Homework 1

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1. Prove that $\sqrt{2}$ is not a rational number.

   (a) Write down a descripton of the rational numbers $\mathbb{Q}$ in set builder notation.

   $$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\} \tag{1}$$

   (b) Write down a definition of $\sqrt{2}$.

   $$\sqrt{2} = \frac{x}{y} \tag{2}$$

   (c) Use (a) and (b) to derive a contradiction proving that $\sqrt{2}$ cannot be rational.
   As written in (a) a rational number can be expressed as a ratio of two integers. To show that $\sqrt{2}$ is not rational we must show that it cannot be expressed as a ratio of two integers. We start by assuming $\sqrt{2}$ can be written as a ratio of two integers. If those integers share a common factor, then the fraction can be reduced to lowest terms by the Euclidean algorithm. Then $\sqrt{2}$ can be written as $\frac{a}{b}$ where a, b are coprime. b are two coprime integers, which by definition means one will be odd. This means that $\frac{a^2}{b^2} = 2 \to a^2 = 2b^2$. Therefore a^2 is even because it is 2 times an integer. Because a^2 is even, a must also be even. By the definition of even numbers a can be written as $2k$ where k is an integer. Substituting this into the equation $a^2 = 2b^2$ we get $(2k)^2 = 2b^2$ which is equivalent to $b^2 = 2k^2$. This means that b^2 is even because it is 2 times an integer. Because b^2 is even, b must also be even. Because both a and b are both even they are both divisible by 2 which contradicts the fact that they are coprime. Therefore $\sqrt{2}$ cannot be written as a ratio of two integers.

2. Use the Division Algorithm to prove that every odd integer is either in the form $4k+1$ or $4k+3$ for some $k \in \mathbb{Z}$. Let a be an odd integer, meaning $a = 2n+1$ for some $n \in \mathbb{Z}$. We will prove that a is either in the form $4k+1$ or $4k+3$ for some $k \in \mathbb{Z}$. Using the division algorithm to divide a by 4 we get $a = 4q + r$ where $q \in \mathbb{Z}$ and $r \in \{0, 1, 2, 3\}$.

   If $r = 0$ then $a = 4q$ meaning a is even because $a = 2(2q)$ which is 2 times and integer.

   If $r = 1$ then $a = 4q + 1$ meaning a is in the form $4k+1$ for some $k \in \mathbb{Z}$. $4k+1$ is odd because it is equivalent to $2(2k)+1$ which is 2 times and integer plus 1.

   If $r = 2$ then $a = 4q + 2$ meaning a is even because $a = 2(2q+1)$ which is 2 times and integer.

   If $r = 3$ then $a = 4q + 3$ meaning a is in the form $4k+3$ for some $k \in \mathbb{Z}$. $4k+3$ is odd because it is equivalent to $2(2k+1)+1$ which is 2 times and integer plus 1.

   Therefore any odd integer is in the form $4k+1$ or $4k+3$ for some $k \in \mathbb{Z}$.

3. Which of the following sets are nonempty? Explain your answer for each.

   (a) $A = \{r \in \mathbb{Q} \colon r^2 = 2\}$
   Is empty because $r^2 = 2 \to r = \sqrt{2}$ which is irrational and not in $\mathbb{Q}$.

   (b) $B = \{r \in \mathbb{R} \colon r^2 + 5r - 7 = 0\}$
   Is nonempty because $\frac{-5 \pm \sqrt{53}}{2} \in B$.

   (c) $C = \{t \in \mathbb{Z} \colon 6t^2 - t - 1 = 0\}$
   Is nonempty because $6t^2 - 5 - 1 = 0 has no integer solutions.$

4. Prove the Extended Division Algorithm.

If $a, b \in \mathbb{Z}$ and $b \neq 0$ then there exist a unique pair
$(q, r) \in \mathbb{Z}^2$ such that $a = bq + r$ and $0 \leq r < b$

5. Explain what is wrong with the following proof that reflexivity is unnecessary in the definition of an equivalence relation.

    Proof: Suppose $\sim$ is an equivalence relation on a nonempty set X and that $a, b \in X$. If $a \sim b$, then by symmetry we must have that $b \sim a$. Now, by transitivity, we have that $a \sim b$ and $b \sim a$ implies that $a \sim a$. Therefore, if $\sim$ is symmetric and transitive, then $\sim$ is reflexive.

    Then give an example of a set X and a relation R on it which is both symmetric and transitive, but not reflexive.

LaTeX Exercises

1. Construct the following displays.

$$1_A = \{ \begin{matrix} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{matrix} \tag{3}$$

$$\sqrt[3]{2} \tag{4}$$

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{5}$$

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{R} \text{ and } b \neq 0 \right\} / \sim \frac{a}{b} \sim \frac{c}{d} \iff ad - bc = 0 \tag{6}$$

2. Use the *ifthen* and *amsmath* packages to write a command called \ **piecewise** that will display a piecewise function.

```
\newcommand {\piecewise }[1][1]{ \ifthenelse {##1=1}{ f(x)=\begin {cases} f_1(x)|
\psi _1(x) \end {cases}}{} \ifthenelse {##1=2}{ f(x)=\begin {cases} f_1(x)| \psi
_1(x)\\ f_2(x)| \psi _2(x) \end {cases}}{} \ifthenelse {##1=3}{ f(x)=\begin {cases}
f_1(x)| \psi _1(x)\\ f_2(x)| \psi _2(x)\\ f_3(x)| \psi _3(x) \end {cases}}{} }
```

    Example: $f(x) = \begin{cases} f_1(x)|\psi_1(x) \\ f_2(x)|\psi_2(x) \end{cases}$