Confidence Sets Controlling for False Discovery Rate

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1 Multiple Testing

In a testing setting where m null hypotheses $\{H_1, \ldots, H_m\}$ are tested simultaneously, the following contingency table can be constructed according to the claimed significance and the true state (H_0 true or false) of the hypothesis:

	Not Significant	Significant (rejected)	
H_0	TN	FP	m_0
H_1	FN	TP	$m - m_0$
	$m - R$	R	m

In the above, m represents the total number of hypotheses tested, R represents the number of rejected (claimed significant) hypotheses, and m_0 represents the number of hypotheses that are truly not significant (true H_0). R and m are observable. Let $p_1, \ldots, p_m$ be the p-values associated with the hypotheses $H_1, \ldots, H_m$, and let I_0 be the set of all true H_0 hypotheses. Then, for $i \in I_0$, $\mathbb{P}(p_i < \alpha) < \alpha$, $0 \leq \alpha \leq 1$.

1.1 Family-Wise Error Rate

Family-wise error rate (FWER) is defined as:

$$\mathbb{P}(FP \geq 1)$$

which signifies the probability of making one or more Type I error.

- Bonferroni correction: Controlling $\mathbb{P}(FP \geq 1) < \alpha$ by rejecting H_i if $p_i < \frac{\alpha}{m}$, as $\mathbb{P}(FP \geq 1) \leq \mathbb{E}(FP) \leq \frac{m_0 \alpha}{m} \leq \alpha$

1.2 False Discovery Rate

False discovery rate (FDR) is defined as:

$$\mathbb{E}\left(\frac{FP}{R} | R > 0\right) \mathbb{P}(R > 0) = \mathbb{E}\left(\frac{FP}{\max(1, R)}\right)$$

which aims to control the expected proportion of false positives out of all the significantly claimed tests.

- Benjamini-Hochberg(BH) procedure: The BH procedure aims to control the FDR below α level by rejecting $\{H_{(1)}, \ldots, H_{(k)}\}$ where $k = \max_i \left\{p_{(i)} < \frac{\alpha i}{m}\right\}$.
- With the BH procedure, $FDR < \alpha \cdot \frac{m_0}{m}$ as

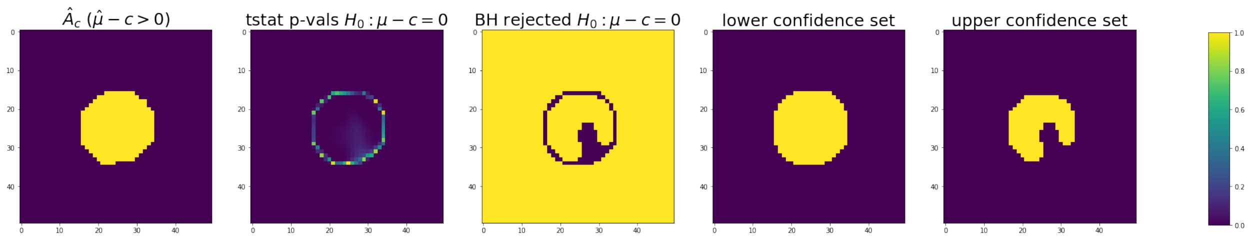
2 FDR controlling confidence sets

The confidence sets for $\{s \in S : \mu(s) - c = 0\}$ can be constructed to control for FDR:

$$\hat{\mathcal{A}}_c^+ = \{\hat{\mathcal{A}}_c^C \cap \mathcal{B}\}^C$$

$$\hat{\mathcal{A}}_c^- = \{\hat{\mathcal{A}}_c \cap \mathcal{B}\}$$

where $\mathcal{B} = \{s \in S : p_s \leq p_k\}$, i.e. a set of voxels rejected by the BH procedure. The voxels in the upper confidence set $\hat{\mathcal{A}}_c^+$ have μ bigger than the threshold c with 95% confidence. The voxels outside of the lower confidence set $\hat{\mathcal{A}}_c^-$ have μ smaller than the threshold c with 95% confidence.



The error rate is given as:

$$\mathbb{E}\left(\frac{|\hat{\mathcal{A}}_c^+ \setminus \mathcal{A}_c \cup \mathcal{A}_c \setminus \hat{\mathcal{A}}_c^-|}{|\hat{\mathcal{A}}_c^+ \cup (\hat{\mathcal{A}}_c^-)^C|}\right) \tag{1}$$

3 Simulation

3.1 Simulation Setting

Take $50 * 50$ and $100 * 100$ Gaussian random field with circular signal where $\mu = 3$:

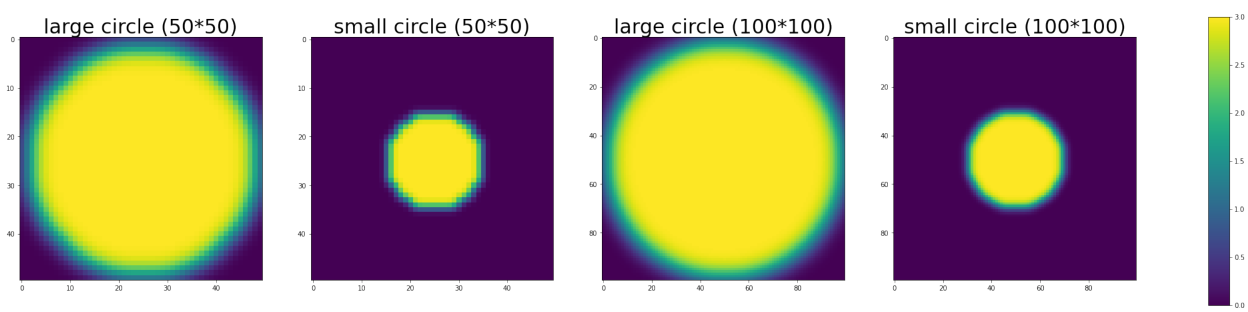


Figure 1: Signal images for 50*50 and 100*100 random fields

The noise can be applied to the the above four to form a random field with different noise settings. For each voxels, the noise is drawn from $\mathcal{N}(0, 3)$. The smoothing was done with FWHM=8.

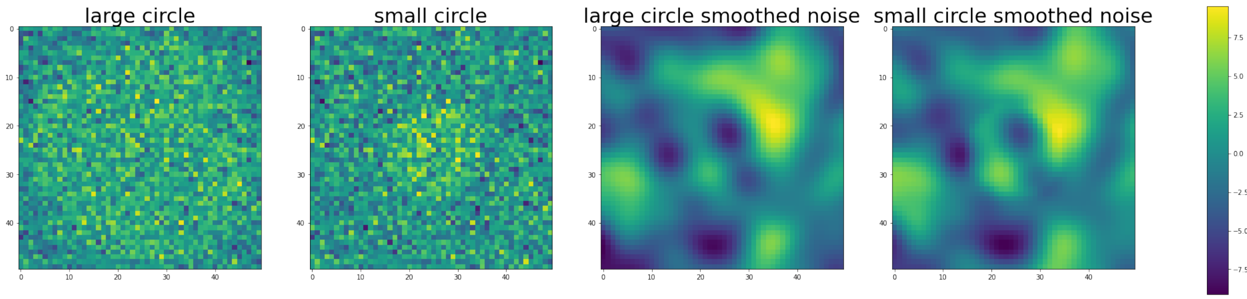


Figure 2: random fields with smoothed or uncorrelated noise for 50*50 image size

3.2 Confidence Sets

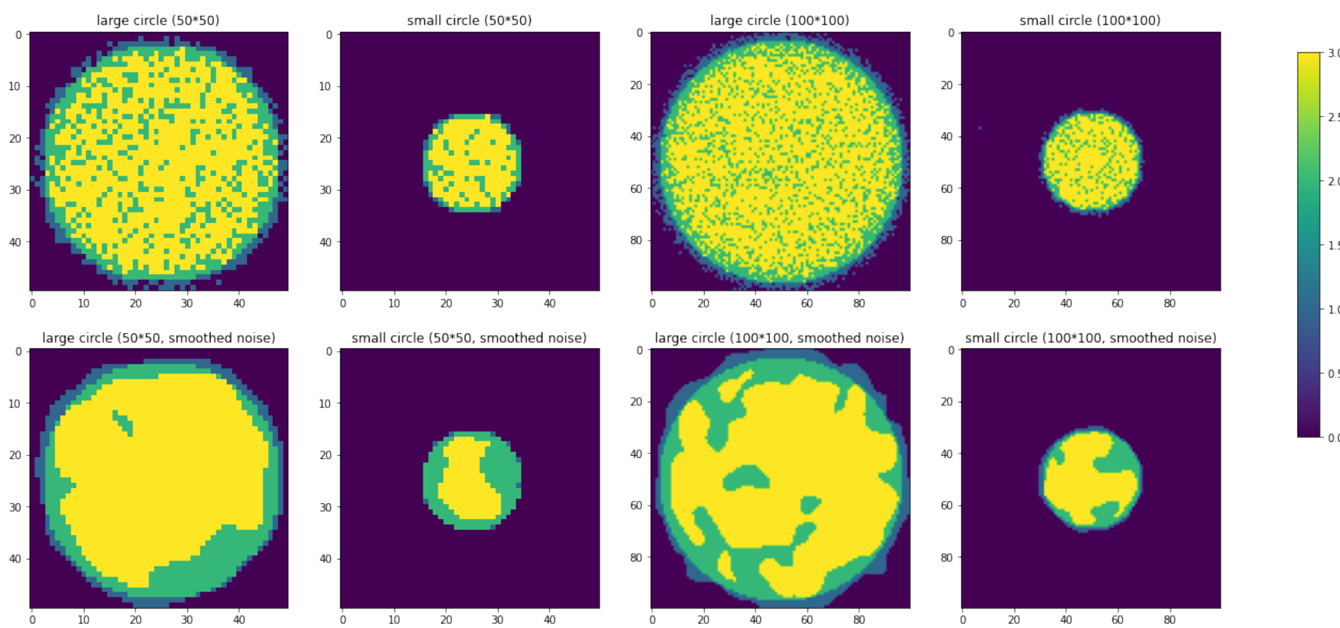


Figure 3: Confidence sets and signal

Figure 3 shows the upper and lower confidence sets drawn on top of the signal. The confidence sets were constructed for 8 different images, each a combination of small/large circle, 100*100/50*50 image size, and smoothed/uncorrelated noise.

3.3 Error Rate

We look at the false discovery rate, expressed as 1, calculated from 1,000 simulations. Recall that $FDR < \alpha \cdot \frac{m_0}{m}$.

- Generally no big difference between image size 50*50 and 100*100.

	small	**small&smth**	$\alpha\frac{m_0}{m}$	**large**	**large&smth**	$\alpha\frac{m_0}{m}$
50*50						
c=0.5	0.0002	0.0002	**0.0**	0.0009	0.0009	**0.0**
c=0.5/std=*3	0.0041	0.0021		0.0026	0.0029	
c=2	0.0002	0.0002	**0.00072**	0.0006	0.0006	**0.00168**
c=2/std=*3	0.0005	0.0006		0.0017	0.0014	
c=3	0.0021	0.0024	**0.00466**	0.012	0.0117	**0.02674**
c=3/std=*3	0.0024	0.0022		0.0135	0.0122	
100*100						
c=0.5	0.0003	0.0003	**0.0**	0.0007	0.0007	**0.0**
c=0.5/std=*3	0.0038	0.0053		0.0023	0.0024	
c=2	0.0001	0.0001	**0.00038**	0.0007	0.0007	**0.00238**
c=2/std=*3	0.0004	0.0004		0.002	0.0015	
c=3	0.0022	0.0023	**0.00461**	0.013	0.0131	**0.02817**
c=3/std=*3	0.0024	0.0026		0.0142	0.0133	

Table 1: FDR

- For $c = 0.5$, since m_0 is 0, the error rates are very close to 0.
- For $c = 2$ and $c = 3$, the error rates are all below $\alpha\frac{m_0}{m}$
- Issue with calculating m_0, as on the grid there was often no voxel exactly of the value of the threshold c. Instead, range was used to capture the number of voxels m_0. In this simulation, used range of $\{c \pm 0.2\}$

Now we turn to the following FWER which is $1 - \mathbb{P}[\hat{\mathcal{A}}_c^+ \subseteq \mathcal{A}_c \subseteq \hat{\mathcal{A}}_c^-]$.The following table is calculated from 1,000 simulations. Here the FWER is not controlled under $\alpha = 0.05$ level as he exclusion rate is well above 0.05. The uncorrelated noise images have higher exclusion rate due to the confidence sets having more erratic shapes in such cases.

	small	small&smth	large	large&smth
50*50				
c=0.5	0.131	0.055	0.809	0.446
c=0.5/std=*3	0.189	0.032	0.89	0.354
c=2	0.364	0.196	0.697	0.352
c=2/std=*3	0.295	0.101	0.086	0.049
c=3	0.992	0.268	1	0.483
c=3/std=*3	0.985	0.267	0.984	0.266
100*100				
c=0.5	0.558	0.238	0.998	0.712
c=0.5/std=*3	0.581	0.102	1	0.409
c=2	0.656	0.372	0.996	0.743
c=2/std=*3	0.674	0.199	0.224	0.075
c=3	1	0.666	1	0.915
c=3/std=*3	1	0.635	1	0.611

Table 2: $FWER$