Reading Summary 4.1

Evan Hughes

March 2023

4.1 Polynomial Arithmetic and Division Algorithm

Defining the Polynomial

We must start with defining polynomials in a way that is an obvious extension of real-number coefficient polynomials.
Let R be any ring. A polynomial with coefficients in R is an expression of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad (1)$$

where n is a non negative integer and $a_i \in R$.
But what is x?

Theorem 4.1

If R is a ring, then there exists a ring T containing an element x that is not in R and has these properties:

1. R is a subring of T.
2. $xa = ax$ for all $a \in R$.
3. The set $R[x]$, read R add x, of all elements of T of from

   $$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \text{ (where } n \geq 0 \text{ and } a_i \in R)$$

   is a subring of T that contains R.
4. The representation of elements of $R[x]$ is unique: if $n \leq m$ and

   $$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m, \text{ then } a_i = b_i \text{ for } i = 1, 2, \cdots, n$$
   $$\text{and } b_1 = 0_R \text{ if and only if } a_i = 0_R.$$

5. $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ if and only if $a_i = 0$ for all i.

Proof of Theorem 4.1

The elements of the ring $R[x]$ in Theroem 4.1 are called polynomials with coefficients in R. and the elements a_i are called coefficients. The special element x is sometimes called an indeterminate. Note:

- Property 2 does not imply that the ring T is commutative, but only that the special element x commutes with each element of the subring R.
- Property 5 is the special case of property 4 when each $b_i = 0_r$
- The first expression in property 5 is not an equation to be solved for x. In this context, asking what value makes $a_0 + a_1x + \cdots + a_nx^n = 0_R$ is meaningless. This is because x is a specific element of a ring, not a variable.

Example(from the book)

Let E be the ring of even integers. Then $4 - 6x + 4x^3 \in E[x]$. However, the polynomial x is not in $E[x]$, because it cannot be written with even coefficients.

Polynomial Arithmetic

The rules for adding and multiplying polynomials follow directly from the fact that $R[x]$ is a ring.

Example

If $f(x) = 2+3x-4x^2$ and $g(x) = 3-2x+x^2$, then $f(x)+g(x) = 5+x-2x^2$ and $f(x)g(x) = 8-x-x^2$.

Theorem 4.2

If R is an integral domain and $f(x), g(x)$ are nonzero polynomials in $R[x]$, then the degree of $f(x)g(x) =$ the degree of $f(x)$ plus the degree of $g(x)$.

Proof of Theorem 4.2

Suppose $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ with $a_n \neq 0$ and $b_m \neq 0$. so that deg$f(x) = n$ and deg$g(x) = m$. then

$$f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots + a_nb_mx^{n+m}$$

The largest exponent of x that can possibly have a nonzero coefficient is $n + m$. But $a_nb_m \neq 0_R$ because R is an integral domain and $a_n \neq 0_R$ and $b_m = 0_R$. Therefore, $f(x)g(x)$ is nonzero and deg$(f(x)g(x)) \leq n + m \leq$deg$f(x)$+deg $g(x)$

Corollary 4.3

If R is an integral domain, then so is $R[x]$.

Proof of Corollary 4.3

Since R is a commutative ring with identity, so is $R(x)$. The proof of Theorem 4.2 shows that the product of nonzero polynomials in $R(x)$ is nonzero. Therefore, $R(x)$ is an integral domain.

Corollary 4.4

Let R be a ring. If $f(x), g(x)$, and $f(x)g(x)$ are nonzero in $R[x]$, then deg$[f(x)g(x)] \leq$deg$f(x)$+deg$g(x)$.

Corollary 4.5

Let R be an integral domain and $f(x) \in R[x]$. Then $f(x)$ is a unit in $R[x]$ if and only if $f(x)$ is a constant polynomial that is a unit in R. In particular, if F is a field, the units in $F[x]$ are the nonzero constants in F.

$$
\begin{array}{r|l}
 & \frac{1}{2}x + \frac{5}{2} \\
\hline
2x + 2 & x^2 + 6x + 9 \\
 & \underline{-\,x^2 + x} \\
 & \quad 5x + 9 \\
 & \underline{\quad -\,5x + 5} \\
 & \qquad\quad 4
\end{array}
$$

$$
\boxed{\begin{array}{c} a = bq + r \\ \deg(r) < \deg(b) \end{array}}
$$

Figure 1: Division Algorithm

The Division Algorithm in $F[x]$

Let F be a field and $f(x), g(x) \in F[x]$ with $g(x) \neq 0_F$. Then there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that

$$f(x) = g(x)q(x) + r(x) \text{ and either } r(x) = 0_F \text{ or } \deg r(x) < \deg g(x).$$