Reading Summary 4.5-4.6

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4.5 Irreduciblity in $\mathbb{Q}[x]$

If $f(x) \equiv [x]$, then $cf(x)$ has integer coefficients for some nonzero integer c.

Example(from the book)

$f(x) = x^5 + \frac{2}{3}x^4 + \frac{3}{4}x^3 - \frac{1}{6}$
The least common denominator of the coefficients is 12
Then $12f(x) = 12[x^5 + \frac{2}{3}x^4 + \frac{3}{4}x^3 - \frac{1}{6}] = 12x^5 + 8x^4 + 9x^3 - 2$

Theorem 4.21 Rational Root Test

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients. If $r \neq 0$ and the rational number r/s is a root of $f(x)$ then $r \mid a_0$ and $s \mid a_n$.

Lemma 4.22

Let $f(x), g(x), h(x) \in \mathbb{Z}[x]$ with $f(x) = g(x)h(x)$. If p is a prime that divides every coefficient of $f(x)$, then either p divides every coefficient of $g(x)$ or p divides every coefficient of $h(x)$.

Theorem 4.23

Let $f(x)$ be a polynomial with integer coefficients. Then $f(x)$ factors as a product of polynomials of degrees m and n in $\mathbb{Q}[x]$ if and only if $f(x)$ factors as a product of polynomials of degree m and n in $\mathbb{Z}[x]$.

Proof of Theorem 4.23

Obviously, if $f(x)$ factors in $\mathbb{Z}[x]$, it factors in $\mathbb{Q}[x]$. Conversely, suppose $f(x) = g(x)l(x)$ in $\mathbb{Q}[x]$. Let c and d be nonzero integers such that $cg(x)$ and $dh(x)$ have integer coefficients. Then $cdf(x) = [cg(x)dh(x)]$ in $\mathbb{Z}[x]$ with deg $cg(x)$ =deg $g(x)$and deg $dh(x)$ = deg $h(x)$. Let p be any prime divisor of cd. Then p divides every coefficient of the polynomial $cdf(x)$. By Lemma 4.22, p divides either every coefficient of $cg(x)$ or every coefficient of $dh(x)$. Then $cg(x) = pk(x)$ with $k(x) \in \mathbb{Z}[x]$ and deg $k(x)$ =deg $g(x)$. Therefore, $ptf(x) = cdf(x) = [cg(x)][dh(x)] = [pk(x)][dh(x)]$. Canceling p on each end, we have $tf(x) = k(x)[chh(x)] \in \mathbb{Z}[x]$.

Section 4.6 Irreduciblity in $\mathbb{R}[x]$ and $\mathbb{C}[x]$

Theorem 4.26

Every nonconstant polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$.
THis theorem is also stated as $\mathbb{C}$ is algebraically closed.

Corollary 4.27

A polynomial is irreducible in $\mathbb{C}[x]$ if and only if it has degree 1.

Proof

A polynomial $f(x)$ of degree ≥ 2 in $\mathbb{C}[x]$ hence a first degree factor by the Factor Theorem. Therefore $f(x)$ is reducible in $\mathbb{C}[x]$. And every irreducible polynomial in $\mathbb{C}[x]$ has degree 1.

Eisenstein's Criterion

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a non constant polynomial with integer coefficients. If there is a prime p such that it divides each coefficient of $f(x)$ and p does not divide a_n and p^2 does not divide a_0, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Example

The polynomial $x^9 + 5$ is irreducible in $\mathbb{Q}[x]$ with Eisenstein's Criterion $p = 5$.