

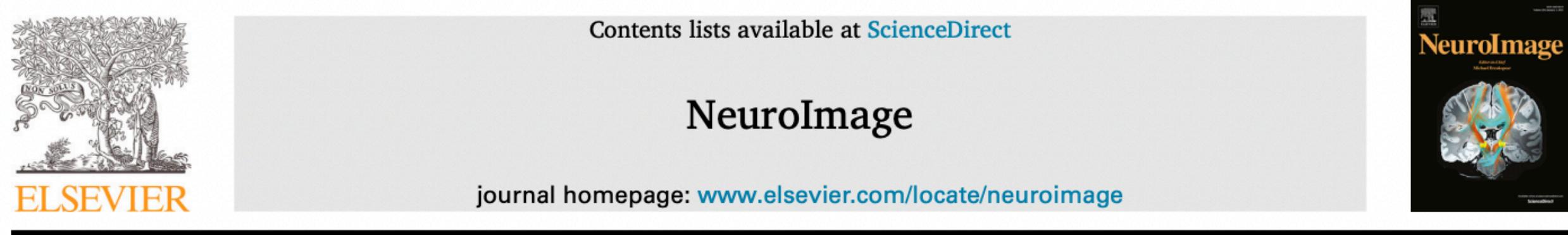
Spatial Bayesian Models for Task fMRI

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A quick aside on icebergs



Highlight results, don't hide them: Enhance interpretation, reduce biases and improve reproducibility

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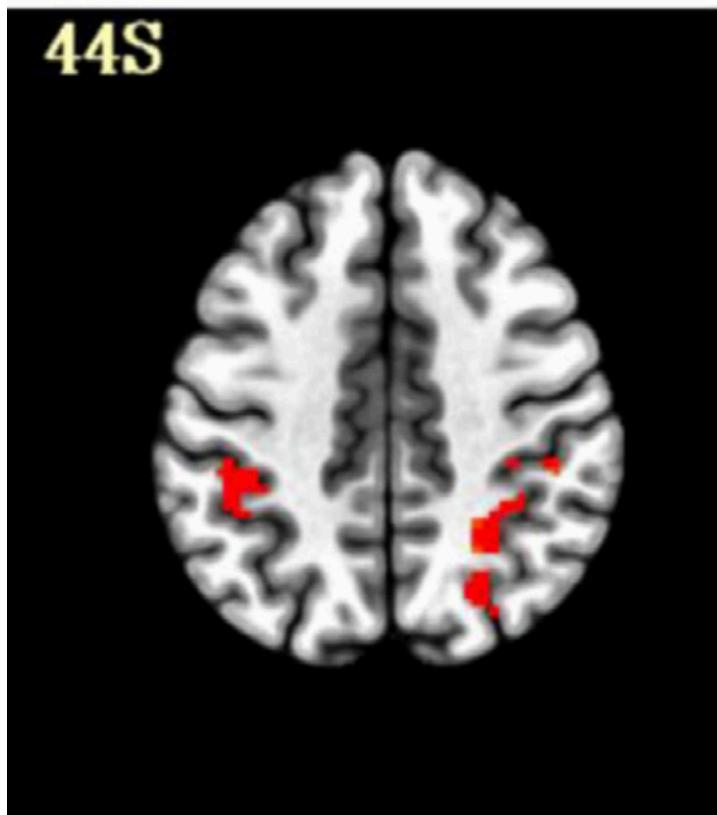
ABSTRACT

Most neuroimaging studies display results that represent only a tiny fraction of the collected data. While it is conventional to present "only the significant results" to the reader, here we suggest that this practice has several negative consequences for both reproducibility and understanding. This practice hides away most of the results of the dataset and leads to problems of selection bias and irreproducibility, both of which have been recognized as major issues in neuroimaging studies recently. Opaque, all-or-nothing thresholding, even if well-intentioned, places undue influence on arbitrary filter values, hinders clear communication of scientific results, wastes data, is antithetical to good scientific practice, and leads to conceptual inconsistencies. It is also inconsistent with

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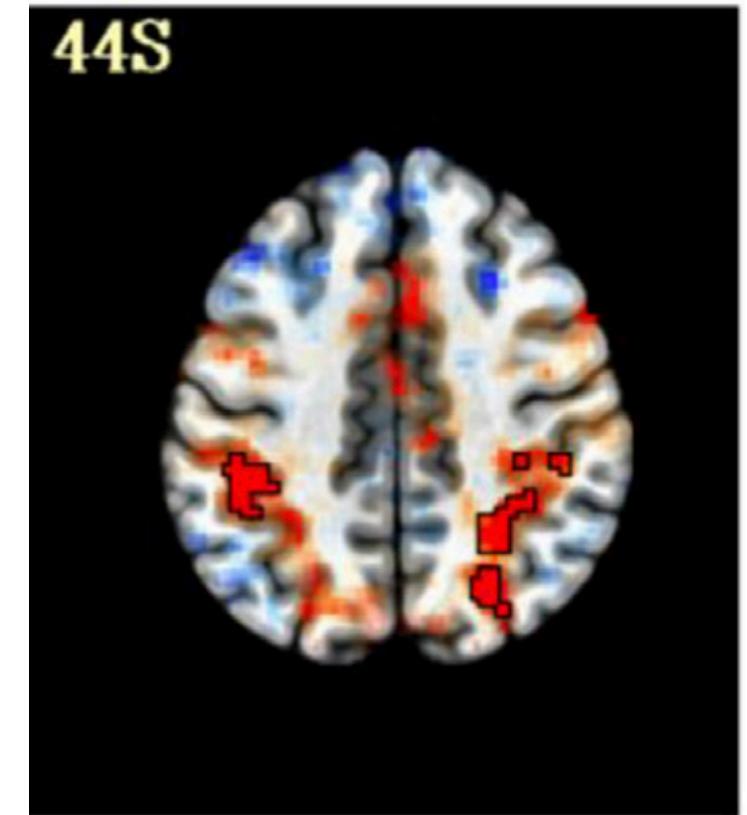
Check out Poster 731!

go from This



Tip of the iceberg

to This



The whole iceberg
with the water
level outlined



Outline

1. Bayesian statistics primer
2. Spatial priors
3. Spatial Bayesian GLM for task fMRI
4. Demo of BayesfMRI R package

Bayesian Statistics Primer

Frequentist vs. Bayesian Statistics

Frequentist Statistics

- Use only the observed data to estimate parameters, typically by maximizing the *likelihood*
- Usually fast & easy to compute  (e.g., sample mean)
- Noise in the data is reflected in the estimates 
- Need to collect more data to reduce noise

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Bayesian Statistics

- Take into account *prior beliefs* about parameter via a *prior distribution*
- Estimation done by maximizing the *posterior distribution*, which reflects updated beliefs based on data
- Results in better (more efficient) estimates 
- Estimation can be difficult and/or slow 

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2. Patterns of task activation should be similar for “proximal” locations

- Encode this belief via a multivariate prior across locations with covariance structure that represents the expected spatial dependence among “neighbors”
- *Spatial Bayesian models* use this kind of prior (a *spatial prior*)

Likelihood, Prior and Posterior

- The **likelihood** $f(\mathbf{y} | \mu)$ is the probability of observing the data \mathbf{y} , given a particular value of a parameter μ
- The **prior distribution** $f(\mu)$ reflects our original or prior beliefs, absent data, about a parameter μ
- The **posterior distribution** $f(\mu | \mathbf{y})$ reflects our updated beliefs about μ based on the data \mathbf{y}

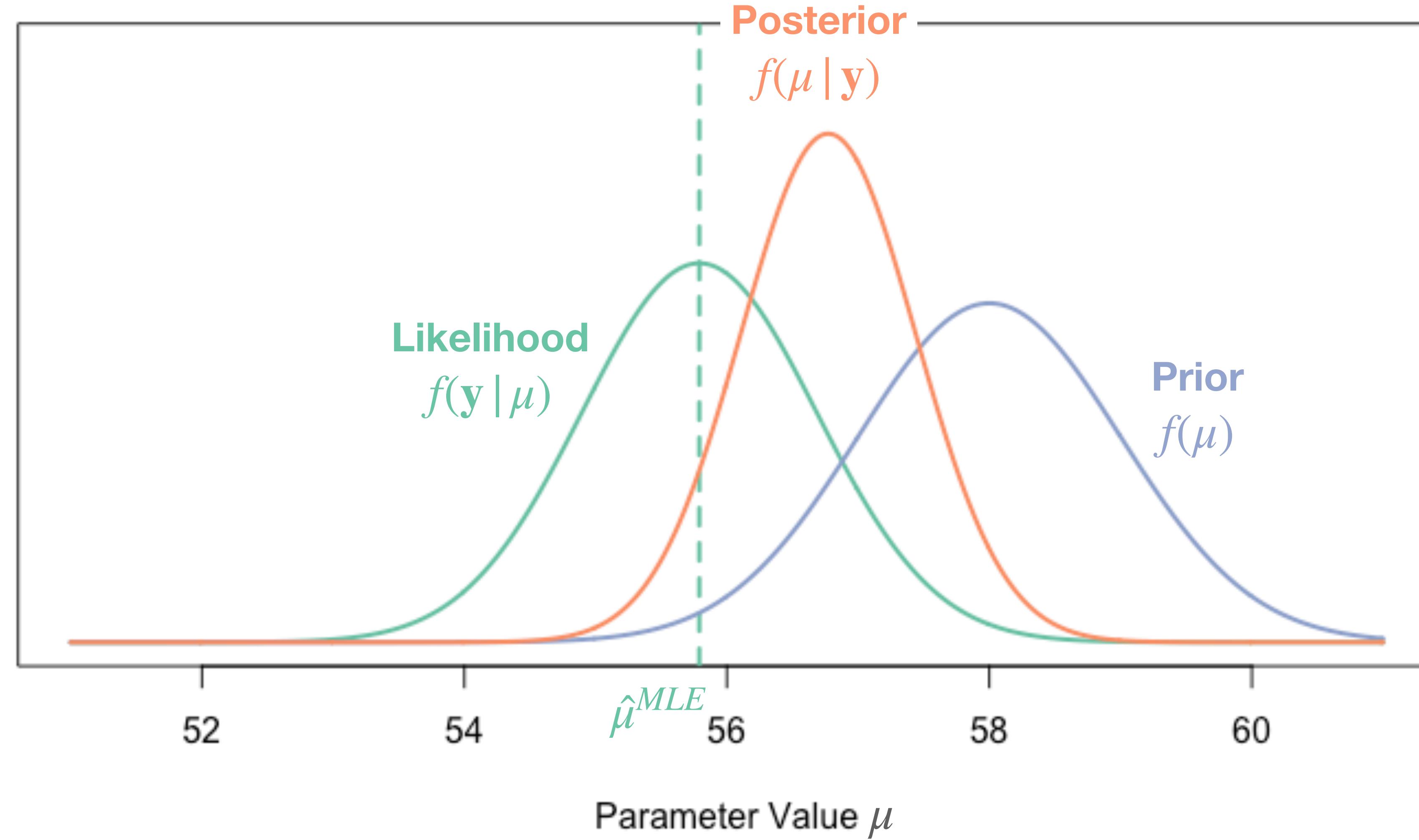
$$f(\mu | \mathbf{y}) = \frac{\underset{\text{Likelihood}}{f(\mathbf{y} | \mu)} \underset{\text{Prior}}{f(\mu)}}{\underset{\text{Marginal Likelihood}}{f(\mathbf{y})}}$$

↑
Bayes Rule

$$\propto \underset{\text{Likelihood}}{f(\mathbf{y} | \mu)} \underset{\text{Prior}}{f(\mu)}$$

Likelihood, Prior and Posterior

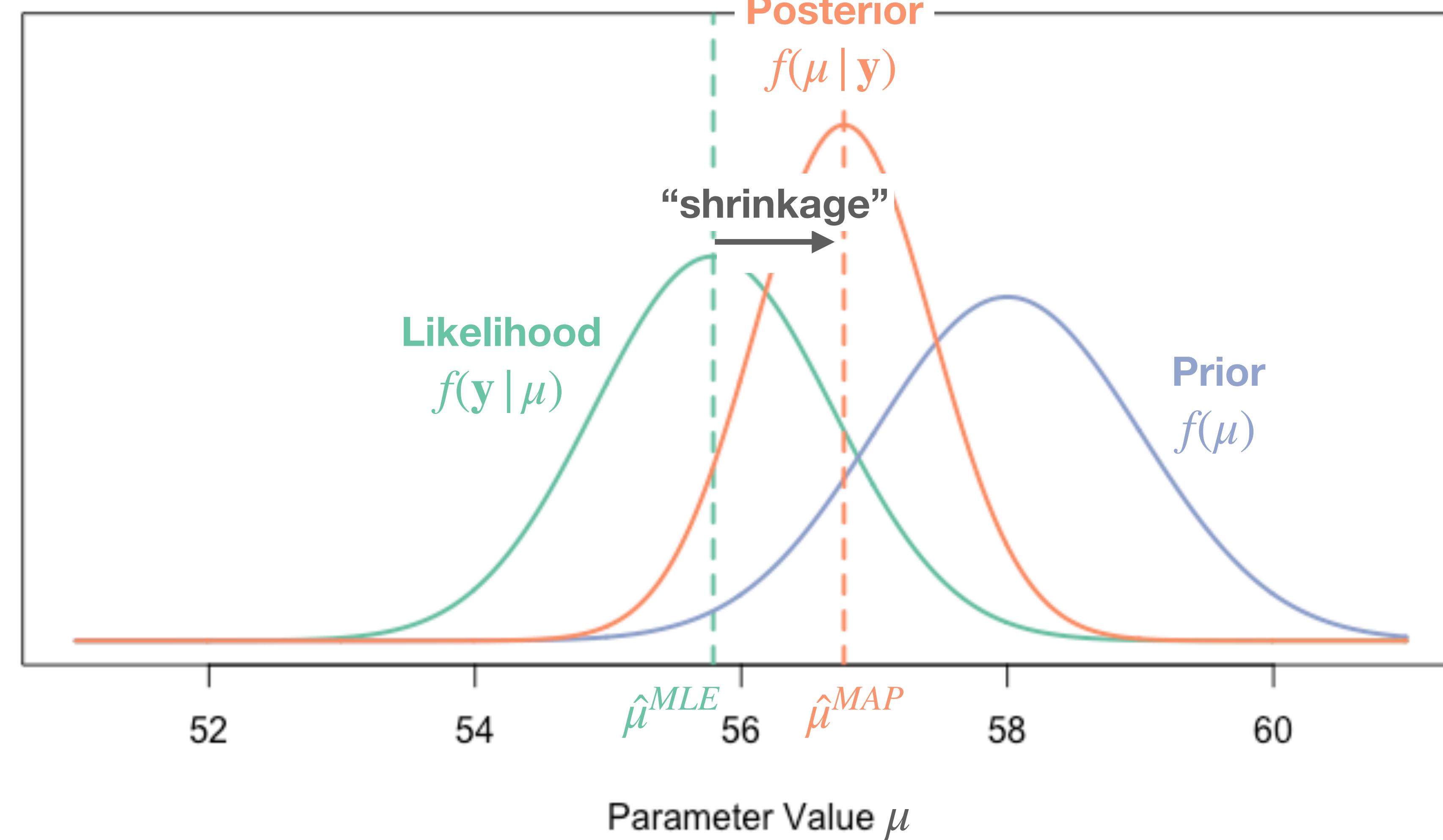
MLE = the value of μ that maximizes the *likelihood*



Maximum-a-posteriori (MAP) estimators

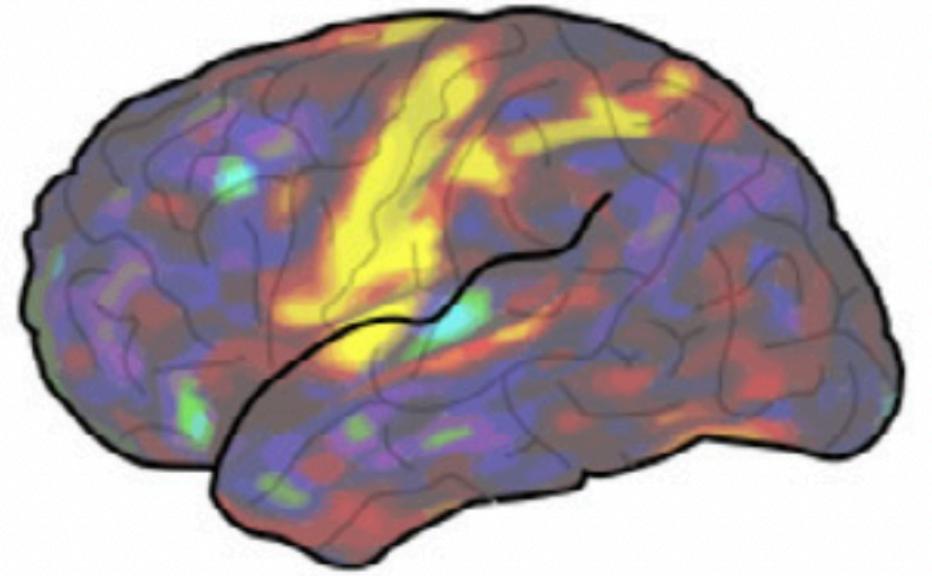
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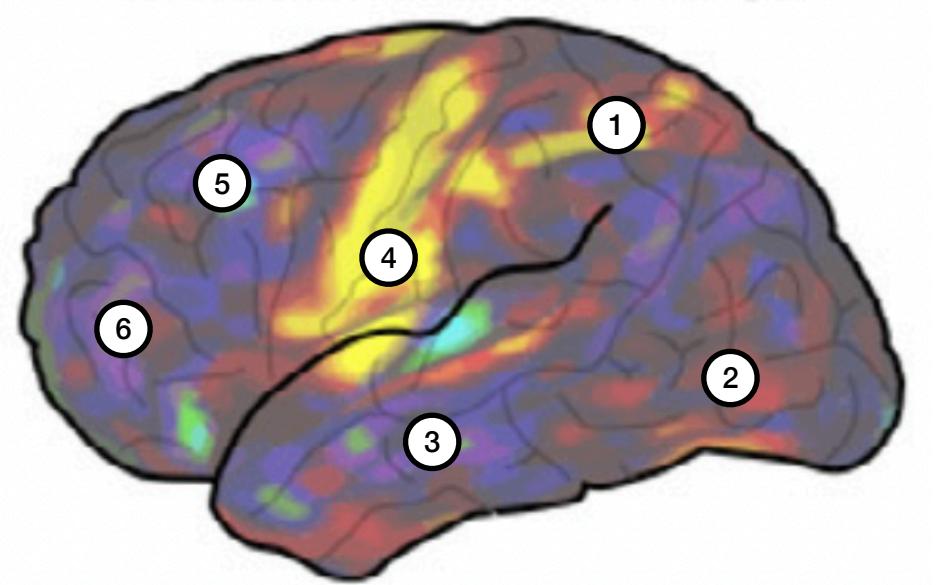
Spatial Priors

What is a spatial prior?



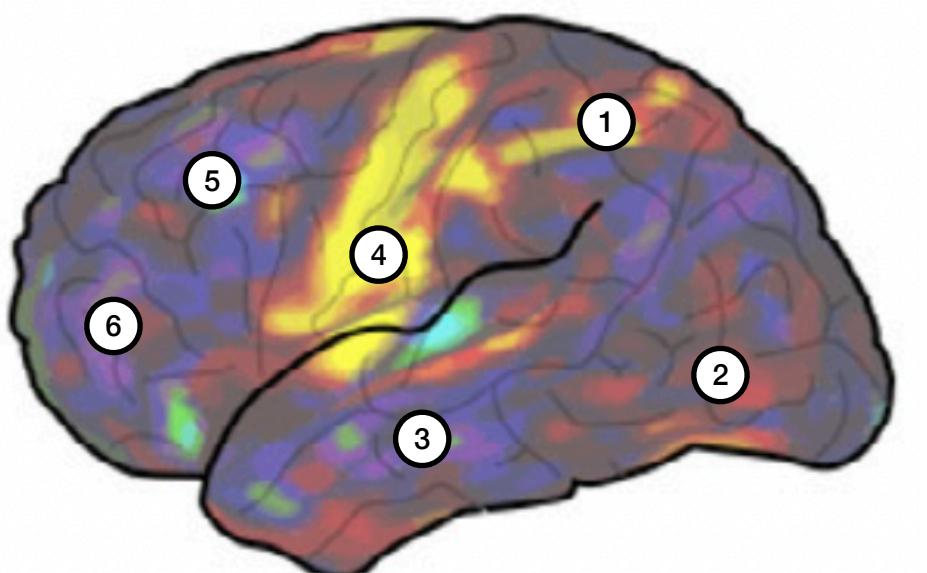
latent spatial map β

What is a spatial prior?



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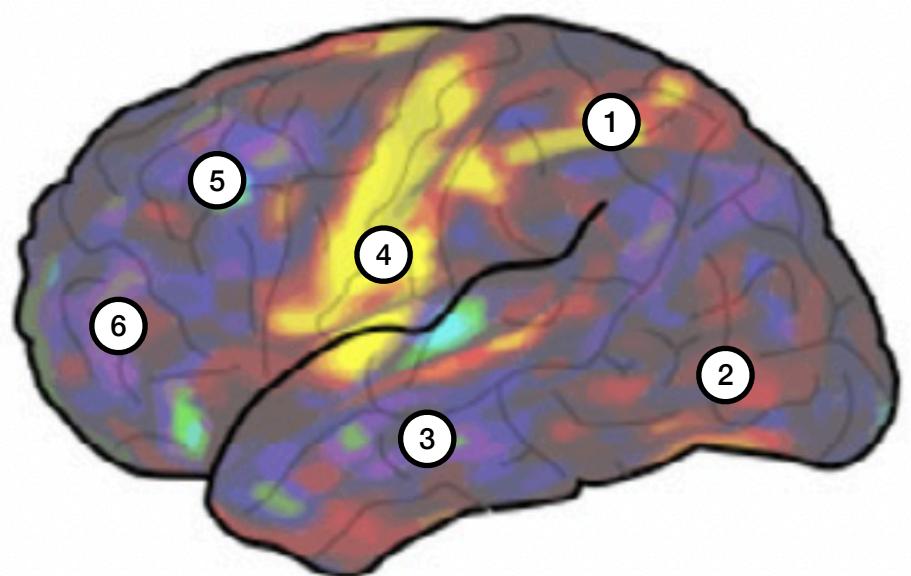


latent spatial map β

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} \sim (\mu_0, \Sigma_0)$$

(6x1) (6x6)

What is a spatial prior?



latent spatial map β

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix}_{(6 \times 1)} \sim (\mu_0, \Sigma_0)_{(6 \times 6)}$$

One example of a spatial prior:

Multivariate Gaussian prior with mean zero and sparse inverse covariance

$$\beta \sim N(\mathbf{0}, Q^{-1})$$

Q contains zeros everywhere except along diagonal & in cells corresponding to “neighbors”

Toy fMRI example

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Consider a single task, and for simplicity let the design column \mathbf{x} be all 1's

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- Likelihood at location $v = 1, \dots, V$: $\mathbf{y}(v) = \beta(v)\mathbf{x} + \mathbf{e}(v), \quad \mathbf{e}(v) \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_T)$
 $(T \times 1) \qquad \qquad \qquad (T \times 1) \qquad (T \times 1)$

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 $(TV \times 1) \qquad (V \times 1)$

$$\mathbf{X} = \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix}_{(TV \times V)}$$

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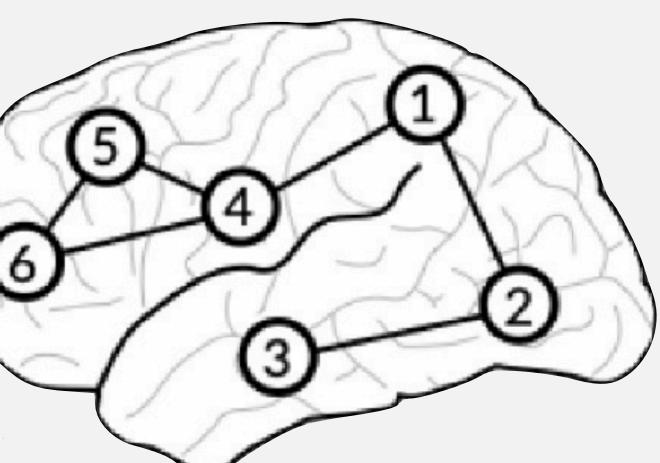
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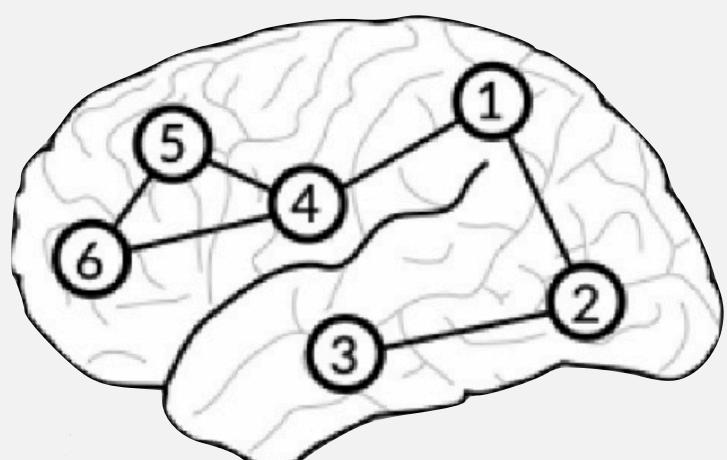
\mathbf{Q} represents our beliefs about the spatial dependence of task activation amplitudes in β .

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adjacency matrix						
①	1	1		1		
②	1	1	1			
③		1	1			
④	1			1	1	1
⑤				1	1	1
⑥				1	1	1
①	②	③	④	⑤	⑥	

precision matrix						
①	█	█		█		
②	█		█	█		
③		█			█	
④	█			█		█
⑤				█	█	█
⑥				█	█	█
①	②	③	④	⑤	⑥	

direct dependence between neighbors only

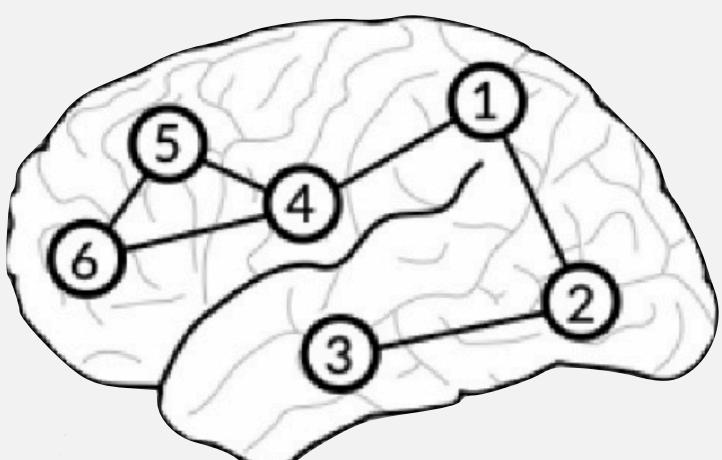
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direct
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between
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some example numbers

$$\mathbf{Q} = \begin{pmatrix} 5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 5 & -2 & -2 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -2 & -2 & 5 \end{pmatrix}$$

(Inverse covariance)

$$\mathbf{Q}^{-1} = \begin{pmatrix} 0.23 & 0.05 & 0.01 & 0.10 & 0.07 & 0.07 \\ 0.05 & 0.22 & 0.04 & 0.02 & 0.01 & 0.01 \\ 0.01 & 0.04 & 0.21 & 0.00 & 0.00 & 0.00 \\ 0.10 & 0.02 & 0.00 & 0.47 & 0.31 & 0.31 \\ 0.07 & 0.01 & 0.00 & 0.31 & 0.45 & 0.30 \\ 0.07 & 0.01 & 0.00 & 0.31 & 0.30 & 0.45 \end{pmatrix}$$

(covariance matrix)

Toy example

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

Frequentist estimate of $\boldsymbol{\beta}$

(MLE equivalent to OLS)

$$\hat{\boldsymbol{\beta}}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

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Bayesian estimate of $\boldsymbol{\beta}$: $\hat{\boldsymbol{\beta}}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$

Toy example

Bayesian estimate of β : $\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}' \mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}' \mathbf{y} \right)$

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What if $\mathbf{Q} \rightarrow$ a matrix of zero's?
(Prior var $\rightarrow \infty$)

$$\hat{\beta}^{MAP} \rightarrow \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right) = \hat{\beta}^{MLE}$$

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What if \mathbf{Q} equals a diagonal matrix, $\frac{1}{\sigma_0^2} \mathbf{I}_V$?
(no dependence information, prior var = σ_0^2)

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$$\left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right) = \left(\frac{T}{\sigma^2} + \frac{1}{\sigma_0^2} \right) \mathbf{I}_V$$

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$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

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$$\mathbf{X} = \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix}$$
$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & T & \end{pmatrix}$$
$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_t y_{t,1} \\ \vdots \\ \sum_t y_{t,V} \end{pmatrix}$$

Bayesian estimate of β : $\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$

What if \mathbf{Q} equals a diagonal matrix, $\frac{1}{\sigma_0^2} \mathbf{I}_V$?
(no dependence information, prior var = σ_0^2)

$$\left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right) = \left(\frac{T}{\sigma^2} + \frac{1}{\sigma_0^2} \right) \mathbf{I}_V$$

$$\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} = \frac{T}{\sigma^2} (\bar{y}_1, \dots, \bar{y}_V)' = \frac{T}{\sigma^2} \hat{\beta}^{MLE}$$

Toy example

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

Frequentist estimate of β

(MLE equivalent to OLS)

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$\begin{aligned} \mathbf{x} &= (1, 1, \dots, 1)' \\ \mathbf{X} &= \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix} \quad \mathbf{X}'\mathbf{X} = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & T & \\ & & & \ddots \end{pmatrix} \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_t y_{t,1} \\ \vdots \\ \sum_t y_{t,V} \end{pmatrix} \end{aligned}$$

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What if \mathbf{Q} equals a diagonal matrix, $\frac{1}{\sigma_0^2} \mathbf{I}_V$?
(no dependence information, prior var = σ_0^2)

$$\left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right) = \left(\frac{T}{\sigma^2} + \frac{1}{\sigma_0^2} \right) \mathbf{I}_V$$

$$\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} = \frac{T}{\sigma^2} (\bar{y}_1, \dots, \bar{y}_V)' = \frac{T}{\sigma^2} \hat{\beta}^{MLE}$$

$$\Rightarrow \hat{\beta}^{MAP} = \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2/T} \hat{\beta}^{MLE} = \lambda \hat{\beta}^{MLE}$$

Toy example

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

Frequentist estimate of β

(MLE equivalent to OLS)

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$\begin{aligned} \mathbf{x} &= (1, 1, \dots, 1)' \\ \mathbf{X} &= \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix} \quad \mathbf{X}'\mathbf{X} = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & T & \\ & & & \ddots \end{pmatrix} \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_t y_{t,1} \\ \vdots \\ \sum_t y_{t,V} \end{pmatrix} \end{aligned}$$

Bayesian estimate of β : $\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$

What if \mathbf{Q} equals a diagonal matrix, $\frac{1}{\sigma_0^2} \mathbf{I}_V$?
 (no dependence information, prior var = σ_0^2)

$$\left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right) = \left(\frac{T}{\sigma^2} + \frac{1}{\sigma_0^2} \right) \mathbf{I}_V$$

$$\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} = \frac{T}{\sigma^2} (\bar{y}_1, \dots, \bar{y}_V)' = \frac{T}{\sigma^2} \hat{\beta}^{MLE}$$

$$\Rightarrow \hat{\beta}^{MAP} = \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2/T} \hat{\beta}^{MLE} = \lambda \hat{\beta}^{MLE}$$

If $T = 100$, $\sigma^2 = 1$, $\sigma_0^2 = 1/5$, then $\lambda = 0.988$

Toy example

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

Frequentist estimate of β

(MLE equivalent to OLS)

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$\begin{aligned} \mathbf{x} &= (1, 1, \dots, 1)' \\ \mathbf{X} &= \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix} \quad \mathbf{X}'\mathbf{X} = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & T & \\ & & & \ddots \end{pmatrix} \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_t y_{t,1} \\ \vdots \\ \sum_t y_{t,V} \end{pmatrix} \end{aligned}$$

Bayesian estimate of β : $\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$

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 (no dependence information, prior var = σ_0^2)

$$\left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right) = \left(\frac{T}{\sigma^2} + \frac{1}{\sigma_0^2} \right) \mathbf{I}_V$$

$$\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} = \frac{T}{\sigma^2} (\bar{y}_1, \dots, \bar{y}_V)' = \frac{T}{\sigma^2} \hat{\beta}^{MLE}$$

$$\Rightarrow \hat{\beta}^{MAP} = \frac{\sigma_0^2}{\sigma_0^2 + \sigma^2/T} \hat{\beta}^{MLE} = \lambda \hat{\beta}^{MLE}$$

If $T = 100$, $\sigma^2 = 1$, $\sigma_0^2 = 1/5$, then $\lambda = 0.988$

Toy example

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

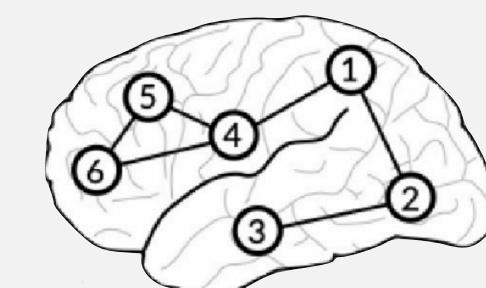
Frequentist estimate of β
(MLE equivalent to OLS)

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$\mathbf{x} = (1, 1, \dots, 1)'$$
$$\mathbf{X} = \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix} \quad \mathbf{X}'\mathbf{X} = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & T & \end{pmatrix} \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_t y_{t,1} \\ \vdots \\ \sum_t y_{t,V} \end{pmatrix}$$

Bayesian estimate of β : $\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$

What if $\mathbf{Q} =$



$$\begin{pmatrix} 5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 5 & -2 & -2 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -2 & -2 & 5 \end{pmatrix} ?$$

Toy example

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

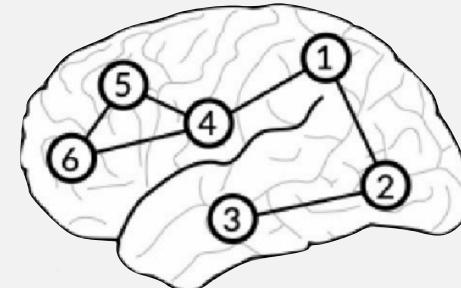
Frequentist estimate of β
(MLE equivalent to OLS)

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

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What if $\mathbf{Q} =$



$$\begin{pmatrix} 5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 5 & -2 & -2 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -2 & -2 & 5 \end{pmatrix} ?$$

We just showed that

$$\hat{\beta}^{MAP} = \frac{T}{\sigma^2} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \hat{\beta}^{MLE} = \Lambda \hat{\beta}^{MLE}.$$

Toy example

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

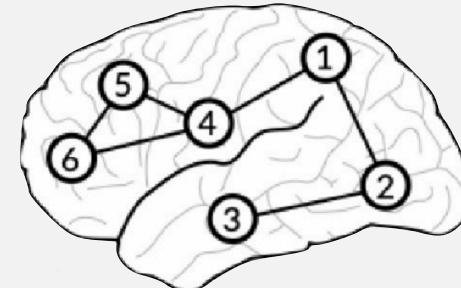
Frequentist estimate of β
(MLE equivalent to OLS)

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$\begin{aligned} \mathbf{x} &= (1, 1, \dots, 1)' \\ \mathbf{X} &= \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix} \quad \mathbf{X}'\mathbf{X} = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & T & \end{pmatrix} \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_t y_{t,1} \\ \vdots \\ \sum_t y_{t,V} \end{pmatrix} \end{aligned}$$

Bayesian estimate of β : $\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$

What if $\mathbf{Q} =$



$$\begin{pmatrix} 5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 5 & -2 & -2 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -2 & -2 & 5 \end{pmatrix} ?$$

We just showed that

$$\hat{\beta}^{MAP} = \frac{T}{\sigma^2} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \hat{\beta}^{MLE} = \Lambda \hat{\beta}^{MLE}.$$

If $T = 100$ and $\sigma^2 = 1$, $\Lambda =$

$$\begin{pmatrix} 0.95 & 0.01 & 0 & 0.01 & 0 & 0 \\ 0.01 & 0.95 & 0.01 & 0 & 0 & 0 \\ 0 & 0.01 & 0.95 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0.95 & 0.02 & 0.02 \\ 0 & 0 & 0 & 0.02 & 0.95 & 0.02 \\ 0 & 0 & 0 & 0.02 & 0.02 & 0.95 \end{pmatrix}$$

Toy example

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

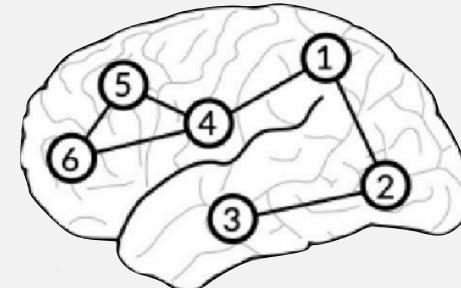
Frequentist estimate of β
(MLE equivalent to OLS)

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$\begin{aligned} \mathbf{x} &= (1, 1, \dots, 1)' \\ \mathbf{X} &= \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix} \quad \mathbf{X}'\mathbf{X} = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & T & \end{pmatrix} \quad \mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_t y_{t,1} \\ \vdots \\ \sum_t y_{t,V} \end{pmatrix} \end{aligned}$$

Bayesian estimate of β : $\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$

What if $\mathbf{Q} =$



$$\begin{pmatrix} 5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 5 & -2 & -2 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -2 & -2 & 5 \end{pmatrix} ?$$

We just showed that

$$\hat{\beta}^{MAP} = \frac{T}{\sigma^2} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \hat{\beta}^{MLE} = \Lambda \hat{\beta}^{MLE}.$$

If $T = 100$ and $\sigma^2 = 1$, $\Lambda =$

$$\begin{pmatrix} 0.95 & 0.01 & 0 & 0.01 & 0 & 0 \\ 0.01 & 0.95 & 0.01 & 0 & 0 & 0 \\ 0 & 0.01 & 0.95 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0.95 & 0.02 & 0.02 \\ 0 & 0 & 0 & 0.02 & 0.95 & 0.02 \\ 0 & 0 & 0 & 0.02 & 0.02 & 0.95 \end{pmatrix}$$

Toy example

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

Frequentist estimate of β
(MLE equivalent to OLS)

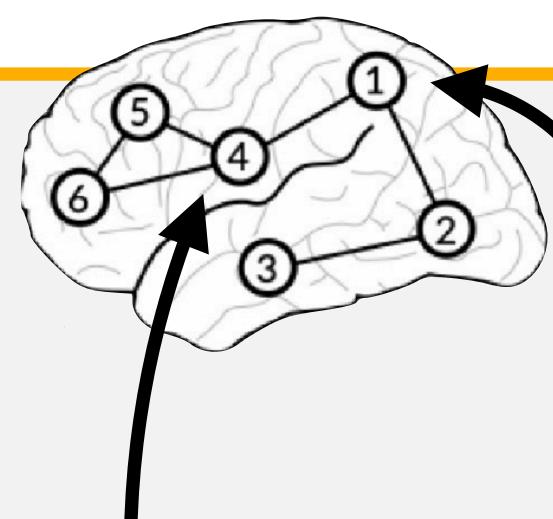
$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$\mathbf{x} = (1, 1, \dots, 1)'$$

$$\mathbf{X} = \begin{pmatrix} \mathbf{x} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{x} \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & T & \end{pmatrix}$$

$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} \sum_t y_{t,1} \\ \vdots \\ \sum_t y_{t,V} \end{pmatrix}$$



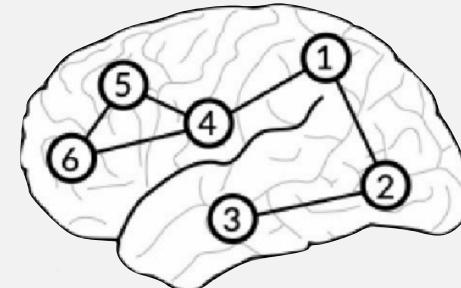
For example:

$$\hat{\beta}_1^{MAP} = 0.95\hat{\beta}_1^{MLE} + 0.01\hat{\beta}_2^{MLE} + 0.01\hat{\beta}_4^{MLE}$$

$$\hat{\beta}_4^{MAP} = 0.95\hat{\beta}_4^{MLE} + 0.01\hat{\beta}_1^{MLE} + 0.02\hat{\beta}_5^{MLE} + 0.02\hat{\beta}_6^{MLE}$$

Bayesian estimate of β : $\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$

What if $\mathbf{Q} =$



$$\begin{pmatrix} 5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 5 & -2 & -2 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -2 & -2 & 5 \end{pmatrix} ?$$

We just showed that

$$\hat{\beta}^{MAP} = \frac{T}{\sigma^2} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \hat{\beta}^{MLE} = \Lambda \hat{\beta}^{MLE}.$$

If $T = 100$ and $\sigma^2 = 1$, $\Lambda =$

$$\begin{pmatrix} 0.95 & 0.01 & 0 & 0.01 & 0 & 0 \\ 0.01 & 0.95 & 0.01 & 0 & 0 & 0 \\ 0 & 0.01 & 0.95 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0.95 & 0.02 & 0.02 \\ 0 & 0 & 0 & 0.02 & 0.95 & 0.02 \\ 0 & 0 & 0 & 0.02 & 0.02 & 0.95 \end{pmatrix}$$

What about the variance?

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

Frequentist estimate of β

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$Cov(\hat{\beta}^{MLE}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

Bayesian estimate of β

$$\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$$

$$Cov(\hat{\beta}^{MLE}) = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1}$$

What about the variance?

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

Let $T = 100$, $\sigma^2 = 1$, $\sigma_0^2 = 1/5$

Frequentist estimate of β

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

$$Cov(\hat{\beta}^{MLE}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

$$Var(\hat{\beta}_{(v)}^{MLE}) = 0.01$$

Bayesian estimate of β

$$\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$$

$$Cov(\hat{\beta}^{MLE}) = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1}$$

What about the variance?

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}, \quad \mathbf{e} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

Let $T = 100$, $\sigma^2 = 1$, $\sigma_0^2 = 1/5$

Frequentist estimate of β

$$\hat{\beta}^{MLE} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\bar{y}_1, \dots, \bar{y}_V)'$$

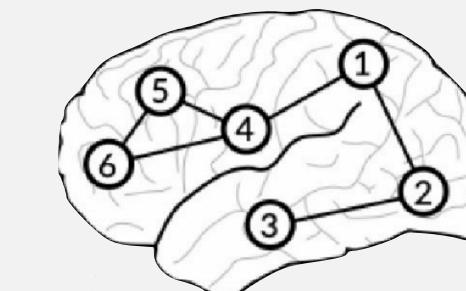
$$Cov(\hat{\beta}^{MLE}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

$$Var(\hat{\beta}_{(v)}^{MLE}) = 0.01$$

Bayesian estimate of β

$$\hat{\beta}^{MAP} = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1} \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{y} \right)$$

$$Cov(\hat{\beta}^{MLE}) = \left(\frac{1}{\sigma^2} \mathbf{X}'\mathbf{X} + \mathbf{Q} \right)^{-1}$$



$$\mathbf{Q} = \begin{pmatrix} 5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 5 & -2 & -2 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & -2 & -2 & 5 \end{pmatrix}$$

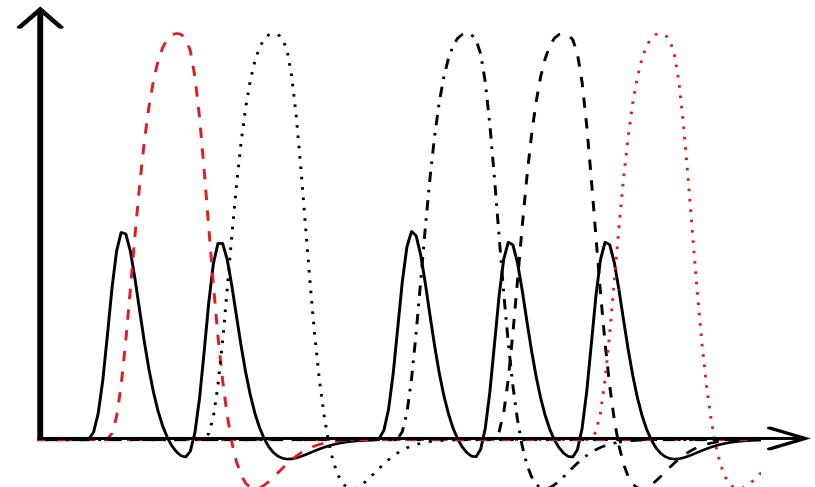
$$Cov(\hat{\beta}^{MAP}) = \begin{pmatrix} 0.0095 & 0.0001 & 0 & 0.0001 & 0 & 0 \\ 0.01 & 0.0095 & 0.0001 & 0 & 0 & 0 \\ 0 & 0.0001 & 0.0095 & 0 & 0 & 0 \\ 0.0001 & 0 & 0 & 0.0095 & 0.0002 & 0.0002 \\ 0 & 0 & 0 & 0.0002 & 0.0095 & 0.0002 \\ 0 & 0 & 0 & 0.0002 & 0.0002 & 0.0095 \end{pmatrix}$$

$$Var(\hat{\beta}_{(4)}^{MLE}) = 0.0095 \rightarrow \text{5% lower variance!}$$

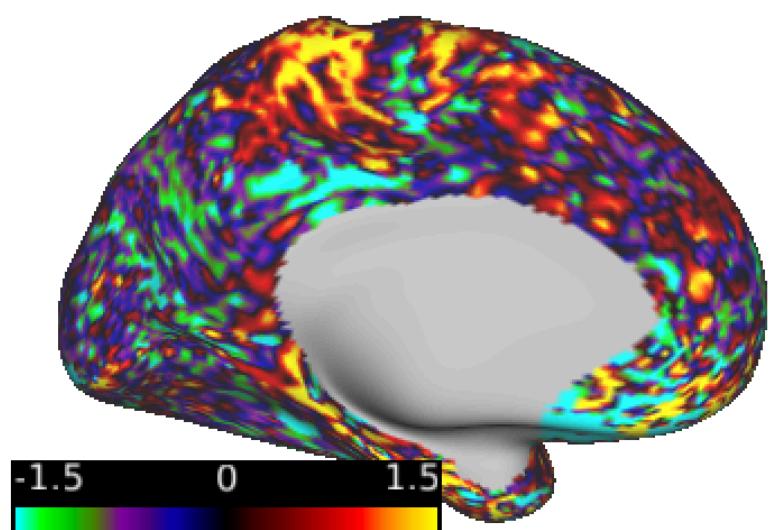
→ more powerful inference (especially if we take into account the covariances between locations)

Spatial Bayesian GLM

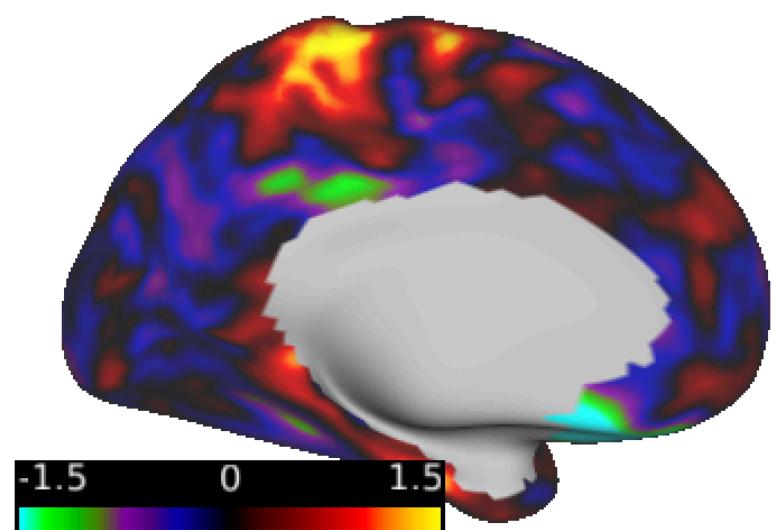
Spatial Bayesian GLM



GLM (General Linear Model): Estimate amplitude and areas of activation during a task or stimulus



Classical GLM: Fit a separate linear model at every voxel or vertex (“massive univariate”) to estimate activation amplitude



Spatial Bayesian GLM: Fit a single Bayesian model accounting for spatial dependence via spatial priors for greater accuracy & power

The Classical GLM

$$\mathbf{y}(\nu) = \sum_{k=1}^K \beta_k(\nu) \mathbf{x}_k + \epsilon(\nu), \quad \epsilon(\nu) \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$$

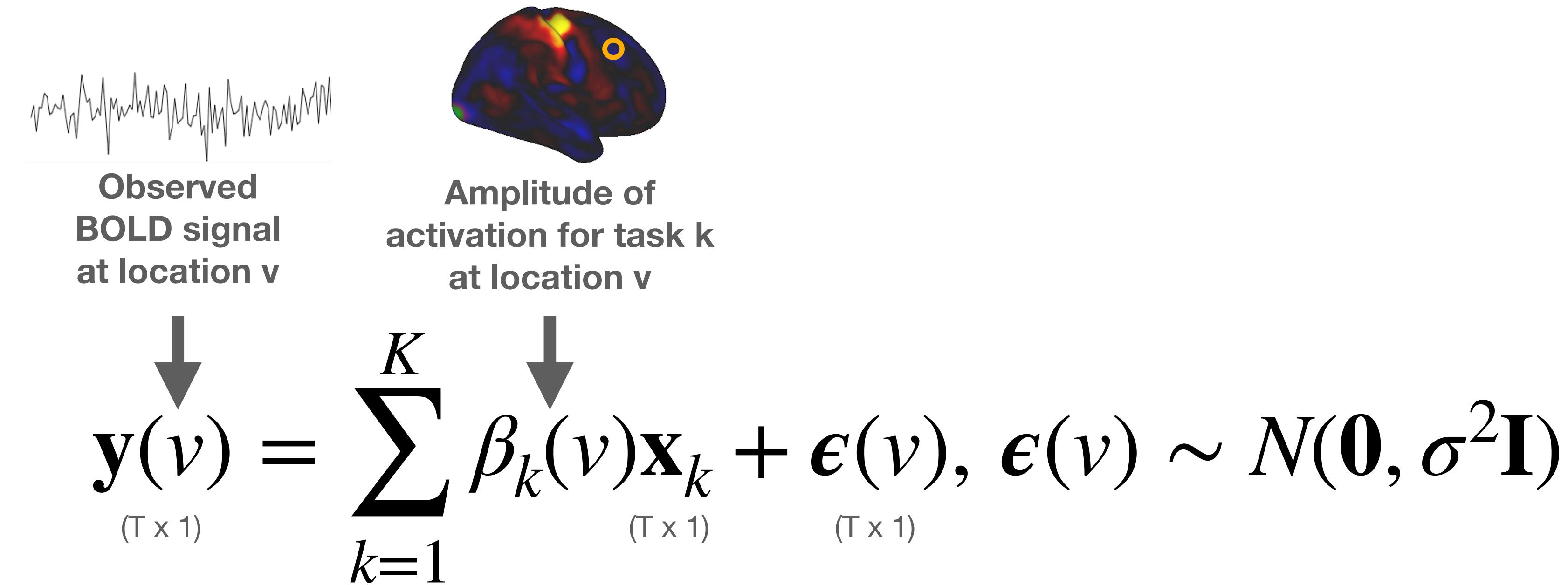
The Classical GLM



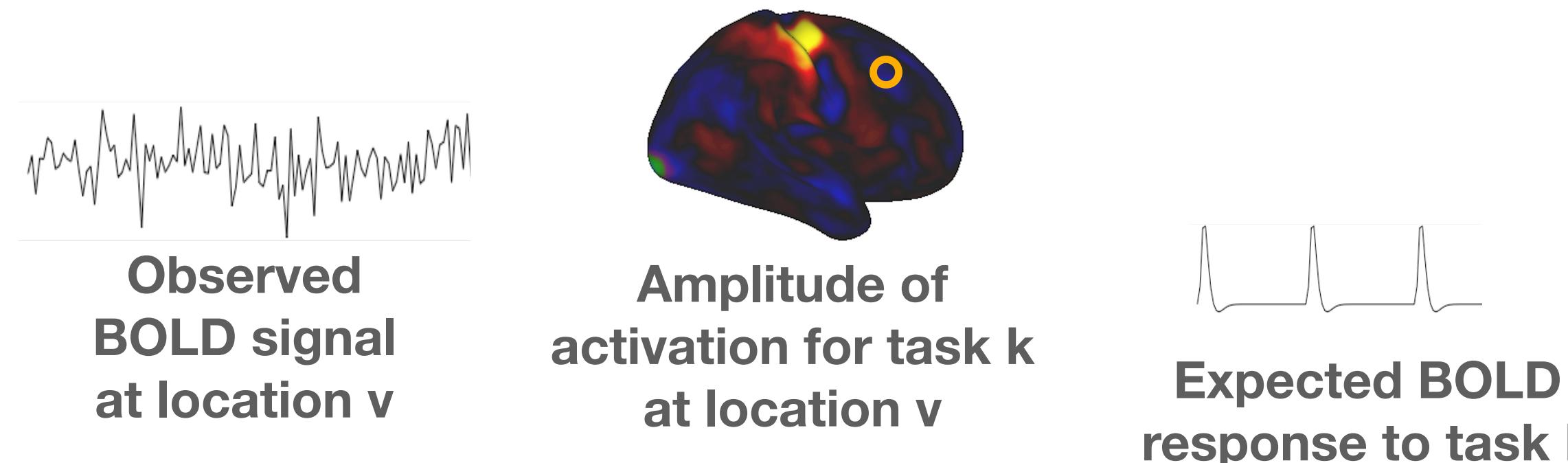
Observed **BOLD** signal at location v

$$\mathbf{y}(\nu) = \sum_{k=1}^K \beta_k(\nu) \mathbf{x}_k + \epsilon(\nu), \quad \epsilon(\nu) \sim N(0, \sigma^2 \mathbf{I})$$

The Classical GLM



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$$\mathbf{y}(v) = \sum_{k=1}^K \beta_k(v) \mathbf{x}_k + \epsilon(v), \epsilon(v) \sim N(\mathbf{0}, \sigma^2 \mathbf{I})$$

Observed BOLD signal at location v

Amplitude of activation for task k at location v

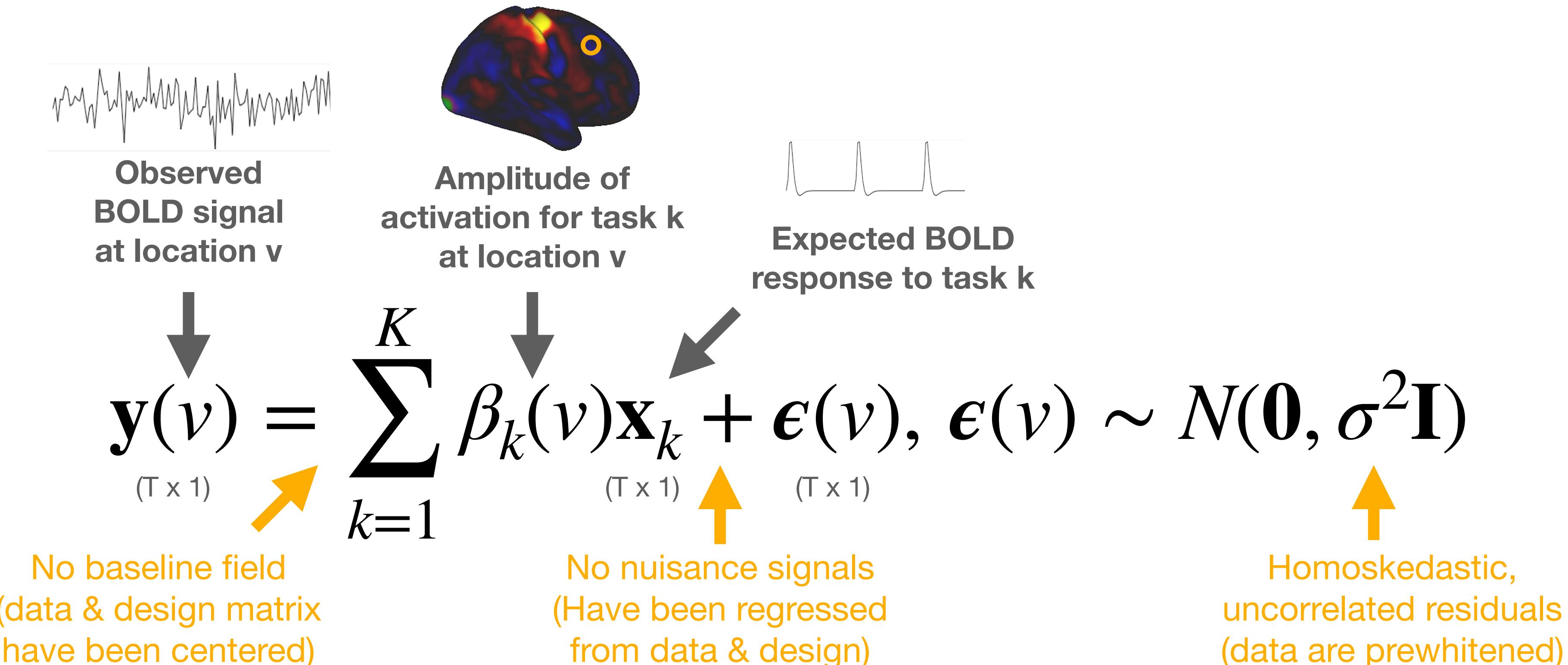
Expected BOLD response to task k

($T \times 1$)

($T \times 1$)

($T \times 1$)

The Classical GLM



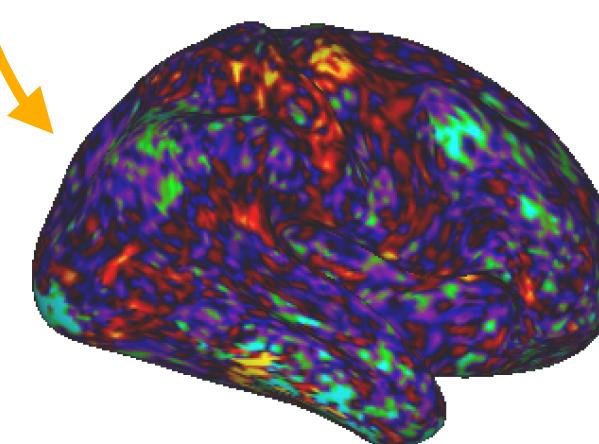
Several assumptions being made for simplicity

Classical GLM Activations

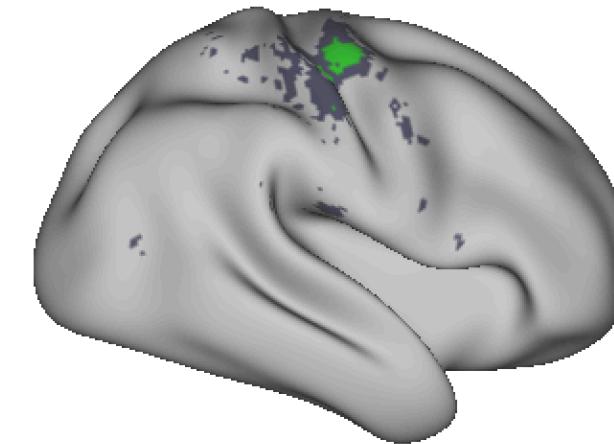
$$\mathbf{y}(1) = \sum_{k=0}^K \beta_k(1) \mathbf{x}_k + \boldsymbol{\epsilon}(1), \quad \boldsymbol{\epsilon}(1) \sim MVN(\mathbf{0}, \sigma^2 \mathbf{I})$$

:

$$\mathbf{y}(V) = \sum_{k=0}^K \beta_k(V) \mathbf{x}_k + \boldsymbol{\epsilon}(V), \quad \boldsymbol{\epsilon}(V) \sim MVN(\mathbf{0}, \sigma^2 \mathbf{I})$$



Hypothesis Testing +
Multiple Comparisons Correction



The Spatial Bayesian GLM

The classical GLM,
concatenated
across all locations

$$\mathbf{y} = \sum_{k=1}^K \mathbf{X}_k \boldsymbol{\beta}_k + \boldsymbol{\epsilon}, \boldsymbol{\epsilon} \sim MVN(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

The Spatial Bayesian GLM

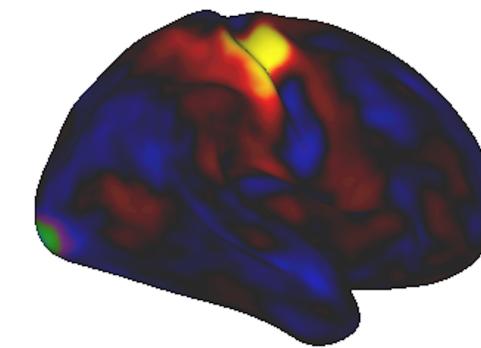
Observed
BOLD signal,
concatenated across
all locations

The classical GLM,
concatenated
across all locations

$$\mathbf{y} = \sum_{k=1}^K \mathbf{X}_k \boldsymbol{\beta}_k + \boldsymbol{\epsilon}, \boldsymbol{\epsilon} \sim MVN(\mathbf{0}, \sigma^2 \mathbf{I}_{TV})$$

The Spatial Bayesian GLM

Observed BOLD signal, concatenated across all locations



Activation amplitude for task k, concatenated across all locations

The classical GLM, concatenated across all locations

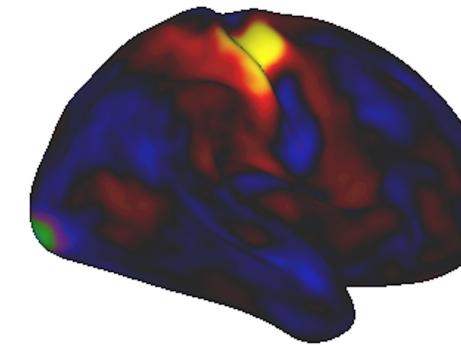
The Spatial Bayesian GLM

Observed BOLD signal, concatenated across all locations

$X_k = \begin{pmatrix} x_k & 0 & \dots & 0 \\ 0 & x_k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_k \end{pmatrix}$

Expected BOLD response to task k

Activation amplitude for task k, concatenated across all locations



The classical GLM, concatenated across all locations

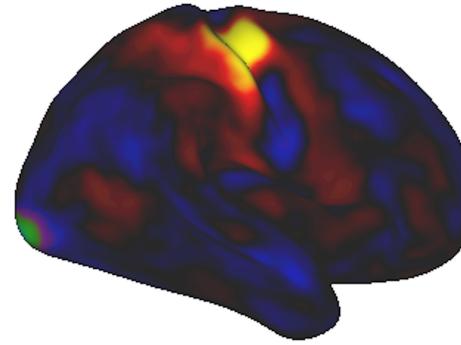
$$y = \sum_{k=1}^K X_k \beta_k + \epsilon, \quad \epsilon \sim MVN(0, \sigma^2 I_{TV})$$

The Spatial Bayesian GLM

Observed BOLD signal, concatenated across all locations

Expected BOLD response to task k

Activation amplitude for task k, concatenated across all locations

$$\mathbf{y} = \sum_{k=1}^K \mathbf{X}_k \boldsymbol{\beta}_k + \boldsymbol{\epsilon}, \boldsymbol{\epsilon} \sim MVN(0, \sigma^2 \mathbf{I}_{TV})$$
$$\mathbf{X}_k = \begin{pmatrix} \mathbf{x}_k & 0 & \cdots & 0 \\ 0 & \mathbf{x}_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{x}_k \end{pmatrix}$$


$\boldsymbol{\beta}_k \sim N(\mathbf{0}, \mathbf{Q}_k^{-1})$

The Spatial Bayesian GLM

The diagram illustrates the spatial Bayesian Generalized Linear Model (GLM) framework. It consists of three main components arranged horizontally:

- Observed BOLD signal, concatenated across all locations:** Represented by a vertical blue bar on the left, labeled y with dimensions $(TV \times 1)$. An arrow points from this bar to the equation.
- Expected BOLD response to task k :** Represented by a vertical light blue bar in the middle, labeled X_k with dimensions $(TV \times V)$. An arrow points from this bar to the equation.
- Activation amplitude for task k , concatenated across all locations:** Represented by a horizontal blue bar on the right, labeled β_k with dimensions $(V \times 1)$. An arrow points from this bar to the equation.

The central equation is:

$$y = \sum_{k=1}^K X_k \beta_k + \epsilon, \quad \epsilon \sim MVN(0, \sigma^2 I_{TV})$$

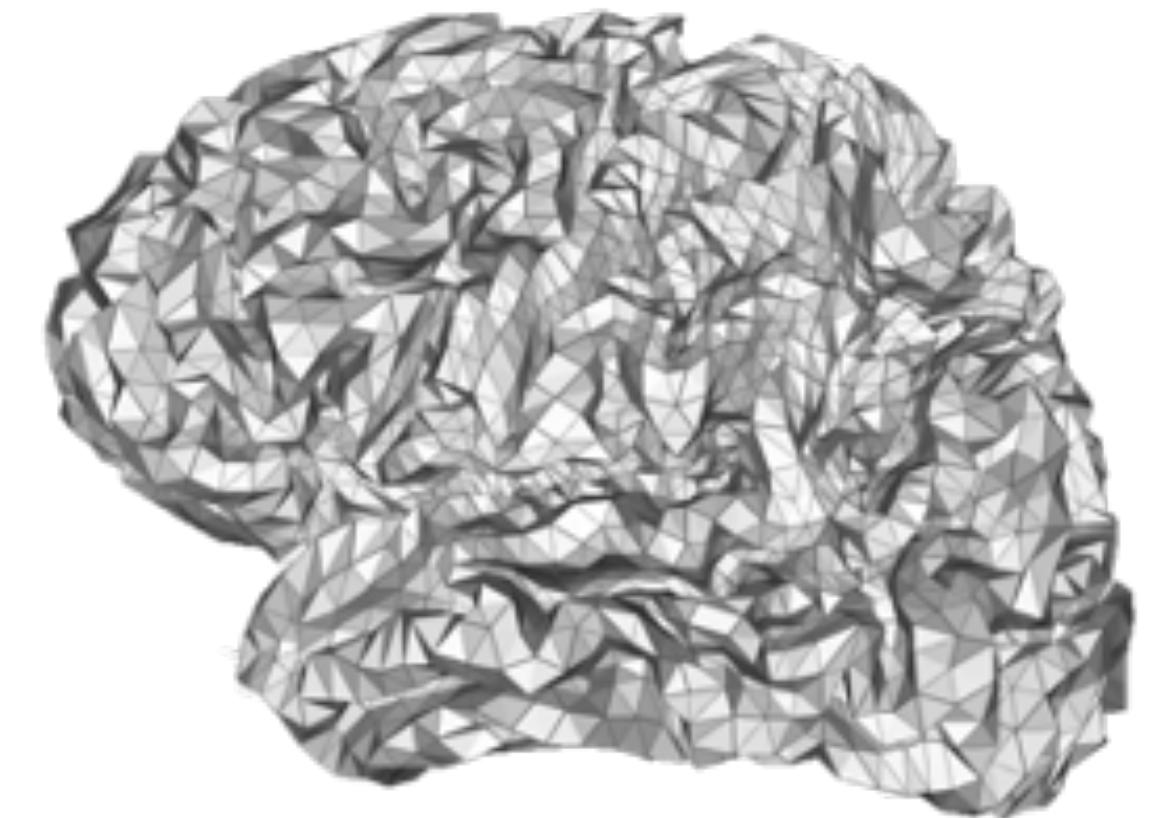
Below the equation, the prior distribution for the activation amplitudes is given as:

$$\beta_k \sim N(0, Q_k^{-1})$$

Spatial priors for cortical surface task fMRI

We need/want a spatial prior that:

- Have sparse inverse covariance
- Is applicable to triangular mesh data
- Has a Gaussian distribution



Stochastic partial differential equation (SPDE) priors (Lindgren et al. 2011)
check all the boxes!

SPDE Prior

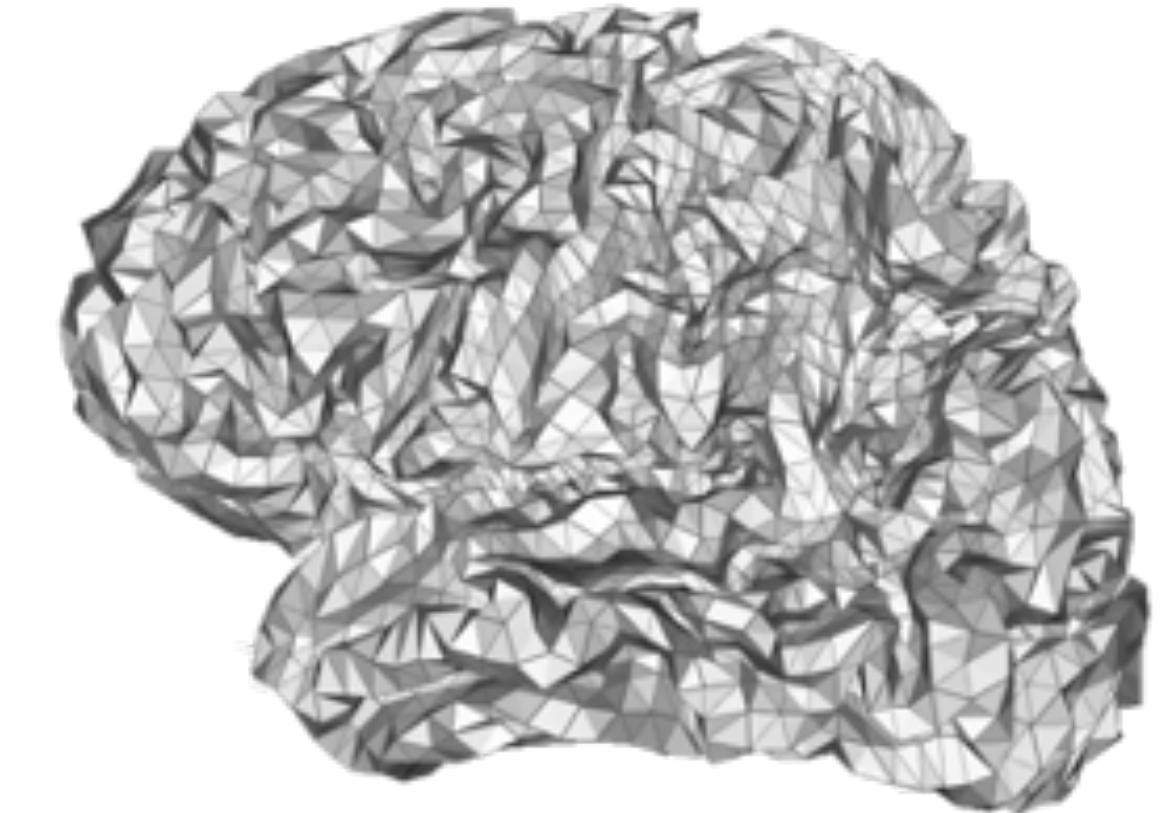
$$\beta \sim MVN(\mathbf{0}, \mathbf{Q}_{\kappa,\tau}^{-1})$$

$$\mathbf{Q}_{\kappa,\tau} = \tau^2(\kappa^4 \mathbf{F} + 2\kappa^2 \mathbf{G} + \mathbf{G}\mathbf{F}^{-1}\mathbf{G})$$



A fixed
diagonal
matrix

A fixed sparse matrix with
non-zero entries for
neighboring locations in mesh



κ controls the spatial correlation (must be estimated)

τ controls the variance (must be estimated)

Areas of Activation via Excursion Sets

Bolin, D., & Lindgren, F. (2015). Excursion and contour uncertainty regions for latent Gaussian models. *Journal of the Royal Statistical Society Series B: Statistical Methodology*.

Mejia, A. F., Yue, Y., Bolin, D., Lindgren, F., & Lindquist, M. A. (2020). A Bayesian general linear modeling approach to cortical surface fMRI data analysis. *Journal of the American Statistical Association*. 22

Areas of Activation via Excursion Sets

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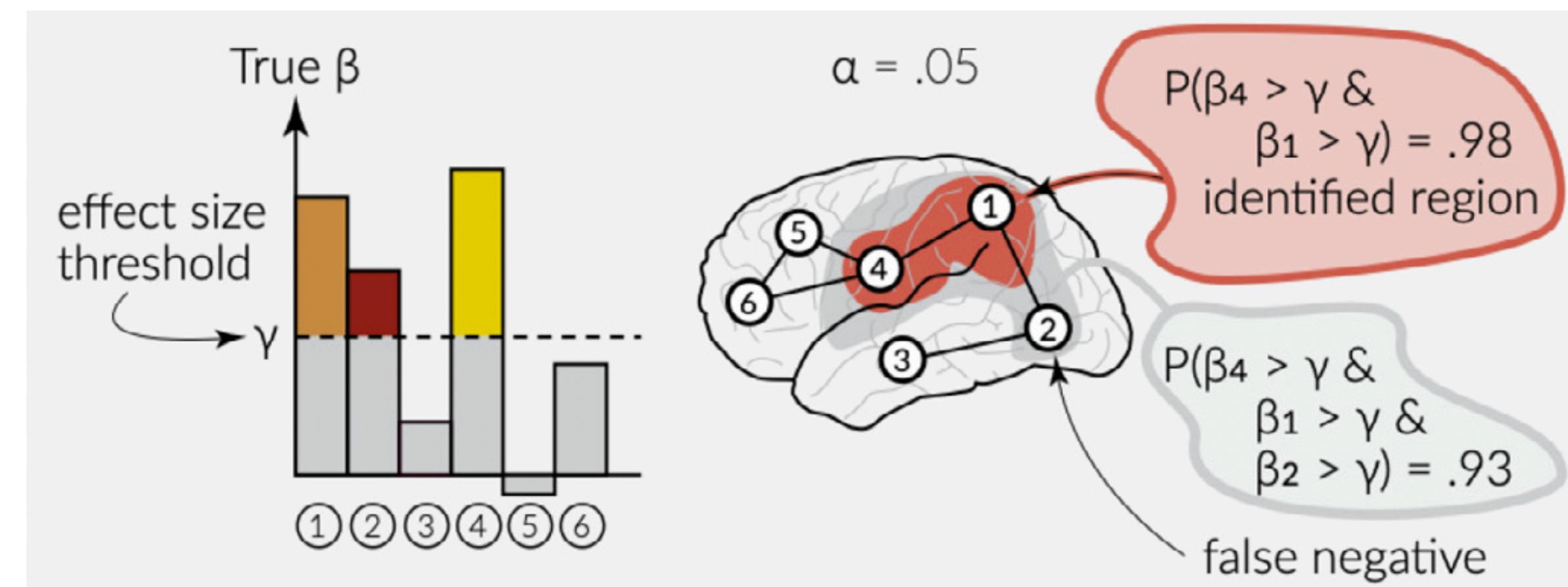
There is a ***dramatic gain in power*** over massive univariate modeling, facilitating the use of a scientifically meaningful minimum effect size

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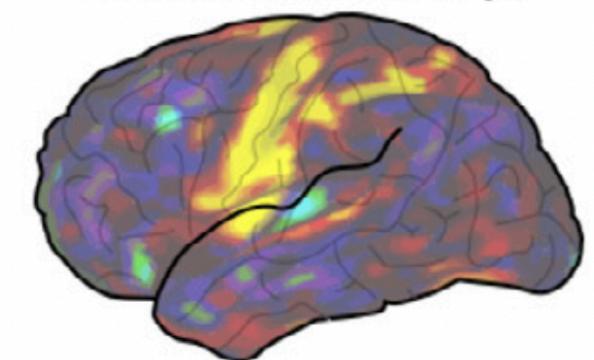
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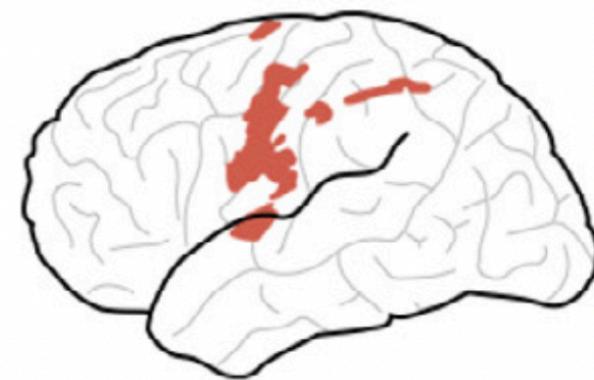


Excursions Set Procedure

$\hat{\beta}_{MAP}$ (based on posterior distribution of β)



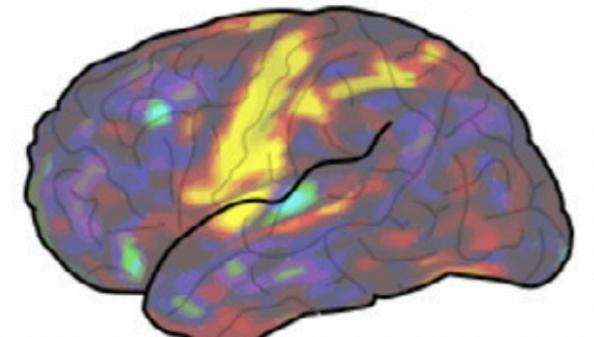
$\gamma = 0.5\%$ signal change



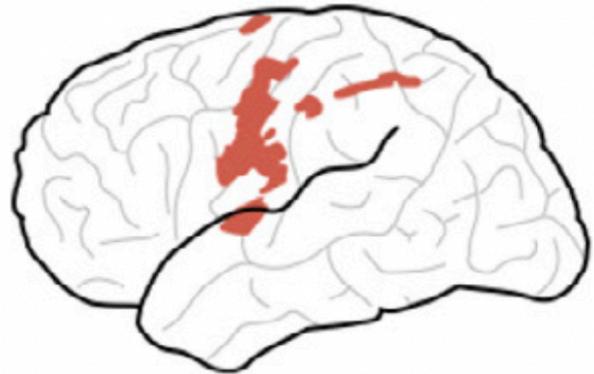
Excursions Set Procedure

- A location v is defined as *truly activated* by task k if $\hat{\beta}_k(v) > \gamma$

$\hat{\beta}_{MAP}$ (based on posterior distribution of β)



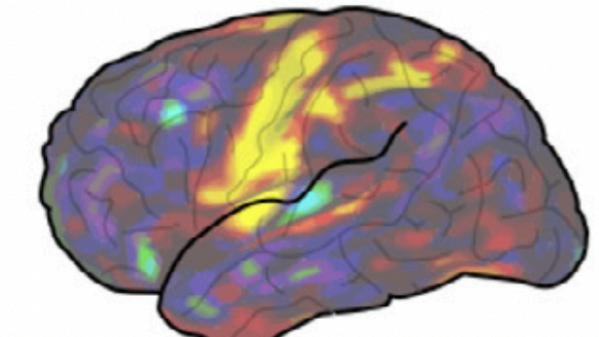
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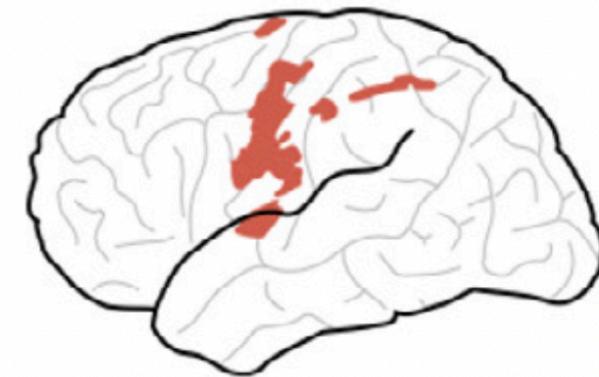
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- A location v is defined as *truly activated* by task k if $\beta_k(v) > \gamma$
- We aim to identify the *excursion region* $E_{\gamma,\alpha}$, the largest set of locations where with joint posterior probability $1 - \alpha$, every location is truly activated (above γ)

$\hat{\beta}_{MAP}$ (based on posterior distribution of β)

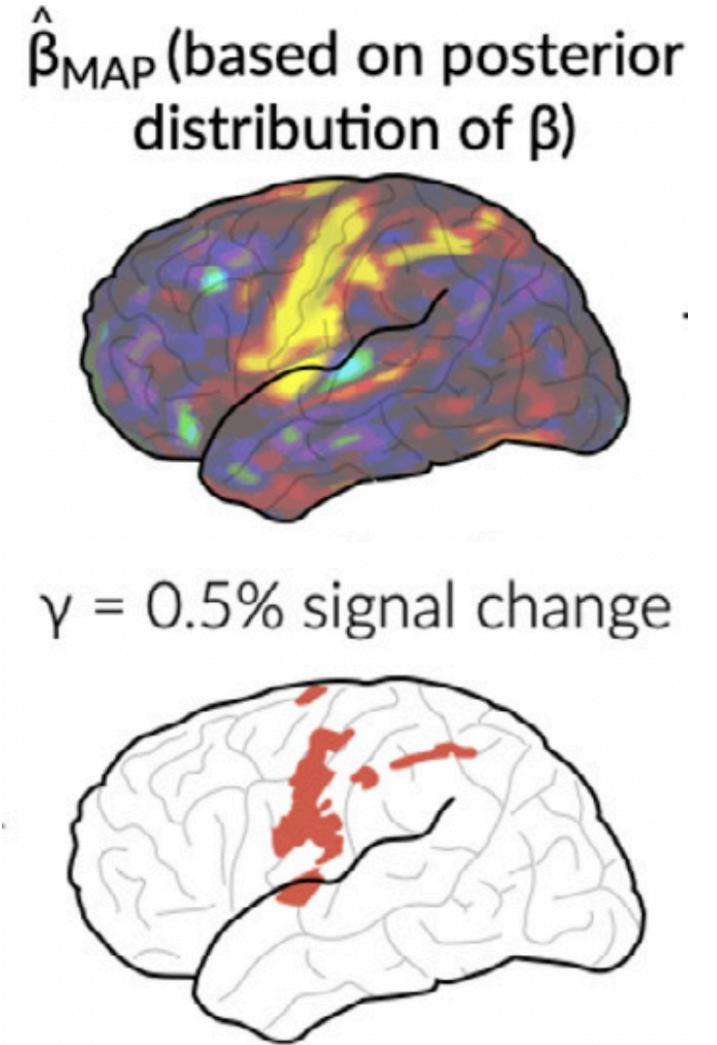


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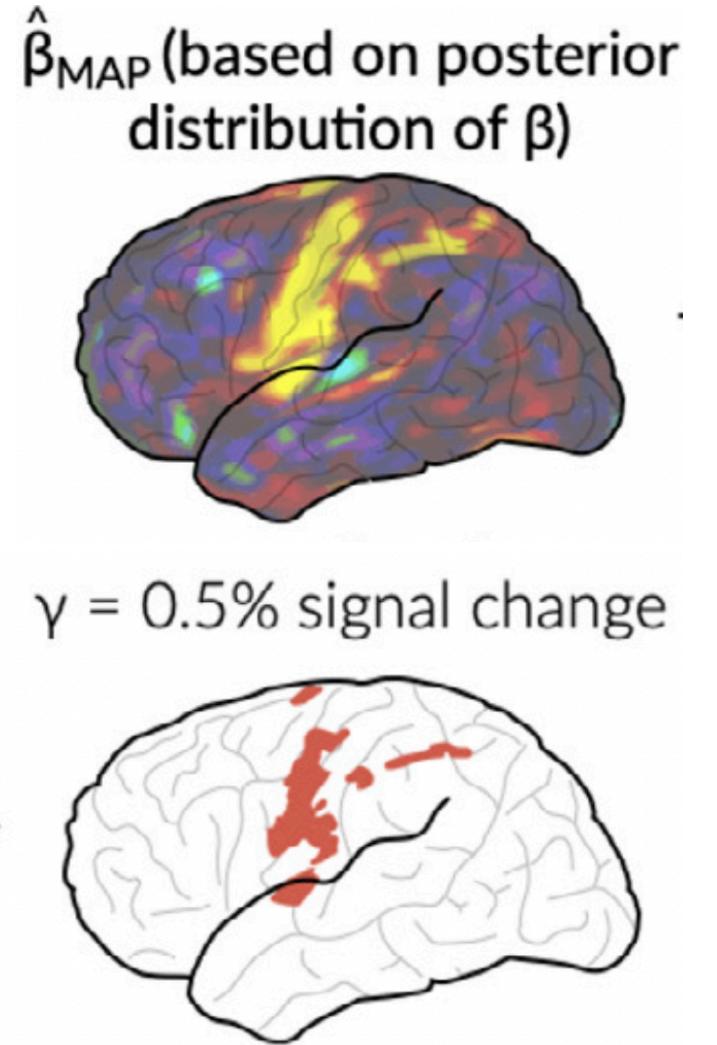
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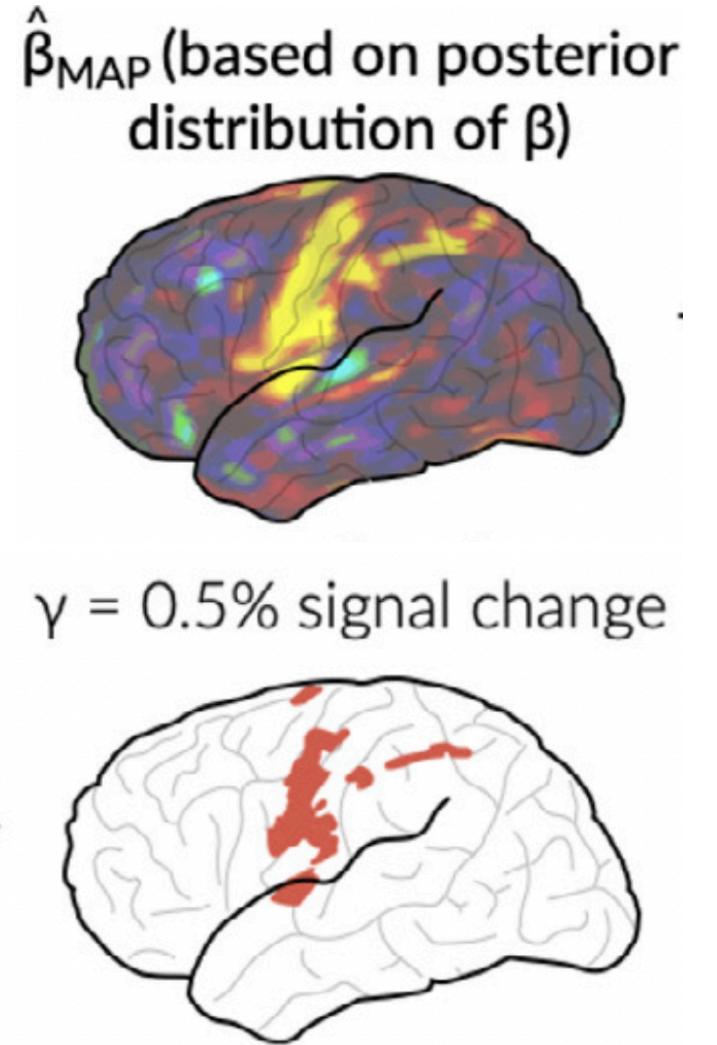
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- Thresholding the excursion function at, e.g., 0.95 yields the excursion region for $\alpha = 0.05$



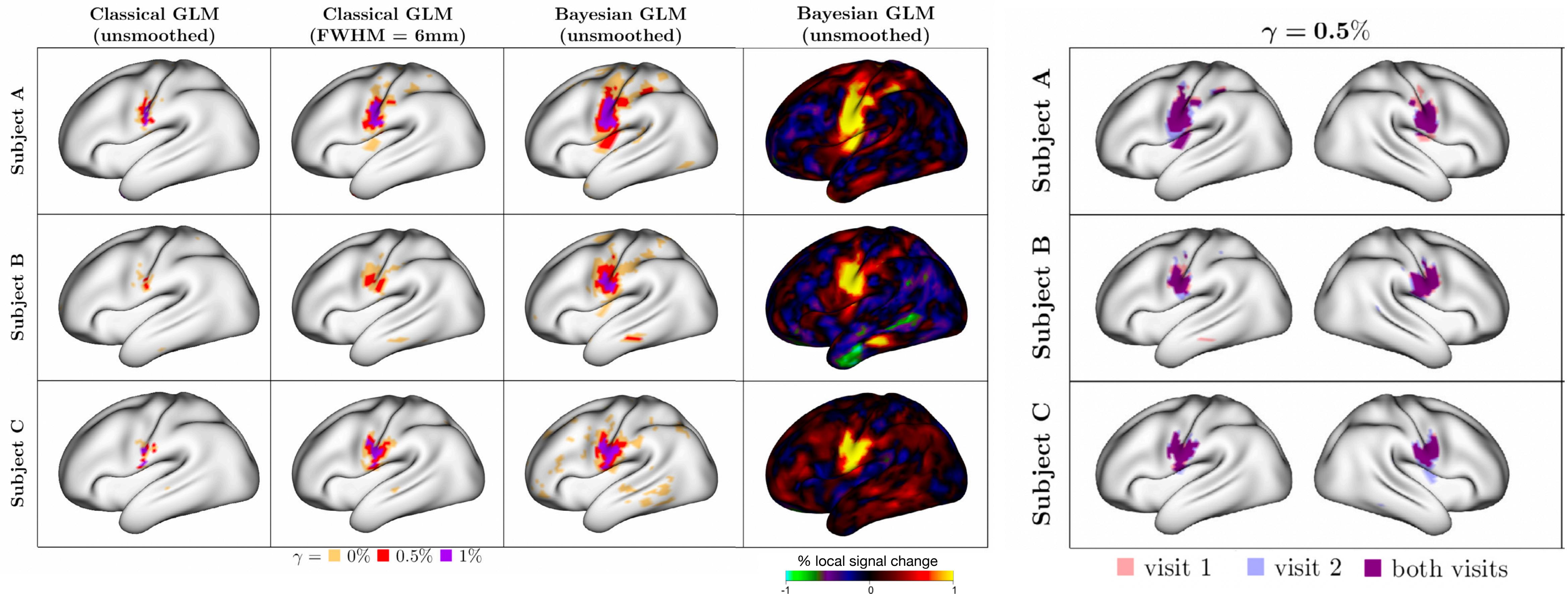
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- Thresholding the excursion function at, e.g., 0.95 yields the excursion region for $\alpha = 0.05$
- **This approach controls the vertex-wise FWER**, since the probability that one or more vertices in the set is *not* activated (i.e., there is at least one false positive) is α



Bayesian GLM is powerful in individuals

Tongue task activations in the motor task fMRI study for 3 individuals in the HCP.
The LR + RL runs for each session were analyzed together for each subject.

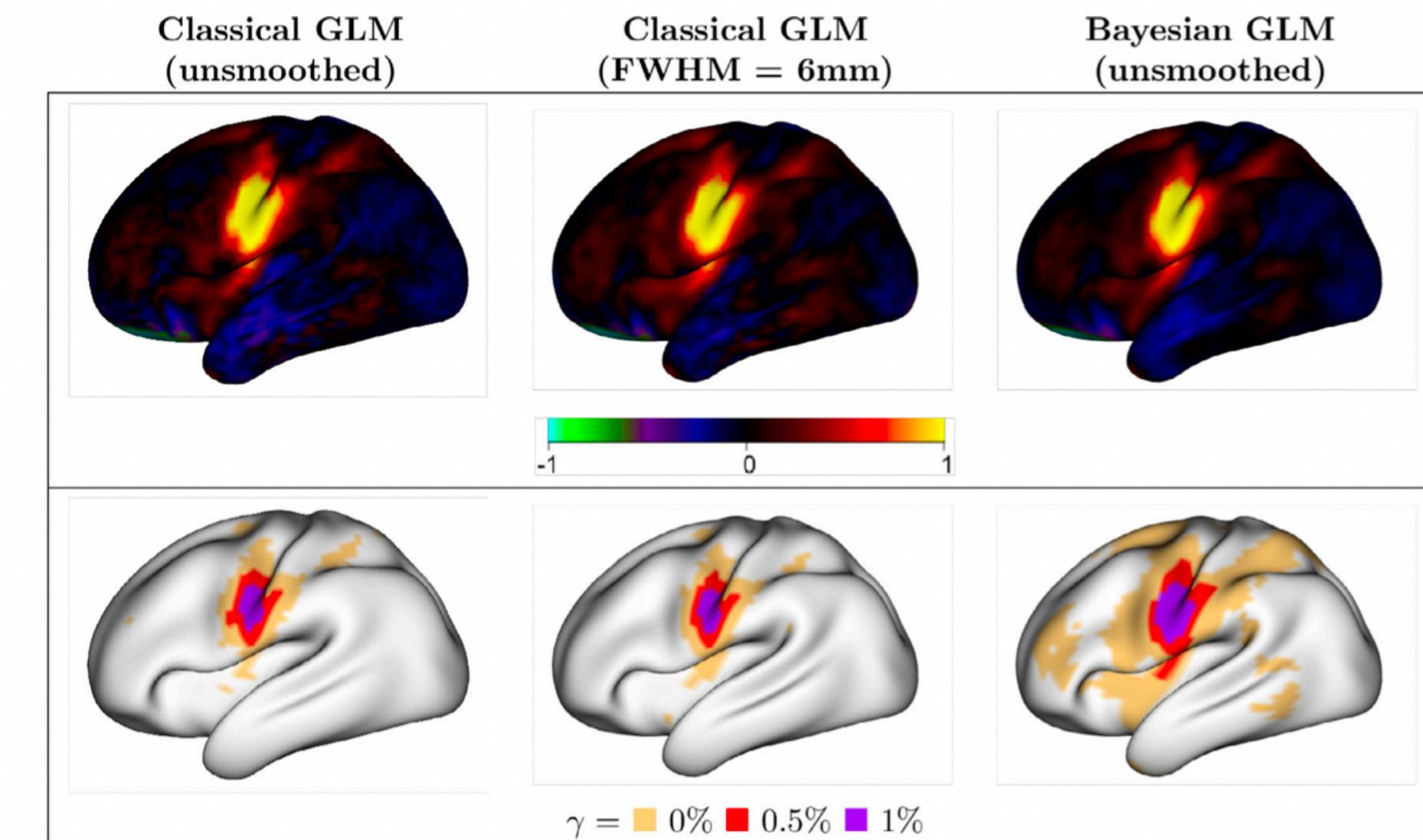


Bayesian Group Averages and Contrasts

We can analyze group averages and contrasts in a principled way after fitting all of the subject-level models, for **scalable & powerful** multi-subject analysis.

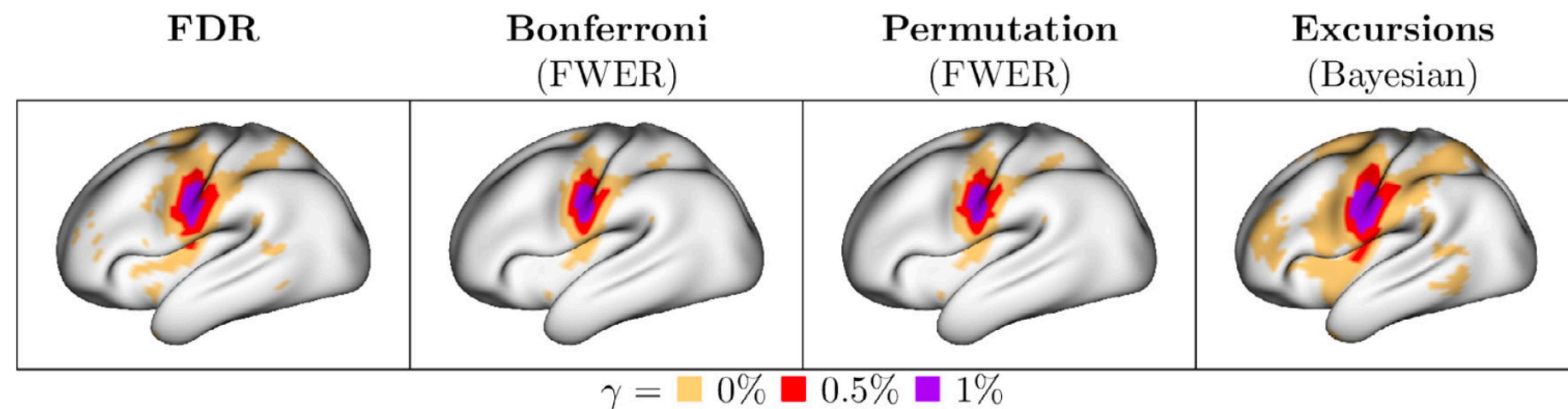
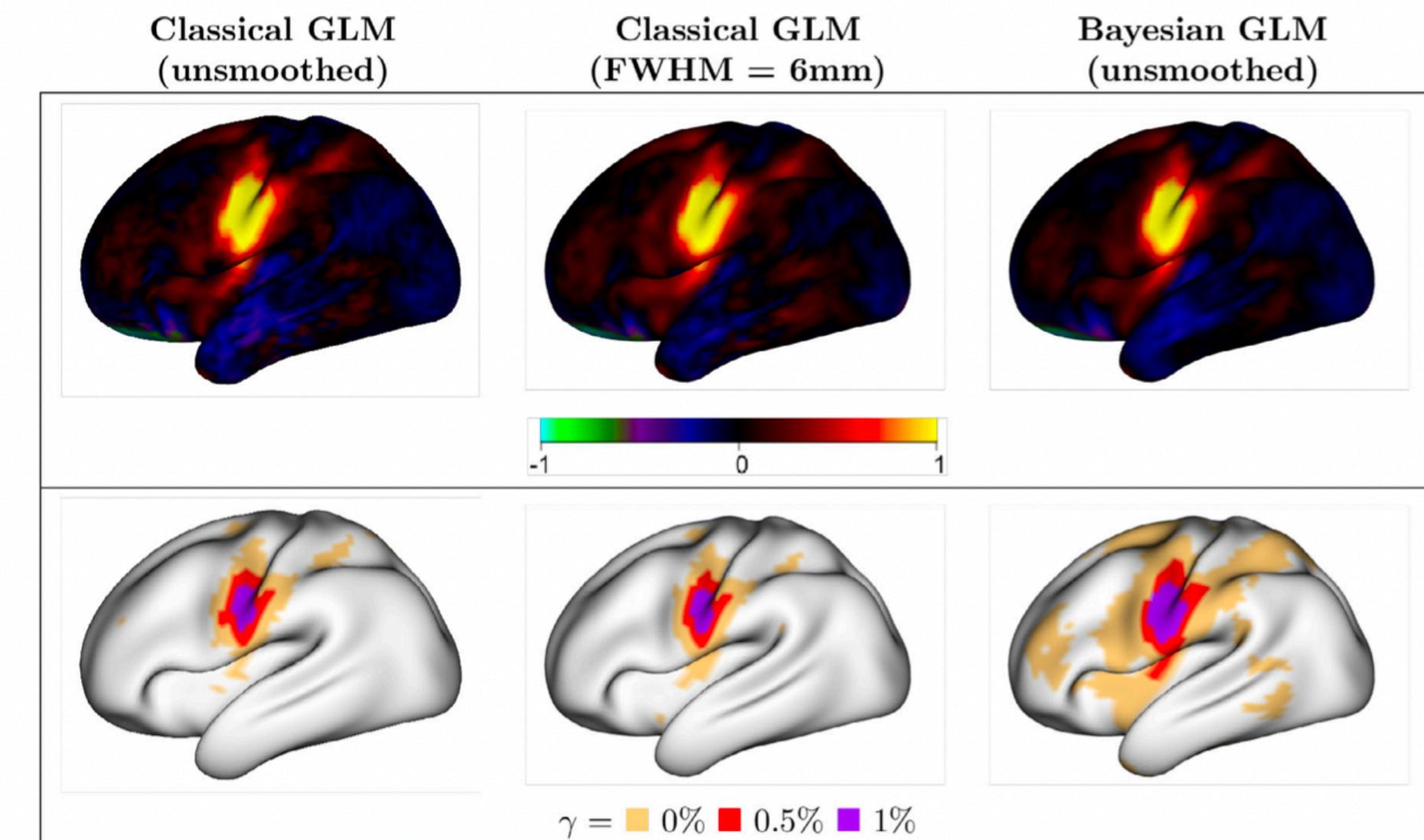
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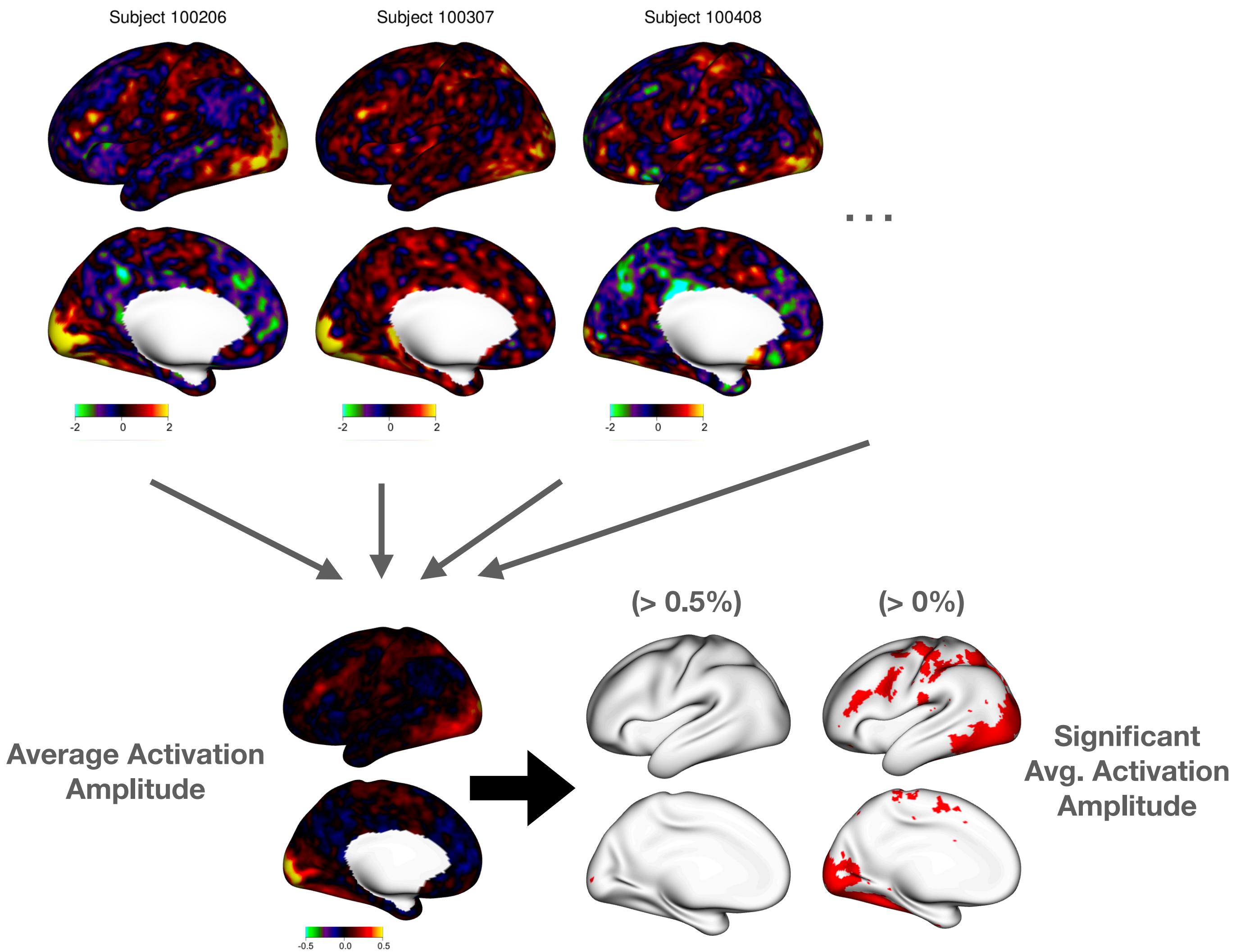


Bayesian Group Averages and Contrasts

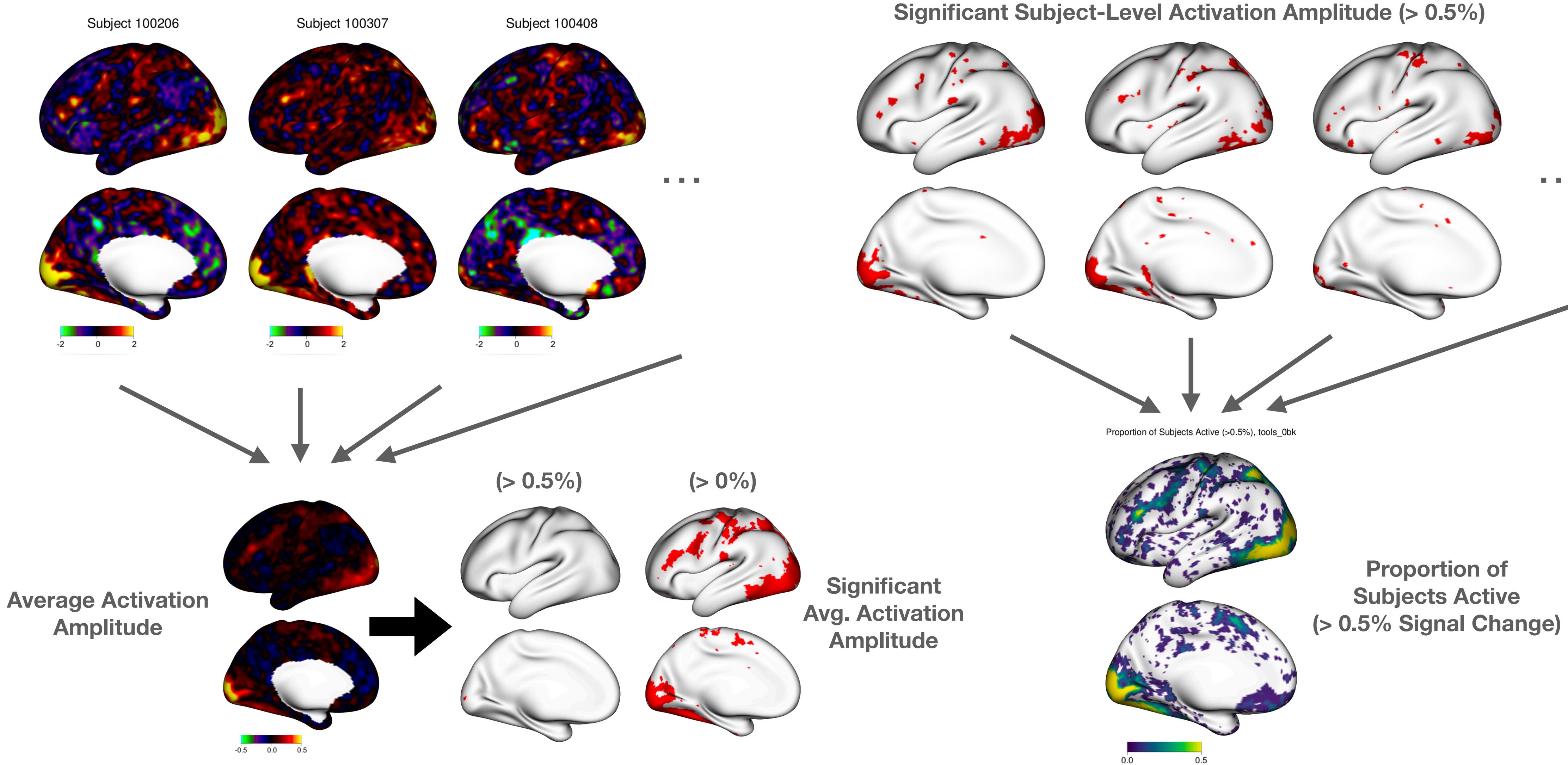
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From Averages to Proportions



From Averages to Proportions

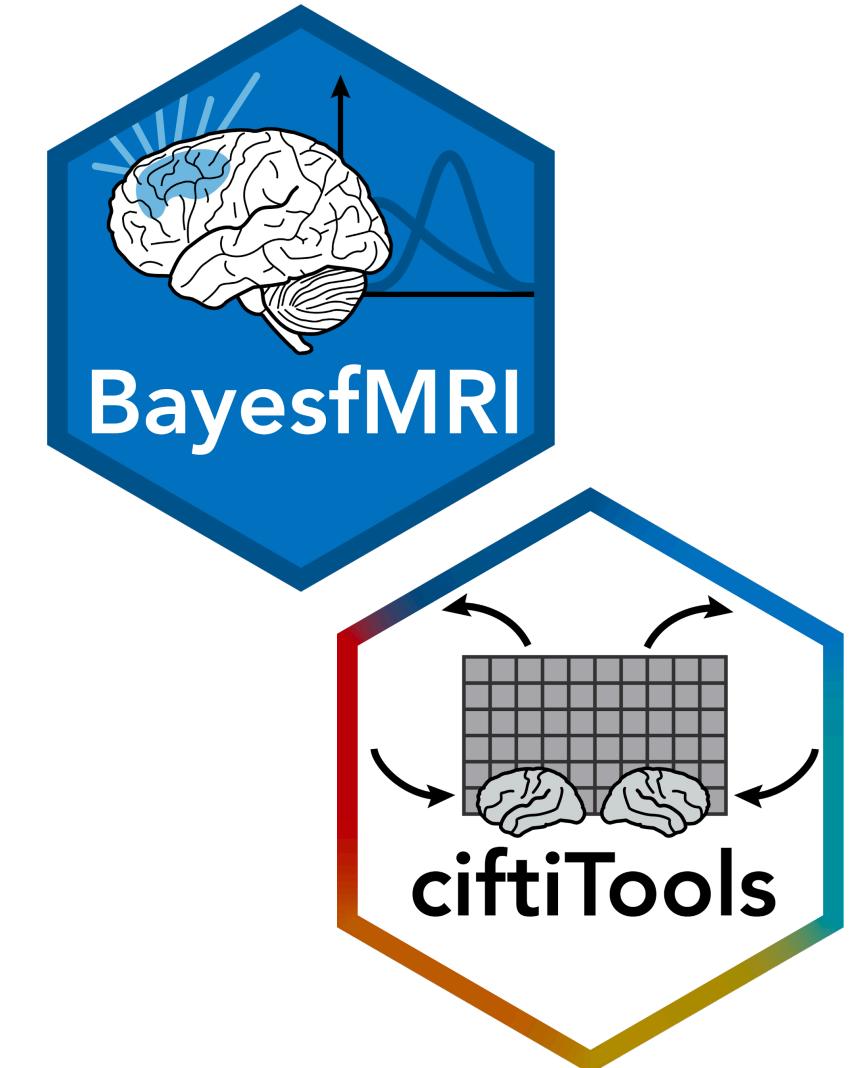


Demo in R

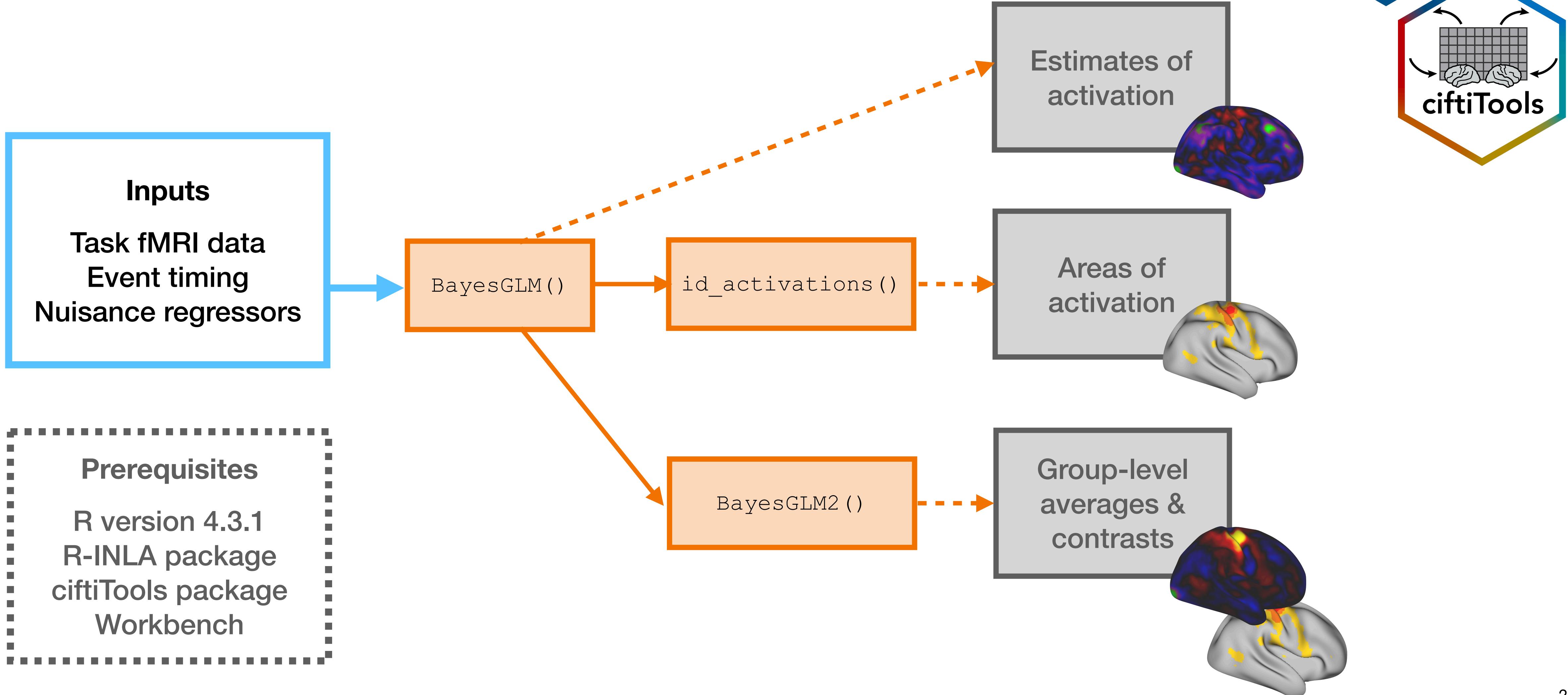
https://github.com/wdweeda/ohbm2023_edu_course/

The screenshot shows a GitHub repository page for 'wdweeda / ohbm2023_edu_course'. The 'Code' tab is selected. On the left, the file structure is shown with 'main' as the active branch. The 'Mejia_SpatialBayes' folder is expanded, showing three files: 'demo.R', 'prerequisites.md', and 'slides.pdf'. On the right, the contents of the 'Mejia_SpatialBayes' folder are listed, including a commit by 'mandymejia' and files: '...', 'demo.R', 'prerequisites.md', and 'slides.pdf'.

Name
...
demo.R
prerequisites.md
slides.pdf



BayesfMRI R Package



Thank you!



www.statmindlab.com

Collaborators:

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University of Utah

Twitter: @mandyfmejia
<https://github.com/mandymejia>

Funding:



Students and Trainees:

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Fatma Parlak, PhD (Poster 536)
Daniel Spencer, PhD