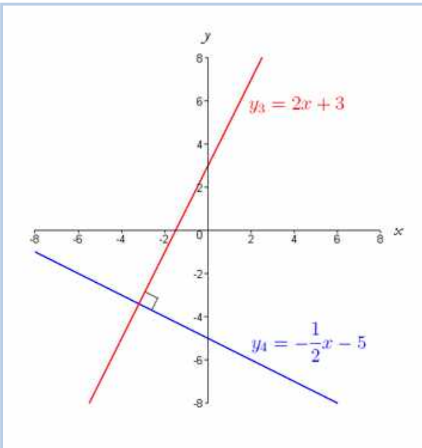
$$
\begin{array}{rcrcrcr}
x & + & y & + & z & = & 6 \\
 & & 2y & + & 5z & = & -4 \\
2x & + & 5y & - & z & = & 27
\end{array}
$$

$$
\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}
$$

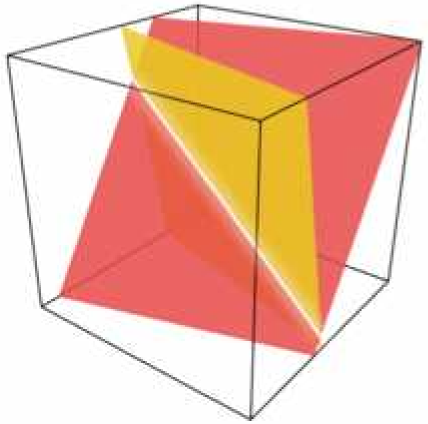
$$AX = B$$

Linear Algebraic Equations

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{cases}
$$



solution set for two equations
in two variables : point



$$
\begin{aligned}
x + 3y - 2z &= 5 \\
3x + 5y + 6z &= 7
\end{aligned}
\qquad x = -7z - 1 \quad \text{and} \quad y = 3z + 2
$$

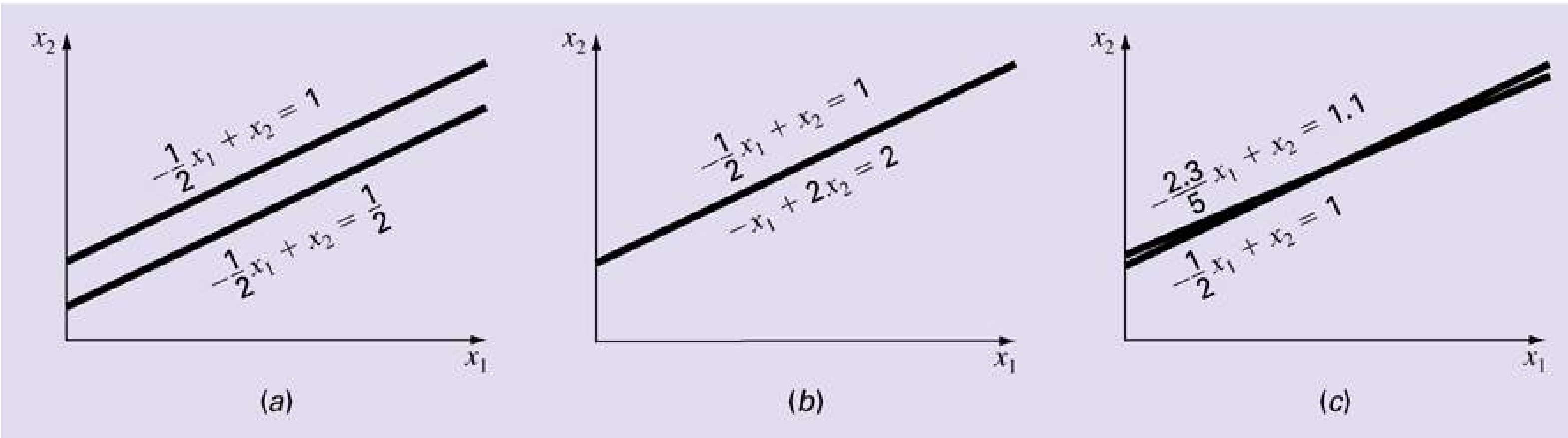
solution for two equations in three variables : line
solution for three equations in three variables : point

Condition of a system

(a) singular system – no solution

(b) singular system – infinite solutions

(c) ill-conditioned system : systems that are extremely sensitive to round-off errors



Direct Solvers

- Gaussian Elimination
- LU factorization, Cholesky Decomposition
- $Ax=b \rightarrow x=A^{-1}b$

  - Analytical, faster
  - Error accumulation.
  - Suitable for small number of unknowns.
  - $O(n^3)$

Iterative Solvers

- Newton-Raphson
- Gauss-Seidel methods
- Relaxation
  - Successive Overrelaxation (SOR)

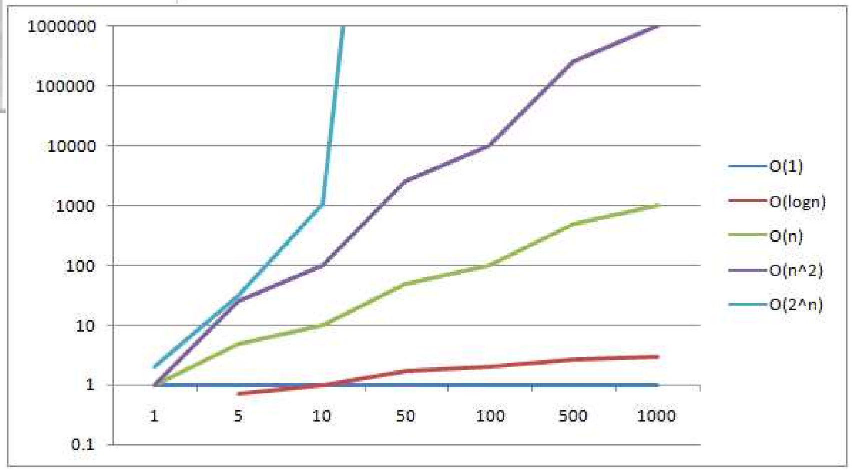
    - Iterative, slower
    - No error accumulation.
    - Suitable for larger number of unknowns.
    - $O(n^3)$

n의 크기	CRAY-1 $3n^3$나노초	TRS-80 19,500,000n나노초
10	3×10^{-6}초	2×10^{-1}초
100	3×10^{-1}초	2초
1,000	3초	20초
2,500	50초	50초
10,000	49분	3.2분
1,000,000	95년	5.4시간

The Cray-1 was succeeded in 1982 by the 800 MFLOPS

0.89MHz (RAM: 4K)

복잡도함수 (μsec)		$33n$	$46n \log n$	$13n^2$	$3.4n^3$	2^n
	10	0.00033초	0.0015초	0.0013초	0.0034초	0.001초
입력	1000	0.003초	0.03초	0.13초	3.4초	4×10^{16}년
n의 크기	1,000	0.033초	0.45초	13초	94시간	
	10,000	0.33초	6.1초	22분	39일	
	100,000	3.3초	1.3분	1.5일	108년	



Matrix Notation : A_{mxn}

- Row vector (m=1), Column vector (n=1, default)
- Square matrix : principal (or main) diagonal
  - Symmetric matrix
  - Diagonal matrix, Identity matrix
  - Upper (lower) triangular matrix
  - Band matrix : tridiagonal matrix

Symmetric

$$[A]=\begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix}$$

Diagonal

$$[A]=\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix}$$

Identity

$$[A]=\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Upper Triangular

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{bmatrix}$$

Lower Triangular

$$[A]=\begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Banded

$$[A]=\begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & a_{23} & \\ & a_{32} & a_{33} & a_{34} \\ & & a_{43} & a_{44} \end{bmatrix}$$

Matrix Operations

- $A_{mxn}+B_{mxn}=C_{mxn}$ $\quad c_{ij} = a_{ij} + b_{ij}$

- $A_{mxk}B_{kxn}=C_{mxn}$ $\quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 9 & -5 \\ 0 & 13 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ 1 & 13 \\ -2 & 6 \end{bmatrix}$$

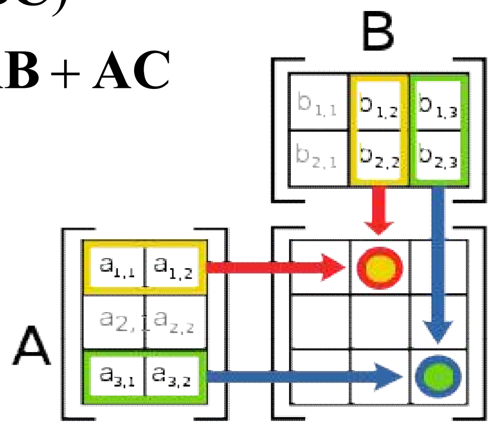
$$4 \begin{bmatrix} 1 & 3 & 5 \\ -1 & -8 & 10 \\ -7 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 20 \\ -4 & -32 & 40 \\ -28 & -20 & 52 \end{bmatrix}$$

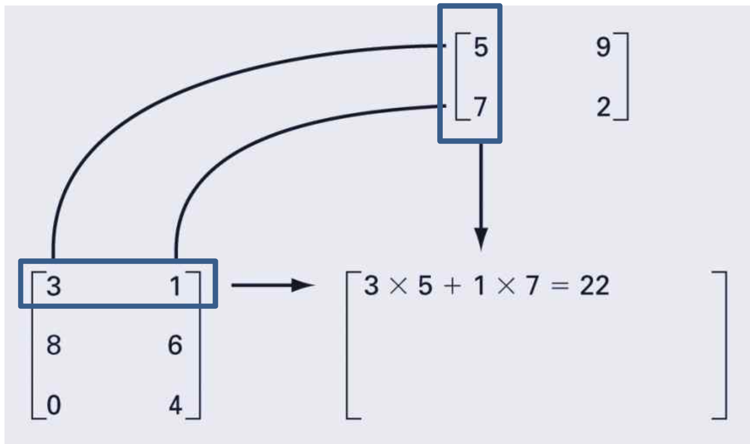
- $\mathbf{AB} \neq \mathbf{BA}$

  $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

  $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

  $\mathbf{AA}^{-1} = \mathbf{I}$





Matrix Application

- Suppose you make the following sales:
  - Monday
    - 3 T-shirts at $10 each
    - 4 hats at $15 each
    - 1 pair of shorts at $20
  - Tuesday
    - 4 T-shirts at $10 each
    - 2 hats at $15 each
    - 3 pairs of shorts at $20.

- Then your total revenue for the two days is

$$\begin{bmatrix} 10 & 15 & 20 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 110 & 130 \end{bmatrix}$$

Price × Quantity = Revenue

대부분의 의사결정 문제는 연립방정식 문제 → matrix 문제

Representing System of Linear Algebraic Equations

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{cases}
$$

$$[A][x]=[b]$$

$$[A]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},\ [x]=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ [b]=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$
\begin{aligned}
x + 3y - 2z &= 5 \\
3x + 5y + 6z &= 7 \\
2x + 4y + 3z &= 8
\end{aligned}
\qquad
\begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}
\begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}
$$

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

MATLAB Matrix Operations

- Creation

```
>> A=[1 2 3; 4 5 6; 7 8 9]
A =
  1      2      3
  4      5      6
  7      8      9
```

- Matrix augmentation, column-wise

```
>> [A A]
ans =
  1    2    3    1    2    3
  4    5    6    4    5    6
  7    8    9    7    8    9
```

- Matrix augmentation, row-wise

```
>> [A;A]
ans =
     1     2     3
     4     5     6
     7     8     9
     1     2     3
     4     5     6
     7     8     9
```

MATLAB Matrix Operations (cont.)

- Creation

```
>> A=[1 2 3; 4 5 6; 7 8 9]
A =
 1      2      3
 4      5      6
 7      8      9
```

- Matrix Multiplication

```
>> A*A
ans =
     30     36     42
     66     81     96
    102    126    150
```

- Transpose

```
>> A'
ans =
     1      4      7
     2      5      8
     3      6      9
```

- Element-by-element Multiplication

```
>> A.*A
ans =
     1      4      9
    16     25     36
    49     64     81
```

Solving With MATLAB

- MATLAB provides two direct ways to solve systems of linear algebraic equations [A]{x}={b}:
  - Left-division : x = A\b
  - Matrix inversion : x = inv(A)*b

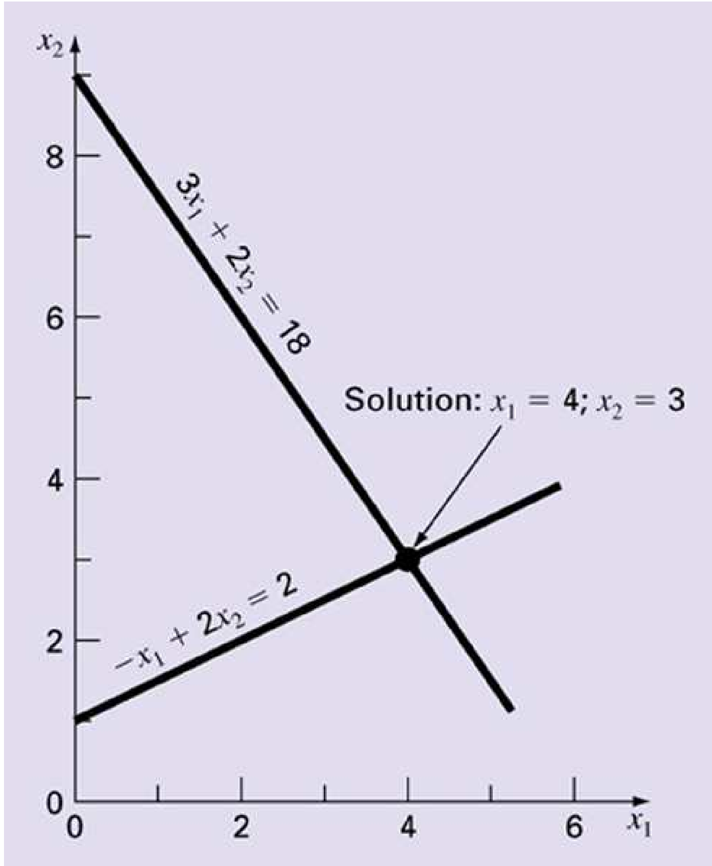
- The matrix inverse is less efficient than left-division and also only works for square, non-singular systems.

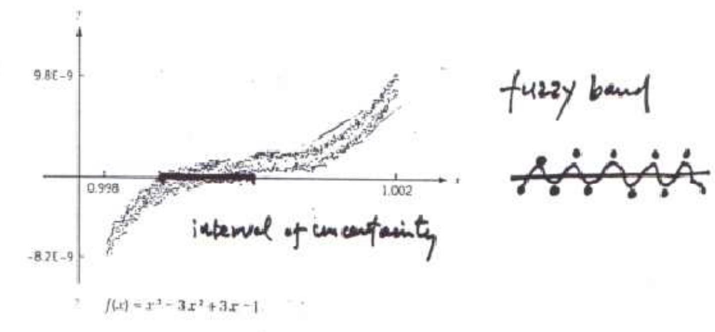
Solving Small Number of Equations

- Graphical methods
- Eliminations of unknowns
- Cramer's rule

Graphical method

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$





Eliminations of unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

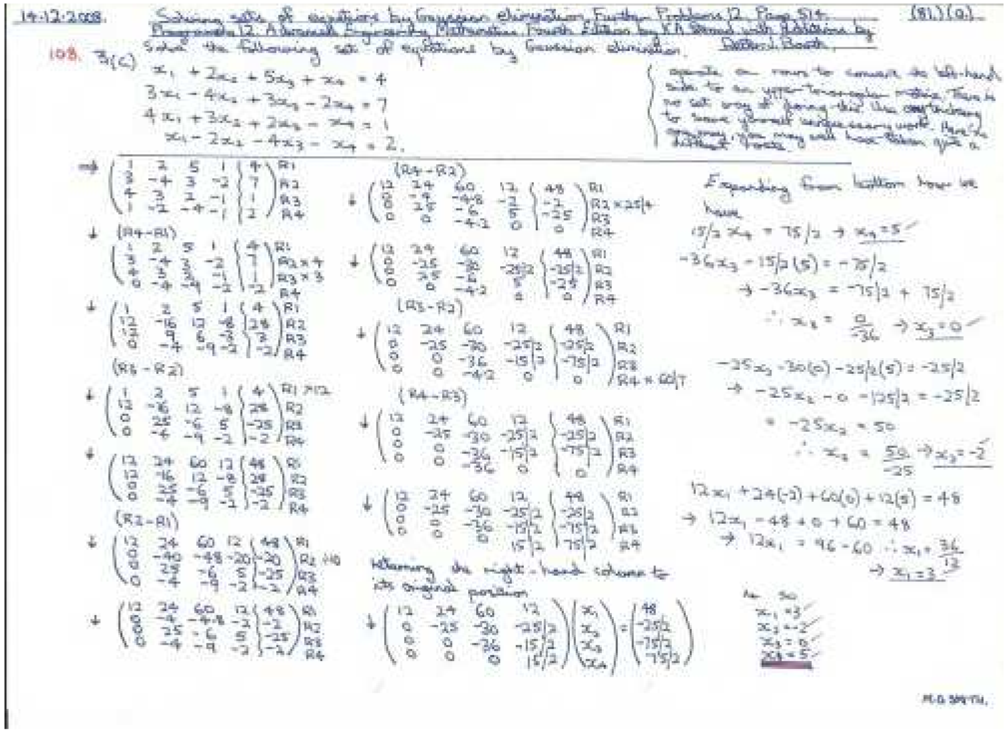
$$\begin{array}{rl} & a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}b_1 \\ -) & \underline{a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}b_2} \end{array}$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}}$$

Chapter 9

Gauss Elimination



Gauss Elimination

- Based on the principle that elementary row operations retain the solution.
  - swaps two rows
  - scales a row (multiply each element in a row by a non-zero number)
  - subtracts a scaled version of one row from another

- Two steps
  - Forward elimination
  - Back substitution

$$R_i <--> R_j$$

$$sR_i --> R_i$$

$$sR_i + R_j --> R_j$$

$$\left.\begin{matrix} \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \\ \downarrow \\ \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right] \end{matrix}\right\} \text{(a) Forward elimination}$$

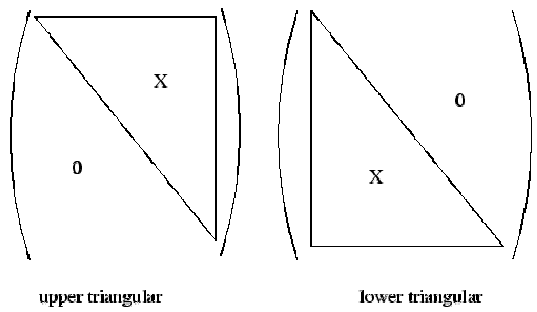
$$\downarrow$$

$$\left.\begin{matrix} x_3 = b''_3/a''_{33} \\ x_2 = (b'_2 - a'_{23}x_3)/a'_{22} \\ x_1 = (b_1 - a_{13}x_3 - a_{12}x_2)/a_{11} \end{matrix}\right\} \text{(b) Back substitution}$$

Forward Elimination

- Goal: to trianglize the matrix by elementary row operations

- Procedure
  - Remove column 1 from row *i by*
    - *subtracting* a_{i1}/a_{11}*(row 1) *from row i for* $2 \le i \le n$.
    - *(or by adding* $-a_{i1}/a_{11}$*(row 1))
  - Repeat for column 2 for *i for* $3 \le i \le n$.
  - ...
  - Repeat for column *n-1 for i for i=n.*



MULTIPLIERS

$$\begin{array}{lrrrrcr}
 & 4x_0 & +6x_1 & +2x_2 & -2x_3 & = & 8 \\
-(2/4) & 2x_0 & & +5x_2 & -2x_3 & = & 4 \\
-(-4/4) & -4x_0 & -3x_1 & -5x_2 & +4x_3 & = & 1 \\
-(8/4) & 8x_0 & +18x_1 & -2x_2 & +3x_3 & = & 40
\end{array}$$

$$4x_0 \quad +6x_1 \quad +2x_2 \quad -2x_3 \quad = \quad 8$$

MULTIPLIERS

$$-3x_1 \quad +4x_2 \quad -1x_3 \quad = \quad 0$$

$-(3/-3)$

$$+3x_1 \quad -3x_2 \quad +2x_3 \quad = \quad 9$$

$-(6/-3)$

$$+6x_1 \quad -6x_2 \quad +7x_3 \quad = \quad 24$$

$$
\begin{array}{rrrrcr}
4x_0 & +6x_1 & +2x_2 & -2x_3 & = & 8 \\
 & -3x_1 & +4x_2 & -1x_3 & = & 0 \\
 & & 1x_2 & +1x_3 & = & 9 \\
 & & 2x_2 & +5x_3 & = & 24
\end{array}
$$

!!

$$
\begin{array}{rrrrcr}
4x_0 & +6x_1 & +2x_2 & -2x_3 & = & 8 \\
 & -3x_1 & +4x_2 & -1x_3 & = & 0 \\
 & & 1x_2 & +1x_3 & = & 9 \\
 & & & 3x_3 & = & 6
\end{array}
$$

Forward Elimination (cont.)

- Pseudo code

```
for k = 1:(n-1) % column index
        for i = (k+1):n % row index
                A(row i) = A(row i) – a_ik/a_kk * A(row k);
        end
end
```

- MATLAB Code

```
[m,n] = size(A);
if m~=n, error('Matrix A must be square'); end
nb = n+1;
Aug = [A b];
% forward elimination
for k = 1:n-1
  for i = k+1:n
    factor = Aug(i,k)/Aug(k,k);
    Aug(i,k:nb) = Aug(i,k:nb)-factor*Aug(k,k:nb);
  end
end
```

Back Substitution

$$
\begin{array}{lllllll}
1x_0 & +1x_1 & -1x_2 & +4x_3 & = & 8 & \\
 & -\ 2x_1 & -3x_2 & +1x_3 & = & 5 & \\
 & & 2x_2 & -\ 3x_3 & = & 0 & \\
 & & & 2x_3 & = & 4 & x_3 = 2
\end{array}
$$

Back Substitution

$$
\begin{array}{rrrrcr}
1x_0 & +1x_1 & -1x_2 & & = & 0 \\
 & -\,2x_1 & -3x_2 & & = & 3 \\
 & & 2x_2 & & = & 6 \\
 & & & 2x_3 & = & 4
\end{array}
$$

Back Substitution

$$
\begin{array}{llllcl}
1x_0 & +1x_1 & -1x_2 & & = & 0 \\
 & -\ 2x_1 & -3x_2 & & = & 3 \\
 & & 2x_2 & & = & 6 \\
 & & & 2x_3 & = & 4
\end{array}
$$

$x_2 = 3$

Back Substitution

$$
\begin{array}{ccccc}
1x_0 & +1x_1 & & & = & 3 \\
& -2x_1 & & & = & 12 \\
& & 2x_2 & & = & 6 \\
& & & 2x_3 & = & 4
\end{array}
$$

Back Substitution

$$1x_0 \quad +1x_1 \quad = \quad 3$$

$$-\,2x_1 \quad = \quad 12 \qquad x_1 = -6$$

$$2x_2 \quad = \quad 6$$

$$2x_3 \quad = \quad 4$$

Back Substitution

$$
\begin{array}{ccccccc}
1x_0 & & & & = & 9 \\
& -\ 2x_1 & & & = & 12 \\
& & 2x_2 & & = & 6 \\
& & & 2x_3 & = & 4
\end{array}
$$

Back Substitution

$$1x_0 = 9 \qquad x_0 = 9$$

$$-2x_1 = 12$$

$$2x_2 = 6$$

$$2x_3 = 4$$

Back Substitution

- Goal: to find $\{x_1, x_2, x_2, \ldots, x_n\}$
- Procedure:
  - From row $i=n$ to 1

$$x_i = \frac{b_i' - \sum_{j=i+1}^{n} a_{ij}' x_j}{a_{ii}'}$$

```
% back substitution
x = zeros(n,1);
x(n) = Aug(n,nb)/Aug(n,n);
for i = n-1:-1:1
  x(i) = (Aug(i,nb)-Aug(i,i+1:n)*x(i+1:n))/Aug(i,i);
end
```

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2		Naïve Gauss elimination for 3 by 3 matrices										
3												
4		Starting Matrix										
5			1	2	3			1				
6		A=	4	5	6		b=	7				
7			7	8	-9			2				
8												
9		Step 1										
10			1	2	3			1				
11			0	-0.75	-1.5		b=	0.75				
12			0	-0.85714	-4.28571			-0.71429				
13												
14												
15		Step 2										
16			1	2	3			1				
17			0	-0.75	-1.5		b=	0.75				
18			0	0	-2.25			-1.375				
19												
20												
21						Verification						
22		x3=	0.611111									
23		x2=	-2.22222			-1.72222	0.777778	-0.05556			3.611111	
24		x1=	3.611111		A^(-1)=	1.444444	-0.55556	0.111111		x=	-2.22222	
25						-0.05556	0.111111	-0.05556			0.611111	
26												

Operation Counting

- Forward Elimination: O(*n³*)
- Back substitution: O(*n²*)

Forward Elimination	$\frac{2n^3}{3}+O(n^2)$
Back Substitution	$n^2+O(n)$
Total	$\frac{2n^3}{3}+O(n^2)$

add/sub: $\sum_{i=1}^{i=n-1} i(i+1)=\sum_{i=1}^{i=n-1}(i^2+i)=\frac{(n-1)n(2n-1)}{6}+\frac{n(n-1)}{2}\approx\frac{n^3}{6}$

mul/div: $\sum_{i=1}^{i=n-1} i(i+2)=\sum_{i=1}^{i=n-1}(i^2+2i)=\frac{(n-1)n(2n-1)}{6}+n(n-1)\approx\frac{n^3}{6}$

Outer Loop k	Inner Loop i	Addition/Subtraction flops	Multiplication/Division flops
1	$2, n$	$(n-1)(n)$	$(n-1)(n+1)$
2	$3, n$	$(n-2)(n-1)$	$(n-2)(n)$
⋮	⋮		
k	$k+1, n$	$(n-k)(n+1-k)$	$(n-k)(n+2-k)$
⋮	⋮		
$n-1$	n, n	$(1)(2)$	$(1)(3)$

Forward Elimination

Pitfalls of Elimination Methods

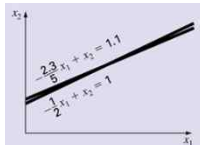
- Round-off errors.

Forward Elimination: $O(n^3)$
Back substitution: $O(n^2)$

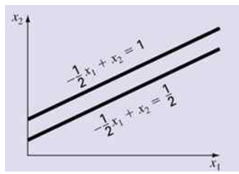
- Division by zero.
  - It is possible that during both elimination and back-substitution phases a division by zero can occur.

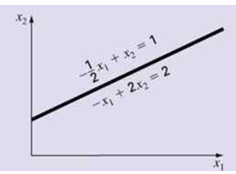
$$x_i = \frac{b_i' - \sum_{j=i+1}^{n} a_{ij}' x_j}{a_{ii}'}$$

- Ill-conditioned systems.
  - Systems where small changes in coefficients result in large changes in the solution.



- Singular systems.
  - When two equations are identical, we would loose one degree of freedom and be dealing with the impossible case of n-1 equations for n unknowns.
  - For large sets of equations, it may not be obvious.
    - The fact that the determinant of a singular system is zero can be used.





Techniques for Improving Solutions

- Use of more significant figures (pivoting)

- Pivoting.
  - Errors occur in the divisions of forward elimination due to limited number of significant digits.
    - The larger the divider the smaller the error.

  - If a pivot element is zero(or close to zero), normalization step leads to division by zero. Problem can be avoided by

    - Complete (full) pivoting : Searching for the largest element in all rows and columns then switching.

    - Partial pivoting. : Switching the rows so that the largest element is the pivot element.

Example

$$0.0003\,x_1 + 3.0000\,x_2 = 2.0001$$
$$1.0000\,x_1 + 1.0000\,x_2 = 1.0000$$

Significant Figures	x_2	x_1	Absolute Value of Percent Relative Error for x_1
3	0.667	−3.33	1099
4	0.6667	0.0000	100
5	0.66667	0.30000	10
6	0.666667	0.330000	1
7	0.6666667	0.3330000	0.1

After pivoting

$$1.0000\,x_1 + 1.0000\,x_2 = 1.0000$$
$$0.0003\,x_1 + 3.0000\,x_2 = 2.0001$$

Significant Figures	x_2	x_1	Absolute Value of Percent Relative Error for x_1
3	0.667	0.333	0.1
4	0.6667	0.3333	0.01
5	0.66667	0.33333	0.001
6	0.666667	0.333333	0.0001
7	0.6666667	0.3333333	0.0000

MATLAB M-File: `GaussPivot`

```
function x = GaussPivot(A,b)
% GaussPivot(A,b) :
% Solve Ax =b using Gaussian elimination with pivoting
% Input:
%     A = coefficient matrix
%     b = right-hand-side matrix
%
% Output:
%     x = solution matrix

% compute the matrix sizes
[m, n] = size(A);
if m ~= n, error('Matrix A must be square'); end
nb = n + 1;
Aug = [A b];

% forward elimination
for k = 1 : n-1
    % partial pivoting
    [big, i] = max(abs(Aug(k:n,k)));
    ipr = i+k-1;
    if ipr ~= k
        %pivot the rows
        Aug([k,ipr],:) = Aug([ipr,k],:);
    end
  for i = k+1 : n
     factor = Aug(i,k) / Aug(k,k);
     Aug(i,k:nb) = Aug(i,k:nb) - factor*Aug(k,k:nb);
  end;
end

% back-substitution
x = zeros(n,1);
x(n) = Aug(n,nb) / Aug(n,n);
for i = n-1 : -1 : 1
     x(i) = (Aug(i,nb) - Aug(i,i+1:n)*x(i+1:n)) / Aug(i,i);
end
```

Partial Pivoting

Partial Pivoting
(switch rows)

largest element in {x}

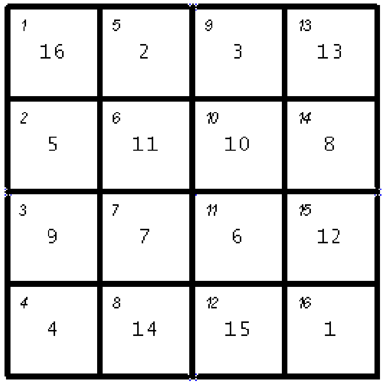
`[big,i] = max(x)`

index of the largest element
(from diagonal, not from 1)

MATLA Matrix Indexing

```
v = [16 5 9 4 2 11 7 14];

v(3)     % Extract the third element
ans = 9

v([1 5 6])      % Extract the first, fifth, and sixth elements
ans = 16   2   11

v(1:2:end)   % Extract all the odd elements
ans = 16   9   2   7

v(end:-1:1)   % Reverse the order of elements
ans = 14   7   11   2   4   9   5   16

v([2 3 4]) = [10 15 20]   % Replace some elements of v
v = 16   10   15   20   2   11   7   14

v([2 3]) = 30   % Replace second and third elements by 30
v = 16   30   30   20   2   11   7   14
```

MATLA Matrix Indexing



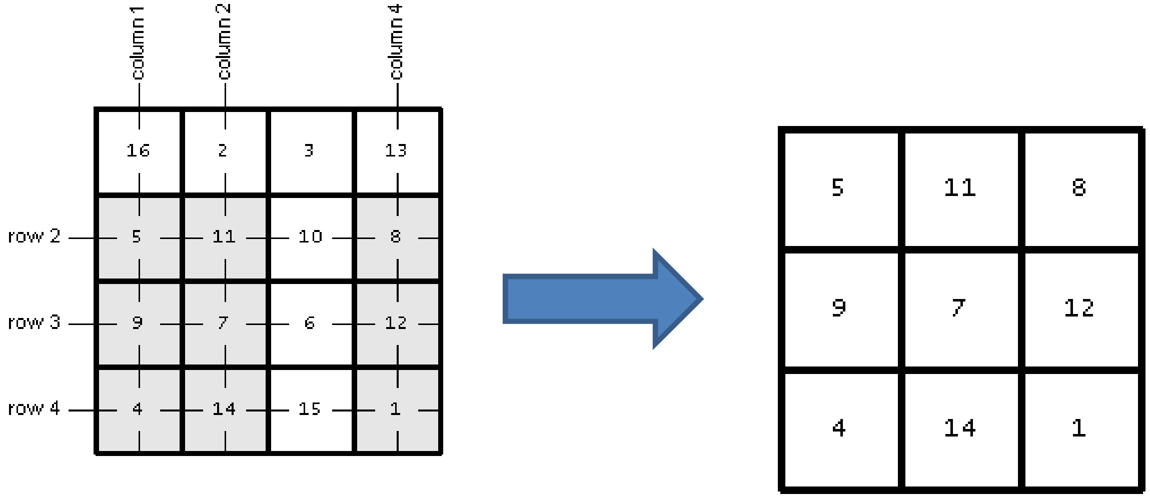
A

From the diagram you can see that `A(14)` is the same as `A(2,4)`.

The single subscript can be a vector containing more than one linear index, as in:

```
A([6 12 15])
ans =
     11   15   12
```

```
A([2 3 4], [1 2 4])
```



A

```
>> format short
>> x=GaussPivot0(A,b)
Aug =
     1     0     2     3     1
    -1     2     2    -3    -1
     0     1     1     4     2
     6     2     2     4     1
big =
     6
i =
     4
ipr =
     4
Aug =
     6     2     2     4     1
    -1     2     2    -3    -1
     0     1     1     4     2
     1     0     2     3     1
factor =
   -0.1667
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
    1.0000         0    2.0000    3.0000    1.0000
factor =
     0
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
    1.0000         0    2.0000    3.0000    1.0000
factor =
    0.1667
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
         0   -0.3333    1.6667    2.3333    0.8333
```

Aug = [A b]

Find the first pivot element and its index

Interchange rows 1 and 4

Eliminate first column

No need to interchange

```
big =
    2.3333
i =
     1
ipr =
     2
```

Second pivot element and index

No need to interchange

```
factor =
    0.4286
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0         0    5.0000    2.3571
         0   -0.3333    1.6667    2.3333    0.8333
factor =
   -0.1429
```

Eliminate second column

```
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0         0    5.0000    2.3571
         0         0    2.0000    2.0000    0.7143
big =
     2
i =
     2
```

Third pivot element and index

```
ipr =
     4
```

Interchange rows 3 and 4

```
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0    2.0000    2.0000    0.7143
         0         0         0    5.0000    2.3571
factor =
     0
```

Eliminate third column

```
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0    2.0000    2.0000    0.7143
         0         0         0    5.0000    2.3571
```

Back substitution

```
x =
         0
         0
         0
    0.4714
x =
         0
         0
   -0.1143
    0.4714
x =
         0
    0.2286
   -0.1143
    0.4714
x =

   -0.1857
    0.2286
   -0.1143
    0.4714
```

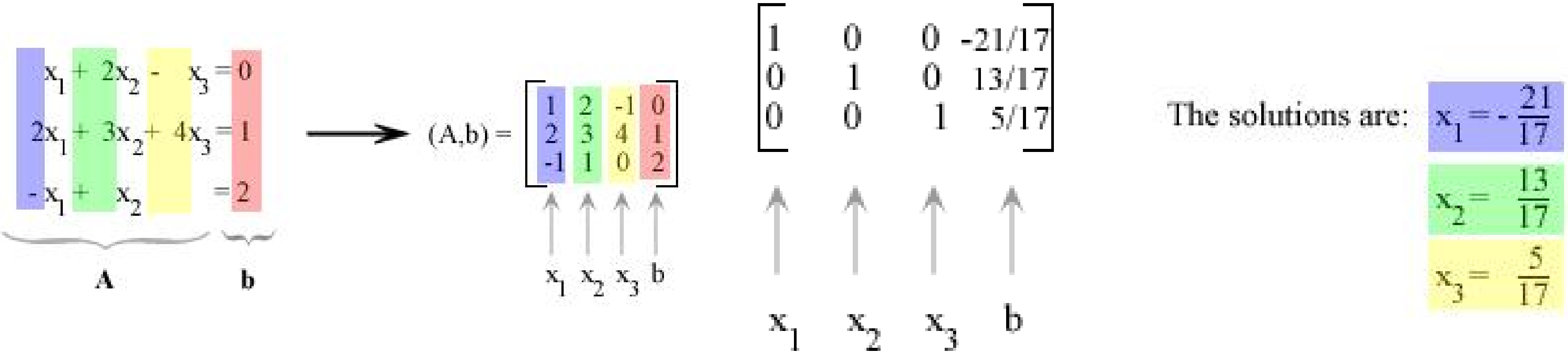
Tridiagonal Systems

- Sparse, only a few elements each row.
- Tridiagonal systems can be solved using the same method as Gauss elimination,
  - but with much less effort because most of the matrix elements are already 0.
  - O(*n*)

$$\begin{bmatrix} f_1 & g_1 & & & & & \\ e_2 & f_2 & g_2 & & & & \\ & e_3 & f_3 & g_3 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & e_{n-1} & f_{n-1} & g_{n-1} \\ & & & & & e_n & f_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \cdot \\ \cdot \\ \cdot \\ r_{n-1} \\ r_n \end{Bmatrix}$$

Gauss-Jordan

- It is a variation of Gauss elimination. The major differences are
  - All rows are normalized by dividing them by their pivot elements.
  - Elimination step results in an identity matrix.
  - Consequently, it is not necessary to employ back substitution to obtain solution.



B	C	D	E	F	G	H
Gauss-Jordan elimination for 3 by 3 matrices						
Starting Matrix						
	1	2	3			1
A=	4	5	6		b=	7
	7	8	-9			2

Step 1					
	1	2	3		1
	0	-0.75	-1.5	b=	0.75
	0	-0.85714	-4.28571		-0.71429

$(1/4)R_2$ - R_1

Step 2					
	-0.375	0	0.375		-1.125
	0	1	2	b=	-1
	0	0	-2.25		-1.375

Step 3					
	2.25	0	0		8.125
	0	-1.125	0	b=	2.5
	0	0	-2.25		-1.375

x3=	3.611111
x2=	-2.22222
x1=	0.611111

	Verification				
	-1.72222	0.777778	-0.05556		3.611111
A^(-1)=	1.444444	-0.55556	0.111111	x=	-2.22222
	-0.05556	0.111111	-0.05556		0.611111

Gauss-Jordan elimination for 3 by 3 matrices

Starting Matrix

	1	2	3		1
A=	4	5	6	b=	7
	7	8	-9		2

Step 1 (Normalize Pivot)

1	2	3		1
4	5	6	b=	7
7	8	-9		2

$R_2 - (4/1)R_1$

Step 2 (eliminate)

1	2	3		1
0	-3	-6	b=	3
0	-6	-30		-5

Step 3 (Normalize pivot)

1	2	3		1
0	1	2	b=	-1
0	-6	-30		-5

Step 4 (Eliminate)

1	0	-1		3
0	1	2	b=	-1
0	0	-18		-11

Step 5 (Normalize pivot)

1	0	-1		3
0	1	2	b=	-1
0	0	1		0.611111

Step 5 (Normalize pivot)

1	0	0		3.611111
0	1	0	b=	-2.22222
0	0	1		0.611111

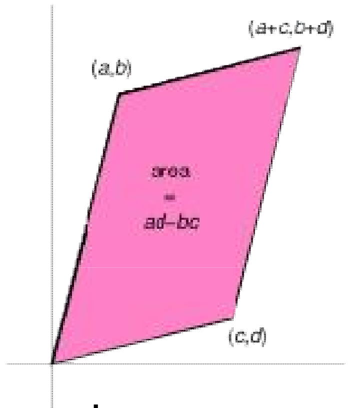
x1=	3.611111
x2=	-2.22222
x3=	0.611111

Verification

	-1.72222	0.777778	-0.05556		3.611111
A^(-1)=	1.444444	-0.55556	0.111111	x=	-2.22222
	-0.05556	0.111111	-0.05556		0.611111

Determinant

- Unique number associated with each square matrix
- Required for many multivariate procedures
- Determinant is a scalar
- Determinant denoted as det X or |X|



- Geometrically, the determinant of a 2 x 2 matrix corresponds to the area spanned by the two (e.g., column) vectors of the matrix.
  – When the vectors are aligned (i.e., linearly dependent), the determinant is 0;
  – when the vectors have some angle between each other, the determinant grows as the area spanned between the vectors grows.
  – At 90^o, the determinant is maximal (because the area of the rectangle spanned by the two vectors is maximal).

Determinant

- Rank of matrix
  - Number of linearly independent rows (or columns)
  - Matrix is full rank if all rows (columns) linearly independent
  - Rank reduced if rows linearly dependent
  - Quick check of rank of matrix,
    - calculate determinant,
    - if 0, then matrix is not of full rank

Input matrix:

```
1.000 2.000 3.000 2.000
2.000 3.000 4.000 2.000
3.000 4.000 5.000 2.000
1.000 1.000 1.000 1.000
```

Trace:

10.000

Determinant:

0.000

Rank:

3

Theorem

- if A has a row of zeros or a column of zeros, then det(A) = 0.

- det(A) = det(A^T)

- If A is an nxn triangular(upper, lower) or diagonal, then det(A) is the product of the entries on the main diagonal of the matrix ; that is, det(A) = $a_{11}a_{22}\ldots a_n$.

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = (2)(-3)(6)(9)(4) = -1296$$

Theorem

- If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, than det(B) = k det(A)

- If B is the matrix that results when two rows or two columns of A are interchanged, then det(B) = - det(A)

- If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple column is added to another column, then det(B) = det(A). (pivoting)

$$\begin{vmatrix} ka_{11} & \mathrm{k}a_{12} & \mathrm{k}a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

det(B) =$k$det(A)

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

det(B) =-det(A)

$$\begin{vmatrix} a_{11}+ka_{21} & \mathrm{a}_{12}+\mathrm{k}a_{22} & \mathrm{a}_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

det(B) = det(A)

$$\boxed{\det(kA) = k^n \det(A)}$$

Theorem

- If A is a square matrix with two proportional rows or two proportional column, then $\det(A) = 0$.

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = 0$$

The second row is 2 times the first, so we added -2 times the first row to the second to introduce a row of zeros

$$\begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix}, \quad \begin{vmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{vmatrix}$$

Theorem

- If A and B are square matrices of the same size, then
  $\det(AB)=\det(A)\det(B)$

- If A and B are square matrices of the same size, then
  $\det(A+B)\neq\det(A)+\det(B)$

$$A=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix},\quad B=\begin{bmatrix}-1 & 3\\ 5 & 8\end{bmatrix},\quad AB=\begin{bmatrix}2 & 17\\ 3 & 14\end{bmatrix}$$

$$\det(A)=1 \qquad \det(B)=-23 \qquad \det(AB)=-23$$

$$A=\begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix},\quad B=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix},\quad \mathrm{A}+\mathrm{B}=\begin{bmatrix}4 & 3\\ 3 & 8\end{bmatrix}$$

$$\det(\mathrm{A})=1 \qquad \det(\mathrm{B})=8 \qquad \det(\mathrm{A}+\mathrm{B})=23$$

Theorem

- Let A, B, and C be nxn matrices that differ only in a single row, say the r th, and assume that the r th row of C can be obtained by adding corresponding entrues in the r th rows of A and B. Then

$$\det(C) = \det(A)+\det(B)$$

The same result holds for columns.

$$\begin{array}{ccc} A & B & C \\ \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} & \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} \end{array}$$

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Theorem

- A square matrix A is invertible if and only if det(A) ≠ 0

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

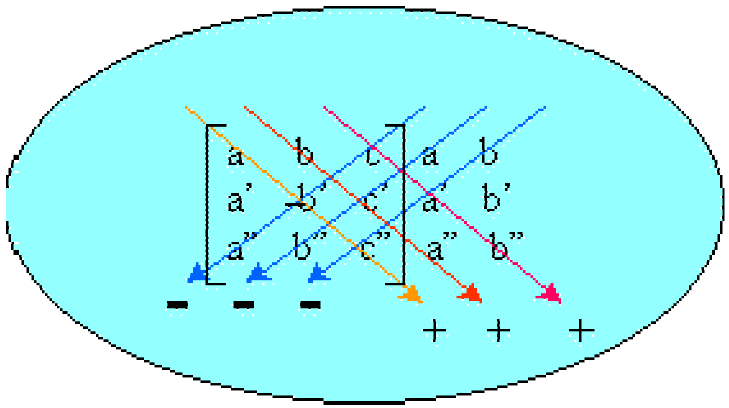
Since the first and third rows of A are proportional, det(A) = 0. Thus, A is not invertible

- If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Determinant

- Computation
  - Mnemonic method
  - Permutation method
  - Gauss Jordan method
  - Minor method (method of Cofactors)



Mnemonic Method

- The determinant is computed by summing the products on the rightward arrows and subtracting the products o n the leftward arrows.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

$$A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{array}\right]$$

$\det(A)$ = (3)(-2) - (1)(4) = -10

$\det(B)$ = (45) + (84) + (96) - (105) - (-48) - (-72) = 240

Permutation method : Permutation

- The set of integers {1,2,…,n} is an arrangement of these integers in some order without omission and repetition
  - Permutations of three integers
    - (ex) six different permutations of the set of integers {1,2,3}.
      - (1,2,3) (2,1,3) (3,1,2) (1,3,2) (2,3,1) (3,2,1)
  - Permutations of four integers
    - (1,2,3,4) (2,1,3,4) (3,1,2,4) (4,1,2,3) (1,2,4,3) (2,1,4,3) (3,1,4,2) (4,1,3,2) (1,3,2,4) (2,3,1,4) (3,2,1,4) (4,2,1,3) (1,3,4,2) (2,3,4,1) (3,2,4,1) (4,2,3,1) (1,4,2,3) (2,4,1,3) (3,4,1,2) (4,3,1,2) (1,4,3,2) (2,4,3,1) (3,4,2,1) (4,3,2,1)

Permutation method : Inversion

- In a permutation (j_1, j_2, j_3, … ,j_n) a larger integer precedes a smaller one. (2,1,3)

- Counting Inversions
  - (6,1,3,4,5,2) : 5+0+1+1+1=8.
  - (2,4,1,3) : 1+2+0=3.
  - (1,2,3,4) : no inversions in this permutation

- A permutation is called even if the total number of inversions is an even integer and is called odd if the total inversions is an odd integer.

Permutation	Number of Inversions	classification
(1,2,3)	0	even
(1,3,2)	1	odd
(2,1,3)	1	odd
(2,3,1)	2	even
(3,1,2)	2	even
(3,2,1)	3	odd

Permutation method : Elementary Products

- From an n xn matrix A, any product of n entries from A, no two of which come from the same row or same column

- An n xn matrix A has n! elementary products.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad a_{1\_}\, a_{2\_} \qquad a_{11} a_{22} \quad a_{12} a_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad a_{1\_}\, a_{2\_}\, a_{3\_} \qquad \begin{matrix} a_{11}a_{22}a_{33} & a_{12}a_{21}a_{33} & a_{13}a_{21}a_{32} \\ a_{11}a_{23}a_{32} & a_{12}a_{23}a_{31} & a_{13}a_{22}a_{31} \end{matrix}$$

Permutation method : Signed Elementary Products

- *Signed elementary product* is an elementary product multiplied by +1 or -1 .
  - (+1) : if elementary product is an even permutation
  - (– 1) : if elementary product is an odd permutation.

Elementary Product	Associated Permutation	Even or Odd	Signed Elementary Product
$a_{11}a_{22}$	(1,2)	even	$a_{11}a_{22}$
$a_{12}a_{21}$	(2,1)	odd	$-a_{12}a_{21}$

Elementary Product	Associated Permutation	Even or Odd	Signed Elementary Product
$a_{11}a_{22}a_{33}$	(1,2,3)	even	$a_{11}a_{22}a_{33}$
$a_{11}a_{23}a_{32}$	(1,3,2)	odd	$-a_{11}a_{23}a_{32}$
$a_{12}a_{21}a_{33}$	(2,1,3)	odd	$-a_{12}a_{21}a_{33}$
$a_{12}a_{23}a_{31}$	(2,3,1)	even	$a_{12}a_{23}a_{31}$
$a_{13}a_{21}a_{32}$	(3,1,2)	even	$a_{13}a_{21}a_{32}$
$a_{13}a_{22}a_{31}$	(3,2,1)	odd	$-a_{13}a_{22}a_{31}$

Permutation method : Determinant

- The *determinant* is denoted by |A| or det(A) to be the sum of all signed elementary products from A.

$$\det(A) = \sum \pm a_{1j_1} a_{2j_2} \ldots a_{nj_n}$$

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31}a + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Gauss Elimination Method

- Evaluate det(A) where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

The first and second rows of A are interchanged.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign

.

Gauss Elimination Method

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

-2 times the first row was added to the third row.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

-10 times the second row was added to the third row

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

.

- Gauss-Jordan 소거법

  - 단위행렬 변환

(ex) $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$$

$$\frac{1}{4}|A| = \begin{vmatrix} 1 & \frac{3}{4} \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \frac{3}{4} \\ 0 & -\frac{1}{2} \end{vmatrix}$$

$$(-2)\left(\frac{1}{4}\right)|A| = \begin{vmatrix} 1 & \frac{3}{4} \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore |A| = \left(-\frac{1}{2}\right)\cdot(4) = 2$$

Minor/Cofactor Method : Minors

- The determinant of a 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\ \Rightarrow \det(A) &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Minor/Cofactor Method : Cofactors

- If A is a square matrix, then the minor of a_{ij} is denoted by M_{ij} and
  - is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A.
  - The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the cofactor of a_{ij}

Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

The minor of entry a_{11} is $M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$

+	-	+
-	+	-
+	-	+

The cofactor of a_{11} is $C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$

Minor/Cofactor Method : Cofactor Expansions

- The expression can be written in terms of minors and cofactors as

$$\begin{aligned}\det(A) &= a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}M_{13} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\end{aligned}$$

- This method is called *cofactor expansion* along the first row of A.

- By rearranging, we can obtain

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}\end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor/Cofactor Method : Cofactor Expansions

Let $A=\begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ $\det(A)=\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}=3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix}-1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix}+0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$

$$=3(-4)-(1)(-11)+0=-1$$

$$\det(A)=\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}=3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix}-(-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix}+5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$=3(-4)-(-2)(-2)+5(3)=-1$$

Sparse Matrices

$$\begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 8 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 7 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \end{pmatrix}$$

Cramer's Rule

- If $Ax=b$ is a system of n linear equations in n unknowns such that det(A)≠0, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad x_3 = \frac{\det(A_3)}{\det(A)}$$

Where A_j is the matrix obtained by replacing the entries in the column of A by vector b.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ coefficient matrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|\mathbf{A}|}, x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|\mathbf{A}|}, x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|\mathbf{A}|}$$

Cramer's Rule

- Use Cramer's rule to solve

$$\begin{aligned} x_1 + \quad\;\; + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11} \qquad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11} \qquad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Matlab : Determinant

- d = det(X)
  - Using det(X) == 0 as a test for matrix singularity (==, ~=)
  - The function cond(X) can check for singular and nearly singular matrices.
  - The determinant is computed from the triangular factors obtained by Gaussian elimination
  - Examples
    - A = [1 2 3; 4 5 6; 7 8 9]
    - d = det(A) produces d = 0 (singular matrix)
    - Changing A(3,3) with A(3,3) = 0
    - d = det(A) produces d = 27

Inverse Matrix

- if $AB = I$, *then* $A = B^{-1}$ or $B = A^{-1}$ $(\det(A) \neq 0)$

- Gauss-Jordan 소거법

$$A \cdot X = I$$

$$(A^{-1})A \cdot X = (A^{-1})I$$

$$I \cdot X = A^{-1}$$

$$\Rightarrow \quad [A \rightarrow I;\ I \rightarrow A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 9 & 2 & 0 & 1 & 0 \\ 1 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.8 & 0.6 & -1 \\ 0 & 1 & 0 & -0.4 & 0.2 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

the second part is the INVERSE MATRIX of the blue part (ex) $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{([A|B]에서 A,B가 같이 변화하므로 무방)}]{R_2 \rightleftarrows R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & -5 & -3 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & 1 & -3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array}\right].$$

(Ref) $$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{3} & -1 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{5}{3} & -1 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{3} & 0 & -\frac{1}{3} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{5}{3} & -1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array}\right]$$

Inverse Matrix : Adjoint Matrix

- The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij}

- If A is any nxn matrix and C_{ij} is the cofactor of a_{ij} then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

  is called the *matrix of cofactors from A*.

- The transpose of this matrix is called the *adjoint of A* and is denoted by adj(A)

Inverse Matrix : Adjoint Matrix

Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

The cofactors of A are

$C_{11} = 12 \qquad C_{12} = 6 \qquad C_{13} = -16$

$C_{21} = 4 \qquad C_{22} = 2 \qquad C_{23} = 16$

$C_{31} = 12 \qquad C_{32} = -10 \qquad C_{33} = 16$

cofactor

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

$$adj\,(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Inverse Matrix : Adjoint Matrix

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

$\det(A) = 64$

$$A^{-1} = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & \frac{-10}{64} \\ \frac{-16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

Inverse Matrix : Matlab

- y = inv(X) returns the inverse of the square matrix X.

- A warning message is printed if X is badly scaled or nearly singular.

$$Ax=b$$
$$x=A^{-1}b$$

- Matrix inversion

```
>> x=inv(A)*b
```

- Left division

```
>> x=A\b
```

/ (slash) : matrix right division, *A/B* is *A*inv(B)*
₩ (backslash) : matrix left division, *A₩B is inv(A)*B*

3w-2y+4z=8
5w+8y-6z=-5
9w-2y+7z=-17

```
b=        A=
   8       3  -2   4
  -5       5   8  -6
 -17       9  -2   7
```

```
>> X=inv(A)*b
```

or

```
>> X=A\b
```

```
X=
-36.7778
71.2778
65.2222
```

Gaussian elimination

THE END

Homework : 추후 공지

Report : 추후 공지