Reading Summary: Appendix C, D

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Appendix C Well Ordering and Induction

Well Ordering

The subset of nonnegative integers will be denoted by N. Thus

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Well Ordering Axiom *Every nonempty set of* $\mathbb{N}$ *contains a smallest a smallest element* This axiom might not hold if $\mathbb{N}$ is replaced with some other set of numbers. Well Ordering Axiom also creates the proof method known as **Induction**. Induction can be used to prove statements like

A set of n elements has 2^n subsets.

Denote this statement by the symbol $P(n)$ and see that there are infinitely many statements, one for each possible value of n:
$P(0)$: a set of 0 elements has $2^0 = 1$ subsets
$P(1)$: a set of 1 elements has $2^1 = 2$ subsets
$P(2)$: a set of 1 elements has $2^4 = 4$ subsets
$P(3)$: a set of 1 elements has $2^3 = 8$ subsets
Then you can prove

$$P(n) \text{ is a true statement for } n \in \mathbb{N}$$

Theorem C.1 The principle of Mathematical Induction

Assume that for each nonnegative Integer n, a statement $P(n)$ is given. If

1. $P(0)$ is a true statement; and
2. Whenever $P(k)$ is true, then $P(k+1)$ is also true

then $P(n)$ is a true statement for every $n \in \mathbb{N}$.
To use Induction you must make sure that your set satisfies both cases. Without satisfing both Induction does not work
Example: Proof that $\frac{n(n+1)}{2}$ is the sum of the first n nonnegative integers First you need to prove the base case $P(0)$

$$P(0) = \frac{0(0+1)}{2} = 0 \tag{1}$$

this holds true, the sum of the first 0 nonnegative integers is 0.
Next the inductive step, you need to prove that when $P(k)$ is true then $P(k+1)$ is also true

$$P(k) = \frac{k(k+1)}{2} \tag{2}$$

$$P(k+1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2} \tag{3}$$

then you need to prove that that is equal to $P(k) + (k+1)$

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2} = P(k+1) \tag{4}$$

this completes the induction because you have proven the when $P(k)$ is true then $P(k+1)$ is true. ■

Theorem C.2 The principle of Complete Induction

1. $P(0)$ is a true statement; and
2. Whenever $P(j)$ is true for all j such that $0 \leq j < t$, then $P(t)$ is also true

then $P(n)$ is a true statement for every $n \in \mathbb{N}$.
Example: (from the book) Proof of Theorem C.2
For each $n \in \mathbb{N}$, let $Q(n)$ be the statement

$$P(J) \text{ is true for all } j \text{ such that } O \leq j \leq n.$$

We will use Theorem C.1 to prove that $Q(n)$ is true for every $n \in \mathbb{N}$. Now $Q(O)$ is the statement

$$P(J) \text{ is true for all } j \text{ such that } 0 \leq j \leq 0.$$

This says, $Q(O)$ is just the statement "$P(O)$ is true". Assume that Q(k) is true, that is,

$$P(J) \text{ is true for all } j \text{ such that } O \leq j \leq k.$$

By hypothesis (2) (with $t = k + 1$), we conclude the $P(k + 1)$ is also true. Therefore, $P(J)$ is true for all j such that $O \leq j \leq k + 1$, that is, $Q(k + 1)$ is a true statement. Thus we have shown that whenever $Q(k)$ is true, then $Q(k + 1)$ is also true. By the Principle of Mathematical Induction, $Q(n)$ is true for every $n \in \mathbb{N}$, and the proof is complete. ■

Theorem C.3

Let r be a positive integer and assume that for each $n \geq r$ a statement $P(n)$ 1s given. If (i) $P(r)$ is a true statement; and either (ii) Whenever $k \geq r$ and $P(k)$ is true, then $P(k + 1)$ is true; or (ii') Whenever $P(j)$ is true for all j such that $r \leq j < t$, then $P(t)$ is true, then $P(n)$ is true for every $n \geq r$.

Theorem C.4

The following are equivalent

1. The Well Ordering Axiom
2. The principle of Mathematical Induction
3. The principle of Complete Induction

Appendix D Equivalence Relations

If A is a set, then any subset of $A * A$ is called a relation on A. A relation T on A is called an **equivalence relation** provided that the subset T is
(i) **Reflexive**: $a\ a \in T$ for every $a \in A$.
(ii) **Symmetric**: If $a\ b \in T$, then $b\ a$.
(iii) **Transitive**: If $a\ b \in T$ and $b\ c \in T$, then $a\ c$.