Reading Summary 4.4

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4.4 Polynomial Functions, Roots, and Zeros

Throughout this section R is a commutative ring. Associated with each polynomial $a_n x^n + \cdots + a_1 x + a_0$ in $R[x]$ is a function $f : R \to R$ whose rule is $f(r) = a_n r^n + \cdots + a_1 r + a_0$ for each $r \in R$. This is called a polynomial function.

Example 1(from the book)

The polynomial $x^2 + 5x + 3 \in R[x]$ induces the function $f : R \to \mathbb{R}$ whose rule is $f(r) = r^2 + 5r + 3$ for each $r \in \mathbb{R}$.

Example 2

The polynomial $x^3 + 2x + 1 \in \mathbb{Z}_3[x]$ induces the function $f : \mathbb{Z}_3 \to \mathbb{Z}_3$ whose rule is $f(r) = r^3 + 2r + 1$ for each $r \in \mathbb{Z}_3$. Thus $f(0) = 1$, $f(1) = 0$, $f(2) = 2$

Definition of Roots

Let R be a commutative ring and $f(x) \in R[x]$. An element a of R is said to be a root of the polynomial $f(x)$ if $f(a) = 0_R$, that is, if the induced function $f : R \to R$ maps a to 0_R.

Theorem 4.15 The remainder theorem

Let F be a field, $f(x) \in F[x]$, and $a \in F$. The remainder when $f(x)$ is divided by the polynomial $x - a$ is $f(a)$.

Proof

By the Division algorithm $f(x) = (x - a)q(x) + r(x)$ where the $r(x)$ either is 0_f or has smaller degree than the divisor $x - a$. Thus degree $r(x) = 0$ or $r(x) = 0_f$, In either case, $r(x) = c$ for some $c \in F$. Hence, $f(x) = (x - a)q(x) + c$ so that $f(a) = (a - a)q(a) + c = 0_f + c = c$.

Theorem 4.16 The Factor Theorem

Let F be a field, $f(x) \in F[x]$, and $a \in F$. Then a is a root of the polynomial $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

Proof

First assume that a is a root of $f(x)$. Then we have

- $f(x) = (x - a)q(x) + r(x)$
- $f(x) = (x - a)q(x) + f(a)$
- $f(x) = (x - a)q(x)$

Therefore, $x - a$ is a factor of $f(x)$. Conversely, assume $x - a$ is a factor of $f(x)$, say $f(x) = (x - a)g(x)$. Then a is a root of $f(x)$ because $f(a) = (a - a)g(a) = 0_f g(x)$.

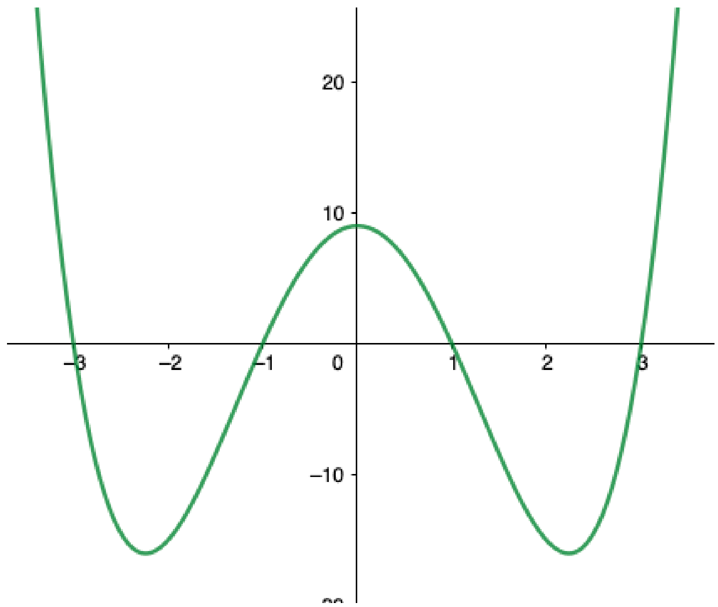

Figure 1: Zeros in polynomials

Corollary 4.18

Let F be a field, $f(x) \in F[x]$, and deg $f(x) \geq 2$. If $f(x)$ is irreducible in $F[x]$ then $f(x)$ has no roots in F.

Proof

If $f(x)$ is irreducible, then it has no factor in the form $x - a$ in $F[x]$. Therefore it has no roots.

Corollary 4.20

Let F be an infinite field and $f(x), g(x) \in F[x]$. Then $f(x)$ and $g(x)$ induce the same function from F to F if and only if $f(x) = g(x)$ in $F[x]$.

Proof

Suppose that $f(x)$ and $g(x)$ induce the same function from F to F. Then $f(a) = g(a)$ so that $f(a) - g(a) = 0_F$. this means that every element of F is a root of the polynomial $f(x) - g(x)$. Since F is infinite, this is impossible by Corollary 4.17 unless $f(x) - g(x)$ is the zero polynomial, that is, $f(x) = g(x)$.