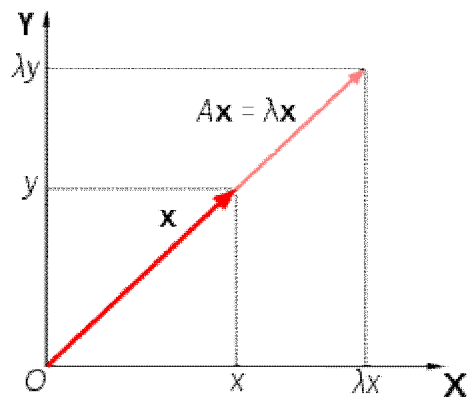
Part 3
Chapter 13

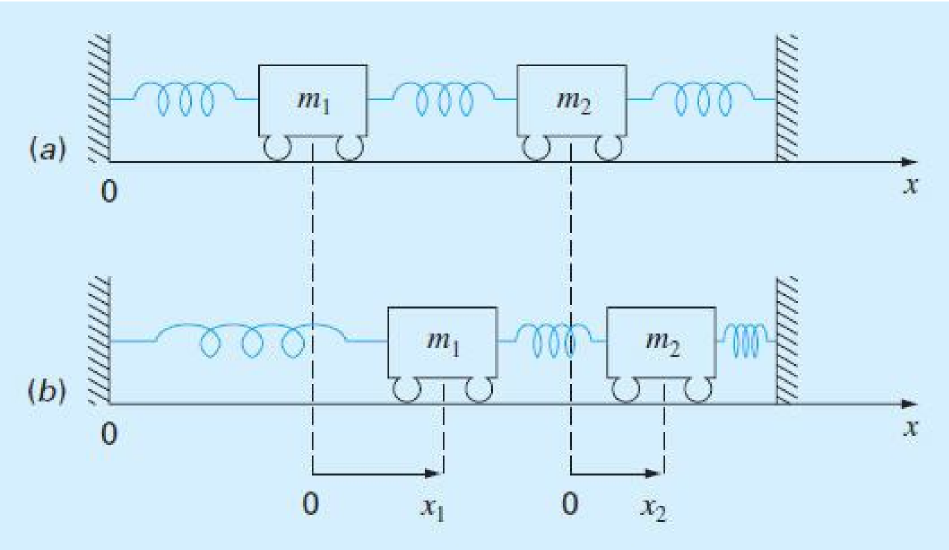
Eigenvalues

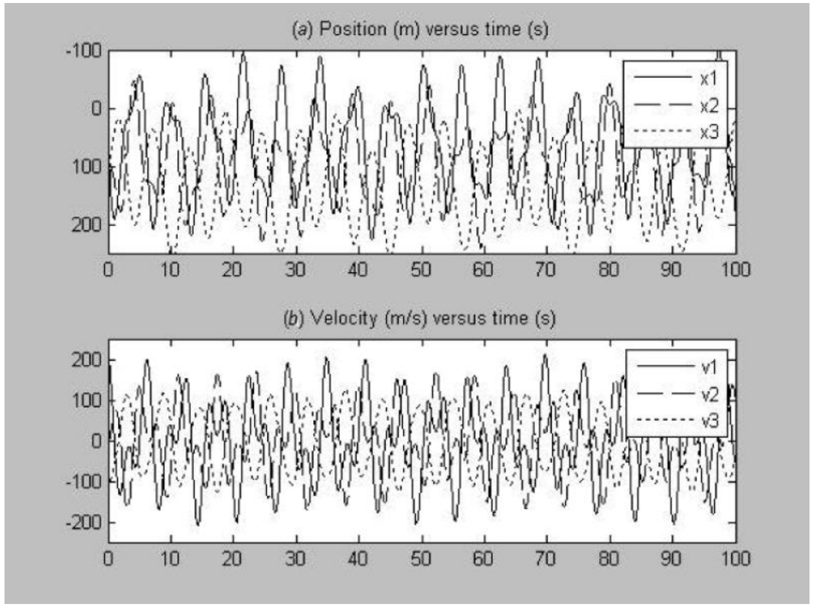


Chapter Objectives

- Understanding the mathematical definition of eigenvalues and eigenvectors.
- Knowing how to implement the polynomial method.
- Knowing how to implement the power method to evaluate the largest and smallest eigenvalues and their respective eigenvectors.
- Knowing how to use and interpret MATLAB's `eig` function.

Oscillations or Vibration of Mass-Spring Systems
Dynamics of Three Coupled Bungee Jumpers in Time





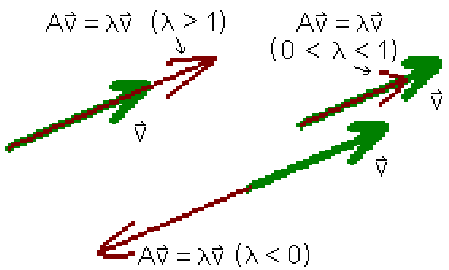
Is there an underlying pattern ?

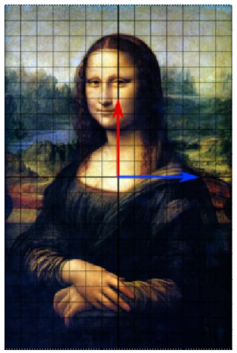
Eigenvalue, Eigenvector

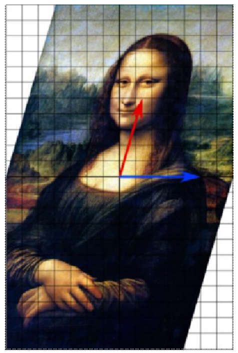
- 고유벡터(eigenvector)
  - 어떤 선형 변환이 일어난 후에도 그 방향이 변하지 않는 벡터
    - 연산자(operator)를 적용시켜 크기와 방향이 모두 변하는 basis 에서 문제를 푸는것 보다
    - 방향은 전혀 변하지 않고 크기만 변하게 하는 적절한 좌표계를 선택하여 문제를 푸는 것은 큰 이득

- 고유값(eigenvalue)
  - 어떤 선형 변환 후에 고유벡터의 크기가 변하는 비율

- 모든 선형변환은 고유벡터와 그 고유값만으로 설명 가능







원래 이미지가 옆으로 기울어진 모양으로 변하는 선형 변환. 수평축(푸른색 화살표)은 방향이 변하지 않지만 붉은색 화살표는 방향이 변함. 푸른색 화살표는 이 변환의 고유벡터가 되고 붉은색 화살표는 고유벡터가 아님. 푸른색 화살표의 크기가 변하지 않았으므로 이 벡터의 고유값은 1.

Determining Eigenvalues

- Consider a nxn matrix A, and equation: $Ax = \lambda x$
  - Seek solutions for x and λ
  - λ satisfying the equation are the eigenvalues
  - x satisfying the equation are the eigenvectors

- Nomenclature
  - The set of all eigenvalues is called the spectrum
  - The largest eigenvalue (λ_1) is called the dominant eigenvalue
  - Then is and $|\lambda_1|$ is the spectral radius of A, denoted $\rho(A)$
    - (cf.) Spectral norm $\|A^k\|^{1/k}$

$$\|A\|_2 = \sqrt{\mu_{\max}} \text{ where } \mu_{\max} \text{ is the largest eigenvalue of } \mathbf{A}^t\mathbf{A}$$

Determining Eigenvalues

- Vector equation $Ax = \lambda x \rightarrow (A-\lambda I)x = 0$
  - $(A-\lambda I)$ is called the characteristic matrix
  - Non-trivial (non-zero) solutions exist if and only if

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

  - This is called the characteristic equation
    - n^{th}-order polynomial in λ
    - Roots are the eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$

Eigenvalue Example

- Characteristic matrix

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{bmatrix}$$

- Characteristic equation

$$\left| A - \lambda I \right| = (1-\lambda)(-4-\lambda) - (2)(3) = \lambda^2 + 3\lambda - 10 = 0$$

- Eigenvalues : $\lambda_1 = -5,\ \lambda_2 = 2$

- Eigenvalues

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \quad \begin{aligned} \lambda_1 &= -5 \\ \lambda_2 &= 2 \end{aligned}$$

- Determine eigenvectors : $Ax = \lambda x$

$$\begin{aligned} x_1 + 2x_2 &= \lambda x_1 \\ 3x_1 - 4x_2 &= \lambda x_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (1-\lambda)x_1 + 2x_2 &= 0 \\ 3x_1 - (4+\lambda)x_2 &= 0 \end{aligned}$$

- Eigenvector for $\lambda_1 = -5$

$$\begin{aligned} 6x_1 + 2x_2 &= 0 \\ 3x_1 + x_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} -0.3162 \\ 0.9487 \end{bmatrix} \text{or } \mathbf{x}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

- Eigenvector for $\lambda_1 = 2$

$$\begin{aligned} -x_1 + 2x_2 &= 0 \\ 3x_1 - 6x_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix} \text{or } \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The Dominant Eigenvalue

- A $n \times n$ matrix A will have n eigenvalues
  - The dominant eigenvalue is the biggest in terms of absolute value.
  - This would include any eigenvalues that are complex.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -7 \\ 0 & -4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

- eigenvalues : {1,−4, 2} (upper triangular)
- dominant eigenvalue : −4

Power Method

1. Start with an initial guess for x

2. Calculate $w = Ax$

3. Largest value (magnitude) in w is the estimate of eigenvalue

4. Get next x by rescaling w

5. Continue until converged

- Note that the smallest eigenvalue and its associated eigenvector can be determined by applying
  - the power method to the inverse of A

Example: Power Method

Consider

$$A = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix}$$

Start with

$$z^{(1)} = x_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Assume all eigenvalues are equally important, since we don't know which one is dominant

$$Az^{(1)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 15 \\ 21 \end{Bmatrix} = (21) \begin{Bmatrix} 0.9524 \\ 0.7143 \\ 1.0 \end{Bmatrix}$$

Eigenvalue estimate

Eigenvector

- Current estimate for largest eigenvalue is 21
- Rescale *w* by eigenvalue to get new *x*

$$x = \frac{w}{\max(abs(w))} = \left(\frac{1}{21}\right)\begin{Bmatrix} 20 \\ 15 \\ 21 \end{Bmatrix} = \begin{Bmatrix} 0.9524 \\ 0.7143 \\ 1.0 \end{Bmatrix}$$

- Check Convergence (Norm < tol)

$$Ax - \lambda x = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix}\begin{bmatrix} 0.9524 \\ 0.7143 \\ 1.0 \end{bmatrix} - 21 * \begin{bmatrix} 0.9524 \\ 0.7143 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -2.3812 \\ -1.2382 \\ -1.6188 \end{bmatrix}$$

$$\left\| Ax - \lambda x \right\| = \sqrt{(-2.3812)^2 + (-1.2382)^2 + (-1.6188)^2} = 3.1343$$

- Update the estimated eigenvector and repeat

$$Az^{(2)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{Bmatrix} 0.9524 \\ 0.7143 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 17.619 \\ 13.762 \\ 19.381 \end{Bmatrix}$$

- New estimate for largest eigenvalue is 19.381 Rescale *w* by eigenvalue to get new *x*

$$x = \frac{w}{\max(abs(w))} = \left(\frac{1}{19.381}\right) \begin{Bmatrix} 17.619 \\ 13.762 \\ 19.381 \end{Bmatrix} = \begin{Bmatrix} 0.9091 \\ 0.7101 \\ 1.0 \end{Bmatrix}$$

$$Ax - \lambda x = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.9091 \\ 0.7101 \\ 1.0 \end{bmatrix} - 19.381 * \begin{bmatrix} 0.9091 \\ 0.7101 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.1203 \\ -0.3594 \\ -0.4496 \end{bmatrix}$$

$$\left\| Ax - \lambda x \right\| = \sqrt{(-0.1203)^2 + (-0.3594)^2 + (-0.4496)^2} = 0.5880$$

- One more iteration

$$Az^{(3)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{Bmatrix} 0.9091 \\ 0.7101 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 17.499 \\ 13.403 \\ 18.931 \end{Bmatrix} = (18.931) \begin{Bmatrix} 0.9243 \\ 0.7080 \\ 1.0 \end{Bmatrix}$$

$$Ax - \lambda x = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{bmatrix} 0.9243 \\ 0.7080 \\ 1.0 \end{bmatrix} - 18.931 * \begin{bmatrix} 0.9243 \\ 0.7080 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.0147 \\ -0.1153 \\ -0.1440 \end{bmatrix}$$

$$\left\| Ax - \lambda x \right\| = \sqrt{(-0.0147)^2 + (-0.1153)^2 + (-0.1440)^2} = 0.1851$$

Convergence criterion -- Norm (or relative error) < tol

$$Az^{(4)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{Bmatrix} 0.9243 \\ 0.7080 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 17.513 \\ 13.519 \\ 19.075 \end{Bmatrix} = (19.075) \begin{Bmatrix} 0.9181 \\ 0.7087 \\ 1.0 \end{Bmatrix}$$

$$Az^{(5)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{Bmatrix} 0.9181 \\ 0.7087 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 17.506 \\ 13.471 \\ 19.016 \end{Bmatrix} = (19.016) \begin{Bmatrix} 0.9206 \\ 0.7084 \\ 1.0 \end{Bmatrix}$$

$$Az^{(6)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{Bmatrix} 0.9206 \\ 0.7084 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 17.508 \\ 13.490 \\ 19.040 \end{Bmatrix} = (19.040) \begin{Bmatrix} 0.9196 \\ 0.7085 \\ 1.0 \end{Bmatrix}$$

$$Az^{(7)} = \begin{bmatrix} 2 & 8 & 10 \\ 8 & 3 & 4 \\ 10 & 4 & 7 \end{bmatrix} \begin{Bmatrix} 0.9196 \\ 0.7085 \\ 1.0 \end{Bmatrix} = \begin{Bmatrix} 17.507 \\ 13.482 \\ 19.030 \end{Bmatrix} = (19.030) \begin{Bmatrix} 0.9200 \\ 0.7085 \\ 1.0 \end{Bmatrix}$$

Script file: Power_eig.m

```
function [z, m] = Power_eig(A, max_it, tol)
[ n, nn ] = size(A); z = ones(n,1);
it=0;   error = 100;
disp('    it      m      z(1)      z(2)      z(3)      z(4)      z(5)')
while (it < max_it & error > tol)
    w = A*z;    ww = abs(w);
    [k, kk] = max(ww);  % kk is index of max element of ww
    m = w(kk);          % estimate of eigenvalues
    z = w/w(kk);        % estimate of eigenvector
    out = [  it+1    m    z' ];   disp(out)
    error = norm(A * z - m * z);
    it=it + 1;
end
error
```

$$\boxed{Norm = \|Ax - \lambda x\|}$$

MATLAB Example:

Power Method

```
» A=[2 8 10; 8 3 4; 10 4 7]
A =
     2     8    10
     8     3     4
    10     4     7
» [z,m] = Power_eig(A,100,0.001);

   it     m      z(1)     z(2)     z(3)     z(4)     z(5)
    1.0000   21.0000    0.9524    0.7143    1.0000
    2.0000   19.3810    0.9091    0.7101    1.0000
    3.0000   18.9312    0.9243    0.7080    1.0000
    4.0000   19.0753    0.9181    0.7087    1.0000
    5.0000   19.0155    0.9206    0.7084    1.0000
    6.0000   19.0396    0.9196    0.7085    1.0000
    7.0000   19.0299    0.9200    0.7085    1.0000
    8.0000   19.0338    0.9198    0.7085    1.0000
    9.0000   19.0322    0.9199    0.7085    1.0000
error =
  8.3175e-004
» z
z =
    0.9199
    0.7085
    1.0000
» m
m =
   19.0322

» x=eig(A)
x =
   -7.7013
    0.6686
   19.0327
```

eigenvector

the largest eigenvalue

MATLAB function

eigenvalues

MATLAB

- d = eig(A)
  - returns a vector of the eigenvalues of matrix A.

- [V,D] = eig(A)
  - produces matrices of eigenvalues (D) and eigenvectors (V)
  - Matrix V is the modal matrix
    - Its columns are the eigenvectors of A
  - Matrix D is the canonical form of A
    - Diagonal matrix with A's eigenvalues on the main diagonal.

```
>> A=[ 1 2; 3 -4];
>> e=eig(A)
e =
    2
   -5
```

```
>> [V,D] = eig(A)
V =
   0.8944  -0.3162
   0.4472   0.9487
D =
    2    0
    0   -5
```

THE END

Homework/Report

- Homework
  - 예제 13.4

- Report
  - 13.2,