Stochastic Analysis Cheatsheet All in ONE

Hoyori@github/冷泉望海遥@知乎

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1.1 Gaussian Distribution

1.1.1 Univariate Gaussian Distribution

Def. Univariate Gaussian Distribution. X satisfies Univariate Gaussian Distribution, if CDF $F_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}\exp\{-\frac{(y-\mu)^2}{2\sigma^2}\}dy$, denote as $X \sim N(\mu,\sigma^2)$

Property.

(1) Closed under linear transformation: If $X \sim N(\mu,\sigma^2)$, (a) $aX+b \sim N(a\mu+b, a^2\sigma^2)$, (b) if $Y \sim N(0,1)$, then $X \stackrel{d.}{=} \sigma Y + \mu$

(2) Sum of independent Gaussian random variables is still Gaussian: if $Y \sim N(\mu_1,\sigma_1^2)$, $Z \sim N(\mu_2,\sigma_2^2)$, $Y \perp Z$ (independent), then $Y + Z \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

(3) Central Limit Theorem: $\{X_n\}_{n=1}^{\infty}, iid, \mathbb{E}[X_1] = \mu, Var(X_1) = \sigma^2, \mathbb{E}[X^2] < \infty$, define $S_n := X_1 + .. + X_n$, then

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \overset{d.}{\sim} N(0,1), n \to \infty$$

1.1.2 Convergence Types

Def. Weakly Convergence. $\{X_n\}_{n=1}^{\infty}$ converges to X in distribution, or $\{X_n\}_{n=1}^{\infty}$ weakly converges to X, if $\forall f$ continuous and bounded, $\mathbb{E}[f(X_n)] \to [f(X)]$

Properties. Equivalent Statement of Convergence.

(1) $X_n \overset{d.}{\to} X \Leftrightarrow F_{X_n}(x) \to F_X(x)$ for $\mathbb{P}(X = x) = 0$ (X is continuous at x)

(2) If all X_n admits densities as in (1), then $f_{X_n}(x) \overset{a.s.}{\to} f_X(x) \Leftrightarrow X_n \overset{d.}{\to} X$

(3) Characteristic function of the distribution (Fourier Transformation of CDF) : $\mathcal{L}_n(t) = \mathbb{E}(e^{itX_n}) \to \mathcal{L}(t) = \mathbb{E}(e^{itX})$

1.1.3 Multivariate Gaussian Distribution

Def. Joint Distribution. On probability space $(\Omega, \mathcal{F}, \mathbb{P})$, denote X as a vector of random variable, $X : \Omega \to \mathcal{B}(\mathbb{R}^n)$,

(1) $\mathbb{P}_X(B) = \mathbb{P}(X \in B) = \mathbb{P}(\{\omega : X(\omega) \subseteq B \in \mathcal{B}(\mathbb{R}^n)\}) = \mathbb{P}((X_1, .., X_n) \in B)$

(2) $F_X(x) = F_X(x_1, .., x_n) = \mathbb{P}(X_1 \leq x_1, .., X_n \leq x_n)$

(3) PDF (if exists) $F_X(x) = \int_{-\infty}^{x_n} .. \int_{-\infty}^{x_1} f_X(y_1, .., y_n) dy_1..dy_n$

Def. Multivariate Gaussian Distribution. $X = (X_1, .., X_n)$ is standard Gaussian if $X_i \overset{iid}{\sim} N(0,1)$

Def. Gaussian Dist. Random variable $X \in \mathbb{R}^n$ is Gaussian if it is linear transformation of the standard Gaussian random variables $X = AZ + b$, where A is a matrix and b is a vector, Z is standard Gaussian. Specifically, $X_i = \sum_{j=1}^{n} A_{i,j} Z_j + b_i$.

Specially, $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} z = (z, z)$, where $z \sim N(0,1)$, X is Gaussian but not standard Gaussian since not iid.

Def. Covariance Matrix. Covariance matrix for $y = [y_1, .., y_n]^\top$ is

$$\Sigma = \mathbb{E}[(y - \mathbb{E}(y))(y - \mathbb{E}(y))^\top], \Sigma_{i,j} = \mathbb{E}[(y_i - \mathbb{E}(y_i))(y_j - \mathbb{E}(y_j))]$$

Theorem. Gaussian Determiend by Cov. For any $b \in \mathbb{R}^n$, and symmetric positive-semi-definite matrix Σ, there exists unique $\exists!$ Gaussian distribution with mean b and covariance Σ, denoted as $N(b, \Sigma)$

Theorem. Cramer-Wold Device. The distribution of a random vector $X \in \mathbb{R}^n$ is completely determined by the set of all one-dimensional distributions of linear combination of the form $t^\top X$, where $t \in \mathbb{R}^n$ as

$$y = t^\top X, \Phi_y(s) = \mathbb{E}(e^{isy}) = \mathbb{E}(e^{ist^\top X})$$

, which is similar to moment generating function/characteristic function, and uniquely determined the distribution of y

Remark: use the Device to prove that Gaussian distribution can be determined by b, Σ

Remark. PDF of Multivariate Gaussian. if Σ is invertible, then the PDF w.r.t. Lebesgue measure exists, denote as $f_X(x) = \frac{1}{(2\pi)^{n/2}\sqrt{|\Sigma|}} \exp(-\frac{1}{2}(X - \mu)^\top \Sigma^{-1}(X - \mu))$

Property.

(1) Invariant (distribution) under linear transformation, $C_{n\times n} X_{n\times 1} + b_{n\times 1} \sim N(C \cdot \mu + b, C\Sigma C^\top)$, where $X \sim N(\mu, \Sigma)$

(2) Any linear transformation to 1-D random variable from Gaussian r.v. is still Gaussian, a r.v. X is Gaussian iff for any λ, $\lambda_1 X_1 + .. + \lambda_i X_i$ is Gaussian

(3) Limit (if exists) of Gaussiance r.v. is Gaussian, as stated in **Theorem. Cvg in d.** below

(4) Pair (X, Y) is joint Gaussian (must be joint Gaussian, otherwise, statement fails), and $Cov(X, Y) = 0$, then X, Y are independent. *CounterExample* $X \sim N(0,1), Y = WX, W \perp X, \mathbb{P}(W = 1) = \frac{1}{2} = \mathbb{P}(W = -1)$, then $Y \sim N(0,1)$, but (X, Y) is not Gaussian, so even $Cov(X, Y) = Cov(X, WX) = \mathbb{E}(WX^2) = 0$, X, Y are not independent.

(5) Cauchy Schwarz Inequality, $Cor(x,y) \in [-1,1]$
Theorem. Convergence to Gaussian $\{X^n : X^n = (X_1^n, .., X_m^n)\}_{n=1}^{\infty}$ is a Gaussian sequence, if there exists such X such that $X_n \overset{d.}{\sim} X$, then X is Gaussian.
Remark: Above theorem implies that the sequence of distribution is tight, i.e. if sequence converges, then there is a further sub-sequence that converges, or i.e., $\forall \varepsilon, \exists$ a compact set K such that $\mathbb{P}(X_n \in K) > 1-\varepsilon$
Remark. $\{X_n\}$ Gaussian, and is tight, iff, $\sup_n(|\mu_n| + \sigma_n^2) < \infty$ with mean+variance bounded uniformly

1.2 Stochastic Process

1.2.1 Distribution of Stochastic Process

Def. Modification of a Stochastic Process $X = (X_t)_{t\geq 0}, Y = (Y_t)_{t\geq 0}$, then Y is a "modification" of X if $\mathbb{P}(X_t = Y_t) = 1, \forall t$
Def. Indistinguishable of two Stochastic Processes $X = (X_t)_{t\geq 0}, Y = (Y_t)_{t\geq 0}$, then X and Y are indistinguishable if $\mathbb{P}(X_t = Y_t, \forall t) = 1$
Remark. indistinguishable (strong) $\Rightarrow$ modification (weak), modification $\nRightarrow$ indistinguishable
Remark. Uncountable set can be reduced to countable dense set by continuity, for easier proof [Ref:Stochastic Processes, Indistinguishability and Modifications
Theorem. If X, Y have RCLL (right-continuous and left-limit) paths, then modification $\Leftrightarrow$ indistinguishable
Theorem. Kolmogorov's Extension Theorem The distribution of stochastic processes is uniquely determined by its finite dimensional distribution, i.e. if X, Y have the same finite dimensional distributions, then

$$\mathbb{E}[F(X_t)_{t\geq 0}] = \mathbb{E}[F(Y_t)_{t\geq 0}]$$

for any function F that is measurable.
Remark. If we know the distribution of $X_{t_1}, X_{t_2}, .., X_{t_n}$, then we know the unique distribution of $X := (X_{t_1}, X_{t_2}, .., X_{t_n})$, then we know $(X_{t_1}, X_{t_2}, .., X_{t_n}) \overset{d.}{\sim} (Y_{t_1}, Y_{t_2}, .., Y_{t_n})\ \forall t_i$, so X, Y have the same distribution

1.2.2 Gaussian Processes

Def. Gaussian Processes A processes is Gaussian if its finite dimensional distribution are Gaussian, with
mean function $\mu(t) = \mathbb{E}(X_t), t \geq 0$
covariance function $Cov(t,s) = \mathbb{E}[(X_t - \mu(t))(X_s - \mu(s))]$, where covariance matrix is symmetric and positive semi-definite function.
Theorem. Kolmogorov's Consistency Condition. Add new r.v. X_{t+1} to finite dimensional r.v. $(X_1, .., X_t)$, then CDF $F(X_1, .., X_{t+1})$ is just marginal CDF of $(X_1, .., X_t)$
Property. Operators. X is Gaussian with mean $\mu(t)$ and covariance $C(t,s)$, then
(1) Stability under linear transformation. If $Y_t = a(t)X_t + b(t)$, where $a(t), b(t)$ are deterministic function, then Y_t is Gaussian with mean $a(t)\mu(t) + b(t)$ and covariance $a(t)C(t,s)a(s)$
(2) Stability w.r.t. time change/different speed: $Z_t = X_{f(t)}$, where $f(t)$ is deterministic function, then Z_t is Gaussian with mean $\mu(f(t))$ and covariance $C(f(t), f(s))$
(3) Any linear combination of its value is Gaussian, $\lambda_1 X_{t_1} + \lambda_2 X_{t_2} + .. + \lambda_n X_{t_n} \sim N(\sum_{i=1}^n \lambda_i \mu(t_i), \sum_{i,j} \lambda_i \lambda_j C(t_i, t_j))$

1.3 Brownian Motion

1.3.1 Brownian Motion

Def. Brownian Motion A Gaussian process $\{B_t\}_{t\geq 0}$ is a Brownian Motion, if (1) $\mathbb{E}(B_t) = 0$ (2) $Cov(B_t, B_s) = t \wedge s = \min(t,s)$ (3) its paths are continuous
Remark. $C(t,s) = t \wedge s$ is positive definite

Theorem. Kolmogorov's Continuity Theorem. X is a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$, if $\mathbb{E}[|X_t - X_s|^\alpha] \leq c(t-s)^{1+\beta}, \forall 0 \leq s \leq t \leq T$, where $\alpha, \beta, c \geq 0$, then X is almost surely Hölder's continuous, i.e. $|X_t - X_s| \leq c|t-s|^\gamma$ where the order $\gamma \in [0, \frac{\beta}{\alpha})$
Remark. For a process B that satisfies $\mathbb{E}[|B_t - B_s]^{2n} = C_n|t-s|^n$, so B is almost surely Hölder's continuous of all orders strictly less than $\frac{1}{2}$. Further, B has a continuous modification (making path continuous) to be Brownian Motion.

1.3.2 Properties

Property. Operators. Inherit all properties from Gaussian Process, i.e. $(B_{t_1}, .., B_{t_n}) \sim N(0, \Sigma)$ where $\Sigma(t_i, t_j) = t_i \wedge t_j$
Property. Time Integral.
(1) Expectation

$$\mathbb{E}[\int_0^t B_u du] = \mathbb{E}[\lim_{n\to\infty} \sum_{i=1}^n \frac{t}{n} B_{t_i/n}] = \lim_{n\to\infty} \sum_{i=1}^n \frac{t}{n} \mathbb{E}[B_{t_i/n}] = \int_0^t \mathbb{E}[B_u] du = 0$$

by Fubini's Theorem
Justification: *Theorem* if $|X_n| < Y$, $\mathbb{E}(Y) < \infty$, $X_n \overset{a.s.}{\to} X$, then $\mathbb{E}[X_n] \to \mathbb{E}[X]$
Theorem $\mathbb{E}[(\max_{u\in[0,1]} |B_u|)^n] < \infty$
Remark $|\sum_{i=0}^{n-1} \frac{t}{n} B_{t_i/n}| \leq t \max_{u\in[0,1]} |B_u|$, by Doob's Martingale Inequality, $\max_{[0,1]} |B_u| < \infty$, so it is okay to exchange the expectation and the limit. Further, by Fubini Theorem, it's okay to exchange integral and expectation
(2) Variance

$$\begin{aligned}
Var(\int_0^t B_u d_u) &= \mathbb{E}[(\int_0^t B_u d_u)^2] = \mathbb{E}[\int_0^t \int_0^t B_u B_v d_u d_v] \\
&= \int_0^t \int_0^t \mathbb{E}[B_u B_v] d_u d_v \\
&= \int_0^t \int_0^t u \wedge v d_u d_v, \min(u,v) = \begin{cases} u, 0 \leq u \leq v \\ v, v \leq u \leq t \end{cases} \\
&= \int_0^t \int_0^v u du dv + \int_0^t \int_v^t v du dv \\
&= \frac{1}{6} t^3 + (\frac{1}{2} t^3 - \frac{1}{3} t^3) = \frac{1}{3} t^3
\end{aligned}$$

(3) Since limite of Gaussian Process is Gaussian, so limit of Reimann Sum of $\int_0^t B_u du$ is Gaussian
(4) Independence Increment. Given Brownian Motion Process $B = (B_{t_1}, B_{t_2}, .., B_{t_n})$ and matrix $A = \begin{bmatrix} 1 & 0 & 0 & .. \\ -1 & 1 & 0 & .. \\ 0 & -1 & 1 & .. \end{bmatrix}$, then $AB^\top = (B_{t_1}, B_{t_2} - B_{t_1}, .., B_{t_n} - B_{t_{n-1}})$ is still Gaussian. Further,

$$\begin{aligned}
&\mathbb{E}[(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})], i < j, t_i \text{ ordered} \\
&= \mathbb{E}[B_{t_i} B_{t_j} - B_{t_i} B_{t_{j-1}} - B_{t_{i-1}} B_{t_j} + B_{t_{i-1}} B_{t_{j-1}}] \\
&= t_i - t_i - t_{i-1} + t_{i-1} = 0, t_i < t_j, t_i \leq t_j
\end{aligned}$$

so, $B_{t_1}, B_{t_2} - B_{t_1}, .., B_{t_n} - B_{t_{n-1}}$ are Gaussian with covariance 0, so mutually independent
(5) Stationary Increments. $B_t - B_s \sim N(0, t-s)$ where covariance only depends on length of interval as $|t-s|$
(6) Equivalent characterization.
(Form A) Stochastic Process B is Brownian Motion, iff (i) increments are stationary (ii) $B_t - B_s \sim N(0, t-s), \forall t, s, B_0 = 0$ (iii) paths are continuous
(Form B) Stochastic Process B is Brownian Motion, iff (i) increments are stationary and independent (ii) paths are continuous (iii) (automatically from i and ii, by CLT) $B_0 = 0, \mathbb{E}[B_1] = 0, Var(B_1) = 1$

Remark. There is only ONE process that are continuous and have stationary independent increment, as $X_t = a + b \cdot t + \sigma B_t$
(7) Law property of Brownian Motion
(i) Scaling invariant, $\{\frac{1}{\sqrt{c}} B_{ct}\}_{t\geq 0}$ is also a Brownian Motion (with different mean and variance)
(ii) Time inversion, $\{tB_{1/t}\}_{t\geq 0}$ is also a Brownian Motion
(iii) Symmetric around 0, $\{-B_t\}_{t\geq 0}$ is also a Brownian Motion
(8) Path Property. r.v. $\mathbb{E}[\frac{B_{t+\Delta}-B_t}{\Delta}] = \frac{1}{\Delta^2}\Delta \to \infty$ as $\Delta \to 0$

1.3.3 Convergence

Theorem. Sequence $(\xi_n)_{n=0}^{\infty}$ with $\xi_n \sim N(0, \sigma_n^2), \sigma_n^2 \to \infty$ does not converge to Gaussian Process everywhere.
Remark. Sample paths of Brownian Motion $(B_t)_{t\geq 0}$ are no-where differentiable
Remark. From Kolmogorov Continuity Theorem, since $B_{t+\Delta} - B_t = \xi\sqrt{\Delta}, \xi \sim N(0,1)$, so each path of Brownian Motion is locally Hölder's continuous with order $\alpha \in (0, \frac{1}{2})$, i.e. for any $t > 0$, $\exists c > 0$ such that $|B_{t+\Delta}(\omega) - B_t(\omega)| \leq c|\Delta|^{\alpha}, \forall \omega, \Delta, t$
Theorem. Times on Hitting 0. Brownian Motion hits zero infinitely many times on any time interval $(0, \xi], \forall \xi > 0$
Theorem. Times on Hitting 0. Brownian Motion hits zero infinitely many times on any time interval $[N, \infty), \forall N > 0$, by applying time reversal property onto previous Theorem.
Theorem. Times on Hitting Infinity. $\mathbb{P}(\sup_t B_t = +\infty, \inf_t B_t = -\infty) = 1$
Proof sketch. Taking $Z := \sup_t B_t$, since $c > 0, \mathbb{E}(F(cB_t)) = F(\mathbb{E}(cB_t)) = F(0) = \mathbb{E}(F(B_t))$ as from scaling property, $cB_t \overset{d.}{=} B_t$ so $cZ \overset{d.}{=} Z$. So, $Z(\omega) \in \{0, \infty\}$. Then Suffix To Show that $\mathbb{P}(Z = 0) = 0$
Fix a specific type of path that we don't care how it moves when $t < 1$, but $t > 1$ all Brownian motion are non-positive, then $\mathbb{P}(Z = 0) \leq \mathbb{P}(B_v \in \mathbb{R}, \forall t \in [0,1]; B_1 \leq 0; B_u \leq 0, \forall u \geq 1)$ (inequality can hold, since $\sup_t B_t$ is now larger than $\sup_{t\in[0,1]} B_t > 0$)
$= \mathbb{P}(B_1 \leq 0, \sup_{t\geq 0}(B_{t+1-B_1}) = 0) = \mathbb{P}(B_! \leq 0)\mathbb{P}(\sup_{t\geq 0}(B_{t+1-B_1}) = 0) = \frac{1}{2}\mathbb{P}(Z = 0)$, so $\mathbb{P}(Z = 0) = 0$.
So $Z = \infty$ a.s.
By symmetric around 0, $\inf_f B_t = -\infty$

1.3.4 Approximate Brownian Motion

Theorem. Donsker's Theorem./Functional Central Limit Theorem. Given $\{X_n\}_{n=1}^{\infty} iid, \mathbb{E}(X_n) = 0, Var(X_n) = \sigma^2 < \infty$, define $S_n = 0, S_n = \sum_{i=0}^{n} X_i$, then define linear interpolation $\bar{S}_t = (1 - (t - \lfloor t \rfloor))S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)S_{\lfloor t \rfloor + 1}$, then

$$\left(\frac{\bar{S}_{tn}}{\sigma\sqrt{n}}\right)_{t\geq 0} \xrightarrow{d.} (B_t)_{t\geq 0}, n \to \infty$$

i.e., for funciton $f \in \mathcal{C}_b$ (in space of continuous and bounded) $\mathbb{E}(F(\frac{\bar{S}_{tn}}{\sigma\sqrt{n}})_{t\geq 0}) \to \mathbb{E}(F(B_t)_{t\geq 0}), n \to \infty$

1.3.5 Quadratic Variation

Def. Partition of Time Interval Π is a collection of $0 = t_0 < t_1 < .. < t_n = T$, and $||\Pi|| = \max_{i=0,..,n-1}(t_{i+1} - t_i)$
Def. First Order Variation of Function For function $f : [0, T] \to \mathbb{R}$, First order Variation is $FV_T(f) = \lim_{n\to\infty} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$, where $\lim_{n\to\infty} = \lim_{||\Pi||\to 0}$
Lemma. If f is first-order continuous, i.e. $f \in \mathcal{C}^1([0, T])$, then $FV_T(f) = \int_0^T |f'(t)|dt$
Prop. BM is nowhere differentiable Taking partition with length $\frac{t}{2^n}$, then
$FV_T(B_t) = \lim_{n\to\infty} \sum_{k=1}^{2n} |B_{kt/2^n}(\omega) - B_{(k-1)t/2^n}(\omega)| = +\infty$ as it is path-wise none-differentiable
Def. Quadratic Variation of Function For function $f : [0, T] \to \mathbb{R}$, second order Variation is $[f, f](T) = \lim_{||\Pi||\to 0} \sum_{i=0}^{n-1} [f(t_{i+1}) - f(t_i)]^2$
Lemma. If $f \in C^1([0, T])$, then $[f, f](T) = 0$

Prop. BM has paths of finite quadratic variation $\lim_{n\to\infty}\sum_{k=1}^{2n}[B_{kt/2^n}(\omega) - B_{(k-1)t/2^n}(\omega)]^2 = t$ (non-trivial, i.e. not just $\in \{0, \infty\}$)
Proof sketch by showing that under finite partition, $\mathbb{Q}_n = \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_i)^2$, $\mathbb{E}(\mathbb{Q}_n) = T, Var(\mathbb{Q}_n) = 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \le 2||\Pi||T \to 0$
Lemma. Finite Total Variation. Let $f : [0, t] \to \mathbb{R}$ be a continuous functions, if its total variation is finite, the its quadratic variation is zero.
Proof sketch by.... week 2
Lemma. Quadratic Variation. In general (1) consider random interval split, $\Pi = (t_0, t_1, ..)$, and $||\Pi|| = \max_i(t_{i+1} - t_i)$, $\lim\limits_{||\Pi||\to 0}\sum\limits_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2$ converges in probability to t. (2) If $\sum\limits_{k=1}^{\infty}||\Pi||_k < \infty$ for k split, then $\lim\limits_{||\Pi||\to 0}\sum\limits_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2$ converges almost surely to t.

2 SA02: Probability Theory

2.1 Common Inequality

2.1.1 Inequality

Lemma. Markov's Inequality. X is a random variable, function f is increasing and positive, if $f(X) \ge f(b)\mathbb{I}_{X\ge b}$, then $\mathbb{E}[f(x)] \ge f(b)\mathbb{P}(X \ge b) \Rightarrow \mathbb{P}(X \ge b) \le \frac{\mathbb{E}[f(X)]}{f(b)}$.
Remark. Markov's Inequality denotes the concentration bound, or how fast the probability decays.
Lemma. Chebyshev's Inequality. $\mathbb{P}[(X-a)^2 > \varepsilon^2] \le \frac{\mathbb{E}[(X-a)^2]}{\varepsilon^2}$
Remark. When $a = \mathbb{E}(X)$, then $\mathbb{P}[|X - a| > \varepsilon] \le \frac{\mathrm{Var}(X)}{\varepsilon^2}$, $\varepsilon > 0$

2.1.2 Borel-Cantelli Lemma

Lemma. Borel-Cantelli Lemma. Suppose for each $\varepsilon > 0$, if $\sum_n \mathbb{P}[|X_n - X]| > \varepsilon] < \infty$, then $X_n \to X$ a.s. (path-wise)
Sketch proof. By the corollary just below, if $\sum_n \mathbb{P}[|X_n - X]| > \varepsilon] < \infty$, then $\sum_n \mathbb{I}_{|X_n - X]|>\varepsilon} < \infty$ a.s., then ...? see week 2
Lemma. On Bernoulli r.v.. For Bernoulli random variables $Y_1, .., Y_n$, $\mathbb{P}(Y_n = 1) = p_n = 1 - \mathbb{P}(Y_n = 0)$, denote $S_n = Y_1 + .. + Y_n$ as binomial random variable, increasing with running sum. Denote $S = \lim S_n = \sum_{n=1}^{\infty} Y_n$. (1) If $\sum_n p_n < \infty$, then $S < \infty$ a.s. (2) $\mathbb{E}[S] < \infty \Rightarrow S < \infty$ a.s.
Corollary. Denote $H_1, .., H_n$ as events, then $\sum_n \mathbb{P}(H_n) < \infty \Rightarrow \sum_n \mathbb{I}_{H_n} < \infty$

2.2 Probability Space

Def. Probability Space. Denote as $(\Omega, \mathcal{F}, \mathbb{P})$
Def. Probability Measure. Function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure if (i) $\mathbb{P} = \emptyset$ (ii) $\mathbb{P}(\Omega) = 1$ (iii) $\mathbb{P}(\biguplus_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty}\mathbb{P}(A_i)$, for $A_i \in \mathcal{F}$ and A_i disjoint.
Def. Smallest $\sigma-$algebra. Denote $\mathcal{E}$ as collection of some sets in Ω. The smallest $\sigma-$algebra containing $\mathcal{E}$, or the $\sigma-$algebra generated by $\mathcal{E}$, is written as $\sigma(\mathcal{E}) = \cap\mathcal{A}$, where $\mathcal{A}$ is any $\sigma-$algebra containing $\mathcal{E}$
Def. Measurable Function. Given two measurable spaces $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$, function $f : \Omega \to \Omega'$ is $(\mathcal{F} - \mathcal{F}')-$measurable if for any image $A \in \mathcal{F}'$, the preimage $f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in A\} \in \mathcal{F}$
Lemma. Composition of Measurable Function. Given three measurable spaces $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ and $(\Omega'', \mathcal{F}'')$, function $f : \Omega \to \Omega', g : \Omega' \to \Omega''$, and f is $(\mathcal{F} - \mathcal{F}')-$measurable, g is $(\mathcal{F}' - \mathcal{F}'')-$measurable, denote $h = g \circ f$ is $(\mathcal{F} - \mathcal{F}'')-$measurable, where $h : \Omega \to \Omega''$.
Remark. Random Variable. Set $\Omega' = \mathbb{R}$, $\mathcal{F}'$ as Borel-$\sigma-$algebra, then mapping function is then X, called random variable. $X : \mathcal{F} \to \mathcal{B}(\mathbb{R})$, $X^{-1}(A \in \mathcal{B}(\mathbb{R})) \in \mathcal{F}$.
Def. $\sigma-$algebra generated by r.v.. On $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$, denote $X : \Omega \to \Omega'$, then $\sigma(X) = \sigma(\{X^{-1}(A) : A \in \mathcal{F}'\})$ is the smallest $\sigma-$algebra that makes X be $(\mathcal{F} - \mathcal{F}')-$measurable, then we have new but smaller measurable space $(\Omega, \sigma(X))$
Lemma. $\sigma(X) \subset \sigma(Y) \iff X = g(Y)$ for some measurable function g, as surjection.

2.3 Expectation

Def. Expectation.
Given random variable X, then its expectation $\mathbb{E}(X)$ is
(1) when X is simple function, i.e. $X = \sum_{n=1}^{m} a_n \mathbb{I}_{A_n}$, where $a_n \in \mathbb{R}^+$, and $\mathbb{I}_{A_n}(\omega) = \begin{cases} 1, \omega \in A_n \\ 0, \text{otherwise} \end{cases}$.
Then $\mathbb{E}(X) = \sum_{n=1}^{m} a_n \mathbb{P}(A_n)$
(2) when X is nonnegative function, $\mathbb{E}(X) = \sup\{\mathbb{E}(Y) : \text{Y is simple, positive r.v.}, Y \leq X\}$, since simple function can approximate nonnegative function from below, then it is okay to exchange sup and $\mathbb{E}$
(3) when X is real-valued function, i.e. $X \in \mathcal{L}_0$ (measurable), then denote $X = X^+ - X^-$, where $X^+ = \max(X, 0), X^- = \max(-X, 0)$, then $|X| = X^+ + X^-$. Then, $\mathbb{E}(X) = \mathbb{E}(X^+) - \mathbb{E}(X^-)$. If either of $\mathbb{E}(X^+)$ or $\mathbb{E}(X^-)$ if finite, then $\mathbb{E}(X)$ is well defined.
Otherwise, like Cauchy r.v., then it has expectation, but not well defined, have case $\infty - \infty$
Remark. If $\mathbb{E}(|X|) = \mathbb{E}(X^+) + \mathbb{E}(X^-) < \infty$, then X is absolutely integrable, denote as $X \in \mathcal{L}$
Remark. If $\mathbb{E}(|X|^p) < \infty$, $p \geq 1$, then $X \in \mathcal{L}$

2.4 Mode of Convergence

Def. Almost Surely Convergence. $X_n \overset{a.s.}{\to} X$ almost surely/ almost everywhere, if $\mathbb{P}(\{\omega : X_n(\omega) \to X(\omega)\}) = 1$, as pointwise convergence up to a null set
Def. Converge in Probability. $X_n \overset{p.}{\to} X$ converge in probability, if $\mathbb{P}(|X - X_n| \geq \varepsilon) \overset{n\to\infty}{\to} 0$, $\forall \varepsilon > 0$. Also called converge in L_1.
Lemma. $X_n \overset{a.s.}{\to} X \Rightarrow X_n \overset{p.}{\to} X$, but $X_n \overset{p.}{\to} X \not\Rightarrow X_n \overset{a.s.}{\to} X$.
Def. Converge in Lp. L_p convergent ($p \geq 1$) $X_n \overset{L_p}{\to} X$ if $\mathbb{E}(|X_n - X|^p) \to 0$, where L_p norm is $||X_n - X||_p := [...]^{1/p}$
Lemma. $X_n \overset{L_p}{\to} X \Rightarrow X_n \overset{p.}{\to} X$, but $X_n \overset{p.}{\to} X \not\Rightarrow X_n \overset{L_p}{\to} X$.
Def. Uniformly Integrability. Denote $\mathcal{C}$ as infinite collection of random variables, then $\mathcal{C}$ is called Uniformly Integrable, if given $\varepsilon > 0$, $\exists$ a constant $K < \infty$, such that $\mathbb{E}(|X| \cdot \mathbb{I}_{|X|>K} : \forall X \in \mathcal{C}) < \varepsilon$, where $\mathbb{I}_{|X|>K}$ controls the tail, and the uniformly comes from $\forall X \in \mathcal{C}$
Property. Uniformly Integrable $\Rightarrow$ L_1-boundedness, but L_1-boundedness ($\sup_{X\in\mathcal{C}} \mathbb{E}(|X|) < \infty$) $\not\Rightarrow$ Uniformly Integrable (unless $|\mathcal{C}| = 1$, prove by take compliment as $\{|X| < K\}$
Property. Sufficient condition for Uniformly Integrable (1) L_p-boundedness $\Rightarrow$ Uniformly Integrable, as ($p > 1$), because tails converge faster (2) $\forall X \in \mathcal{C}$, $|X| \overset{a.s.}{\leq} Y \in L^1 \Rightarrow \mathcal{C}$ is Uniformly Integrable, by DCT.
Theorem on Uniformly Integrable. $X_n \overset{L_1}{\to} X \iff (X_n)$ is Uniformly Integrable and $X_n \overset{p.}{\to} X$
Remark. $X_n \overset{p.}{\to} X$ and (X_n) is Uniformly Integrable $\Rightarrow \lim_n \mathbb{E}[|X_n - X|] \to 0$, thus $\lim_x \mathbb{E}[X_n] = \mathbb{E}[X]$ (interchange expectation and limit)
Theorem. Fatou Lemma. If $f_n \in L^+, n \in \mathbb{N}$, then $\int \liminf_n f_n \leq \liminf_n \int f_n$.
Theorem. Fatou Lemma for Conditional Expectation. On $(\Omega, \mathcal{F}, \mathbb{P})$, given sequence of nonnegative r.v. $X_1, X_2, .., \sigma-$subalgebra $\mathcal{G} \subset \mathcal{F}$, then $\mathbb{E}[\liminf_{n\to\infty} X_n|\mathcal{G}] \leq \liminf_{n\to\infty} \mathbb{E}[X_n|\mathcal{G}]$ a.s.
Theorem. Monotone Convergence Theorem. If $X_n \nearrow X$, then $\lim_n \mathbb{E}(X_n) = \mathbb{E}(\lim_n X_n) = \mathbb{E}(X)$
Theorem. Dominate Convergence Theorem. If $|X_n| \leq Y \in L^1$, and $X_n \overset{a.s.}{\to} X$, then $\lim_n \mathbb{E}(X_n) = \mathbb{E}(\lim_n X_n) = \mathbb{E}(X)$
Def. Convergence in Distribution. $X_n \overset{w.}{\to} X$, where X_n can be defined on other probability space, if for any $F \in C_b$ (continuous and bounded function), $\mathbb{E}(F(X_n)) \to \mathbb{E}(F(X))$
Def. Summary of Convergence.
(1) $X_n \overset{a.s.}{\to} X \Rightarrow X_n \overset{p.}{\to} X$
(2) $X_n \overset{L^1}{\to} X \iff X_n \overset{p.}{\to} X$ and (X_n) is Uniformly Integrable
(3) $X_n \overset{p.}{\to} X \Rightarrow X_n \overset{w.}{\to} X$, if X is also defined on the same $(\Omega, \mathcal{F}, \mathbb{P})$, then $X_n \overset{p.}{\to} X \Rightarrow$ characteristic function $F(X_n) \overset{p.}{\to} F(X)$ since F is bounded $\Rightarrow$ Uniformly Integrable $\Rightarrow X_n \overset{w.}{\to} X$

2.5 Independence

Def. Independence of sigma algebra. On probability space $(\Omega, \mathcal{H}, \mathbb{P})$, there is further sub-$\sigma$-algebra $\mathcal{F}, \mathcal{G}$, then if $\forall A \in \mathcal{F}, B \in \mathcal{G}, \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, then $\mathcal{F}$ and $\mathcal{G}$ are independent.
Def. Independence of random variable. X and Y are random variables, if $\sigma(X)$ and $\sigma(Y)$ are independent, then X and Y are independent.
Remark. If X and Y are independent, then any function f, $f(X)$ and $f(Y)$ are independent
Def. Independence of random variable and algebra. Random variable X is independent of $\mathcal{F}$, if $\sigma(X)$ and $\mathcal{F}$ are independent.

2.6 Conditional Expectation

Def. Conditional Expectation on Random Variable
Given random variable $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G}$ is $\sigma - sub-$algebra of $\mathcal{F}$, then random variable $Z \in L^1(\mathcal{G})$ is said to be the conditional expectation (a version) of Y given G, if

$$\mathbb{E}(Y\mathbb{I}_G) = \mathbb{E}(Z\mathbb{I}_G), \forall G \in \mathcal{G}$$
$$\int_G Y d\mathbb{P} = \int_G Z d\mathbb{P} = \int_\Omega Z\mathbb{I}_G d\mathbb{P}$$

or equivalently, $\mathbb{E}(Y\xi) = \mathbb{E}(X\xi)$ given $X\xi$ and $Y\xi$ are in $L^1(\mathcal{G})$
Theorem. Kolmogorov's conditional expectation theorem. The conditional expectation exists and almost surely unique, thus denote
(1) condition on $\mathcal{G}$ as $Z = \mathbb{E}(Y|\mathcal{G})$
(2) condition on $X_1, .., X_n$ as, then $\mathbb{E}(Y|X_1, .., X_n) = \mathbb{E}(Y|\sigma(X_1, .., X_n)) = \phi(X_1, .., X_n)$ as a function of random variable $X_1, .., X_n$
(3) X is a stochastic process, define $\sigma(X_s : s \leq t)$ as the smallest $\sigma-$algebra, that makes all X_s $(s \leq t)$ measurable, then $\mathbb{E}(Y|X) = \phi(X)$, where ϕ is continuous function on r.v. X
Properties.
Setting $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$
(1) (L^1-projection) $\min\limits_{Z \in L^2(\mathcal{G})} \mathbb{E}[(Y - Z)^2] = \mathbb{E}[Y|\mathcal{G}]$ as projection to smaller vector space (with min L^2 distance)
(2) If $\mathcal{G} = \{\emptyset, \Omega\}$, then $\mathbb{E}(Y|\mathcal{G}) = \mathbb{E}(Y)$
(3) If $\mathcal{G} = \mathcal{F}$, then $\mathbb{E}(Y|\mathcal{F}) = Y$
(4) $Y \geq 0, \mathbb{P}-$a.s., then $\mathbb{E}(Y|\mathcal{G}) \geq 0$ a.s.
(5) (**what is known comes out**) $X \in L^1(\mathcal{G})$, and $XY \in L^1(\mathcal{F})$, then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$ $\mathbb{P}-$a.s.
(6) Given Y and $\mathcal{G}$ independent, then $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathbb{E}(Y|\mathcal{G}) = \mathbb{E}(Y)$
(7) Given Y and $\mathcal{G}$ independent, function $\phi : \mathbb{R}^2 \to \mathbb{R}$, $X \in L^1(\mathcal{G})$, then $\mathbb{E}(\phi(X, Y)|\mathcal{G}) = \mathbb{E}(\phi(x, Y))|_{X=x}$ as a function of x
(8) (**Tower Property**) On probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there is further sub-$\sigma$-algebra $\mathcal{G}, \mathcal{H}$, $\mathcal{H} \subset \mathcal{G}$, then $\mathbb{E}(Y|\mathcal{H}) = \mathbb{E}(\mathbb{E}(Y|\mathcal{G})|\mathcal{H})$, also, $\mathbb{E}(Y|\mathcal{H}) = \mathbb{E}(\mathbb{E}(Y|\mathcal{H})|\mathcal{G})$
(9) (**Jensen's Inequality**) $\phi : \mathbb{R} \to \mathbb{R}$, ϕ is convex function, $\phi \in L^1$, then $\mathbb{E}(\phi(X)|\mathcal{G}) \geq \phi(\mathbb{E}(Y|\mathcal{G}))$

2.7 Filtered Probability Space

Def. Filtration. On $(\Omega, \mathcal{F}, \mathbb{P})$, a family of sub-$\sigma$-algebra of $\mathcal{F}$ as $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, if $\mathcal{F}_t \subset \mathcal{F}_s \forall t \leq s$, then $\mathbb{F}$ is called filtration.
Def. Filtered Probability Space. $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a filtered probability space.
Def. Natural Filtration. For given random variable $(X_t)_{t \geq 0}$, then $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$ is called natural filtration.
Def. Adapted to Filtration. X is adapted to the filtration $\mathbb{F}$ if $X_t \in L^0(\mathcal{F}_t)$ ($\mathcal{F}_t$-measurable)

2.8 Stopping Time

Def. Stopping time. On $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a random variable $\tau : \Omega \to [0, \infty]$ is a stopping time if $\{\tau \le t\} \in \mathcal{F}_t, \forall t$

Remark. Hitting time $\mathcal{T}_a = \inf\{t \ge 0 : B_t = a\}$ is a stopping time, since $\{\mathcal{T}_a \le t\} = \{B_t = a\} \cap (\cap_{n\in\mathbb{N}} \cup_{q\in\mathbb{Q}\cap[0,t]} \{B_q \in A_n\})$, where $A_n = \{x \in \mathbb{R} : |x-a| < \frac{1}{n}\}$, and $\{B_q \in A_n\}$ is $\mathcal{F}_t$-measurable, so $\mathcal{T}_a$ is $\mathcal{F}_t$-measurable. **Note, to prove stuff in continuous, reduce to discrete series and take limit.**

Property. τ_1, τ_2 are stopping time, then pointwise operator (1) $\tau_1 \vee \tau_2$ is stopping time, (2) $\tau_1 \wedge \tau_2$ is stopping time, (3) $\tau_1 + \tau_2$ is stopping time

Def. σ-algebra defined on Stopping time. τ is a stopping time, $\mathcal{F}_t := \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t\}$, then (1) $\mathcal{F}_\tau$ is a σ-algebra (2) $\tau \in L^0(\mathcal{F}_t)$, adapted.

Theorem. If τ, θ are stopping time, $\tau(\omega) \le \theta(\omega), \forall \omega \in \Omega$, then $\mathcal{F}_\tau \subset \mathcal{F}_\theta$

Property. $\mathcal{F}_{\tau\wedge\theta} = \mathcal{F}_\tau \cap \mathcal{F}_\theta$, since $\tau \wedge \theta \le \tau \Rightarrow \mathcal{F}_{\tau\wedge\theta} \subset \mathcal{F}_\tau$, similarly, $\mathcal{F}_{\tau\wedge\theta} \subset \mathcal{F}_\theta$.

Theorem. if X is adapted continuous (and left-/right- continuous), then $X_\tau \in L^0(\mathcal{F}_\tau)$

Def. X is a stochastic process, then $X^\tau_t = X_{t\wedge\tau}, \forall t$, then $X^\tau_t \in L^0(\mathcal{F}_{t\wedge\tau}), \forall t$ i.e. X^τ is adapted to $(\mathcal{F}_{t\wedge\tau})_{t\ge0}$

2.9 Martingale on Stochastic Process

Def. Generalized Brownian Motion. B is a Brownian Motion with respect to $\mathbb{F} = (\mathcal{F}_t)_{t\ge0}$ if it is a Brownian Motion if (1) B is adapted to $\mathbb{F}$ (2) $B_t - B_s$ is independent of $\mathcal{F}_s, t \ge s$

Def. Stochastic Process. X is a stochastic process given $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, then X is a martingale with respect to $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ if (1) $X_t \in L^1(\mathcal{F}_t), \forall t$, i.e. $X_t \in L^0(\mathcal{F}_t)$ adapted and $\mathbb{E}(|X_t|) < \infty$ (2) and then $\mathbb{E}(X_t|\mathcal{F}_s) = X_s, 0 \le s \le t$

Property. Brownian Motion is a Martingale. Let $B = (B_t)_{t\ge0}$ be a Brownian Motion with respect to $\mathbb{F}$, then B is a martingale with respect to $\mathbb{F}$ since (1) $B_t \in L^0(\mathcal{F}_t)$ (2) $\mathbb{E}(|B_t|) = \sqrt{(\mathbb{E}(B_t))^2} \overset{Jensen}{\le} \sqrt{\mathbb{E}(B_t^2)} = \sqrt{t} < \infty$ so $B_t \in L^1(\mathcal{F}_t)$ (3) $\mathbb{E}(B_t|\mathcal{F}_s) = \mathbb{E}(B_s + B_t - B_s|\mathcal{F}_s) = B_s + \mathbb{E}(B_t - B_s) = B_s$

Def. Super Martingale. $(X_t)_{t\ge0}$ is adapted, then X is a super-martingale if (1) $X_t \in L^1(\mathcal{F}_t), \forall t$ (2) $\mathbb{E}(X_t|\mathcal{F}_s) \le X_s$, the expectation decreasing over time.

Def. Sub Martingale. $(X_t)_{t\ge0}$ is adapted, then X is a sub-martingale if (1) $X_t \in L^1(\mathcal{F}_t), \forall t$ (2) $\mathbb{E}(X_t|\mathcal{F}_s) \ge X_s$, the expectation increasing over time.

Theorem. Joe Doob's Optional Sampling Theorem/Stopping Theorem I. Let M be a continuous martingale/sub-/super-martingale, τ be a stopping time, then M^τ is also a martingale/sub-/super-martingale. $\mathbb{E}(M_{\tau\wedge t}) = \mathbb{E}(M_0)$ constant or a random variable w.r.t. M_0.

Remark. $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$ if random variable τ is bounded (can be infinite). $\tau_a = \inf\{t > 0 : B_t = a\}$ is not bounded , so $\mathbb{E}(B_{\tau_a}) = \mathbb{E}(a) = a \ne 0 = \mathbb{E}(B_0)$

Theorem. Doob's Optional Sampling Theorem II. M is a continuous martingale/sub-/super- martingale, if $\tau < \infty$ almost surely, and $|M_{t\wedge\tau}| \le X$ where X is a nonnegative random variable and $X \in L^1$, then $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$ (martingale) / $\mathbb{E}(M_\tau) \ge \mathbb{E}(M_0)$ (sub-martingale) / $\mathbb{E}(M_\tau) \le \mathbb{E}(M_0)$ (super-martingale). Explain: $t \wedge \tau \overset{t\to\infty}{\to} \tau$, and use DCT to prove.

Theorem. Martingale Convergence Theorem I. X is a sub-martingale, and X is right-continuous, if $\sup_t \mathbb{E}(X_t^+) < \infty$, then $\exists$ a random variable $X_\infty \in L^1$ such that $\lim_{t\to\infty} X_t = X_\infty$ a.s.

Remark. Brownian Motion does not converge.

Theorem. Martingale Convergence Theorem II. $(X_t)_{t\ge0}$ is a continuous sub/super-martingale, if $\mathbb{E}(|X_t|^p) \le C$ for some constant $C < \infty, p \ge 1$ (uniform L^p-bound, or X is L^p-bounded), then $X_t \overset{t\to\infty}{\to} X_\infty \in L^p$

Remark. If X is a martingale or a non-positive sub-martingale, then (1) $||X_t - X_\infty||_p \to 0, p > 1$ (2) $\mathbb{E}[|X_\infty|^p] \le \liminf_t \mathbb{E}[|X_t|^p] < \infty$ by Fatou's Lemma

Corollary. A positive super-martingale converges. Proof, $\mathbb{E}(X_t^+) = \mathbb{E}(X_t) \le \mathbb{E}(X_0) = 0 < \infty$, so it converges. Proof.

Theorem. $(M_t)_{t\ge0}$ be a martingale, if $M_t \overset{a.s., L^1}{\to} M_\infty$ (i.e., almost surely convergent, so uniformly convergent, and L^1 convergent, so $M_t \in L^1$ and $M_\infty \in L^\infty$), then $M_t = \mathbb{E}(M_\infty|\mathcal{F}_t)$.

Sketch Proof: set $\mathbb{E}(M_s\mathbb{I}_F) = \mathbb{E}(M_t\mathbb{I}_F), s \geq t, F \in \mathcal{F}_t$, then $|\mathbb{E}(M_t\mathbb{I}_F) - \mathbb{E}(M_\infty\mathbb{I}_F)| = |\mathbb{E}(M_s\mathbb{I}_F) - \mathbb{E}(M_\infty\mathbb{I}_F)| \leq |\mathbb{E}(M_s - M_\infty)| \leq \mathbb{E}(|M_s - M_\infty|) \xrightarrow{L^1} 0$

Theorem. Doob's Maximal Inequalities. (X_t) is a continuous nonnegative submartingale, then $\mathbb{P}(\sup_{u\in[0,t]} X_u \geq \lambda) \leq \frac{\mathbb{E}(X_t^p)}{\lambda^p}\ \forall p \geq 1$, by Jensen's Inequality. When $\forall p > 1$, $||\sup_{u\in[0,t]} X_u||_p \leq \frac{p}{p-1}||X_t||_p$.

3 SA03: Special Brownian Motion

3.1 Property of Brownian Motion

3.1.1 Exit Time

Def. Exit Times. Given standard Brownian Motion B_t, exit time is $\tau_a = \inf\{t \geq 0 : B_t = a\}$

Property. $\mathbb{P}(\tau_a < \infty) = 1$ given $\tau_a = \inf\{t \geq 0 : B_t = a, a < 0, B_0 = 0\}$

Proof. 根据optional sampling theorem，则$B_{t\wedge\tau_a}$为martingale，则定义

$$S_t := B_{t\wedge\tau_a} - a, S_t \geq 0$$

故，$\forall t, S_t$非负、supermartingale (因为当$t > \tau_a > s$，则$S_t = 0, \mathbb{E}[S_t|\mathcal{F}_s] \leq S_s, S_s \geq 0$)，根据Martingale Convergence Theorem，根据folded gaussian distribution，则$\mathbb{E}[|S_t|] \leq \mathbb{E}[|B_t|] - a = \sqrt{\frac{\pi}{2}} - a < \infty$，则有

$$S_{t\wedge\tau_a} \overset{t\to\infty}{\to} S_{\infty\wedge\tau_a} \in L^1$$

则不妨假设$\mathbb{P}(\tau_a = \infty) > 0$，for $\omega \in \{\tau_a = +\infty\}$，则

$$B_{t\wedge\tau_a}(\omega) = B_t(\omega) \overset{t\to\infty}{\to} S_\infty \in L^1$$

则,

$$\lim_{n\to\infty} \frac{B_t}{\sqrt{t}} = \frac{S_\infty}{\infty} = 0$$

也就是$\mathbb{P}\left(\lim_{n\to\infty} \frac{B_t}{\sqrt{t}} = 0\right) > 0$，而由于$Var\left(\frac{B_t}{\sqrt{t}}, \forall t\right) = 1 < \infty$，根据Central Limit Theorem，

$$\lim_{n\to\infty} \frac{B_1 + (B_2 - B_1) + .. + (B_n - B_{n-1})}{\sqrt{n}} \xrightarrow{d} N(0,1)$$

如果他是正态分布的话，任何一个点的概率都是0（只有density），所以$\mathbb{P}\left(\lim_{n\to\infty} \frac{B_t}{\sqrt{t}} = 0\right) > 0$矛盾，所以$\mathbb{P}(\tau_a = \infty) = 0$

所以$\tau_a < \infty$ almost surely。

Property. $\mathbb{P}(\tau_a < \tau_b) = \frac{b}{b-a}, a < 0 < b$ ，对于martingale过程都成立。

这个property是由上一个property推出的，也就是当$\tau_a < \infty$ almost surely，则我们可以确定其分布了。

Remark1. 如果过程为standard Brownian Motion，则根据OPT，$\mathbb{E}[B_{t\wedge\tau_a\wedge\tau_b}] = 0, \tau_a \wedge \tau_b < \infty$ almost surely and thus $\lim_{t\to\infty} B_{t\wedge\tau_a\wedge\tau_b} = B_{\tau_\wedge\tau_b}$.

Remark2. $|B_{t\wedge\tau_a\wedge\tau_b}| \leq \max(|a|, b)$

Proof. 根据DCT，或者更一般而言，根据uniform bound by constant，有

$$\begin{aligned}
\mathbb{E}[t\wedge\tau_a\wedge\tau_b] &\overset{t\to\infty}{\to} \mathbb{E}[B_{\tau_a\wedge\tau_b}] = B_0 = 0 \\
\mathbb{E}[B_{\tau_a\wedge\tau_b}] &= \mathbb{E}[B_{\tau_a\wedge\tau_b}\mathbb{I}_{\tau_a<\tau_b}] + \mathbb{E}[B_{\tau_a\wedge\tau_b}\mathbb{I}_{\tau_b<\tau_a}] \\
&= a\mathbb{P}(\tau_a < \tau_b) + b\mathbb{P}(\tau_b < \tau_a), \tau_i = \inf\{t \geq 0 : X_t = i\} \\
\Rightarrow \mathbb{P}(\tau_a < \tau_b) &= \frac{b}{b-a} = 1 - \mathbb{P}(\tau_b < \tau_a)
\end{aligned}$$

3.1.2 Markov Property

Def. Markov Property. Stochastic Process X has Markov Property if $\mathbb{E}(f(X_t)|\mathcal{F}_s) = g(X_s)$ for all f such that $\mathbb{E}[|f(X_t)|] < \infty, \exists$ such g.
Theorem. Brownian Motion is Markov Process. $\mathbb{E}(f(B_t)|\mathcal{F}_s) = g(B_s)$
Proof. $\mathbb{E}[f(B_t)|\mathcal{F}_s] = \mathbb{E}[f(B_s + B_t - B_s)|\mathcal{F}_s] = \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} f(B_s + y)e^{-\frac{y^2}{2(t-s)}} dy = g(B_s)$ where $g(\cdot)$ is a martingale CDF.
Theorem. Strong Markov Property of Brownian Motion. B is a Brownian Motion, and τ is stopping time, $\mathbb{P}(\tau < \infty) > 0$, define $W_t := B_{t+\tau} - B_\tau$ given $\{\tau < \infty\}$, then (1) W_t is also a Brownian Motion, (2) W_t is independent of $\mathcal{F}_\tau$.
Remark. Strong Markov Property helps to simplify the Brownian Motion process.
Theorem. Hitting 0 infinite times. For any fixed $N \geq 0$, Brownian Motion B will hits 0 infinitely many times on $t \in [N, \infty)$
Proof. Consider $W_t = B_{t+N} - B_N$, then W_t is a B.M. and $W_t \perp \mathcal{F}_N$ by Strong Markov Property. Consider stopping time $\tau_x^W := \inf\{t \geq 0 : W_t = x\}$, then $\tau_x^W \perp \mathcal{F}_N$ and its law is also independent of $\mathcal{F}_N$, then

$$
\begin{aligned}
&\mathbb{P}[B_t \text{ hits zero after } N \text{ infinitely many times}] \\
&= \mathbb{P}[B_t \text{ hits zero after } N \text{ at least once a.s.}] \\
&= \mathbb{P}[W_t \text{ hits } -B_N \text{ at least once}, t \geq 0] \\
&= \int_{\mathbb{R}} \mathbb{P}[W_t \text{ hits } x \text{ at least once}| - B_N = x]\mathbb{P}(-B_N \in dx) \\
&= \int_{\mathbb{R}} \mathbb{P}[W_t \text{ hits } x \text{ at least once}]\mathbb{P}(-B_N \in dx) \\
&= \int_{\mathbb{R}} \mathbb{P}[\tau_x < \infty]\mathbb{P}(-B_N \in dx) \\
&= \int_{\mathbb{R}} \mathbb{P}(-B_N \in dx), \mathbb{P}[\tau_x < \infty] = 1 \\
&= 1, B_N \sim N(0, N)
\end{aligned}
$$

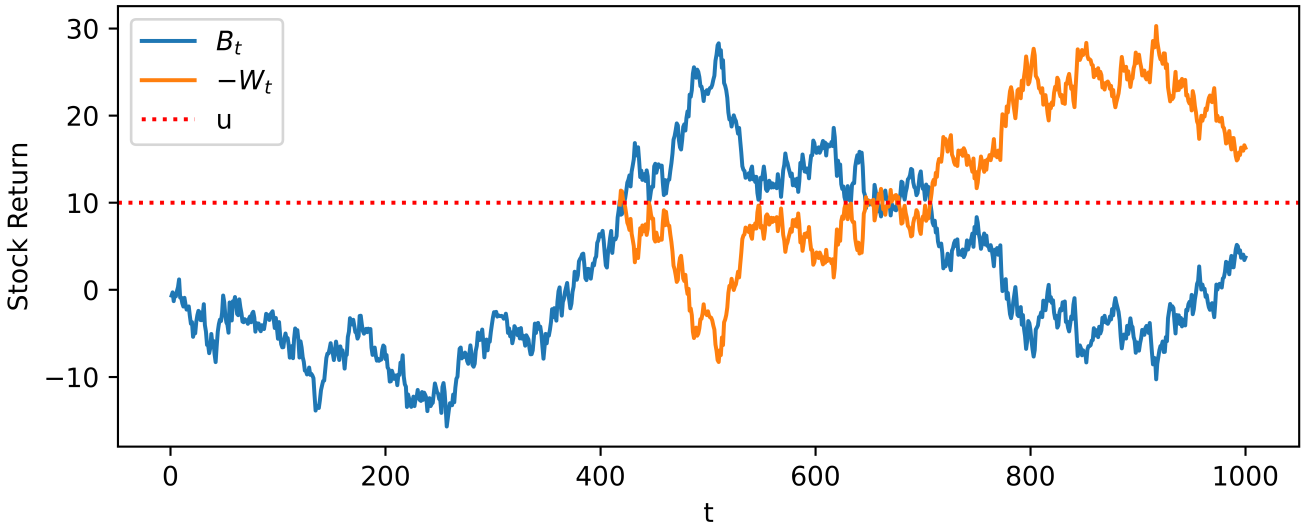
Then, every time B_t hits 0, we can start a new process by Strong Markov Property, then the new process will hits 0 a.s./$\mathbb{P} = 1$, and start again. So, B_t reach 0 countable/infinitely many times.

3.1.3 Strong Reflection Principle

Theorem 2.21. Reflection Principle. [GTM274 p.35] Given $b \leq a, a \geq 0$, $\mathbb{P}(\max_{u\in[0,t]} B_u \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b) = 1 - \Phi(2a - b)$.

```python
import numpy as np
import matplotlib.pyplot as plt
# BM
t = np.linspace(1,1000,1000)
ts = np.random.normal(0,1,1000)
BM = np.cumsum(ts)
# Wt
u = 10
tau = np.min(np.argwhere(BM>=u))
u_tau = BM[tau]
t_new = t[tau:]
Ws = BM[tau:]-u
nWs = -Ws+u
# Plot
plt.figure(figsize=(8,3),dpi=1200)
plt.plot(t,BM)
plt.plot(t_new,nWs)
plt.axhline(u,linestyle=":",c="red")
plt.legend(["$B_t$","$-W_t$","u"])
```

```
plt.xlabel("t")
plt.ylabel("Stock Return")
```



Proof. Set $S_t = \sup_{u\in[0,t]} B_u$. Consider stopping time r.v. $\tau_a = \inf\{t \geq 0 : B_t = a\}$, by 1.1., $\tau_a < \infty$ a.s. Then following event are same: $S_t \geq a \iff \tau_a \leq t$, and $B_t \leq b \iff B_t - a \leq b - a \iff B_t - B_{\tau_a} \leq b - a$, set $B^{\tau_a}_{t-\tau_a} = B_t - B_{\tau_a}$ by Strong Markov Property $B^{\tau_a} \perp \mathcal{F}_{\tau_a}$ (new process starting from τ_a, end at t, with time length $t - \tau_a$). Then $\mathbb{P}[S_t \geq a, B_t \leq b] = \mathbb{P}[\tau_a \leq t, B^{\tau_a}_{t-\tau_a} \leq b - a]$.

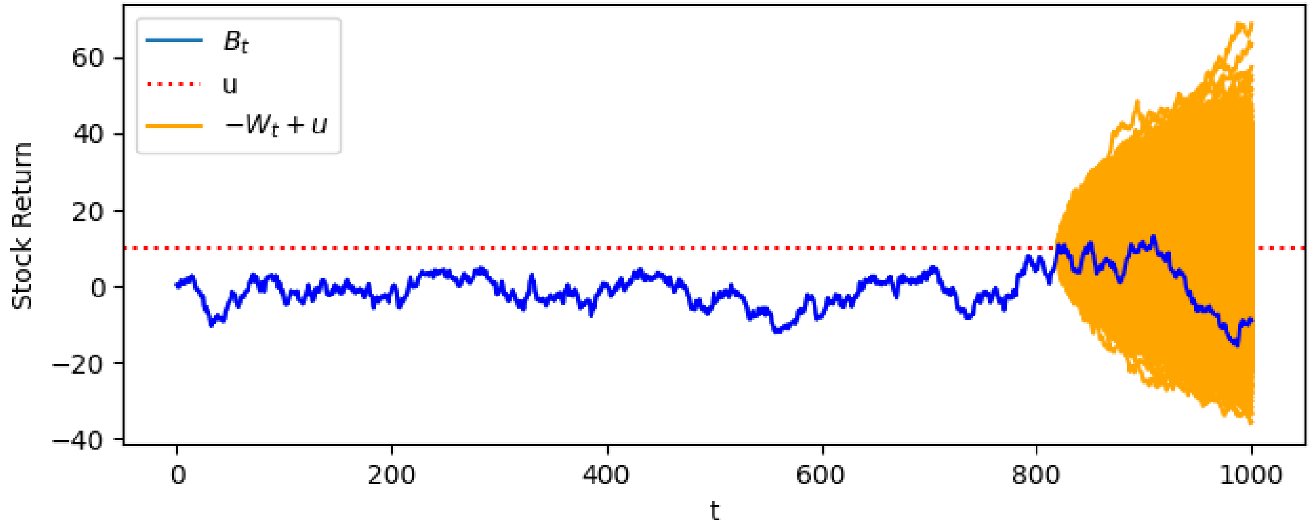
New B.M. start from τ_a as B^{τ_a}, note $B^{\tau_a} \stackrel{d.}{=} -B^{\tau_a}$. Since $B^{\tau_a} \perp \mathcal{F}_{\tau_a}$, then $B^{\tau_a} \perp \tau_a$. Then can conclude $(\tau_a, B^{\tau_a}) \stackrel{d.}{=} (\tau_a, -B^{\tau_a})$, then

$$
\begin{aligned}
\mathbb{P}[S_t \geq a, B_t \leq b] &= \mathbb{P}[\tau_a \leq t, B^{\tau_a}_{t-\tau_a} \leq b - a] \\
&= \mathbb{P}[\tau_a \leq t, -B^{\tau_a}_{t-\tau_a} \leq b - a] \\
&= \mathbb{P}[\tau_a \leq t, B^{\tau_a}_{t-\tau_a} \geq a - b] \\
&= \mathbb{P}[\tau_a \leq t, B^{\tau_a}_{t-\tau_a} + a \geq 2a - b] \\
&= \mathbb{P}[\tau_a \leq t, B^{\tau_a}_{t-\tau_a} + B_{\tau_a} \geq 2a - b] \\
&= \mathbb{P}[\tau_a \leq t, B_t \geq 2a - b] \\
&\text{,where } B_t \geq 2a - b, a \geq b \iff B_t \geq a \iff \tau_a \leq t \\
&\text{, so } \{\omega | B_t(\omega) \geq 2a - b\} = \{\omega | \tau_a(\omega) \leq t\} \ a.s. \\
&= \mathbb{P}[B_t \geq 2a - b]
\end{aligned}
$$

, since $B_t \sim N(0, 1)$, then $\mathbb{P}(B_t \geq 2a - b) = 1 - \Phi((2b - a)/\sqrt{t})$.

```
## gen distribution simulation on every t after tau_u}
u = 10
tau = np.min(np.argwhere(BM>=u))
t_new = t[tau:]
Ws = BM[tau:]-u

Ws0 = Ws[0]
WsSim = [-(np.cumsum(np.random.normal(0,1,1000-tau))+Ws0)+u for i in range(5000)]
## plot}
plt.figure(figsize=(8,3))
plt.plot(t,BM)
plt.axhline(u,linestyle=":",c="red")
[plt.plot(t_new,WsSim[i],c="orange") for i in range(len(WsSim))]
plt.plot(t,BM, c="blue")
plt.legend(["$B_t$","u","$-W_t+u$"])
plt.xlabel("t")
plt.ylabel("Stock Return")
```



We can see that for current time t, $W_s = B_t - u, s + \tau_u = t$ is BM, $W_s \stackrel{d.}{=} -W_s$, $W_s + u$ symmetric around $u = 10$ here, so does $-W_s + u$.

Corollary. Marginal of Probability/Bachelier's Theorem. $\mathbb{P}(\max_{u \in [0,t]} B_u \geq a) = 2\mathbb{P}(B_t \geq a)$

Proof. $\mathbb{P}(S_t \geq a) = \mathbb{P}(S_t \geq a, B_t \geq a) + \mathbb{P}(S_t \geq a, B_t \leq a)$
$= \mathbb{P}(B_t \geq a) + \mathbb{P}(B_t \geq 2a - a) = 2\mathbb{P}(B_t \geq a)$

Corollary. $\mathbb{P}(|B_t| \geq a) = 2\mathbb{P}(B_t > a)$ given $a > 0$

Proof. Since $B_t \sim N(0, t)$, then $\mathbb{P}(|B_t| \geq a) = \int_a^\infty x dF(x) + \int_{-\infty}^a x dF(x) = 2\int_a^\infty x dF(x) = 2\mathbb{P}(B_t \geq a)$

Remark. $M_t = \max_{u \in [0,t]} B_u \stackrel{d.}{=} |B_t|$

Proof. $\mathbb{P}(\max_{u \in [0,t]} B_u \geq a) = 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a)$, same CDF, then equal in distribution.

Corollary. Joint distribution of S_t, B_t. For pair $(S_t, B_t) = (x, y)$ then joint cdf is

$$f(x, y) = \frac{2(2x - y)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2x - y)^2}{2t}\right) \mathbb{I}_{x>0, y<x}$$

3.2 Special Case

Given standard Brownian Motion B_t,

3.2.1 Geometry Brownian Motion

Prop. $M_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ is a martingale. [Louis Bachelier, Albert Einstein, 1905]

Proof. (1) M_t is adapted since B_t is adapted

(2) Integrability: $\mathbb{E}[|M_t|] = \mathbb{E}[M_t] = e^{-\frac{1}{2}\lambda^2 t}\mathbb{E}[e^{\lambda B_t}] = 1 < \infty$, since if $\xi \sim N(\mu, \sigma^2)$, then $\mathbb{E}[e^{t\xi}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

(3) Martingality:

$$\begin{aligned}
\mathbb{E}[e^{\lambda B_t - \frac{1}{2}\lambda^2 t}|\mathcal{F}_s] &= e^{-\frac{1}{2}\lambda^2 t}\mathbb{E}[e^{\lambda B_t}|\mathcal{F}_s] \\
&= e^{-\frac{1}{2}\lambda^2 t}\mathbb{E}[e^{\lambda B_s + \lambda(B_t - B_s)}|\mathcal{F}_s] \\
&= e^{-\frac{1}{2}\lambda^2 t + \lambda B_s}\mathbb{E}[e^{\lambda(B_t - B_s)}] \\
&= e^{-\frac{1}{2}\lambda^2 t + \lambda B_s} e^{-\frac{1}{2}\lambda^2 (t-s)}, t > s \\
&= e^{\lambda B_s - \frac{1}{2}\lambda^2 s} =: M_s
\end{aligned}$$

Prop. $\mathbb{E}[e^{\frac{1}{2}\lambda^2 \tau_a}] = e^{-\lambda a}$, where $\tau_a = \inf\{t \geq 0 : M_t = a\}$.

Proof. Since $\mathbb{E}[M_{t\wedge\tau_a}] = \mathbb{E}[M_0] = e^0 = 1$ by OST. Since $\tau_a < \infty$ a.e., then $\lim_{t\to\infty} M_{t\wedge\tau_a} = M_{\tau_a} = e^{\lambda a - \frac{1}{2}\lambda^2 a}$. Since $0 \le M_{t\wedge\tau_a} \le e^{\lambda a}$ for $\tau_a > 0$, by DCT

$$1 = \lim_{t\to\infty} \mathbb{E}[M_{t\wedge\tau_a}] = \mathbb{E}[\lim_{t\to\infty} M_{t\wedge\tau_a}] = \mathbb{E}[M_{\tau_a}] = \mathbb{E}[e^{\lambda a - \frac{1}{2}\lambda^2\tau_a}] \Rightarrow \mathbb{E}[e^{\frac{1}{2}\lambda^2\tau_a}] = e^{-\lambda a}$$

Further, if $a > 0$, then $\mathbb{E}[e^{-\alpha\tau_a}] = e^{-\sqrt{2\alpha}a}$, like Laplace Transform.

3.2.2 Compensated Square of Brownian Motion

Prop. $M_t = B_t^2 - t$ is a martingale.
Proof. (1) M_t is adapted since B_t^2 is adapted
(2) Integrability: $\mathbb{E}[|M_t|] = \mathbb{E}[|B_t^2 - t|] \le \mathbb{E}[B_t^2] + \mathbb{E}[t] = t + t < \infty$
(3) Martingality:

$$\begin{aligned}
\mathbb{E}[B_t^2 - t|\mathcal{F}_s] &= \mathbb{E}[(B_s + B_t - B_s)^2|\mathcal{F}_s] - t \\
&= \mathbb{E}[B_s^2 + (B_t - B_s)^2 + 2B_s(B_t - B_s)|\mathcal{F}_s] - t \\
&= B_s^2 + \mathbb{E}[(B_t - B_s)^2] + 2\mathbb{E}[B_s(B_t - B_s)] - t \\
&= B_s^2 + (t - s) - t \\
&= B_s^2 - s = M_s
\end{aligned}$$

Prop. $\mathbb{E}[\tau_a \wedge \tau_b] = -ab$
Proof.[1] $\mathbb{E}[M_{\tau_a\wedge\tau_b}] = 0$ by martingale, $\mathbb{E}[M_{\tau_a\wedge\tau_b}] = \mathbb{E}[B^2_{\tau_a\wedge\tau_b}] - \mathbb{E}[\tau_a \wedge \tau_b]$, then, $\tau = \tau_a \wedge \tau_b$

$$\begin{aligned}
E(B_\tau^2 - \tau) &= 0 \\
E(B_\tau^2 \mid B_\tau = a)P(B_\tau = a) + E(B_\tau^2 \mid B_\tau = b)P(B_\tau = b) - E(\tau) &= 0 \\
a^2\frac{b}{b-a} + b^2\frac{a}{a-b} - E(\tau) &= 0 \\
E(\tau) &= \frac{a^2b - b^2a}{b-a} \\
E(\tau) &= -ab.
\end{aligned}$$

Prop. $\mathbb{E}[(\tau_a \wedge \tau_b)^2] = -ab(a^2 - 3ab + b^2)$
For a new BM

$$\begin{aligned}
E(B_\tau^3 - 3\tau B_\tau) &= 0 \\
E(B_\tau^3 \mid B_\tau = a)P(B_\tau = a) + E(B_\tau^3 \mid B_\tau = b)P(B_\tau = b)& \\
-3E(\tau B_\tau \mid B_\tau = a)P(B_\tau = a) - 3E(\tau B_\tau \mid B_\tau = b)P(B_\tau = b) &= 0 \\
a^3\frac{b}{b-a} + b^3\frac{a}{a-b} - 3aE(\tau \mid B_\tau = a)\frac{b}{b-a} - 3bE(\tau \mid B_\tau = b)\frac{a}{a-b} &= 0 \\
a^3b - b^3a - 3abE(\tau \mid B_\tau = a) + 3abE(\tau \mid B_\tau = b) &= 0 \\
E(\tau \mid B_\tau = b) - E(\tau \mid B_\tau = a) &= \frac{b^2 - a^2}{3}.
\end{aligned}$$

We have

$$\begin{aligned}
E(\tau \mid B_\tau = a)P(B_\tau = a) + E(\tau \mid B_\tau = b)P(B_\tau = b) &= E(\tau) \\
E(\tau \mid B_\tau = a)\frac{b}{b-a} + E(\tau \mid B_\tau = b)\frac{a}{a-b} &= -ab \\
bE(\tau \mid B_\tau = a) - aE(\tau \mid B_\tau = b) &= -ab(b-a).
\end{aligned}$$

Combine this with the equation earlier to get a system of equations you need to solve.

You will find that $E(\tau \mid B_\tau = a) = \frac{a^2-2ab}{3}$ and $E(\tau \mid B_\tau = b) = \frac{b^2-2ab}{3}$. Now use the second hint in the link.

$$
\begin{aligned}
E(B_\tau^4 - 6\tau B_\tau^2 + 3\tau^2) &= 0 \\
E(B_\tau^4 \mid B_\tau = a)P(B_\tau = a) + E(B_\tau^4 \mid B_\tau = b)P(B_\tau = b) & \\
-6E(\tau B_\tau^2 \mid B_\tau = a)P(B_\tau = a) - 6E(\tau B_\tau^2 \mid B_\tau = b)P(B_\tau = b) & \\
+3E(\tau^2) &= 0 \\
a^4 \frac{b}{b-a} + b^4 \frac{a}{a-b} - 6a^2 \frac{b}{b-a} E(\tau \mid B_\tau = a) - 6b^2 \frac{a}{a-b} E(\tau \mid B_\tau = b) + 3E(\tau^2) &= 0 \\
a^4 \frac{b}{b-a} + b^4 \frac{a}{a-b} - 6a^2 \frac{b}{b-a} \frac{a^2-2ab}{3} - 6b^2 \frac{a}{a-b} \frac{b^2-2ab}{3} + 3E(\tau^2) &= 0 \\
ab(a^2 - 3ab + b^2) + E(\tau^2) &= 0.
\end{aligned}
$$

So, $E(\tau^2) = -ab(a^2 - 3ab + b^2)$

3.2.3 Integral of Brownian Motion

Prop. $M_t = \int_0^t B_u du - tB_t$ is a martingale.
Proof. (1) M_t is adapted obviously
(2) Integrability: $\mathbb{E}[|M_t|] = \mathbb{E}[|\int_0^t B_u du - tB_t|] \leq \int_0^t \mathbb{E}[|B_u|]du + t\mathbb{E}[|B_t|]$ by funibi and triangle inequality, since $\mathbb{E}[|B_u|] = \mathbb{E}[\sqrt{B_u^2}] \leq \sqrt{\mathbb{E}[B_u^2]} = \sqrt{u}$ by Jensen Inequality, then $\mathbb{E}[|M_t|] \leq \frac{2}{3}t^{3/2} + t^{3/2} < \infty$
(3) Martingality:

$$
\begin{aligned}
\mathbb{E}[\int_0^t B_u du - tB_t | \mathcal{F}_s] &= \mathbb{E}[\int_0^t B_u du | \mathcal{F}_s] - tB_s \\
&= \int_0^s B_u du - tB_s + \mathbb{E}[\int_s^t B_u du | \mathcal{F}_s] \\
&, \text{ where } \mathbb{E}[\int_s^t B_u du | \mathcal{F}_s] = \mathbb{E}[\int_s^t (B_u - B_s)du + \int_s^t B_s du | \mathcal{F}_s] \\
&\qquad = \mathbb{E}[\int_s^t (B_u - B_s)du] + \mathbb{E}[\int_s^t B_s du] \\
&\qquad = 0 + B_s(t-s) \\
&= \int_0^s B_u du - sB_s =: M_s
\end{aligned}
$$

3.2.4 Selling a house

Example. sell a house when a target is reached, the target grows inear in time t with rate r, house price is B.M. B_t, find $\mathbb{P}$(we will sell the house after 1 year).
Solution.

$$
\begin{aligned}
\mathbb{P}(B_t > rt, t \geq 1) &= \mathbb{P}(\frac{B_t}{t} > r, \exists t \geq 1) \\
&= \mathbb{P}(tB_{\frac{1}{t}} > r, \exists t \in [0,1]), tB_{\frac{1}{t}} \overset{d.}{=} B_t \\
&= \mathbb{P}(\max_{u \in [0,1]} W_u \geq r), W_t = tB_{\frac{1}{t}} \\
&= 2\mathbb{P}(W_1 \geq r), \text{Bachelier's Theorem} \\
&= 2(1 - \Phi(r))
\end{aligned}
$$

3.2.5 Credit Rating/Black-Cox Model

Example. Black-Cox Model on credit rating. suppose company value satisfies $S_t = S_0 \exp(\sigma B_t)$ (historical return 0, variance ¿0), define barrier D, $D < S_0$, and stopping time $\tau_D = \inf\{t \geq 0 : S_t = D\}$ as company value drop to D from above. What is $\mathbb{P}(\tau_D > t)$?

Solution.

$$\begin{aligned}\mathbb{P}(\tau_D > t) &= \mathbb{P}(\min_{u\in[0,t]} S_u > D)\\ &= \mathbb{P}(S_0 \exp(\min_{u\in[0,t]} \sigma B_u) > D)\\ &= \mathbb{P}(\min_{u\in[0,t]} \sigma B_u > \log \frac{D}{S_0})\\ &= 1 - \mathbb{P}(\max_{u\in[0,t]} -B_u \geq -\frac{1}{\sigma} \log \frac{D}{S_0})\\ &= 1 - 2\mathbb{P}(-B_t \geq -\frac{1}{\sigma} \log \frac{D}{S_0}), \text{Bachelier's Theorem}\\ &= 1 - 2\Phi(\frac{\log(D/S_0)}{\sigma\sqrt{t}})\end{aligned}$$

3.2.6 Pricing and Hedging Barrier Options

Example. Lookback option. $\mathbb{E}[F(\max_{u\in[0,T]} S_u, \min_{u\in[0,T]} S_u, S_T)]$ =current price

Example. Barrier option.[2][3] [4] Up and out Barrier Option $F(S_T)\mathbb{I}_{\max_{t\in[0,T]} S_t<u}$ where u is up barrier. Knock out/option dead if go above u. Can use static hedge to replicate. Consider stock $S_t = B_t$, and value function $\hat{F}(x) = \begin{cases} F(x), x \leq u \\ 0, x \geq u \end{cases}$ since we sell out/option dead after $x \geq u$. What is current price, i.e. $\mathbb{E}[\hat{F}(B_T)\mathbb{I}_{\max_{t\in[0,T]} B_t<u}] \stackrel{?}{=} \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(2u - B_T)]$.

Solution. First, we know $\mathbb{E}[\hat{F}(B_T)\mathbb{I}_{\max_{t\in[0,T]} B_t<u}] = \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(B_T)\mathbb{I}_{\max_{t\in[0,T]} B_t\geq u}]$

Consider stopping time $T_u := \inf\{t \geq 0 : B_t = u\}$, and new BM $W_s := B_{s+T_u} - B_{T_u} = B_{T_u+s} - u$, $B_{T_u+s} = B_{T_u} - (B_{T_u+s} - B_{T_u})$, $W_s \perp \mathcal{F}_{T_u}$, that is, suppose BM cross u and W_s is new BM starting from T_u. Then,

$$\begin{aligned}\mathbb{E}[\hat{F}(B_T)\mathbb{I}_{\max_{t\in[0,T]} B_t<u}] &= \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(B_T)\mathbb{I}_{\max_{t\in[0,T]} B_t\geq u}]\\ &= \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(W_{T-T_u} + B_{T_u})\mathbb{I}_{T_u\leq T}], s + T_u = T\\ &= \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(W_{T-T_u} + u)\mathbb{I}_{T_u\leq T}]\\ &= \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(-W_{T-T_u} + u)\mathbb{I}_{T_u\leq T}]\\ &\quad, W_t \stackrel{d.}{=} -W_t \text{ symmetric around reflection line+LS msr translation invariant}\\ &= \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(-B_T + u + u)\mathbb{I}_{\max_{t\in[0,T]} B_t\geq u}]\\ &= \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(2u - B_T)\mathbb{I}_{B_T\geq u}], \{B_T \geq u\} \cap \{\max_{t\in[0,T]} B_t \geq u\} = \{B_T \geq u\}\\ &= \mathbb{E}[\hat{F}(B_T)] - \mathbb{E}[\hat{F}(2u - B_T)], \text{by construction of } \hat{F}\end{aligned}$$

Remark: Then to replicate the Barrier Option, buy the European option with payoff $\hat{F}(B_T)$ and $-\hat{F}(2u - B_T)$. **Static Hedging**
Trading Strategy. If the underlying asset hits u before time T, then we sell the portfolio, and value is,

$$\begin{aligned}&\mathbb{I}_{T_u<T}\mathbb{E}[\hat{F}(B_T) - \hat{F}(2u - B_T)|\mathcal{F}_{T_u}]\\ &= \mathbb{I}_{T_u<T}\mathbb{E}[\hat{F}(W_{T-T_u} + u) - \hat{F}(u - W_{T-T_u})|\mathcal{F}_{T_u}], \text{strong Markov}\\ &= \mathbb{I}_{T_u<T}\mathbb{E}[\hat{F}(W_{T-x} + u) - \hat{F}(u - W_{T-x})]|_{x=T_u}\\ &= \mathbb{I}_{T_u<T}\mathbb{E}[\hat{F}(W_{T-x} + u) - \hat{F}(u + W_{T-x})]|_{x=T_u}, W_{T-x} \stackrel{d.}{=} -W_{T-x}\\ &= 0\end{aligned}$$

If the underlying asset never hits u until time T, i.e. $B_t \le u$, then the value is $\mathbb{E}[\hat{F}(B_T)\mathbb{I}_{\max\limits_{t\in[0,T]} B_t<u}] = \mathbb{E}[\hat{F}(B_T)]$.

Example. Digital Binary option. Consider stopping time $T_{-1} = \inf\{t \ge 0 : B_t = -1\}$, payoff structure $\mathbb{I}_{\{T_{-1}<T, \max\limits_{t\in[T_{-1},T]} B_t\ge 1\}}$, compute the price $\mathbb{E}[\mathbb{I}_{\{T_{-1}<T, \max\limits_{t\in[T_{-1},T]} B_t\ge 1\}}] = \mathbb{P}(T_{-1} < T, \max\limits_{t\in[T_{-1},T]} B_t \ge 1\})$.

Solution. denote $W_t = B_{T_{-1}+t} + 1$, then W_t is a B.M., $W_t \perp \mathcal{F}_{T_{-1}}$, we can clearly see again $\mathbb{E}(W_t) = \mathbb{E}(W_0) = \mathbb{E}[B_{T_{-1}+t} + 1] = -1 + 1 = 0$. Then,

$$\begin{aligned}
&\mathbb{P}(T_{-1} < T, \max_{t\in[T_{-1},T]} B_t \ge 1) \\
&= \mathbb{P}(T_{-1} < T, \max_{t\in[0,T-T_{-1}]} W_t \ge 2) \\
&= \mathbb{P}(T_{-1} < T, \max_{t\in[0,T-T_{-1}]} -W_t \ge 2), \text{reflection} \\
&= \mathbb{P}(T_{-1} < T, \min_{t\in[0,T-T_{-1}]} W_t \le -2) \\
&= \mathbb{P}(T_{-1} < T, \min_{t\in[T_{-1},T]} B_t \le -3) \\
&= \mathbb{P}(\min_{t\in[0,T]} B_t \le -1, \min_{t\in[T_{-1},T]} B_t \le -3) \\
&= \mathbb{P}(\min_{t\in[T_{-1},T]} B_t \le -3) \\
&= \mathbb{P}(\max_{t\in[T_{-1},T]} -B_t \ge 3) \\
&= \mathbb{P}(\max_{t\in[T_{-1},T]} B_t \ge 3), \text{reflection} \\
&= 2\mathbb{P}(B_T \ge 3), \text{Bachelier's Theorem} \\
&= 2(1 - \Phi(\frac{3}{\sqrt{T}}))
\end{aligned}$$

3.2.7 Brownian Bridge

Example. Brownian Bridge. consider dynamic $X_t^{0,0} = B_t - \frac{t}{T}B_T$, then $X_t^{0,0}$ is a Brownian Bridge, its covariance is $c(s,t) = s \wedge t - \frac{st}{T}$. And process $X_t^{a,b} = \underbrace{a + \frac{b-a}{T}t}_{\text{drift term}} + \underbrace{X_t^{0,0}}_{\text{noise term}}$.

Theorem. Model informed trader. For dynamic $Z_t = a + B_t$, B is BM, condition Z on $Z_T = b$, then $\mathbb{E}[F((Z_t)_{t\in[0,T]})|Z_T = b] = \mathbb{E}[F((X_t^{a,b})_{t\in[0,T]})]$, i.e. same law.

3.2.8 Fractional Brownian Motion

Example. Fractional Brownian Motion. Consider process X_t s.t. its covariance is $c(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), H \in (0,1)$ as Horst factor.

Property. (1) $H = \frac{1}{2}$, then X_t is a B.M.

(2) $H > \frac{1}{2}$, then X_t increments that positively correlated, X_t has long-range dependence, can be served as volatility model

(3) $H < \frac{1}{2}$, then X_t increments negatively correlated.

4 SA04: Ito Integral

4.1 Ito integrals for simple process

4.1.1 Ito integral

Def. Multi-dimensional Brownian Motion. $B_t = (B_t^1, .., B_t^d) \in \mathbb{R}^d$ is a Brownian Motion in $\mathbb{R}^d$ if each coordinate is Brownian Motion and mutually independent.

Def. Itô/Stochastic Integration. $I[\Delta](t) = \int_0^t \Delta_u dB_u$ or $\int_0^t \Delta_u dW_u$, where dB_u has infinite variation, so Reimann-Stietljes inegral doesn't work here, but its finite quadratic variation allows SDE by Itô Lemma.

4.1.2 Integrant is simple process

Def. Simple Stochastic Process. A stochastic process $(\Delta_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is simple if $\Delta_t = \sum_{i=0}^{n-1} a_i \mathbb{I}_{(t_i,t_{i+1}]}$ on partition $\Pi = \{0 \le t_0 \le t_1, .., \le t_n = T\}$, and random variable $a_i \in \mathcal{L}_0(\mathcal{F}_{t_i})$.
Remark. Simple process is adapted by definition.
Def. Itô integrals for simple process. When Δ_u is simple function, then $I[\Delta](t) = \int_0^t \Delta_u dB_u = \sum_{t_i<t} a_i(B_{t_{i+1}\wedge t} - B_{t_i})$.

4.1.3 6 Properties

Theorem. If Δ is simple, then $\{I[\Delta](t)\}_{t\ge 0}$ has following property:
(1) adapted, (2) it's linear operator w.r.t. Δ, (3) continuous,
(4) martingale,
(5) Itô isometry (same length in different Hilbert Space), i.e. $\mathbb{E}[(I[\Delta](t))^2] = \mathbb{E}[\int_0^t \Delta_u^2 du]$, LHS is norm of integral (on $d\mathbb{P}$), RHS is integral of norm of integrand (on $dt \otimes d\mathbb{P}$, where dt is Lebesgue measure),
(6) quadratic variation, bi-linear form $[I[\Delta], I[\Delta]](t) = \int_0^t \Delta_u^2 du$, where RHS can still be random variable,
(6') quadratic co-variation, $[I[\Delta^1], I[\Delta^2]](t) = \int_0^t \Delta_u^1 \Delta_u^2 du$.
$Proof-(1)(2)$ True by definition
$Proof-(3)$ Since integration w.r.t. Lebesgue measure is continuous. And summation preserve continuous.
$Proof-(4)$ WLOG, consider a partition $\Pi = \{0 = t_0 < t_1 < .. < t_l < t_{l+1} < .. < t_n = t\}$ then $\mathbb{E}[I[\Delta](t)|\mathcal{F}_s] = \mathbb{E}[\sum_{i=1}^n a_i(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_s]$, where
(1) $t_l < s < t_{l+1}$, $\mathbb{E}[a_l(B_{t_{l+1}} - B_{t_l})|\mathcal{F}_s] = a_l(B_s - B_{t_l})$
(2) $t_l > s$, $\mathcal{F}_s \subset \mathcal{F}_{t_l}$, $\mathbb{E}[a_l(B_{t_{l+1}} - B_{t_l})|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[a_l(B_{t_{l+1}} - B_{t_l})|\mathcal{F}_{t_l}]|\mathcal{F}_s]$, Tower property
$= \mathbb{E}[a_l(\mathbb{E}[B_{t_{l+1}}|\mathcal{F}_{t_l}] - B_{t_l})|\mathcal{F}_s] = 0$
Then, $\mathbb{E}[I[\Delta](t)|\mathcal{F}_s] = \sum_{i=1}^{l-1} a_i(B_{t_{i+1}} - B_{t_i}) + a_l(B_s - B_{t_l}) = I[\Delta][s]$
$Proof-(5)$ $\mathbb{E}[(I[\Delta](t))^2] = \mathbb{E}[\sum a_i a_j (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})]$, then
(1) for $i < j$, $\mathbb{E}[a_i a_j (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})]$
$= \mathbb{E}[\mathbb{E}[a_i a_j (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})]|\mathcal{F}_{t_j}]$
$= \mathbb{E}[a_i a_j (B_{t_{i+1}} - B_{t_i})\mathbb{E}[B_{t_{j+1}} - B_{t_j}|\mathcal{F}_{t_j}]] = 0$ by independent increment, similar for $j < i$
(2) for $i = j$, $\mathbb{E}[a_i a_i (B_{t_{i+1}} - B_{t_i})^2] = \mathbb{E}[\mathbb{E}[a_i^2 (B_{t_{i+1}} - B_{t_i})^2]|\mathcal{F}_{t_i}]$
$= \mathbb{E}[a_i^2 \mathbb{E}[(B_{t_{i+1}} - B_{t_i})^2]]$, independent
$= \mathbb{E}[a_i^2 (t_{j+1} - t_j)]$, Gaussian distribution
So, $\mathbb{E}[\sum a_i a_j (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})] = \mathbb{E}[\sum a_i^2 (t_{j+1} - t_j)] =: \mathbb{E}[\int_0^t (\Delta_u)^2 du]$ (by (6))
性质5的意义在于，可以从求和推到积分，而且根据(3)在时间上连续，则性质5亦对一个process成立。
$Proof-(6)$ Recall quadratic variation $[X,X](s,t) = \lim_{||\Pi||\to 0} \sum (X_{t_{i+1}} - X_{t_i})^2$, Π is a mesh on time itnerval $[s,t]$, (and quadratic covariation $[X,Y](s,t) := \lim_{||\Pi||\to 0} \sum (X_{t_{i+1}} - X_{t_i})(Y_{t_{i_1}} - Y_{t_i})$), and Lemma $[X,X](0,t) + [X,X](t,T) = [X,X](0,T)$,
Then, $X_t := I[\Delta](t)$, so $[X,X](t) = \sum_{i=0}^{n-1} [X,X](t_i, t_{t+1})$. Now $(X_t - X_{t_i})\mathbb{I}_{t\in(t_i,t_{i+1}]}(t) = a_i(B_{t\wedge t_{i+1}} - B_{t_i})$
Thus, $[X,X](t_i, t_{i+1}) = [a_i B, a_i B](t_i, t_{i+1}) \overset{p.}{=} a_i^2 (t_{i+1} - t_i)$ by finite Quadratic Variation, so $[X,X](0,t) = \sum a_i^2 (t_{i+1} - t_i) =: \int_0^T \Delta_u^2 du$

4.2 Ito integral for square integrable integrant

4.2.1 Progressively Measurable

Def. Measurable. A stochastic process $\Delta = (\Delta_t)_{t\in[0,T]}$ is measurable if $(\omega, u) \mapsto \Delta_u(\omega)$ is in $\mathcal{L}^0(\mathcal{F} \otimes \mathcal{B}([0,T]))$, i.e. $\mathcal{F} \otimes \mathcal{B}([0,T])-$measurable, where Δ_u is to extend space by u, and $(\omega, u) \in \Omega \times [0,T]$.
Def. Progressively Measurable. A stochastic process $\Delta = (\Delta_t)_{t\in[0,T]}$ is progressively measurable if $(\omega, u) \mapsto \Delta_u(\omega)$ is in $\mathcal{L}^0(\mathcal{F} \otimes \mathcal{B}([0,t]))$, i.e. no look into the future, where $(\omega, u) \in \Omega \times [0,t]$.

Property. A simple process is progressively measurable.
Property. If a process is progressively measurable, then it's adapted
Property. If a process is adapted and left/right-continuous, then it is progressively measurable
Property. X is progressively measurable, then $\int_0^t f(s, X_s) ds$ is progressively measurable, and $X_{t\wedge T}$ is progressively measurable.

4.2.2 Class of stochastic integrant - H2

Def. Space of Ito integral. $\mathcal{H}^2 := \{\Delta \text{ is progressively measurable, s.t. } \mathbb{E}[\int_0^T \Delta_u^2 \, d\mu] < \infty\}$, and $\mathcal{H}_0^2 := \{\Delta \text{ is simple, s.t. } \mathbb{E}[\int_0^T \Delta_u^2 \, d\mu] < \infty\}$.
Remark. $\mathcal{H}_0^2 \subset \mathcal{H}^2$, and $\mathcal{H}_0^2$ is dense in $\mathcal{H}^2$
Theorem. Take $\Delta \in \mathcal{H}^2$, then $\exists(\Delta^n) \subset \mathcal{H}_0^2$ such that $\mathbb{E}[\int_0^T (\Delta_t - \Delta_t^n)^2 dt] \overset{n\to\infty}{\to} 0$ [Shreve, Chap 3.]. Meaning, can approximate Δ in $L^2(dt \times d\mathbb{P})$ sense. Prove by approximate by simple function.
Def. Ito Integral. Take $\Delta \in \mathcal{H}^2$, consider $(\Delta^n) \subset \mathcal{H}_0^2$ such that $\mathbb{E}[\int_0^T (\Delta_t - \Delta_t^n)^2 dt] \overset{n\to\infty}{\to} 0$, then $I[\Delta](t) = \int_0^t \Delta_u dB_u := \lim_{n\to\infty} I[\Delta^n](t)$ in $\mathcal{L}^2$ ($\mathcal{L}^2$ limits)

4.2.3 Ito Integral is nice

Theorem. Ito Integral is well defined. The Ito integral exists and doesn't depend on the approximate sequence, i.e., well defined, or, any sequence that approximate Δ can also approximate the integral.
Proof. Limit Exists. Take a sequence $(\Delta^n)_{n\geq 0} \subset \mathcal{H}_0^2$ such that $\mathbb{E}[\int_0^T (\Delta_t - \Delta_t^n)^2 dt] \overset{n\to\infty}{\to} 0$, then
WantToShow: for $\forall n, m$, they converge to same value.
Then, take $n, m, n \neq m$, $\mathbb{E}[(I[\Delta^n](t) - I[\Delta^m](t))^2] = \mathbb{E}[(I[\Delta^n - \Delta^m])^2]$, linearity
$= \mathbb{E}[\int_0^t (\Delta_u^n - \Delta_u^m)^2 d\mu]$, Ito Isometry

> **Theorem. Minkowski's Inequality/Triangle Inequality.** Given $\xi, \eta \in \mathcal{L}_+^p, p \geq 1$, and μ is bounded measurable or $\sigma-$finite measurable. Then $(\int |\xi + \eta|^p d\mu)^{1/p} \leq (\int \xi^p d\mu)^{1/p} + (\int \eta^p d\mu)^{1/p}$, or $||\xi + \eta||_p \leq ||\xi||_p + ||\eta||_p$.

Then, $(\mathbb{E}[\int_0^t (\Delta_u^n - \Delta_u^m)^2 d\mu])^{1/2} \leq (\mathbb{E}[\int_0^t (\Delta_u^n - \Delta_u)^2 d\mu])^{1/2} + (\mathbb{E}[\int_0^t (\Delta_u^m - \Delta_u)^2 d\mu])^{1/2} \overset{n,m\to\infty}{\to} 0 + 0 = 0$, by property above $\mathbb{E}[\int_0^T (\Delta_t - \Delta_t^n)^2 dt] \overset{n\to\infty}{\to} 0$
So, $(I[\Delta^n])_{n\geq 0}$ is a Cauchy Sequence, then limit exists.
Proof. Limit Unique. Take two different sequences $(\Delta^{1,n}), (\Delta^{2,n})$ such that $\mathbb{E}[\int_0^T (\Delta_t - \Delta_t^{1,n})^2 dt] \overset{n\to\infty}{\to} 0$ and $\mathbb{E}[\int_0^T (\Delta_t - \Delta_t^{2,n})^2 dt] \overset{n\to\infty}{\to} 0$, then

$$\begin{aligned}
&\left(\mathbb{E}[(I[\Delta^{1,n}](t) - I[\Delta^{2,n}](t))^2]\right)^{1/2} \\
&= \left(\mathbb{E}[\int ((\Delta^{1,n} - \Delta_u) + (\Delta_u - \Delta^{2,n}))^2]\right)^{1/2}, \text{Ito Isometry} \\
&\leq (\mathbb{E}\int_0^t (\Delta^{1,n} - \Delta_u)^2)^{1/2} + (\mathbb{E}\int_0^t (\Delta^{1,n} - \Delta_u)^2)^{1/2}, \text{Triangle Inequality} \\
&\overset{n\to\infty}{\to} 0
\end{aligned}$$

> **Lemma. Completeness of $\mathcal{L}^p$.** On $(\Omega, \mathcal{F}, \mathbb{P})$, for any sequence $(\xi_n)_{n\geq 0}$ such that $\mathbb{E}[|\xi_n - \xi_m|^p] \overset{n,m\to\infty}{\to} 0$ or $||\xi_n - \xi_m||_p \to 0$, then $\exists$ a random variable ξ as the limit, i.e. $||\xi_n - \xi||_p \to 0$

Say $I[\Delta]$ is the L^2-limit of $I[\Delta^{1,n}]$, then for $I[\Delta^{2,n}]$ we have

$$\begin{aligned}
&(\mathbb{E}[(I[\Delta^{2,n}] - I[\Delta])^2])^{1/2} \\
&\leq (\mathbb{E}[(I[\Delta^{1,n}] - I[\Delta^{2,n}])^2])^{1/2} + (\mathbb{E}[(I[\Delta^{1,n}] - I[\Delta])^2])^{1/2} \\
&\overset{n\to\infty}{\to} 0 + 0 = 0
\end{aligned}$$

, where $(\mathbb{E}[(I[\Delta^{1,n}]-I[\Delta^{2,n}])^2])^{1/2} \overset{n\to\infty}{\to} 0$ by the existence property above, and $(\mathbb{E}[(I[\Delta^{1,n}]-I[\Delta])^2])^{1/2} \overset{n\to\infty}{\to} 0$ by the property above as $\mathbb{E}[\int_0^T(\Delta_t - \Delta_t^n)^2 dt] \overset{n\to\infty}{\to} 0$
So, $\Delta^{2,n}$ and $\Delta^{1,n}$ converge to same limit in $\mathcal{L}^2$.

4.2.4 Doob-Meyer Decomposition

Theorem. Doob-Meyer Decomposition. $(M_t)_{t\geq 0}$ is square integrable continuous martingale, $M_t^2 = N_t + [M,M]_t$ where N_t is martingale and $[M,M]_t$ is quadratic martingale.
Remark. Approximate $\int_0^t B_u dB_u$. Given $B \in \mathcal{H}^2$, so $\mathbb{E}\int_0^t B_u^2 du = \int_0^t \mathbb{E}B_u^2 du = \int_0^t udu < \infty$. Approximation:

$$\begin{aligned}
\int_0^t B(s)dB_s &= \lim_{n\to\infty}\sum_{k=1}^{n} B_{t_k'}\left(B_{t_k} - B_{t_{k-1}}\right)\\
&= \begin{cases} -\frac{1}{2}t + \frac{1}{2}B_t^2 & \text{if } B_{t_k'} = B_{k-1}, \text{ Ito Integral}\\ \frac{1}{2}t + \frac{1}{2}B_t^2 & \text{if } B_{t_k'} = B_k\\ \frac{1}{2}B_t^2 & \text{if } B_{t_k'} = \frac{B_{t_{k-1}}+B_{t_k}}{2}, \text{ Stratonovich Integral}\end{cases}
\end{aligned}$$

Ito integral is martingale, but Stratonovich integral is not martingale.
Ito Integral. Approximate BM with simple process. Take left end point, $B_u^n = \sum_{j=0}^{n-1} B_{jt/n}\mathbb{I}_{(jt/n,(j+1)t/n]}(u)$
(1) Check: $B^n \in \mathcal{H}_0^2$, that is,

$$\begin{aligned}
\mathbb{E}[\int_0^t (B_u^n - B_u)^2 du] &= \sum_{j=0}^{n-1}\int_{jt/n}^{(j+1)t/n} \mathbb{E}(B_{jt/n} - B_u)^2 du\\
&= \sum_{j=0}^{n-1}\int_{jt/n}^{(j+1)t/n} u - jt/n \; du\\
&= \sum_{j=0}^{n-1}\int_0^{t/n} wdw, \text{ change of variable}\\
&= \sum_{j=0}^{n-1}\frac{1}{2}\frac{t^2}{n^2} = \frac{t^2}{2n} \overset{n\to\infty}{\to} 0
\end{aligned}$$

So, B_u^n can approximate B_u in L^2.
(2) Approximation Limit/Integral,

$$\begin{aligned}
I[B^n](t) &= \sum_{j=0}^{n-1} B_{jt/n}(B_{(j+1)t/n} - B_{jt/n})\\
&= \sum_{j=0}^{n-1} B_{jt/n}B_{(j+1)t/n} - B_{jt/n}^2\\
&= \sum_{j=0}^{n-1}\frac{1}{2}(B_{(j+1)t/n}^2 - B_{jt/n}^2) - \frac{1}{2}(B_{(j+1)t/n} - B_{jt/n})^2\\
&= \frac{1}{2}B_t^2 - \frac{1}{2}\sum_{j=0}^{n-1}(B_{(j+1)t/n} - B_{jt/n})^2\\
&\overset{n\to\infty}{\to} \frac{1}{2}(B_t^2 - t)
\end{aligned}$$

Not adapted/Martingale - Stratonovich Integral. Define $\tilde{B}_u^n := \sum_{j=0}^{n-1}\frac{B_{jt/n}+B_{(j+1)t/n}}{2}\mathbb{I}_{(jt/n,(j+1)t/n]}$,
(1) can check $\tilde{B}_u^n \in \mathcal{H}^2$
(2) $I(\tilde{B}^n) = \sum_{j=0}^{n-1}\frac{1}{2}(B_{(j+1)t/n} + B_{jt/n})(B_{(j+1)t/n} - B_{jt/n}) = \sum_{j=0}^{n-1}\frac{1}{2}(B_{(j+1)t/n}^2 - B_{jt/n}^2) = \frac{1}{2}B_t^2$. Larger class than $\mathcal{H}^2$, not a martingale.

4.2.5 Properties, Again

Theorem. If $\Delta \in \mathcal{H}^2$, then $\{I[\Delta](t)\}_{t\geq 0}$ has following property:
(1) adapted, (2) it's linear operator w.r.t. Δ, (3) continuous in t,
(4) martingale,
(5) Itô isometry, i.e. $\mathbb{E}[(I[\Delta](t))^2] = \mathbb{E}[\int_0^t \Delta_u^2 du]$,
(6) quadratic variation, $[I[\Delta], I[\Delta]](t) = \int_0^t \Delta_u^2 du$.
$\underline{Proof-(3)}$ following section 2.3., consider simple approximation Δ^n, $X^n = I[\Delta^n](t)$, observe $|X^n - X^m|$ is a submartingale. Then,
$\mathbb{P}(\sup_{u\in[0,t]} |X_u^n - X_u^m| \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbb{E}[|X_t^n - X_t^m|^2] = \frac{1}{\varepsilon^2}\mathbb{E}[\int_0^t (\Delta_u^n - \Delta_u^m)^2 du], Ito\ Isometry \overset{n,m\to\infty}{\to} 0.$
Since Ito Integral is Nice by 2.3. Then pick subsequence $\{n_k\}$ such that $\mathbb{E}[\int_0^t (\Delta_u^{n_k} - \Delta_u^{n_{k+1}})^2 du] \leq 2^{-3k}$, by let $\varepsilon = 2^{-k}$, then $\mathbb{P}(\sup_{u\in[0,t]} |X_u^n - X_u^m| \geq 2^{-k}) \leq 2^{2k}\mathbb{E}[\int_0^t (\Delta_u^{n_k} - \Delta_u^{n_{k+1}})^2 du] \leq 2^{2k}2^{-3k} = 2^{-k}$

> **Theorem. Borel Cantalli Lemma** If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then with probability 1, only a finitly many of $\{A_n\}$ can occur.

With probability 1, only finitely many event $\{\sup_{u\in[0,t]} |X_u^{n_k} - X_u^{n_{k+1}}| \geq 2^{-k}\}$ can occur. Then choose $C = C_\omega > 0$ such that with probability 1, $\sup_{u\in[0,t]} |X_u^{n_k} - X_u^{n_{k+1}}| < 2^{-k}, k \geq C$, i.e., after C, the event cannot occur (since it only occur finitely many times).
Then, for $K' > K \geq C$, $\sup_{u\in[0,t]} |X_u^{n_K} - X_u^{n_{K'}}| \leq \sum_{i=1}^{K'-K} \sup_{u\in[0,t]} |X_u^{n_{K+i-1}} - X_u^{n_{K+i}}| < \sum_{i=1}^{K'-K} 2^{-K-i+1} < 2^{-K+1} \overset{K\to\infty}{\to} 0$

> **Theorem.** Consider a sequence of continuous functions $\{f_n\}$, $f_n : [0,T] \to \mathbb{R}$, $\sup_{u\in[0,T]} |f^n(u) - f^m(u)| \to 0$, then, there is a continuous function f such that $\sup_{u\in[0,T]} |f^n(u) - f(u)| \to 0$. i.e., in continuous space, Cauchy sequence has a limit. [Continuous function space is complete]

Since we have shown, the constructed approximation sequence is such that with probability 1, $\sup |X_u^{n_K} - X_u^{n_{K'}}| \to 0, K, K' \to \infty$. Then, there is a process X as Ito integral with continuous path, such that $\sup_{u\in[0,T]} |X_u^{n_K} - X_u| \to 0$.
Since $X^{n_K} \to I[\Delta](t)$, so X is a continuous modification of $I[\Delta]$.
$\underline{Proof-(4)}$ WTS: each of approximation sequence is a martingale, or, WTS, $\mathbb{E}[X_t^n|\mathcal{F}_s] = X_s^n$.

> **Lemma.** Let $\{X_n\}$ be a sequence of integrable random variable converging to X in L^1, then $\mathbb{E}[X_n|\mathcal{G}] \to \mathbb{E}[X|\mathcal{G}]$ in L^1 for any $\sigma-$algebra $\mathcal{G}$.
> *Proof.* $\mathbb{E}[\mathbb{E}[X_n - X|\mathcal{G}]] \leq \mathbb{E}[\mathbb{E}[|X_n - X||\mathcal{G}]] = \mathbb{E}[|X_n - X|] \to 0$ by tower property.

Then, by 2.3., X_t^n, X_s^n converge to X_t, X_s in L^2, thus converge in L^1, then there is a further subsequence $\{n_k\}$ such that $X_t^{n_k}, X_s^{n_k}$ converge almost surely to X_t, X_s in L^1. Thus LHS=$\mathbb{E}[X_t^{n_k}|\mathcal{F}_s] \to \mathbb{E}[X_t|\mathcal{F}_s]$, and RHS=$X_s^{n_k} \to X_s$. ■ Q.E.D.

4.2.6 More properties

Remark. $I[\Delta](s,t) = \int_s^t \Delta_u dB_u$, $I[\Delta](s,t) = -\int_t^s \Delta_u dB_u = I[\Delta](t) - I[\Delta](s)$ for $s > t$
Remark. If $s \leq t$, $I[\Delta](0,s) = \int_0^t \mathbb{I}_{[0,s]}(u)\Delta_u dB_u$
Theorem. Let $\Delta \in \mathcal{H}^2$, for any $0 \leq s \leq t$, $\mathbb{E}[(\int_s^t \Delta_u\ dB_u)^2|\mathcal{F}_s] = \mathbb{E}[\int_s^t \Delta_u^2\ du|\mathcal{F}_s]$
Theorem. Let $\Delta \in \mathcal{H}^2$, then $(I[\Delta](t))^2 - \int_0^t \Delta_s^2 ds$ is a martingale.
Remark. We can extend Ito integral to $(0,\infty)$, as $\int_0^\infty \Delta_u\ dB_u$

Theorem. Let $\Delta \in \mathcal{H}^2$, $0 \le \sigma \le \tau \le T$ be two stopping times, then $I[\Delta](\sigma,\tau) := \int_0^T \Delta_u \mathbb{I}_{\sigma \le u \le \tau} dB_u$.

5 SA05: Ito Formula and Local Martingale

5.1 Hloc class

5.1.1 Hloc class

Def. $\mathcal{H}^2_{Loc}$ On $(\Omega, \mathcal{F}, \mathbb{P})$ and $[0,T]$, $\mathcal{H}^2 := \{\Delta$ is progressively measurable, s.t. $\mathbb{E}[\int_0^T \Delta_u^2 \, d\mu] < \infty\}$, and $\mathcal{H}^2_{Loc} := \{\Delta$ is progressively measurable, s.t. $\mathbb{P}[\int_0^t \Delta_u^2 \, du < \infty] = 1, \forall t \in T, \forall \omega \in \Omega\}$
Remark. Any continuous process Δ is in $\mathcal{H}^2_{Loc}$
Remark. $\mathcal{H}^2_{Loc} \Rightarrow \mathcal{H}^2$, but $\mathcal{H}^2 \not\Rightarrow \mathcal{H}^2_{Loc}$
Corollary. If $X \in \mathcal{H}^2$ martingale, then $X^2 - [X,X]$ is a martingale by Doob's decomposition theorem.
Corollary. If $X, Y \in \mathcal{H}^2$ martingale, then $XY - [X,Y]$ is a martingale. Further, $Cov(X,Y) = 0$ when $[X,Y] = 0$ uncorrelated.

5.1.2 Stopping time

Def. Stopping time sequence. $\theta_n = T \wedge \inf\{t \ge 0 : \int_0^t \Delta_u^2 \, du \ge n\}$
Lemma. $\theta_n \nearrow T \wedge \infty$, for $T < \infty$, i.e. $\exists N(\omega)$ such that $\theta_n = T, \forall n \ge N(\omega)$ a.s.
Lemma. $(\Delta_t \mathbb{I}_{[t \le \theta_n]})_{t \in [0,T]} \in \mathcal{H}^2$
Def. Stopped Martingale Sequence. $M_t^n := I[\Delta_t \mathbb{I}_{[t \le \theta_n]}]$, we define $I[\Delta](t) = \lim_{n \to \infty} M_t^n$ a.s.
Theorem. If $\Delta \in \mathcal{H}^2_{Loc}$, then $I[\Delta]$ is well defined and nice
Remark. $M_{t \wedge \theta_n}^{n+1} = M_t^n$

5.1.3 Local Ito Integral 6 Properties

Theorem. 6 Properties. If $\Delta \in \mathcal{H}^2_{Loc}$, then $I[\Delta]$ satisfies,
(1) adapted, (2) linear in Δ, (3) continuity in t, i.e. $I[\Delta]$ has a modification with continuous path, (4)quadratic covariation $[I[\Delta^1], I[\Delta^2]](t) = \int_0^t \Delta_u^1 \Delta_u^2 du$, where $I[\Delta^1], I[\Delta^2]$ may not be martingale but local martingale.

5.2 Local Martingale

5.2.1 Local Martingale

Def. Localized Stopping time sequence. (1) For $T < \infty$, a sequence of stopping time $\{\tau_n\}$ is a localizing sequence, if with proability 1, $\exists$ random variable $N(\omega)$ s.t. for $\forall n \ge N$, $\tau_n(\omega) = T$. (2) For $T = \infty$, a sequence of stopping time $\{\tau_n\}$ is a localizing sequence, if $\tau_n(\omega) \to \infty, n \to \infty, \forall \omega \in \Omega$
Remark. Here we consider locally/$\sigma-$ finite, i.e. $\int_0^t H_t du < \infty$ instead of $\int_{-\infty}^{\infty} H_t du < \infty$. So, on $[0,T]$ the process may not be martingale, but on some $[0,\tau]$, it can be. One example for strict local martingale is $B_{r/t}$.
Def. Local Martingale. (M_t) is a local martingale if $\exists$ τ_n is a localizing sequence type-(2) ($\tau_n \nearrow \infty$), sequence $(M_{t \wedge \tau_n})$ is a (uniformly integrable) martingale, with $M_0 = 0$ a.s.
Def. Local Super/Sub-Martingale. (M_t) is a local super/sub-martingale if for τ_n is a localizing sequence, $\exists$ sequence $(M_{t \wedge \tau_n})$ is a true super/sub-martingale.
Remark. M_t may not be in L^1. M_0 can be any $\mathcal{F}_0-$measurable r.v.
Def. Stopping Time reduces Local Martingale. If $\tau_n \nearrow \infty$, and M_{τ_n} is a uniformly integrable martingale $\forall n$, then stopping time τ_n **reduces** M.

5.2.2 Local Martingale to Martingale

Prop. [GTM274, Prop4.7-(i)] A nonnegative continuous local martingale M s.t. $M_0 \in L^1$ is a supermartingale.
Proof. By definition, $\exists$ localizing sequence τ_n s.t. $M_{s \wedge \tau_n} = \mathbb{E}[M_{t \wedge \tau_n} | \mathcal{F}_s]$. By Fatou Lemma,

$\mathbb{E}[\liminf_{n\to\infty} M_{t\wedge\tau_n}|\mathcal{F}_s] \leq \liminf_{n\to\infty}\mathbb{E}[M_{t\wedge\tau_n}|\mathcal{F}_s] = \liminf_{n\to\infty} M_{s\wedge\tau_n}$. Since $\tau_n \overset{n\to\infty}{\to} \infty$, then $\liminf_{n\to\infty} M_{s\wedge\tau_n} = M_s$, then $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$.
Next, when $s = 0$, then take expectation on both side, $\mathbb{E}[M_t] \leq \mathbb{E}[M_0] < \infty$, $\forall t$, since $M_0 \in L^1$. Since M nonnegative, then $\mathbb{E}[|M_t|] = \mathbb{E}[M_t] < \infty$. Then by definition, M is a supermartingale.
Prop. [GTM274, Prop4.7-(ii)] If M is bounded continuous local martingale, i.e. if $\exists Z \in L^1(\mu)$ s.t. $|M_t| \leq Z, \forall t \geq 0$, then M is uniformly integrable martingale.
Proof. since (1) $\lim_{n\to\infty} M_{t\wedge\tau_n} = M_t, \tau_n \to \infty$, (2) $|M_{t\wedge\tau_n}| \leq Z$, then by Dominant Convergence Theorem, $M_t \in L^1$.
By definition, $M_{s\wedge\tau_n} = \mathbb{E}[M_{t\wedge\tau_n}|\mathcal{F}_s]$. Take $s = 0$, then $M_s = \mathbb{E}[M_t|\mathcal{F}_s]$. Then M_t is a Martingale.
Prop. [GTM274, Prop4.7-(iii)] If M is a continuous local martingale, $M_0 \in L^1(\mu)$, then special sequence $\tau_n = \inf\{t \geq 0 : |M_t| \geq n\}$ reduces M.
Proof. $|M_{\tau_n}| < n + |M_0|$ a.s., so by above proposition (DCT), M is martingale.
Prop. [Stochastic Integration and Differential Equations, 2nd ed, Protter, p.73-74] Given M_t is a local martingale, then $\mathbb{E}[M_t^2] < \infty, \forall t > 0$, M_t is a martingale $\iff$ $\mathbb{E}[\langle M, M\rangle_t] < \infty, \forall 0 \leq t < \infty$. Further, if $\mathbb{E}[\langle M, M\rangle_t] < \infty$, then $\mathbb{E}[\langle M, M\rangle_t] = \mathbb{E}[M_t^2]$.
Prop. If M is a local martingale, $\mathbb{E}[\langle M, M\rangle_t] < \infty, \forall t > 0$, then M is a square-integrable martingale, i.e. $\sup_t \mathbb{E}[M_t^2] = \mathbb{E}[M_\infty^2] < \infty$. Further, $\mathbb{E}[\langle M, M\rangle_t] = \mathbb{E}[M_t^2], \forall 0 \leq t \leq \infty$.
Prop. Positive local martingales are weak and unique solution to some SDEs. [SA09]
Prop. Novikov's Condition in SA06.
Def. Class DL. T_a is the class of all stopping times τ such that $\mathbb{P}(\tau \leq a) = 1$. DL is the class of all right-continuous processes $\{X_t\}_t$ such that $\{X_\tau\}_{\tau\in T_a}$ is uniformly integrable for every $0 < a < \infty$. Then X_t is of class DL.
Prop. M is a local martingale of class DL, $\iff$ M is a martingale.

5.2.3 Ito Integral as Local Martingale

Theorem. For $\Delta \in \mathcal{H}^2_{Loc}$, $I[\Delta]$ is a local martingale.
Theorem. $(I[\Delta](t))^2 - \int_0^t \Delta_u^2 du$ is a local martingale. Note $(\int_0^t \Delta_u du)^2 \neq \int_0^t \Delta_u^2 du$, w.g. take $\Delta_t = t$, but clearly $B_t^2 \neq t$.
Theorem. If (M_t) is a local submartingale and $\mathbb{E}[\sup_{t\in[0,T]} |M_t|] < \infty$, then M_t is a true submartingale.
Proof by DCT.

5.3 Ito Formula

5.3.1 Ito Formula

Theorem. Ito Formula. Given B_t a Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$, if $f : (t, x) \mapsto f(t, x)$, $f \in C^{1,2}([0, T]\times \mathbb{R})$, for any $t \in [0, T]$, $f(t, B_t) = f(0, 0) + \int_0^t f_t(u, B_u)du + \int_0^t f_x(u, B_u)dB_u + \frac{1}{2}\int_0^t f_{xx}(u, B_u)du$, where f_t, f_x, f_{xx} are partial derivative of $f(t, x)$.
Proof. Fix time, then for $f(x)$, $f(B_t) = \sum_{i=0}^{n-1}[f(B_{t_{i+1}}) - f(B_{t_i})] + f(B_0)$, take partition on t with mesh goes to 0, by Taylor Expansion, $f(B_t) = \sum_{i=0}^{n-1} f_x(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2}f_{xx}(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2 \overset{||\Pi||\to 0}{\to} \frac{1}{2}f_{xx}(B_{t_i})(t_{i+1} - t_i)$
Example. Expectation of Geometry BM. For $S_t = S_0 e^{\lambda t+\sigma B_t}$, take Ito's formula, then $dS_t = S_t\lambda dt+ S_t\sigma dB_t + \frac{1}{2}S_t\sigma^2 dt$. So, $S_t = S_0 + \int_0^t S_u(\lambda + \frac{\sigma^2}{2})du + \int_0^t S_u\sigma dB_u$.
Here, $\int_0^t S_u\sigma dB_u$ is a true martingale since $\sup_{0\leq u\leq t} |B_u|$ is integrable, so it is true martingale. Then, take continuous function $f(t) := \mathbb{E}(S_t)$. So, $f(t) = \mathbb{E}[S_0]+\int_0^t f(u)(\lambda+\frac{\sigma^2}{2})du$. Solving ODE $\begin{cases} f'(t) = (\lambda + \sigma^2/2)f(t) \\ f(0) = S_0 \end{cases}$, we have $f(t) = S_0 e^{(\lambda+\sigma^2/2)t}$.

5.3.2 Ito Process

Def. $\mathcal{H}^1$ space. $\mathcal{H}^1 := \{\Delta$ is progressively measurable, s.t. $\mathbb{P}[\int_0^T |\Delta_u|\ du < \infty] = 1\}$

Def. Ito Process. An adapted process X is an Ito process if $X_t = X_0 + \int_0^t \theta_u du + \int_0^t \Delta_u dB_u$ a.s., where B is a Brownian motion, $\theta \in \mathcal{H}^1$, and $\Delta \in \mathcal{H}^2_{Loc}$.
Note. $\int_0^t \theta_u du$ is a finite variation process, $\int_0^t \Delta_u dB_u$ is local martingale.
Note. $dX_t = \theta_t dt + \Delta_t dB_t$
Theorem. If X is Ito Process, $[X, X]_t = \int_0^t \Delta_u^2 du$
Theorem. If X^1, X^2 are Ito Processes w.r.t. same Brownian Motion, the covariation $[X^1, X^2] = \int_0^t \Delta_u^1 \Delta_u^2 du$
Def. Stochastic Integral. Given X is an Ito Process, $\mathcal{T}$ is progressively measurable, the stochastic intgral s.t. $\mathcal{T}$ w.r.t. X, $\mathcal{T} \cdot \theta \in \mathcal{H}^1$, $\mathcal{T} \cdot \Delta \in \mathcal{H}^2_{Loc}$, is that $\int_0^t \mathcal{T}_u dX_u = \int_0^t \mathcal{T}_u \theta_u du + \int_0^t \mathcal{T}_u \Delta_u dB_u$ a.s..
Theorem. Ito Process X_t is a local martingale if $\theta \equiv 0$.
Proof. Otherwise, if $\theta \neq 0$, since $\tilde{X}_t = X_0 + \int_0^t \Delta_u dB_u$ is a local martingale, then $X - \tilde{X}$ is also a local martingale, then $A_t := \int_0^t \theta_u du$ is a local martingale by linearity. Then by Doob's Meyer Decomposition, $A_t^2 - \langle A \rangle_t$ is a martingale ($\langle A \rangle_t = 0$ by finite variation). But $A_0 = 0$, so $\mathbb{E}[A_t^2] = 0 \Rightarrow A_t = 0$ a.s..

5.3.3 Ito Formula w.r.t. Ito Process

Theorem. Ito Formula w.r.t. Ito Process. If X is Ito process, $f \in C^{1,2}$, then $f(t, X_t) = f(0, 0) + \int_0^t f_t(u, X_u) du + \int_0^t f_x(u, X_u) dX_u + \frac{1}{2} \int_0^t f_{xx}(u, X_u) d[X, X]_u$
Remark. if $f \in C^2$ where f is defined a.s., then above theorem works only within where f is defined.
Theorem. if dt term is 0, then $f(t, X_t)$ is martingale since $\int_0^t \Delta_u dB_u$ is a martingale.

5.3.4 Example of Ito Process

Example. Exit probability of BM with drift. Consider dynamic $X_t = \mu t + \sigma B_t$, fix $-B < 0 < A$, consider $T_A = \inf\{t \geq 0 : X_t = A\}$, $T_B = \inf\{t \geq 0 : X_t = B\}$, and $T = T_A \wedge T_B = \inf\{t \geq 0 : X_t \in (-B, A)\}$. What is $\mathbb{P}(T_A < T_B)$?
To compute, can find smooth and bounded function $h : [-B, A] \to \mathbb{R}$ s.t. $h(-B) = 0, h(A) = 1$, and $h(X)$ is a martingale. Then by OST, $\mathbb{E}[h(X_{t \wedge T})] = \mathbb{E}[h(0)]$. Since X_t is either super-or-sub martingale by μ direction, thus $\mathbb{P}[T < \infty] = 1$. To find h, take Ito formula,

$$\begin{aligned} dh(X_t) &= h'(X_t) dX_t + \frac{1}{2} h''(X_t) d[X, X]_t \\ &= h'(X_t) \mu dt + h'(X_t) \sigma dB_t + h''(X_t) \frac{\sigma^2}{2} dt \end{aligned}$$

Since $h(X_t)$ is martingale, thus its local martingale, thus its dt term should vanish. So,

$$\begin{cases} h'(x)\mu + h''(x)\frac{\sigma^2}{2} = 0 \\ h(A) = 1, h(B) = 0 \end{cases}$$

then we can solve $h(x) = \frac{e^{-2\mu x/\sigma^2} - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}$. And it's easy to check $\mathbb{E}[\int_0^t (h'(X_u))^2 du] < \infty$. So indeed, h is a true martingale, then

$$\mathbb{E}[h(X_{t \wedge T})] = \mathbb{E}[h(0)] = h(0) = h(A)\mathbb{P}(T_A < T_B) + h(B)\mathbb{P}(T_B < T_A) \Rightarrow \mathbb{P}(T_A < T_B) = \frac{h(0) - h(-B)}{h(A) - h(-B)}$$

Example. Solve Linear SDE linear SDE $dS_t = \sigma_t S_t dB_t$, where σ_t is square integrable and progressively measurable, then $S_t = S_0 e^{\int_0^t \sigma_u dB_u - \frac{1}{2}\int_0^t \sigma_u^2 du}$. As $t \to \infty$, S_t converges by property of supermartingale of S_t (it's supermartingale since it's positive local martingale).
If $|\sigma_t| < \sigma$ bounded, then S_t is a martingale, by Novikov's condition/Kazamaki's condition
Example. Vasicek model/OU process given interest rate process as a mean-reversion between α, β (more stable) as $dr_t = (\alpha - \beta r_t)dt + \sigma dB_t$. Then $r_t = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dB_s$.
Here, $\sigma e^{-\beta t} \int_0^t e^{\beta s} dB_s$ is Gaussian, so r_t is Gaussian.
Example. CIR model/Feller process define $dr_t = (\alpha - \beta r_t)dt + \sigma \sqrt{r_t} dB_t$

then by Chain Rule, $d(e^{\beta t}r_t) = \alpha e^{\beta t}dt + \sigma e^{\beta t}\sqrt{r_t}dB_t$
Integrate on both side, $e^{\beta t}r_t = r_0 + \int_0^t \alpha e^{\beta s}ds + \int_0^t \sigma e^{\beta s}\sqrt{r_s}dB_s$
Take expectation on both side, $\mathbb{E}[e^{\beta t}r_t] = r_0 + \int_0^t \alpha e^{\beta s}ds + 0$, then $\mathbb{E}[r_t] = \frac{1}{e^{\beta t}}(r_0 + \frac{\alpha}{\beta}(e^{\beta t}-1))$. And then easy to get variation by Ito's Formula.

5.4 Levy Characterization

5.4.1 Quadratic Variation of Local Martingale

Theorem 4.9. Quadratic Variation of Local Martingale. [GTM274, p.79; GTM113, p.31] Let $M = (M_t)_{t\geq 0} \in \mathcal{M}^{c,Loc}$ (continuous and is Local martingale), then $\exists$ a unique (up to indistinguishability), adapted, continuous and increasing process $(\langle M, M\rangle_t)_{t\geq 0}$ such that $\langle M\rangle_0 = 0$ and $M_t^2 - \langle M, M\rangle_t \in \mathcal{M}^{c,Loc}$.

Proof. Since $M \in \mathcal{M}^{c,Loc}$, then there $\exists$ a stopping time process $\{S_n\} \nearrow \infty$ such that $M_t^{(n)} := M_{t\wedge S_n}$ is a martingale.

> Note. Martingale $\subset$ Local Martingale. e.g., Stratonovich Integral, $\frac{B_{t_{i+1}}+B_{t_i}}{2}$ is a local martingale but not martingale.

Idea: truncate local martingale to be bounded local martingale, then by DCT, it's true martingale.
Define stopping time $R_n = S_n \wedge \inf\{t \geq 0 : |M_t| = n\}$, $R_n \overset{n\to\infty}{\to} \infty$, and the stopped process is bounded by n.
Define stopped martingale $\widetilde{M}^{(n)} = M_{t\wedge R_n} \in M_c^2$ (continuous and square integrable martingale, true since it's bounded by n.) For any $m > n$, $\widetilde{M}_{t\wedge R_n}^{(m)} = \widetilde{M}_{t\wedge R_n}^{(n)}$ since they coincide before R_n.
Then by Doob-Meyer Decomposition, $(\widetilde{M}_{t\wedge R_n}^{(m)})^2 - \langle \widetilde{M}^{(m)}\rangle_{t\wedge R_n} = (\widetilde{M}_{t\wedge R_n}^{(n)})^2 - \langle \widetilde{M}^{(n)}\rangle_{t\wedge R_n}$ both martingale.
Since $(\widetilde{M}_{t\wedge R_n}^{(m)})^2 = (\widetilde{M}_{t\wedge R_n}^{(n)})^2$, so $\langle \widetilde{M}^{(m)}\rangle_{t\wedge R_n} = \langle \widetilde{M}^{(n)}\rangle_{t\wedge R_n}$. Since for $\forall t \leq R_n$ sequnce coincide with each other, then denote $\langle M\rangle_t := \langle \widetilde{M}^{(n)}\rangle_t$, we have $M_{t\wedge R_n}^2 - \langle M\rangle_{t\wedge R_n} = (\widetilde{M}_{t\wedge R_n}^{(n)})^2 - \langle \widetilde{M}^{(n)}\rangle_{t\wedge R_n}$ is a martingale.
So, $M^2 - \langle M\rangle \in \mathcal{M}^{c,Loc}$ with a localizing sequence of stopping time as R_n.
Remark. For every fixed $t > 0$, given partition, there is a further increasing subdivision as $0 = t_0^n < t_1^n < .. < t_{p_n}^n = t$ with mesh$\to 0$, then $\langle M, M\rangle_t \overset{p.}{=} \lim_{t\to\infty} \sum_{i=1}^{p_n}(M_{t_i^n} - M_{t_{i-1}^n})^2$. And process $\langle M, M\rangle$ is quadratic variation of M.
Lemma. Like R_n but set $S_n = n$, if M is a local martingale, then $\mathcal{T}_n := \inf\{t \geq 0 : |M_t| \geq n\} \wedge n$ is a localizing sequence.
Lemma. If $M \in \mathcal{M}^{c,Loc}$, S is stopping time, then $\mathbb{E}[M_s^2] \leq \mathbb{E}[\langle M\rangle_s]$ where $M_\infty^2 = \lim\inf_{t\to\infty} M_t^2$ in case limit not exist.
Proof. Since $M_t^2 - \langle M\rangle_t$ is a local martingale, define such $\mathcal{T}_n$ above as a localizing sequence. Then

$$\begin{aligned}
\mathbb{E}[M_{s\wedge\mathcal{T}_n}^2] &\overset{OST}{=} \mathbb{E}[\langle M\rangle_{s\wedge\mathcal{T}_n}], \text{since bounded in t and process} \\
\lim_{n\to\infty} \mathbb{E}[\langle M\rangle_{s\wedge\mathcal{T}_n}] &\overset{MCT}{=} \mathbb{E}[\langle M\rangle_s] \\
\mathbb{E}[M_{s\wedge\mathcal{T}_n}^2] &\geq \mathbb{E}[\liminf_n M_{s\wedge\mathcal{T}_n}^2], M_{s\wedge\mathcal{T}_n}^2 \text{ positive, and Fatou Lemma} \\
&=: \mathbb{E}[M_s^2]
\end{aligned}$$

Lemma/Def. Class D Given $M \in \mathcal{M}^{c,Loc}$, if $\mathbb{E}[\langle M\rangle_\infty] < \infty$, then M is a Uniformly Integrable Martingale. Then M is of class D, i.e. U.I. and martingale. U.I. is nice since the process will converge a.s. as $t \to \infty$.
Proof. Use Lemma just above, since $\mathbb{E}[M_s^2] \leq \mathbb{E}[\langle M\rangle] \leq \mathbb{E}\langle M\rangle_\infty < \infty, \forall s$ as stopping time, since M is L^2 bounded.

5.4.2 Levy Characterization of BM

Theorem. Levy Characterization of BM Let M be a local martingale, i.e. $M \in \mathcal{M}^{c,Loc}, M_0 = 0$, $[M, M]_t = t$ iff M is a Brownian Motion.

Proof. Recall

> Ito Lemma, given semi-martingale $X_t = M_t + A_t$, function $f \in C^2$, then $f(X_t) = f(X_0) + \int_0^t f'(X_t)dX_t + \frac{1}{2}\int_0^t f''(X_t)d[X,X]_t$.
> Note, $dX_t = dM_t + dA_t$ and $d[X,X]_t = (dX_t)^2 = d[M,M]_t$

'Suffix To Show (STS)' (1) $M_t - M_s \perp \mathcal{F}_s$ (2) $M_t - M_s$ is Gaussian with mean 0 and variance $t-s$.
'STS' $\mathbb{E}(e^{iu(M_t-M_s)}|\mathcal{F}_s) = e^{-1/2u^2(t-s)}$ where function e^{iux} is continuous and second order differentiable.
Define $f(x) := e^{iux}$, then $f'(x) = iue^{iux}$, $f''(x) = -u^2e^{iux}$, by Ito's Formula,
$e^{iuM_t} = e^{iuM_0} + iu\int_s^t e^{iuM_v}dM_v - \frac{1}{2}u^2\int_s^t e^{iuM_v}dv$, where $|e^{iuM_v}| < 1$ bounded so $\int_s^t e^{iuM_v}dM_v$ is martingale up to t.
By Lemma in 4.1., $s \mapsto M_{s\wedge t} \in \mathcal{M}_c^2$ (continuous, L^2-finite), so $\mathbb{E}[_s^t e^{iuM_v}dM_v|\mathcal{F}_s] = 0$. Then for $\forall A \in \mathcal{F}_s$, multiply both side by $e^{-iuM_s}\mathbb{I}_A$, then

$$\underbrace{\mathbb{E}[e^{iu(M_t-M_s)}\mathbb{I}_A]}_{g(t)} = \underbrace{\mathbb{E}[e^{iuM_0}]}_{g(s)} - \frac{1}{2}u^2\int_s^t \underbrace{\mathbb{E}[e^{iu(M_v-M_s)}\mathbb{I}_A]}_{g(v)}dv$$
$$\mathbb{E}[e^{iuM_0}] = \mathbb{P}(A)$$

, which is equivalent to solve ODE and get $g(t) = \mathbb{P}(A)e^{-1/2u^2(t-s)}$, then $\mathbb{E}(e^{iu(M_t-M_s)}|\mathcal{F}_s) = e^{-1/2u^2(t-s)}$

5.5 Multi-dimensional Ito Formula

5.5.1 Definition

Def. Multi-dimensional Ito Formula. Given Ito Process $X := (X_t^1, .., X_t^n)_{t\geq 0}$, and $f \in C^{1,2}$, then $df(X) = \sum_{i=1}^n \frac{\partial f}{\partial X_i}dX_i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial X_i \partial X_j}dX_i dX_j$, or $dX_i dX_j = d(X_i, X_j)$
Def. Product of Two Ito Process. Given $f(x,y) = x \cdot y$, then $f_X = y, f_{XX} = f_{YY} = 0$, $f_Y = X, f_{XY} = f_{YX} = 1$, then $X_tY_t = X_0Y_0 + \int_0^t X_u dY_u + \int_0^t Y_u dX_u + [X,Y]_t$ where $[X,Y]_t$ is correlation part besides integration by parts.
Example. For $e^{\beta t}r_t$ where $e^{\beta t}$ is of finite variation, then covariance goes to 0 as above.

5.5.2 Bessel Process

Example for Levy's Process.
Def. Bessel Process. Let $W = (W^{(1)}, .., W^{(d)})$ be standard Brownian Motion in $d-$dimension, define Bessel Process $R_t := ||W_t|| = \sqrt{(W_t^{(1)})^2 + .. + (W_t^{(d)})^2}$. Note when $d \geq 3$, then W is transient.
Theorem. Bessel Process Reduce to 1D. Bessel Process R_t can be reduce to 1D-SDE, as $R_t = \int_0^t \frac{d-1}{R_s}ds + B_t$, where B_t is another Brownian Motion beside $(W_t^{(i)})_{i=1}^d$.
Proof. By multivariate-dimensional Ito Formula, define $f(x) = ||x||_2$, then
$\frac{\partial f}{\partial X_i} = \frac{X_i}{||X||}, \frac{\partial^2}{\partial X_i \partial X_j} = \frac{\delta_{ij}}{||X||} - \frac{X_iX_j}{||X||^3}$, where $\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$.

> **Lemma. Covariation is 0 when independent.** Given $W, \widehat{W}$ two independent Brownian Motion, then $[W, \widehat{W}] = 0$.
> *Proof.* Note, $\frac{W+\widehat{W}}{\sqrt{2}}$ is a Wiener Process, then $(*)[\frac{W+\widehat{W}}{\sqrt{2}}, \frac{W+\widehat{W}}{\sqrt{2}}]_t = t$, note quadratic variation $[\cdot,\cdot]$ is symmetric and bilinear operator, then $[\lambda X + Y, Z] = \lambda[X,Z] + [Y,Z]$.
> So, $(*) = \frac{1}{2}[W,W] + [W,\widehat{W}], \frac{1}{2}[\widehat{W},\widehat{W}]_t = t + [W,\widehat{W}] = t$, then only when $[W,\widehat{W}] = 0$

So, $R_t = f(W_t) = \sum_{i=1}^d \int_0^t \frac{W_s^{(i)}}{R_s}dW_s^{(i)} + \frac{1}{2}\int_0^t \sum_{i,j=1}^d(\frac{\delta_{ij}}{R_s} - \frac{W_s^{(i)}W_s^{(j)}}{R_s^3})\delta_{ij}ds$, since $d(W_s^{(i)}, W_s^{(j)}) = \delta_{ij}ds$ by the Lemma.
(1) For finite variation term, $\frac{1}{2}\int_0^t \sum_{i,j=1}^d(\frac{\delta_{ij}}{R_s} - \frac{W_s^{(i)}W_s^{(j)}}{R_s^3})\delta_{ij}ds = \int_0^t \frac{d}{2}\frac{1}{R_s}ds - \frac{1}{2}\int_0^t \frac{R_s^2}{R_s^3}ds = \int_0^t \frac{d-1}{2R_s}ds$

(2) For local martingale term, define $B_t^i := \int_0^t \frac{W_s^{(i)}}{R_s} dW_s^{(i)}$, then $[B_t^i, B_t^j] = \int_0^t \frac{W_s^{(i)} W_s^{(j)}}{R_s^2} dW_s^{(i)} dW_s^{(j)} = \int_0^t \frac{W_s^{(i)} W_s^{(j)}}{R_s^2} \delta_{ij} ds$, then $B_t := \sum_{i=1}^d B_t^i$, we have $[B, B]_t = \sum_{i=1}^d [B_t^i, B_t^j] = t$. Then by Levy's characterization, B is a Brownian Motion.

5.5.3 Correlation of BM

Fact. As we have show, given $W_t, \widehat{W}_t$ as standard Brownian Motion, then (1) $(dW_t)^2 = dt$ (2) $dW_t dt = 0$ (3) $dW_t d\widehat{W}_t = 0$ if $W \perp \widehat{W}$ (4) $dt^2 = 0$

Lemma. consider Brownian motion $W^{(1)} \perp W^{(2)}$, then given $\rho \in [-, 1, 1]$, $\rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}$ is a Brownian Motion, or further $\rho_t W_t^{(1)} + \sqrt{1-\rho_t^2} W_t^{(2)}$ is a Brownian Motion.

Proof. Easy to prove with $[\cdot, \cdot]_t = t$, then by Levy Char., it's B.M.

Lemma. In general, consider two Brownian Motion, $\exists |\rho|_t \leq 1$ such that $[W_t^1, W_t^2] = \int_0^t \rho_u du$, where ρ is correlation between two Brownian Motion.

Lemma. Moreover, define $\sum_t = \sigma_t \sigma_t^\top$ in high-dimension by Cholesky decomposition, then $W_t^{(i)} = \sum_{j=1}^n \int_0^t \sigma_u(i,j) dB_u^i$, where B_u^i is standard Brownian Motion, and W is with covariance given by Σ.

6 SA06: Risk Neutral Measure

6.1 SDE to PDE

6.1.1 Black Scholes Equation to PDE/Feyman Kac Principle

Market. Consider,

$$\begin{aligned} dB_t &= rB_t dt, B_0 = 0, \text{ bond/riskless} \\ dS_t &= rS_t dt + \sigma S_t dW_t, S \text{ is autonomous} \end{aligned}$$

Goal. Want to price contract $V_t := V(t, S_t)$.

SDE for Price. Since by FTAP I, if the contract V is tradable, then there is an EMM, i.e. discounted value $e^{-rt}V_t$ is a $\mathbb{Q}-$martingale. Then,

$$\begin{aligned} de^{-rt}V(t, S_t) &= -re^{-rt}V dt + e^{-rt} dV(t, S_t), \text{Chain Rule} \\ &= -re^{-rt}V dt + e^{-rt}(\partial_t V dt + \partial_S V dS + \frac{1}{2}\partial_{SS}^2 V [dS]^2), \text{ Ito's Lemma} \\ &= e^{-rt}(\partial_t V + rS_t \partial_s V + \frac{1}{2}\sigma^2 S_t^2 \partial_{SS}^2 V - rV)dt + (\cdots)dW_t \end{aligned}$$

Since, $e^{-rt}V_t$ is a martingale, then dt term is 0. So, we have PDE

$$\begin{cases} \partial_t V + rS\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_{SS}^2 V - rV = 0, t \in [0, T], S \in \mathbb{R}^+ \\ V(T, S) = F(S) \end{cases}$$

, where $V = V(t, S), S = S_t$. Solution is,

$$V(t, S) = \mathbb{E}_{t,S}[e^{-r(T-t)} F(S_T)]$$

Verifying solution: Plug back the solution to PDE via Ito's Formula, then it's indeed the solution. In fact, if the BS PDE has a smooth solution then it has to be given by $V(t, S) = \mathbb{E}_{t,S}[e^{-r(T-t)} F(S_T)]$, uniqueness is by change of variable to heat equation.

6.1.2 Heston Model/Stochastic Volatility Model

Market. Heston model capture change in volatility (mean reverse and counter movement between price and volatility), closer to market than BS. Consider,

$$\begin{aligned} dB_t &= rB_t dt, B_0 = 0, \text{ bond/riskless} \\ dS_t &= rS_t dt + \sqrt{Y_t} S_t dW_t^{(1)}, S_t \text{ depends on } Y_t \\ dY_t &= a(b - Y_t)dt + \sigma\sqrt{Y_t}(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}) \end{aligned}$$

Sol. Use FFT to solve, no closed form solution, but good computation property.
SDE for Price.

$$
\begin{aligned}
de^{-rt}V(t,S_t,Y_t) &= -re^{-rt}V\,dt + e^{-rt}dV, \text{Chain Rule}\\
&\text{, where } dV = \partial_t V dt + \partial_S V dS + \partial_Y V dY + \frac{1}{2}\partial^2_{SS}V(dS)^2 + \partial_{SY}V dSdY + \frac{1}{2}\partial^2_{YY}V(dY)^2\\
&\text{, where } \begin{cases} dS = rSdt + \sqrt{Y_t}SdW_t^{(1)}\\ dY = a(b-Y)dt + \sigma\sqrt{Y}d(\rho W_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)})\\ (dS)^2 = Y_t S_t^2 dt\\ (dSdY) = \sigma Y S\rho\ dt\\ (dY)^2 = \sigma^2 Y_t dt \end{cases}
\end{aligned}
$$

, we set dt term to be 0, (and d(Local Martingale) term is smooth s.t. it becomes martingale), then,

$$
\begin{aligned}
\partial_t V + rS\partial_S V + a(b-Y)\partial_Y V + \frac{1}{2}YS^2\partial_{SS}V + \sigma\rho YS\partial_{SY}V +\\
\frac{1}{2}\sigma^2 Y\partial_{YY}V - rV = 0, S, Y \in \mathbb{R}^+, t\in[0,T]\\
V(T,S,Y) = F(S)
\end{aligned}
$$

, where $\sigma\sqrt{Y_t}$ is used to change EMMs.

6.2 Radon Nikodym Theorem

6.2.1 Radon Nikodym Theorem

Def. Radon Nikodym Derivative. Random variable Z is Radon Nikodym derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, $Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$, and $\mathbb{E}^{\widetilde{\mathbb{P}}}[\mathbb{I}_A] = \widetilde{\mathbb{P}}(A) = \int_A d\widetilde{\mathbb{P}}(A) = \int_A \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}d\mathbb{P} = \int_A Z d\mathbb{P} = \mathbb{E}[Z\mathbb{I}_A]$, for $\forall A\in\mathcal{F}$
Def. Absolute Continuous. Given any two measure defined on $(\Omega,\mathcal{F})$, if for any $A\in\mathcal{F}$ s.t. $\mathbb{P}(A) = 0 \Rightarrow \widetilde{\mathbb{P}}(A) = 0$, then $\widetilde{\mathbb{P}} \ll \mathbb{P}$, or $\mathbb{P}$ dominate $\widetilde{\mathbb{P}}$, or $\widetilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$.
Def. Equivelent Measure. If $\widetilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \widetilde{\mathbb{P}}$, then $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent, $\widetilde{\mathbb{P}} \sim \mathbb{P}$.
Theorem. Radon Nikodym Theorem. (1) if $\widetilde{\mathbb{P}} \ll \mathbb{P}$, then $\exists$ a random variable $Z\in L_1^+$ s.t. $\widetilde{\mathbb{P}}(A) = \mathbb{E}[Z\mathbb{I}_A]$. (2) If $\widetilde{\mathbb{P}} \sim \mathbb{P}$, then $Z > 0$ $\mathbb{P}-$a.s. (s.t. Random Nikodym has non-zero denominator).
Corollary. From (1), $\mathbb{E}^{\widetilde{\mathbb{P}}}(X) = \mathbb{E}[XZ]$ prove by simple function, nonnegative function, real function.
Corollary. From (2), $\mathbb{E}[X] = \mathbb{E}^{\widetilde{\mathbb{P}}}[X/Z]$.

6.2.2 Example

Example. Change of Measure. On $(\Omega,\mathcal{F},\mathbb{P})$, $X\sim N(0,1)$, $\theta\in\mathbb{R}$, $Y = X+\theta \sim N(\theta,1)$, define $Z = e^{-\theta X - \frac{1}{2}\theta^2}$, then $\mathbb{E}[Z] = 1$, $Z>0$, and $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z$, then under $\widetilde{\mathbb{P}}$, $Y\sim N(0,1)$.
Solution. First, we show that $\mathbb{E}[Z] = 1$. Since $X\sim N(0,1)$, then $\mathbb{E}[\exp(-\theta X - \frac{1}{2}\theta^2)] = \int_{\mathbb{R}} e^{-\theta x - \theta^2/2}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}e^{-(x+\theta)^2/2}dx = 1$.
Next, it is STS $\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y)] = \mathbb{E}[g(X)]$ for any suitable g integrable, where $X\sim N(0,1)$ under $\mathbb{P}$ measure,

and $Y \sim N(0,1)$ under $\tilde{\mathbb{P}}$ measure.

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{P}}}[g(Y)] &= \mathbb{E}[g(Y)Z], Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \\
&= \mathbb{E}[g(X+\theta)e^{-\theta X - \frac{1}{2}\theta^2}] \\
&= \int_{\mathbb{R}} g(x+\theta)e^{-\theta x-\frac{1}{2}\theta^2}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= \int_{\mathbb{R}} g(x+\theta)\frac{1}{\sqrt{2\pi}}e^{-(x+\theta)^2/2}dx \\
&= \int_{\mathbb{R}} g(u)\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du \\
& \mathbb{E}[g(u)], u \sim N(0,1) \\
&\overset{d.}{=} \mathbb{E}[g(X)]
\end{aligned}$$

6.2.3 Conditional Expectation under New Measure

Theorem. Uniformly Integrable Martingale. On $(\Omega, \mathcal{F}, \mathbb{P})$, random variable $Z \geq 0$, $\mathbb{E}[Z] = 1$, and $Z_t = \mathbb{E}[Z|\mathcal{F}_t]$. Then Z_t is uniformly integrable martingale. Moreover, any uniformly integrable martingale will have such form.

Lemma 5.2.1. For $Y \in L^1(\mathcal{F}_t, \mathbb{P})$, define $\widetilde{\mathbb{E}} = \mathbb{E}^{\tilde{\mathbb{P}}}$, and Z is as above, i.e. random variable $Z \geq 0$, $\mathbb{E}[Z] = 1$, and $Z_t = \mathbb{E}[Z|\mathcal{F}_t]$. Then, $\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ_t]$.

Proof. $\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ] = \mathbb{E}[\mathbb{E}[YZ_T|\mathcal{F}_t]] = \mathbb{E}[Y\mathbb{E}[Z|\mathcal{F}_t]] = \mathbb{E}[YZ_t]$

Lemma 5.2.2. Bayes' Theorem. For conditional expectation on new measure, given $Y \in L^0(\mathcal{F}_t, \mathbb{P})$, $\widetilde{\mathbb{E}}[Y|\mathcal{F}_s] = \frac{\mathbb{E}[YZ_t|\mathcal{F}_s]}{Z_s}$. [Shreve]

Proof. $\forall A \in \mathcal{F}_s$, we have

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[\mathbb{I}_A \frac{\mathbb{E}[YZ_t|\mathcal{F}_s]}{Z_s}\right] &= \mathbb{E}\left[\mathbb{I}_A \frac{\mathbb{E}[YZ_t|\mathcal{F}_s]}{Z_s} Z_s\right], \text{Lemma above} \\
&= \mathbb{E}[\mathbb{E}[\mathbb{I}_A Y Z_t|\mathcal{F}_s]] \\
&= \mathbb{E}[\mathbb{I}_A Y Z_t] = \widetilde{\mathbb{E}}[\mathbb{I}_A Y], \text{Lemma}
\end{aligned}$$

since A is arbitrary, then $\widetilde{\mathbb{E}}[\mathbb{I}_A Y] = \widetilde{\mathbb{E}}\left[\left.\frac{\mathbb{E}[YZ_t|\mathcal{F}_s]}{Z_s}\right| \mathcal{F}_s\right] = \frac{\mathbb{E}[YZ_t|\mathcal{F}_s]}{Z_s}$.

6.3 Girsanov's Theorem

6.3.1 Girsanov's Theorem

Theorem. Girsanov's ver. RN Theorem. On $(\Omega, \mathcal{F}, \mathbb{P})$, random variable $Z \in L_1^+(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{E}[Z] = 1$, then function $\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega) = \mathbb{E}[Z\mathbb{I}_A], \forall A \in \mathcal{F}$, such $\widetilde{\mathbb{P}}$ is a probability measure.

Theorem. Girsanov's Theorem. On natural measure space $(\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\in[0,T]}$ is a Brownian Motion under $\mathbb{P}$. $\mathcal{F}_t = \mathcal{F}_t^W$, process θ_t is adapted, and random variable $Z_t = \exp(-\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du)$, $\widetilde{W}(t) = W_t + \int_0^t \theta_u du$. Assume $\theta_t \in \mathcal{H}^2_{Loc}$ or $\mathbb{E}[\int_0^T \theta_u^2 Z_u^2 du] < \infty$, then Z is a martingale, and $\mathbb{E}[Z] = 1$. If $\frac{d\widetilde{P}}{d\mathbb{P}}|_{\mathcal{F}_T} = Z_T$, then under $\widetilde{\mathbb{P}}$, $\widetilde{W}(t)$ is a Brownian Motion.

Remark. By Ito's Formula, $dZ_t = -Z_t\theta_t dW_t \iff Z_t = \exp(-\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du)$, further, since Z_t is positive local martingale, by Fatou Lemma, Z_t is a supermartingale.

Proof. Levy's Characterization applies here. (1) $\widetilde{W}_t$ is continuous, becuase fix ω, sum of two continuous process is also continuous. (2) $\widetilde{W}_t$ has quadratic variation as t, since $[\widetilde{W}, \widetilde{W}]_t = [W, W]_t = t$. (3) $\widetilde{W}_t$ is $\widetilde{\mathbb{P}}-$local martingale, as followed, under $\mathbb{P}$,

$$\begin{aligned}
dZ_t\widetilde{W}_t &= Z_t d\widetilde{W}_t + \widetilde{W}_t dZ_t + \frac{1}{2}\cdot 2dZ_t d\widetilde{W}_t \\
&= Z_t(dW_t + \theta_t dt) - Z_t\theta_t\widetilde{W}_t dW_t - Z_t\theta_t dt \\
&= (Z_t - Z_t\theta_t\widetilde{W}_t)dW_t
\end{aligned}$$

with no drift term dt, then $Z_t\widetilde{W}_t$ is a local martingale under $\mathbb{P}$. Then by definition, $\exists\ (\tau_n)_{n\geq 0}$ localizing sequence for $(\widetilde{W}_t Z_t)_{t\geq 0}$ s.t.

$$\widetilde{W}_{t\wedge\tau_n} Z_t = \underbrace{\widetilde{W}_{t\wedge\tau_n} Z_{t\wedge\tau_n}}_{\text{martingale}} + \underbrace{\widetilde{W}_{\tau_n\wedge t}(Z_t - Z_{\tau_n})\mathbb{I}_{t\geq\tau_n}}_{\text{martingale,by Z is martingale}}$$

, so $\widetilde{W}_{t\wedge\tau_n} Z_t$ is a martingale under $\mathbb{P}$, then

$$\begin{aligned}\widetilde{E}[\widetilde{W}_{t\wedge\tau_n}|\mathcal{F}_s] &= \frac{\mathbb{E}[\widetilde{W}_{t\wedge\tau_n} Z_t|\mathcal{F}_s]}{Z_s}, \text{by Bayes' Theorem}\\ &= \frac{\widetilde{W}_{s\wedge\tau_n} Z_s}{Z_s} = \widetilde{W}_{s\wedge\tau_n}\end{aligned}$$

So $\widetilde{W}_t$ is a local martingale under $\widetilde{\mathbb{P}}$ with existence of localizing sequence τ_n.

6.3.2 Gronwall's Lemma

<u>Theorem. Gronwall's Lemma.</u> $t \geq a$, $u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds$, $\alpha(t)$ is nondecreasing, then $u(t) \leq \alpha(t)\exp(\int_a^t \beta(s)ds)$.
<u>Prop.</u> From Girsanov's Theorem, $dZ_t = -Z_t\theta_t dW_t$. If $|\theta_t| < c < \infty$ bounded, but $\theta_t Z_t$ may not in $\mathcal{H}^2_{Loc}$, then Z_t is a martingale.
<u>*Proof.*</u> First, observe $Z_t = \exp(-\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du)$ is the solution to SDE, check by Ito's Formula [Shreve, Thm 5.2.3.], define $X(t) = -\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du$ and $f(x) = e^x$, then $f'(x) = e^x = f''(x)$

$$\begin{aligned}dZ_t = d(f(X_t)) &= f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2\\ &= e^{X(t)}(-\theta_t dW_t - \frac{1}{2}\theta_t^2 dt) + \frac{1}{2}e^{X(t)}\theta_t^2 dt\\ &= -\theta_t Z_t dW_t\end{aligned}$$

Then Z_t is local martingale. Then Z_t is a supermartingale (since every positive local martingale is supermartingale), and since $\mathbb{E}[Z_t] = 1$, the $\exists\ (\tau_n)$ a localizing sequence s.t. $Z_{t\wedge\tau_n} = 1 - \int_0^{t\wedge\tau_n}\theta_s Z_s dB_s$, then

$$\begin{aligned}\mathbb{E}[Z^2_{t\wedge\tau_n}] &= \mathbb{E}[1 + \int_0^{t\wedge\tau_n}\theta_s^2 Z_s^2 ds], \text{Ito Isometry}\\ &= 1 + \int_0^t \theta^2_{s\wedge\tau_n}\mathbb{E}[Z^2_{t\wedge\tau_n}]ds, \text{Local integarl,Fubini}\\ &\leq 1 + c^2\int_0^t \mathbb{E}[Z^2_{t\wedge\tau_n}]ds\\ \Rightarrow \mathbb{E}[Z^2_{t\wedge\tau_n}] &\leq c^2 \cdot e^0 = c^2, \text{Gronwall's Lemma}\end{aligned}$$

So, $Z_{t\wedge\tau_n}$ is L^2-bounded, then Z_t is a martingale in n.

6.3.3 Novikov's Condition (1D-case)

<u>Theorem. Novikov's Condition.</u> [Generalized from Gronwall's Lemma] Z is a martingale if $\mathbb{E}[\exp(\frac{1}{2}\int_0^T \theta_u^2 du)] < \infty$. But with Jesen's inequality and Fubini Theorem, then it is equivalent to just check $\theta_t \in \mathcal{H}^2_{Loc}$.

6.3.4 Multidimensional Girsanov's Theorem

<u>Theorem. Multidimensional Girsanov's Theorem.</u> Given $W = (W_t^{(1)}, .., W_t^{(d)})$ as $d-$dimensional Brownian Motion, or use Cholesky decomposition to find independent Brownian Motion from W. Then $Z_t = \exp(-\sum_{i=1}^d \int_0^t \theta_u^{(i)} dW_u^{(i)} - \frac{1}{2}\sum_{i=1}^d \int_0^t (\theta_u^{(i)})^2 du)$, where Z_t is a martingale. Define $Z_t = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}$, then under new measure $\tilde{\mathbb{P}}$, $\tilde{W}_t = W_t + \int_0^t \theta_u du$ is a Brownian Motion.

6.4 Example of Girsanov's Theorem

6.4.1 Change of Numeraire.

Model. $\begin{cases} dB_t = r_t B_t dt, r_t \in \mathcal{H}^1_{Loc}, B_0 = 1 \\ dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \mu_t \in \mathcal{H}^1_{Loc}, \sigma_t \in \mathcal{H}^2_{Loc} \end{cases}$ for $\forall \omega \in \mathcal{F}$.

Discounted Model. Define B_t as numeraire, e.g. risk-free asset, and $\widetilde{S} = \frac{S}{B}$ as discounted value. Then we can pick Z_t s.t. $\tilde{S}$ is a martingale.

Solution. *First*, derive solution for two SDEs as $\begin{cases} B_t = \exp\left[\int_0^t r_s ds\right] \\ S_t = S_0 \exp[\int_0^t \sigma_s dW_s + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) ds] \end{cases}$. We can verify it by Ito's Formula.

Same trick, for S_t, $X_t = \int_0^s \sigma_t dW_s + \int_0^t (\mu_s - \frac{1}{2}\sigma_s^2) ds$, then $f(X) = S_0 e^X$, then $f'(x) = S_0 e^x$, $f''(x) = S_0 e^x$, so apply Ito's Formula,

$$
\begin{aligned}
dS_t = d(f(X_t)) &= f'(X_t) dX_t + f''(X_t)[dX_t]^2 \\
&= S_t[\sigma_t dW_t + (\mu_t - \frac{1}{2}\sigma_t^2) dt] + S_t[\frac{1}{2}\sigma_t^2 dt] \\
&= S_t[\mu_t dt + \sigma_t dW_t]
\end{aligned}
$$

Similarly for B_t.

Observe, $d[B, B]_t = 0$ but $d[S, S]_t = \sigma_t^2 S_t^2 dt$, so B is smooth, but S variate.

Second, discounted stock price $\widetilde{S}_t = S_0 \exp[\int_0^t \sigma_s dW_s + \int_0^t (\mu_s - r_s - \frac{1}{2}\sigma_s^2) ds]$, apply same trick,

$$
d\widetilde{S}_t = \widetilde{S}_t[(\mu_t - r_t) dt + \sigma_t dW_t]
$$

Assume, $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \theta_t dt$, where $W_t^{\mathbb{P}}$ is a $\mathbb{P}-$martingale and $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}-$martingale. Then,

$$
\begin{aligned}
d\widetilde{S}_t &= \widetilde{S}_t[(\mu_t - r_t) dt + \sigma_t dW_t] \\
&= \widetilde{S}_t[(\mu_t - r_t) dt + \sigma_t (dW_t^{\mathbb{Q}} - \theta_t dt)] \\
&= \widetilde{S}_t[(\mu_t - r_t - \sigma_t \theta_t) dt + \sigma_t dW_t^{\mathbb{Q}}]
\end{aligned}
$$

, to make $\widetilde{S}_t$ martingale, we need no drift term, then $\theta_t := \frac{\mu_t - r_t}{\sigma_t}$. Then, we can check

$$
\begin{aligned}
d\widetilde{S}_t &= d[e^{-rt} S_t] = \widetilde{S}_t \sigma_t dW_t^{\mathbb{Q}} \\
d[S_t/B_t] &= -r_t S_t/B_t dt + 1/B_t dS_t \\
&= -r S_t/B_t dt + 1/B_t[\mu_t S_t dt + \sigma_t S_t dW_t] \\
&= S_t/B_t[(\mu_t - r_t) dt + \sigma_t dW_t] \\
&= \widetilde{S}_t \sigma_t dW_t^{\mathbb{Q}} \\
dS_t &= S_t[\mu_t dt + \sigma_t dW_t] \\
&= S_t[\mu_t dt + \sigma_t [dW_t^{\mathbb{Q}} - \theta_t dt]] \\
&= S_t[r_t dt + \sigma_t dW_t^{\mathbb{Q}}]
\end{aligned}
$$

Third, thus $dZ_t = -Z_t \theta_t dW_t$, so $Z_t = \exp(-\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du)$ where $\theta_u = \frac{\mu_u - r_u}{\sigma_u}$. Then new measure is defined as $d\mathbb{Q} = Z_t d\mathbb{P}$. And $S_t = S_0 \exp[\int_0^t \sigma_t dW_t^{\mathbb{Q}} + \int_0^t (r_t - \frac{1}{2}\sigma_t^2) dt]$

6.4.2 Black-Cox Model on Pricing Bond

Model. $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $S_0 = S_0$, then $S_t = S_0 \exp[(\mu - \frac{\sigma^2}{2})t + \sigma W_t]$. Define stopping time $\tau = \inf\{t \geq 0 : S_t \leq D\}$ and assume $S_0 > D$.

Goal. Find default probability before maturity $\mathbb{P}(\tau \leq T)$.

Solution. We first reduce the goal,

$$\begin{aligned}\mathbb{P}[\tau \le T] &= \mathbb{P}[\inf_{t\in[0,T]}\{t : \log\frac{S_t}{S_0} \le \log\frac{D}{S_0}\}] \\ &= \mathbb{P}[\inf_{t\in[0,T]}\{t : \frac{(\mu-\frac{\sigma^2}{2})}{\sigma}t + W_t \le \frac{1}{\sigma}\log\frac{D}{S_0}\}]\end{aligned}$$

Set $\alpha = \frac{(\mu-\frac{\sigma^2}{2})}{\sigma}$ and $a = \frac{1}{\sigma}\log\frac{D}{S_0}$, then $a<0$ by assumption, and $X_t = \alpha t + W_t$. Define $Z_t = e^{-\alpha W_t - \frac{1}{2}\alpha^2 t} = e^{-\alpha X_t + \frac{1}{2}\alpha^2 t}$, and $Z_t = d\tilde{\mathbb{P}}/d\mathbb{P}$. Under $\tilde{\mathbb{P}}$, X is a Brownian Motion (Proof: DIY). Then,

$$\begin{aligned}\mathbb{P}[\inf_{t\in[0,T]} X_t \le a] &= 1 - \mathbb{P}(\inf_{t\in[0,T]} X_t \ge a) \\ &= 1 - \mathbb{E}[\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \ge a\}}] \\ &= 1 - \tilde{\mathbb{E}}[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \ge a\}}] \\ &= 1 - \tilde{\mathbb{E}}[e^{\alpha X_T - \frac{1}{2}\alpha^2 T}\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \ge a\}}] \\ &= 1 - e^{-\frac{1}{2}\alpha^2 T}\tilde{\mathbb{E}}[e^{\alpha X_T}\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \ge a\}}]\end{aligned}$$

Apply strategy from reflection principle of Brownian motion, consider down and out barrier European option $h(X_T) = e^{\alpha X_T}\mathbb{I}_{\{X_T \ge a\}}$, then

$$\begin{aligned}&1 - e^{-\frac{1}{2}\alpha^2 T}\tilde{\mathbb{E}}[e^{\alpha X_T}\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \ge a\}}] \\ &= 1 - e^{-\frac{1}{2}\alpha^2 T}\tilde{\mathbb{E}}[h(X_T)\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \ge a\}}] \\ &= 1 - e^{-\alpha^2 T/2}(\tilde{\mathbb{E}}[h(X_T)] - \tilde{\mathbb{E}}[h(X_T)\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \le a\}}])\ (*)\end{aligned}$$

Define $\tau_a = \inf\{t \ge 0 : X_t = a\}$, then define new B.M. starting from τ_a as $X_{T-\tau_a} = X_T$. By reflection principle, $a - X_{T-\tau_a} \overset{d.}{=} X_T$ and $a - X_{T-\tau_a} = 2a - X_T$. Now since $2a - X_T \ge a \iff X_T \le a$, so we can drop indicator term $\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \le a\}}$ to have $\tilde{\mathbb{E}}[h(X_T)\mathbb{I}_{\{\inf_{t\in[0,T]} X_t \le a\}}] = \tilde{\mathbb{E}}[h(2a - X_T)]$, now

$$\begin{aligned}\tilde{\mathbb{E}}[h(X_T)] &= \tilde{\mathbb{E}}[e^{\alpha X_T}\mathbb{I}_{\{X_T \ge a\}}], X_T \text{ is B.M.} \\ &= \int_a^\infty e^{\alpha x}\frac{1}{\sqrt{2\pi T}}e^{-x^2/(2T)}dx \\ &= \frac{1}{\sqrt{2\pi T}}e^{\alpha^2 T/2}\int_a^\infty e^{-(x-\alpha T)^2/(2T)}dx \\ &= e^{\alpha^2 T/2}(1 - N(\frac{a - T\alpha}{\sqrt{T}})) \\ \tilde{\mathbb{E}}[h(2a - X_T)] &= \tilde{\mathbb{E}}[e^{\alpha(2a - X_T)}\mathbb{I}_{\{X_T \le a\}}], X_T \text{ is B.M.} \\ &= e^{2\alpha a}\tilde{\mathbb{E}}[e^{-\alpha X_T}\mathbb{I}_{\{-X_T \ge -a\}}] \\ &= e^{2\alpha a}\tilde{\mathbb{E}}[e^{\alpha \tilde{X}_T}\mathbb{I}_{\{\tilde{X}_T \ge -a\}}], X_T \overset{d.}{=} \tilde{X}_T \\ &= e^{2\alpha a + \alpha^2 T/2}(1 - N(\frac{-a - T\alpha}{\sqrt{T}})) \\ \Rightarrow \mathbb{P}(\tau \le T) &= N(\frac{a - T\alpha}{\sqrt{T}}) + e^{2a\alpha}(1 - N(\frac{-a - T\alpha}{\sqrt{T}}))\end{aligned}$$

6.4.3 Change of Numeraire w. Risky Asset

Model.

$$\begin{cases} dB_t = rB_t dt \\ dS_t^{(1)} = \mu_t^{(1)} S_t^{(1)} dt + \sigma^{(1)} S_t^{(1)} dW_t^{(1)}, W_t^{(1)} \perp W_t^{(2)} \\ dS_t^{(2)} = \mu_t^{(2)} S_t^{(2)} dt + \sigma^{(2)} S_t^{(2)} [\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)} \end{cases}$$

Goal. Consider spread option payoff $f(S_T^{(1)}, S_T^{(2)}) = (S_T^{(2)} - S_T^{(1)})^+$ for arbitrage on company goes the same direction. Choose $S^{(1)}$ as numerair, find EMM.
Solution. First, find discounted value of SDE solution,

$$\widetilde{S}_t^{(2)} = \frac{S_t^{(2)}}{S_t^{(1)}} = \frac{S_0^{(2)} e^{\int_0^t (\mu_t^{(2)} - \frac{1}{2}(\sigma^{(2)})^2) dt + \sigma^{(2)} (\rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)})}}{S_0^{(1)} e^{\int_0^t (\mu_t^{(1)} - \frac{1}{2}(\sigma^{(1)})^2) dt + \sigma^{(1)} W_t^{(1)}}} = \frac{S_0^{(2)}}{S_0^{(1)}} e^{\int_0^t \tilde{\mu}_u du + (\sigma^{(2)}\rho - \sigma^{(1)}) W_t^{(1)} + \sigma^{(2)} \sqrt{1-\rho^2} W_t^{(2)}}$$

, where $\tilde{\mu}_t = \mu_t^{(2)} - \mu_t^{(1)} + \frac{(\sigma^{(1)})^2}{2} - \frac{(\sigma^{(2)})^2}{2}$, and,

$$\tilde{B}_t = \frac{B_t}{S_t^{(1)}} = e^{\int_0^t (r - \mu_t^{(1)} - \frac{(\sigma^{(1)})^2}{2}) dt - \int_0^t \sigma^{(1)} dW_t^{(1)}}$$

Choose risk-neutral measure by dropping drift term for $\tilde{S}_t^{(2)}$ and $\tilde{B}_t$, then define 2D R-N derivative

$$Z_t = e^{-\sum_{i=1}^2 \int_0^t \theta_u^i dW^i - \frac{1}{2} \sum_{i=1}^2 \int_0^t (\theta_u^i)^2 du}$$
$$\frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_t} = Z_t$$

, by Girsanov Theorem, $\widetilde{W}^{(i)} = W_t^{(i)} + \int_0^t \theta_u^{(i)} du$ is a Brownian motion under $\mathbb{Q}$, then

$$\begin{aligned} d\widetilde{S}_t^{(2)} &= \mu_t \widetilde{S}_t^{(2)} dt + \widetilde{S}_t^{(2)} (\sigma^{(2)}\rho - \sigma^{(1)}) dW_t^{(1)} + \widetilde{S}_t^{(2)} \sigma^{(2)} \sqrt{1-\rho^2} dW_t^{(2)} \\ &\text{, where } \mu_t = \tilde{\mu}_t + \frac{1}{2}(\sigma^{(2)}\rho - \sigma^{(1)})^2 + \frac{1}{2}(\sigma^{(2)})^2 (1-\rho^2) \\ &= \mu_t^{(2)} - \mu_t^{(1)} + (\sigma^{(1)})^2 - \rho\sigma^{(1)}\sigma^{(2)} \\ &= \widetilde{S}_t^{(2)} (\mu_t - \theta_t^{(1)} (\sigma^{(2)}\rho - \sigma^{(1)}) - \sigma^{(2)} \sqrt{1-\rho^2} \theta_t^{(2)}) dt + \\ &\quad \widetilde{S}_t^{(2)} (\sigma^{(2)}\rho - \sigma^{(1)}) d\widetilde{W}_t^{(1)} + \widetilde{S}_t^{(2)} \sigma^{(2)} \sqrt{1-\rho^2} d\widetilde{W}_t^{(2)} \end{aligned}$$

, and,

$$d\tilde{B}_t = B_t (r - \mu_t^{(1)}) dt - \sigma^{(1)} B_t dW_t^{(1)} = B_t (r - \mu_t^{(1)} + \sigma^{(1)} \theta_t^1) dt - \sigma^{(1)} B_t d\widetilde{W}_t^{(1)}$$

Pick θ_1, θ_2 such that both drift term is 0, then

$$\begin{cases} \mu_t^{(2)} - \mu_t^{(1)} + (\sigma^{(1)})^2 - \rho\sigma^{(1)}\sigma^{(2)} - \theta_t^{(1)} (\sigma^{(2)}\rho - \sigma^{(1)}) - \sigma^{(2)} \sqrt{1-\rho^2} \theta_t^{(2)} = 0 \\ r - \mu_t^{(1)} + \sigma^{(1)} \theta_t^1 = 0 \end{cases}$$

, we have $\theta_t^1 = \frac{\mu_t^{(1)} - r}{\sigma^{(1)}}$, and the θ_t^2
Then the price of option is

$$\widetilde{V}_s = \mathbb{E}^{\mathbb{Q}} [V_T / S_T^1 | \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}} [(\widetilde{S}_T^{(2)} - 1)^+ | \mathcal{F}_s]$$

and use B.S. in 2D to solve the distribution. Then true value of the option is thus $V_t = S_t^{(1)} \widetilde{V}_t$.

7 SA07: Martingale Representation Theorem

7.1 Martingale Representation Theorem

7.1.1 Martingale Representation Theorem

Def. Subnull set. On $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $N \subset \Omega$ is negligible if $\exists N' \in \mathcal{F}$ s.t. $\mathbb{P}(N') = 0$, $N \subset N'$
Def. Complete measurable space. On $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F}$ is augmented if $\mathcal{F}_0$ contains all negligible sets of $\mathcal{F}$.
Theorem 5.4.2. Martingale Representation Theorem. On $(\Omega, \mathcal{F}, \mathbb{F}^W, \mathbb{P})$ with natural filtration generated by (d-dimensional) Brownian Motion $W = (W^1, .., W^d)$. We augment $\mathbb{F}^W$. Then any $\mathbb{F}^W$-adapted martingale M , $\exists\ \Gamma_u$ is $\mathbb{F}^W-$adapted, $\Gamma^i_u \in \mathcal{H}^2_{Loc}, \forall i$, then $M_t = M_0 + \sum_{i=0}^d \int_0^t \Gamma^i_u dW^i_u$.
Corollary. Any martingale w.r.t. Brownian Filtration $\mathbb{F}^W$ has continuous modification.

7.2 FTAP I

7.2.1 Example on change of measure

Example. Consider $\begin{cases} dB_t = r_t B_t dt \\ dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \end{cases}$, $\mu_t \in \mathcal{H}^1_{Loc}, \sigma_t \in \mathcal{H}^2_{Loc}$, value process $V \in \mathcal{F}_T$, $V = g(S_T)$. If V_t is tradable, then by FTAP I, $\tilde{V}_t = \frac{V_t}{B_t} = \mathbb{E}^{\mathbb{Q}}[\frac{V_T}{B_T}|\mathcal{F}_t]$. What is the EMM $\mathbb{Q}$?
Solution. $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$, $Z_t = \exp(-\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du) := \xi(-\int_0^t \theta_u dW_u)$, and ξ is called Doléans-Dade exponential.
Apply Girsanov Theoerm, $\theta_t = \frac{\mu_t - r_t}{\sigma_t}$ as market price of risk. Under measure $\mathbb{Q}$, $dS_t = r_t S_t dt + \sigma_t S_t d\tilde{W}_t$, where $\tilde{W}_t = W_t + \int_0^t \theta_u du$. Then $de^{-rt}S_t = \sigma S_t d\tilde{W}_t$.

7.3 Dubins-Dambis-Schwarz Theorem

Theorem Dubins-Dambis-Schwarz Theorem. [GTM274 5.31] On Brownian Motion filtered space, any continuous local martingale is time-changed version of Brownian Motion. That is, on $(\Omega, \mathcal{F}^B, \mathbb{P})$, for $M \in \mathcal{M}^{c,Loc}$ s.t. $\lim\limits_{t\to\infty} \langle M \rangle = +\infty$ $\mathbb{P}-$a.s. Define generalized inverse function (path-wise) $\tau(s) = \inf\{t \geq 0 | \langle M \rangle_t > s\}$. Then define $B_s := M_{T(s)}$, $\mathcal{G}_s = \mathcal{F}_{T(s)}$ as new filtration, then $(B, \mathcal{G})$ is a Wiener-Process. Note that $M_t = B_{\langle M \rangle_t}$, inverse of inverse is itself.
Remark. $(B, \mathcal{G})$ is such manipulation that (1) stretch the M when it is too active by $\langle M \rangle_t > s$, (2) flatten M when it is too flat via inf of time
Remark. Any continuous local martingale is time-changed version of Brownian Motion, where time change is random.

8 SA08: Uniqueness and Existence of SDE Strong Solution

8.1 Strong Solution to SDE

8.1.1 Strong Solution

Def. ODE. $dX_t = b(t, X_t)dt$
Def. SDE. $\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = \xi, r.v.\ \xi \perp W \text{ or } \xi \perp \sigma(W) \end{cases}$, where W is standard Brownian Motion.
Def. Strong Solution to SDE. Given $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a Brownian Motion W, random variable $\xi \perp W$, we say X solve the SDE in strong sense if (1) X is adapted to $\sigma(\xi) \cup \mathbb{F}^W$ (2) X has continuous path (3) $\mathbb{P}(X_0 = \xi) = 1$ (4) $b(t, X_t) \in \mathcal{H}^2_{Loc}$, $\sigma(t, X_t) \in \mathcal{H}^2_{Loc}$ (5) $X_t = X_0 + \int_0^t b(u, X_u)du + \int_0^t \sigma(u, X_u)dW_u$ a.s. $\forall t \in [0, T]$

8.1.2 Simple Cases Example

Example. Exponential Brownian Motion Explode. Given $dX_t = \mu X_t dt + \sigma X_t dW_t$ and $Z_t = \log X_t$, then by Ito's formula

$$
\begin{aligned}
dZ_t = d\log X_t &= \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(dX_t)^2, X_t \neq 0 \\
&= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\
&= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t \\
\Rightarrow X_t &= X_0 \exp[(\mu - \frac{\sigma^2}{2})t + \sigma W_t] \\
&= X_0 \exp(t[(\mu - \frac{\sigma^2}{2}) + \sigma \frac{W_t}{t}])
\end{aligned}
$$

, where $\frac{W_t}{t} \sim N(0,1)$, then taking $t \to \infty$, $\begin{cases} \text{if } \mu < \frac{\sigma^2}{2}, X_t \to 0 \\ \text{if } \mu > \frac{\sigma^2}{2}, X_t \to \infty \end{cases}$

Example. OU process solution. Ornstein-Uhlenbeck Process (Vasicek model) is $dX_t = -\alpha X_t + \sigma dB_t$. Guess solution form in $X_t = a(t)(X_0 + \int_0^t b(s)dW_s)$. Take Ito's formula,

$$
dX_t = a'(t)(X_0 + \int_0^t b(s)dWs)dt + a(t)b(t)dW_t = \frac{a'(t)}{a(t)}X_t dt + a(t)b(t)dWt
$$

Equation formula, (1) $\frac{a'(t)}{a(t)} = -\alpha$, solve ODE to get $a(t) = e^{-\alpha t}$. (2) $a(t)b(t) = \sigma$ get $b(t) = \sigma e^{\alpha t}$. So,

$$
X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dW_s
$$

, where $\int_0^t e^{-\alpha(t-s)} dW_s$ can be treated as convolution.

Example. Brownian Bridge solution. Brownian Bridge, $dX_t = -\frac{X_t}{T-t}dt + dW_t$, like OU process, solution form in $X_t = a(t)(X_0 + \int_0^t b(s)dW_s)$, then Ito Formula gives $dX_t = \frac{a'(t)}{a(t)}X_t dt + a(t)b(t)dWt$. Equating formula, (1) $\frac{a'(t)}{a(t)} = -\frac{1}{T-t}$ then $a(t) = T - t$ (2) $a(t)b(t) = 1$ then $b(t) = \frac{1}{T-t}$, so,

$$
X_t = (T-t)\int_0^t \frac{1}{T-s} dW_s
$$

Example. OU process is Gaussian. We change variable by

$$
e^{\alpha t}X_t = X_0 + \sigma \int_0^t e^{\alpha s} dW_s
$$

, where $M_t := \sigma \int_0^t e^{\alpha s} dW_s$ is martingale. Then $(dM_s)^2 = \sigma^2 e^{2\alpha s}$, thus $\langle M, M \rangle_t = \sigma^2 \int_0^t e^{2\alpha s} ds = \frac{\sigma^2}{2\alpha}(e^{2\alpha t} - 1)$.

> Recall Dubins-Dambis-Schwarz Theorem
> set stopping time as $\tau(s) = \inf\{t \geq 0 | \langle M \rangle_t > s\}$, then $B_s := M_{T(s)}$ is a Brownian Motion

Use it to find law of martingale by $M_t = \widehat{W}_{\frac{\sigma^2}{2\alpha}(e^{2\alpha t}-1)}$ as time changed B.M., then $X_t \sim N(X_0 e^{-\alpha t}, \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}))$ given X_0 constant.

Property. Limiting distribution is Stationary Distribution. $\lim_{t\to\infty} X_t \sim N(0, \frac{\sigma^2}{2\alpha})$. If the initial condition $\xi \overset{set}{\sim} N(0, \frac{\sigma^2}{2\alpha})$, then by $\xi \perp W$, $X_0 e^{-\alpha t} \perp W$ and $X_0 e^{-\alpha t} \sim N(0, \frac{\sigma^2}{2\alpha}e^{-2\alpha t})$. Then by $e^{-\alpha t}M_t \sim N(0, \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}))$, we have,

$$
X_t = X_0 e^{-\alpha t} + e^{-\alpha t}M_t \sim N(0, \frac{\sigma^2}{2\alpha})
$$

So, limiting distirbution is stationary distribution.
Property. Stationary Process. The covariance of OU process is,

$$
\begin{aligned}
\mathbb{E}[X_t X_s] &= \mathbb{E}[e^{-\alpha s}(X_0 + M_s)e^{-\alpha t}(X_0 + M_t)] \\
&= e^{-\alpha(t+s)}(\mathbb{E}[X_0^2] + \mathbb{E}[M_t M_s]) \\
&\qquad , \text{ where } \mathbb{E}[X_0^2] = \frac{\sigma^2}{2\alpha} \\
&\qquad \text{and } \mathbb{E}[M_t M_s]) = \mathbb{E}[\widehat{W}_{\frac{\sigma^2}{2\alpha}(e^{2\alpha t}-1)}\widehat{W}_{\frac{\sigma^2}{2\alpha}(e^{2\alpha s}-1)}]) \\
&\qquad\qquad = \frac{\sigma^2}{2\alpha}(e^{2\alpha(t\wedge s)} - 1), \alpha > 0 \\
&= e^{-\alpha(t+s)\frac{\sigma^2}{2\alpha}e^{2\alpha(t\wedge s)}} \\
&= \frac{\sigma^2}{2\alpha}e^{-\alpha(t-s)}, s < t
\end{aligned}
$$

covariance only depends on $t - s$.
Example. Brownian Bridge is Gaussian. [Homework] BB is time changed Brownian motion by DDS Theorem, then can find its law, and limiting distribution. Then $Cov(X_t, X_s) = s \wedge t - \frac{st}{T}$, so Brownian Bridge Start from 0, end at 0.

8.2 Multidimensional SDE

8.2.1 Multidimensional SDE

Def. Linear SDE in d-Dimension. consider $dX_t = (AX_t + H(t))dt + K(t)dW_t$, where $[X]_{n\times 1}$, $[A]_{n\times n}$, $[H]_{n\times 1}$, $[K]_{n\times d}$, $[W_t]_{d\times 1}$, where $H(t), K(t)$ are time dependent deterministic but not random process.
Remark. Limiting X_t on RHS. Recall exponential matrix $e^{-At} = \sum_i \frac{A^i(-t)^i}{i!}$ by Taylor Expansion, and $de^{-At} = -e^{-At}Adt$. Then by Multidimensional Ito formula/Chain rule,

$$
\begin{aligned}
de^{-At}X_t &= X_t de^{-At} + e^{-At}dX_t \\
&= -e^{-At}AX_t dt + e^{-At}dX_t \\
&= e^{-At}(H(t)dt + K(t)dW_t) \\
\Rightarrow X_t &= e^{At}X_0 + \int_0^t e^{-A(s-t)}H(s)ds + \int_0^t e^{-A(s-t)}K(s)dW_s
\end{aligned}
$$

Marked as SDE solving skills.
Def. Exact SDE. Consider $dX_t = a(t, W_t)dt + b(t, W_t)dW_t$, where $a(t, x) = \partial_t f(t, x) + \frac{1}{2}\partial_{xx}^2 f(t, x)$, and $\sigma(t, x) = \partial_x f(t, x)$. Then the solution is $dX_t = df(t, W_t)$, which is $X_t = X_0 + f(t, W_t)$. Use it to equation special form of SDE. **Marked as SDE solving skills.**
Example. $dX_t = e^t(1 + W_t^2)dt + (1 + 2e^t W_t)dW_t$, then we can check solution is $X_t = X_0 + W_t + e^t W_t^2$.

8.2.2 Integrating Factor

Def. Integrating Factor. For SDE $dX_t = \underbrace{f(t, X_t)}_{\text{nonlinear}}dt + \underbrace{g(t)}_{linear}X_t dW_t$, it can be transferred to ODE via integrating factor $\rho_t = \exp[-\int_0^t g(s)dW_s + \frac{1}{2}\int_0^t g^2(s)ds]$.
How? It is because $d\rho_t = \rho_t(-g(t)dW_t + \frac{1}{2}g^2(t)dt) + \frac{1}{2}\rho_t g^2(t)dt = \rho_t(-g(t))dW_t + \rho_t g^2(t)dt$, then $d\rho_t X_t = \rho_t dX_t + X_t d\rho_t + d[X, \rho]_t = \rho_t f(t, X_t)dt$.
So the procedure is: Define $y_t = \rho_t X_t$, then $dy_t = \rho_t f(t, X_t)dt$, so $\frac{dy_t}{dt} = \rho_t f(t, \frac{y_t}{\rho_t})$, solve y_t, then we can solve $X_t = y_t/\rho_t$.
Marked as SDE solving skills.

8.2.3 Existence and Uniqueness of Solution

<u>Def. SDE.</u> $\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t \\ X_0 = \xi \end{cases}$ where $b(t,X_t)$ is vector and $\sigma(t,X_t)$ is matrix. The SDE is path wise, i.e. $dX_t(\omega) = b(t,X_t(\omega))dt + \sigma(t,X_t(\omega))dW_t(\omega)$ for $\forall\omega \in \Omega$.
<u>*Remark.*</u> Consider

$$\begin{bmatrix} dX_t^{(1)} \\ dX_t^{(2)} \\ \vdots \\ dX_t^{(d)} \end{bmatrix} = \begin{bmatrix} b_1(t,X_t)dt \\ b_2(t,X_t)dt \\ \vdots \\ b_d(t,X_t)dt \end{bmatrix} + \begin{bmatrix} \sigma_{11}(t,X_t)\cdots & \sigma_{1r}(t,X_t) & \\ \vdots & \ddots & \vdots \\ \sigma_{d1}(t,X_t)\cdots & \sigma_{dr}(t,X_t) & \end{bmatrix} \begin{bmatrix} dW_t^{(1)} \\ \vdots \\ dW_t^{(r)} \end{bmatrix}$$

<u>Theorem. Solution exists and unique.</u> The above SDE's solution exists and is unique, if for every <u>row entry i of b and column entry j of σ</u>, that b,σ satisfies Lipschitz Continuous, i.e.
(1) Lipschitz Continuity: $\|b^i(t,x) - b^i(t,y)\| + \|\sigma^{ij}(t,x) - \sigma^{ij}(t,y)\| \le K\|x-y\|$, K is constant
(2) Linear growth condition, $\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \le K^2(1+\|x\|^2)$
(3) initial condition is square integrable, i.e. $\mathbb{E}[\xi^2] < \infty$, $X_0 = \xi$ a.s.

8.2.4 Proof of uniqueness

<u>Proof on Uniqueness.</u> Use (1) Lipschitz continuity.
To ease the notation, no worry of couping, take dimension as only 1-D, then norm reduce to absolute value.
Proof by contradiction. Suppose there exist two solutions X, Y with $X_0 = \xi = Y_0$. Then $X_t - Y_t = \int_0^t (b(t,X_t) - b(t,Y_t))dt + \int_0^t (\sigma(t,X_t) - \sigma(t,Y_t))dW_t$
Since $\mathbb{E}[X_t - Y_t]^2 \le 2\mathbb{E}\left([\int_0^t (b(u,X_u) - b(u,Y_u)du)]^2\right) + 2\mathbb{E}\left([\int_0^t (\sigma(u,X_u) - \sigma(u,Y_u)dW_u)]^2\right)$ by Minkowski's Inequality. Then, note, (1)

$$\begin{aligned} &\left(\int_0^t (b(u,X_u) - b(u,Y_u))du\right)^2 \\ &= t^2 \left(\int_0^t (b(u,X_u) - b(u,Y_u))\frac{du}{t}\right)^2, \frac{du}{t} \text{ is Prob.measure} \\ &\le t^2 \int_0^t ((b(u,X_u) - b(u,Y_u))^2 \frac{du}{t}, \text{ by Jensen's Inequality, } (\cdot)^2 \text{ convex} \\ &= t\int_0^t ((b(u,X_u) - b(u,Y_u))^2 du \end{aligned}$$

(2) Recall ,

> **<u>SA05. Sec4.1. Lemma.</u>** If $M \in \mathcal{M}^{c,Loc}$, S is stopping time, then $\mathbb{E}[M_s^2] \le \mathbb{E}[\langle M\rangle_s]$ where $M_\infty^2 = \lim\inf_{t\to\infty} M_t^2$ in case limit not exist. Proof by Fatou Lemma.

$$\mathbb{E}\left[\left(\int_0^t (\sigma(u,X_u) - \sigma(u,Y_u))dW_u\right)^2\right] \le \mathbb{E}\left[\int_0^t (\sigma(u,X_u) - \sigma(u,Y_u))^2 du\right]$$

We have shown the required. Then by (1) and (2),

$$\begin{aligned}
\mathbb{E}[X_t - Y_t]^2 &\le 2\mathbb{E}\left([\int_0^t (b(u,X_u) - b(u,Y_u)du)]^2\right) + 2\mathbb{E}\left([\int_0^t (\sigma(u,X_u) - \sigma(u,Y_u)dW_u)]^2\right) \\
&\le 2\mathbb{E}\left(T\int_0^t ((b(u,X_u) - b(u,Y_u))^2 du\right) + 2\mathbb{E}\left[\int_0^t (\sigma(u,X_u) - \sigma(u,Y_u))^2 du)\right] \\
&\le 2\mathbb{E}\left(t\int_0^t ((b(u,X_u) - b(u,Y_u))^2 du\right) + 2\mathbb{E}\left[\int_0^t (\sigma(u,X_u) - \sigma(u,Y_u))^2 du)\right], T > 1, 0 \le t \le T \\
&\le 2(T+1)\mathbb{E}\left(\int_0^t ((b(u,X_u) - b(u,Y_u))^2 du + \int_0^t (\sigma(u,X_u) - \sigma(u,Y_u))^2 du)\right) \\
&\le C\mathbb{E}[\int_0^t (X_s - Y_s)^2 ds], \text{Liptschiz} \\
&= C\int_0^t \mathbb{E}[(X_s - Y_s)^2] ds, \text{Fubini}
\end{aligned}$$

Next method (1): By Gronwall's Lemma.

> Recall Gronwall's Lemma: suppose a continuosu function g satisfies, $0 \le g(t) \le \alpha(t) + \beta\int_0^t g(s)ds$, $0 \le t \le T$, $\beta \ge 0$, $\alpha : [0,T] \to \mathbb{R}$, then $\Rightarrow g(t) \le \alpha(t) + \beta\int_0^t \alpha(s)e^{\beta(t-s)}ds$, $0 \le t \le T$.

$\mathbb{E}[(X_t - Y_t)^2] \le 0$, but of course $(X_t - Y_t)^2 \ge 0$. So, $\mathbb{E}[(X_t - Y_t)^2] = 0$.
Or method (2): define $M := \sup\limits_{t\in[0,T]} \mathbb{E}[(X_t - Y_t)^2]$, then $\mathbb{E}[(X_t - Y_t)^2] \le C\int_0^t M dt = CMt$, then again, $\mathbb{E}[(X_t - Y_t)^2] \le C\int_0^t CMt dt = \frac{C^2Mt^2}{2}$, and again, we have $\mathbb{E}[(X_t - Y_t)^2] \le \frac{MC^n t^n}{n!} \overset{n\to\infty}{\to} 0$.
By (1) or (2), since X_t, Y_t continuous, So $X_t = Y_t$ a.s. $\forall t$. So solution is unique a.s., i.e. $\mathbb{P}[X_t = Y_t, \forall t \in [0,T]] = 1$

8.2.5 Proof of existence

Proof on Existence. Use (2) and (3). Use Picard Iteration (maybe big error when t large) from ODE to construct numerical analysis as,

$$\begin{aligned}
X_t^0 &= X_0 = \xi, X_0 \in L^2, \text{ square integrable} \\
&\vdots \\
X_t^{n+1} &= X_0 + \int_0^t b(s, X_s^n)ds + \int_0^t \sigma(s, X_s^n)dW_s
\end{aligned}$$

if the iteration/$\{X_t^{(k)}\}_{k=1}^{\infty}$ converges, then there exist the solution.
Claim.: If $\sup\limits_{t\in[0,T]} \mathbb{E}[(X_t^n)^2] < \infty \Rightarrow \sup\limits_{t\in[0,T]} \mathbb{E}[(X_t^{n+1})^2] < \infty$, X_t^{n+1} is well defined. That is, start in L^2, stays in L^2 after new iteration.
Claim.Proof. As a corollary, by (3), $\mathbb{E}[\xi^2] < \infty$, $b(t, X_t^n) \in \mathcal{H}^1_{Loc}$, $\sigma(t, X_t^n) \in \mathcal{H}^2_{Loc}$, then $(X_t^{(n)})_n$ is in L^2, $\forall t$.
Recall: $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, then

$$\begin{aligned}
\mathbb{E}[(X_t^{n+1})^2] &\le 3\mathbb{E}[(X_0)^2] + 3\mathbb{E}[(\int_0^t b(s,X_s^n)ds)^2] + 3\mathbb{E}[(\int_0^t \sigma(s,X_s^n)dW_s)^2] \\
&\le 3\mathbb{E}[(X_0)^2] + 3\mathbb{E}[t\int_0^t (b(s,X_s^n))^2 ds] + 3\mathbb{E}[(\int_0^t \sigma(s,X_s^n)dW_s)^2] \\
&\le C(1 + \mathbb{E}[(X_t^n)^2]), \text{Linear Growth, Fubini}
\end{aligned}$$

Then $\sup\limits_{t\in[0,T]} \mathbb{E}[(X_t^{n+1})^2] \le C(1 + \sup\limits_{t\in[0,T]} \mathbb{E}[(X_t^n)^2]) < \infty$. ■

> Recall Arzelà–Ascoli theorem, $(C_c^0 \times [0,T], ||\cdot||_2)$ is complete., where C_c^0 is continuous function with compact support on a fixed interval. By definition, a sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions on an interval $I = [a,b]$ is uniformly bounded, if there is a number M such that $|f_n(x)| \leq M$, for $x \in [a,b]$.

Then, since $\mathbb{E}[X^2] < \infty \Rightarrow X < \infty$ a.s. Again, it is okay to look at iteration of $\mathbb{E}[\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2]$.

$$\begin{aligned}
&\mathbb{E}[\sup_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2] \\
&= \mathbb{E}[\sup_{s\in[0,t]}[(\int_0^t b(s,X_s^n) - b(s,X_s^{n-1})ds + \int_0^t \sigma(s,X_s^n) - \sigma(s,X_s^{n-1})dW_s)^2] \\
&\leq 2t\mathbb{E}[\int_0^t (b(s,X_s^n) - b(s,X_s^{n-1}))^2 ds] \\
&+ 2\mathbb{E}[\sup_{s\in[0,t]}\left(\int_0^s (\sigma(u,X_u^n) - \sigma(u,X_u^{n-1}))dW_u\right)^2], \text{as in Uniqueness proof}
\end{aligned}$$

, where consider $Z_s = \int_0^s(\sigma(u,X_u^n) - \sigma(u,X_u^{n-1}))dW_u$, by uniqueness, $\sigma(u,X_u^n) - \sigma(u,X_u^{n-1}) \in \mathcal{H}^2$, Z_s is martingale (local),

$$\begin{aligned}
\mathbb{E}[\sup_{s\in(0,t)} Z_s^2] &\leq 2\mathbb{E}[Z_t^2], \text{by Doob's maximal inequality of submartingale} \\
&= 2\int_0^t \mathbb{E}[(\sigma(u,X_u^n) - \sigma(u,X_u^{n-1}))^2]du, \text{by Ito isometry, Fubini}
\end{aligned}$$

so,

$$\begin{aligned}
&\mathbb{E}[\sup_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2] \\
&\leq 2t\mathbb{E}[\int_0^t (b(s,X_s^n) - b(s,X_s^{n-1}))^2 ds] \\
&+ 4\int_0^t \mathbb{E}[(\sigma(u,X_u^n) - \sigma(u,X_u^{n-1}))^2]du \\
&\leq (2T+4)K\int_0^t \mathbb{E}[(X_u^n - X_u^{n-1})^2 du], \text{Lipschitz Continuity} \\
&\leq K\int_0^t \mathbb{E}[\sup_{s\in[0,u]}(X_s^n - X_s^{n-1})^2 du]
\end{aligned}$$

Here, we cannot use Gronwall's inequality because LHS is $f(X^n, X^{n+1})$ but RHS is $g(X^n, X^{n-1})$, different, to use it, take iteration and look at bounded procedure. Consider the inition one, $g^0(t) = M := \mathbb{E}[\sup\limits_{s\in[0,t]}(X_s^1 - X_s^0)^2] < \infty$, then define $g^n(t) = \mathbb{E}[\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2]$, so $g^n(t) \leq K\int_0^t g^{n-1}(u)du$.
Take iteration, $g^1(t) \leq KMt \Rightarrow g^2(t) \leq K^2 M \frac{1}{2}t^2 \cdots \Rightarrow g^n(t) \leq C^n M \frac{1}{n!}t^n$. That is, $\mathbb{E}[\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2] \leq C^n M \frac{1}{n!}t^n \overset{n\to\infty}{\to} 0$.
Then by Chebyshev Inequality, $\mathbb{P}[\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2 > \frac{1}{2^{n+1}}] \leq 4^{n+1}\mathbb{E}[\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2] \leq 4^{n+1}C^n t^n M \frac{1}{n!}$.
Then to use Borel-Cantelli Lemma, $\sum_{n=1}^\infty \mathbb{P}[\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2 > \frac{1}{2^{n+1}}] \leq \sum_{n=1}^\infty 4^{n+1}C^n t^n M \frac{1}{n!} = 4Me^{4Ct} < \infty$.
So by Borel-Cantelli, $\mathbb{P}[\lim\limits_{n\to\infty}\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2 > \frac{1}{2^{n+1}}] = 0$, so $\mathbb{P}[\lim\limits_{n\to\infty}\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2 < \frac{1}{2^{n+1}}] = 1$, so we can look at pathwise property. Exists $N(\omega) > 1$ large enough, $\forall n \geq N$, $\sup\limits_{s\in[0,t]}(X_s^{n+1} - X_s^n)^2 < \frac{1}{2^{n+1}}$, then by triangle inequality, $\sup\limits_{s\in[0,t]}(X_s^{n+k} - X_s^n)^2 < \frac{1}{2^n}$, $k \geq 1$.
Since $(C_c^0, ||\cdot||)$ is complete. Then every Cauchy Sequence converges, and (X^n) is a Cauchy sequence defined on $(C_c^0, ||\cdot||_\infty)$, so X^n converges, so the limit exists here.

9 SA09: SDE Weak Solution, Explosion Theory, Feller Test

GTM113 Section 5.5.B

9.1 Weak Solution

9.1.1 D-SDE no drift, no explode

Lemma. Assume the solution to SDE $dX_t = \sigma(X_t)dW_t$ exists, then the solution cannot explode.
Proof. Define event $E_t = \{\omega : \int_0^{t\wedge s} \sigma^2(X_s)ds = +\infty\}$ for some $\omega \in \Omega$. WTS $\mathbb{P}(E_t) = 0$.
Define $\lim_{n\to\infty} \sup X_{t\wedge S_n} = \overline{\lim_{n\to\infty}} X_{t\wedge S_n} = X_0 + \overline{\lim_{n\to\infty}} \int_0^{t\wedge S_n} \sigma(X_s)dW_s$, where the dW_s time-changed B.M.
Define stopping time up to the explosion $S_n = \inf\{t \geq 0 : X_t \notin (l_n, r_n)\}$, where $l_n \searrow l = -\infty$, $r_n \nearrow r = +\infty$

> Recall **Lemma.** that B_t is recurrent, thus can diffuse to $\pm\infty$ at $t = \infty$.
> *Proof.* $\lim_{t\to\infty} \frac{B_t}{t} = 0$ by LLN/CLT, $\overline{\lim}_{t\to\infty} \frac{B_t}{\sqrt{2t\log(\log(t))}} = 1$ hard to prove, so B_t will eventually sup-bounded grows like $\sqrt{2t\log(\log(t))}$ (smaller scaled than t), and $\underline{\lim} = \lim_{t\to\infty} \inf \frac{B_t}{\sqrt{2t\log(\log(t))}} = -1$.
> So, $\lim_{t\to\infty} \sup B_t = +\infty$, and $\lim_{t\to\infty} \inf B_t = -\infty$

On E_t, $\lim_{n\to\infty} \sup X_{t\wedge S_n} = +\infty$, and $\lim_{n\to\infty} \inf X_{t\wedge S_n} = -\infty$, but X_t is continuous a.s. on $x \in \bar{\mathbb{R}}$, then

$$\Rightarrow \mathbb{P}(E_t) = 0 \Rightarrow X_{t\wedge S} = X_0 + \int_0^{t\wedge s} \sigma(X_s)dW_s, \text{ real valued} < \infty$$

since $\langle M\rangle_t < \infty$ a.s. $\Rightarrow M_t < \infty$ a.s., so $X_{t\wedge s}$ not explode.
Another way to proof, take X_t as a time changed Brownian Motion by Dubis-Dumbins-Schwartz Theorem, then it will not explode in finite time , by recurrent of Random Walk.

9.1.2 Weak Solution to nonlinear SDE

Def. Weak Solution up to Explosion Time. [GTM113, Sec 5.5, Def 5.1] For 1D SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^r \to \mathbb{R}^d$, a weak solution up to explosion time is a triplet (X, W), $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}$, where
(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\{\mathcal{F}_t\}$ is a filtration of sub-σ-algebra of $\mathcal{F}$. Note, here $\sigma-$algebra is no longer generated by B.M.
(ii) X is an $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{\pm\infty\}$-valued continuous adapted process, (W_t) is a r-dimensional standard Brownian Motion.
(iii) same as strong solution, but define stopping time $S_n := \inf\{t \geq 0 : |X_t| \geq n\}$, (a) $\mathbb{P}[\int_0^{t\wedge S_n} (|b(X_s)| + |\sigma^2(X_s)|)ds < \infty] = 1$, same as $b \in \mathcal{H}^2_{Loc}$, $\sigma \in \mathcal{H}^2$, and (2) $\mathbb{P}(X_{t\wedge S_n} = X + \int_0^t b(X_s)\mathbb{I}_{S\leq S_n}ds + \int_0^t \sigma(X_s)\mathbb{I}_{S\leq S_n}dW_s, \forall t) = 1$, or $X_{t\wedge S_n} = X + \int_0^t b(X_s)\mathbb{I}_{S\leq S_n}ds + \int_0^t \sigma(X_s)\mathbb{I}_{S\leq S_n}dW_s$ a.s.
Def. Explosion Time. $S = \lim_{n\to\infty} S_n = \inf\{t : X_t \notin \mathbb{R}\} = \inf\{t : X_t = \pm\infty\}$

9.2 Explosion Time Set-ups

9.2.1 An ODE example for Ex-plo-sion!

Consider $\begin{cases} dX_t = \frac{1}{\beta-1}|X_t|^\beta dt \\ X_0 = 1, \beta > 1 \end{cases}$, solve ODE we have $X_t = \begin{cases} (1-t)^{1/(1-\beta)}, t \in [0,1) \\ +\infty, t \geq 1 \end{cases}$, then ODE is not global Liptschiz, thesolution is unique up to a explosion time.

9.2.2 Strong Solution to Drifted SDE

Set-up SDE. $dX_t = b(X_t)dt + \sigma(X_t)dW_t$.
Def. Strong Solution of Bounded SDE. Consider 1D SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$, $X_t \in I \subset \mathbb{R}$, $I = (l, r)$ and $\bar{I} = [l, r]$, $-\infty \leq l < r \leq \infty$, and $b, \sigma : [0, T] \times I \to \mathbb{R}$. Given $X_0 \in (l, r)$, then X is a strong solution to the 1D SDE up to an explosion time, given ω, if (1) X is continuous process on $\bar{I} = [l, r]$ (2) for any $[l', r'] \subset [l, r]$, (X_t) is a strong solution on $t \in [0, \tau]$, where $\tau = \inf\{s \geq 0 | X_s \notin (l, r)\}$.

9.2.3 Explosion Time

Def. Explosion Time. $S = \inf\{t \geq 0 | X_t \notin (l, r)\}$, $S \in (0, \infty]$ r.v., adapted.

9.2.4 Scale Function

Def. Scale Function. For 1D SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$, if (non-degenerate) $\sigma^2(X_t) > 0, \forall X_t \in I = (l, r)$, (local integrability) $\forall x \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\int_{x-\varepsilon}^{x+\varepsilon} \frac{|b(y)|}{\sigma^2(y)} dy < \infty$, then scale function is $p(x) := \int_c^x \exp\left(-2\int_c^y \frac{b(z)}{\sigma^2(z)} dz\right) dy$, where c is arbitrary constant.
Remark. Scale function $p(x)$ is a local martingale.
Proof. We can find $\frac{dp(x)}{dx} = \exp\left(-2\int_c^x \frac{b(z)}{\sigma^2(z)} dz\right)$, and $\frac{d^2p(x)}{dx^2} = -2\frac{b(x)}{\sigma^2(x)} p'(x)$. Then take Ito formula, $dp(x) = p'(x)dX - \frac{b(x)}{\sigma^2(x)} p'(x)(dX)^2 = p'(x)\sigma(x)dW_t$.
Prop. Scale Function change lower bound. [GTM113, Sec 5.5.B., Prob 5.12] Lower bound and initial state X_0 will not affect scale function's finiteness, by $p_a(x) = p_a(c) + p_a'(c)p_c(x)$ to change the constant c. Here $p_a(x)$ and $p_c(x)$ have same behavior at ∞.
Prop. Scale Function strict increasing. Scale function $p(x)$ is strictly increasing w.r.t. x. Proof is simple, $\frac{dp(x)}{dx} = \exp\left(-2\int_c^x \frac{b(z)}{\sigma^2(z)} dz\right)$, whatever x is, $p'(x) > 0$, so p increases.
Prop. Not Explode with Scale Function. Define $Y_t = p(X_t)$, then $Y_t = Y_0 + \int_0^t \sigma(Y_s)dW_s$, then even if X explode, Y does not explode.
Lemma. Scale function control explosion. If $p(+\infty) = \infty$ or $p(-\infty) = -\infty$, then X_t does not explode in finite time. Proof: since p increasing, so if Y not explode at finite time, then X not explode.

9.2.5 Green's Function

Def. RN derivative. $m(dx) = \frac{2}{p'(x)\sigma^2(x)} dx$, $x \in I$, as speed measure, i.e. how much process spend in the interval.
Def. Green's Function. $G_{a,b}(x, y) := \frac{(p(x\wedge y) - p(a))(p(b) - p(x \vee y))}{p(b) - p(a)}$, $x \wedge y = \min(x, y)$, $x \vee y = \min(x, y)$, for $x, y \in [a, b]$, where $p(x)$ is the scale function.

Remark. Green's Function G is a solution to the SDE $\begin{cases} b(x)M'(x) + \frac{1}{2}\sigma^2(x)M''(x) = -1, a < x < b \\ M(a) = M(b) = 0 \end{cases}$,
where the solution is $M_{a,b}(x) = \int_a^b G_{a,b}(x, y)m(dy)$, $m(dx) = \frac{2}{p'(x)\sigma^2(x)} dx$ as defined above.
Proof. $M_{a,b}(x) = -\int_a^x (p(x) - p(y))m(dy) + \frac{p(x)-p(a)}{p(b)-p(a)} \int_a^b (p(b) - p(y))m(dy)$ **CHECK WHY???!!!!!**
Remark. Since p strict increasing, then G is nonnegative, and m is nonnegative, then M is nonnegative

9.2.6 Expected Exit Time

Def. Exit time. $T_{a,b} = \inf\{t > 0 : X_t \notin (a, b)\}$.
Def. Expectation of Exit time. $\mathbb{E}_x[T_{a,b}]$ as expectation of exit time to a or b, starting from $X_0 = x \in (a, b)$
Def. Exit Probability. [Generalization of B.M. exit time] $\mathbb{P}(X^x_{T_{a,b}} = a) = \frac{p(b)-p(x)}{p(b)-p(a)}$, and $\mathbb{P}(X^x_{T_{a,b}} = b) = \frac{p(x)-p(a)}{p(b)-p(a)}$, where $p(x)$ is the scale function. Proof in Sec 2.8. Corollary.

9.2.7 Engelbert-Schmidt Condition

Theorem. Engelbert-Schmidt Condition. If $\sigma^2(x) > 0$, $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|b(y)|}{\sigma^2(y)} dy < \infty$ [local integrability] for $\exists\, \varepsilon > 0, \forall x$, then exists unique solution to the SDE in Sec-2. $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ up to a explosion time.

Example. $|b(x)| + |\sigma(x)| \leq K(1 + |x|)$ for linear growth condition in strong solution's existence, then since the solution exists, X does not explode to ∞. In fact, $\mathbb{E}[|X_t|^m] < \infty, \forall m \geq 1, \forall t$, so it has finite moment, then $X < \infty$ a.s. Then, Geometric B.M. is of linear growth.

9.2.8 Weak Solution Exiting (a,b)

Def. Modified Ver. of Weak Solution. Consider SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ on the interval $I = (l, r)$, a weak solution is a trple (X, W), $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}$, where

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $\{\mathcal{F}_t\}$ is a filtration of sub-σ-algebra of $\mathcal{F}$. Note, here $\sigma-$algebra is no longer generated by B.M.

(ii) $X = \{X_t : 0 \leq t < \infty\}$ is continuous, adapted, $[l, r]-$valued process, $X_0 \in I$ a.s., $\{W_t : 0 \leq t < \infty\}$ is a standard 1D-B.M.

(iii) $\{l_n\}_{n=1}^{\infty}$ decreases $l \searrow l$, $\{r_n\}_{n=1}^{\infty}$ increases $r_n \nearrow r$, $l < l_n < r_n < r$, $\lim_{n\to\infty} l_n = l$, $\lim_{n\to\infty} r_n = r$, stopping time is,

$$S_n := \inf\{t \geq 0 : X_t \notin (l_n, r_n)\}, n \geq 1$$

and with limit $S = \lim\limits_{n\to\infty} S_n$ as exit time from I.

Remark. If $l = -\infty, r = +\infty$, then above definition redices to Sec 1.2. Def, where $X_t = X_s, s \leq t < \infty$

Theorem. Exiting from (a, b) of Weak Solution. On interval $I = (l, r)$, $-\infty \leq l < r \leq +\infty$, (X, W) is a weak solution to $dX_t = b(X_t)dt + \sigma(X_t)dW_t$, $X_0 \in (a, b) \subsetneq I$, $l < a < b < r$,

(1) speed measure $m(dx) = \frac{2}{p'(x)\sigma^2(x)} dx$,

(2) Green's function $G_{a,b}(x, y) := \frac{(p(x\wedge y)-p(a))(p(b)-p(x\vee y))}{p(b)-p(a)}$,

(3) solution to SDE $\begin{cases} b(x)M'(x) + \frac{1}{2}\sigma^2(x)M''(x) = -1, a < x < b \\ M(a) = 0 = M(b) \end{cases}$ as $M_{a,b}(x) = \int_a^b G_{a,b}(x, y)m(dy)$, nonegative,

(4) stopping time $\tau_n = \{t \geq 0 : \int_0^t \sigma^2(X_s)ds \geq n\}$, then after n, the process X_t will become martingale,

(5) explosion time, $T_{a,b} = \inf\{t \geq 0 : X_t \notin (a, b)\}$.

Then X will exit from every compact subinterval with $I = (l, r)$ in finite time a.s., i.e. $\mathbb{E}[T_{a,b}] < \infty$

Proof.

Taking Ito's Formula, then

$$M_{a,b}(X_{t\wedge\tau_n\wedge T_{a,b}}) = M_{a,b}(X_0) - (t \wedge \tau_n \wedge T_{a,b}) - \int_0^{t\wedge\tau_n\wedge T_{a,b}} M'_{a,b}(X_s)\sigma(X_s)dW_s$$

Taking expectation, and $\lim\limits_{n\to\infty}$, by Martingale Convergence Theorem, and rearrange terms

$$\mathbb{E}[t \wedge T_{a,b}] = M_{a,b}(X_0) - \mathbb{E}[M_{a,b}(X_{t\wedge T_{a,b}})] \leq M_{a,b}(X_0) < \infty$$

Taking $\lim\limits_{t\to\infty}$,

$$\mathbb{E}[T_{a,b}] \leq M_{a,b}(X_0) < \infty$$

So, $T_{a,b} < \infty$ a.s.

Corollary. $\mathbb{E}[T_{a,b}] = M_{a,b}(x)$, $X_0 = x \in (a, b)$

Proof. Since $T_{a,b} < \infty$ a.s., then $X_{t\wedge T_{a,b}} \overset{t\to\infty}{\to} X_{T_{a,b}}$, then since $M(a) = M(b) = 0$, then $M(X_{t\wedge T_{a,b}}) \overset{t\to\infty}{\to} M(X_{T_{a,b}}) = 0$, so

$$\mathbb{E}[t \wedge T_{a,b}] = M_{a,b}(X_0) - \mathbb{E}[M_{a,b}(X_{t\wedge T_{a,b}})] \Rightarrow \text{take } \lim_{t\to\infty}, \mathbb{E}[T_{a,b}] = M_{a,b}(x), X_0 = x \in (a, b), \text{ by MCT}$$

Corollary. At initial point, $p(x) = \mathbb{E}[p(X_{t\wedge T_{a,b}})] \overset{t\to\infty}{=} \mathbb{E}[p(X_{T_{a,b}})]$.

Proof. Easy, p is a local martingale, and $T_{a,b} < \infty$, use OST, and then $\mathbb{E}[p(X_T)|\mathcal{F}_0] = p(X_0) = p(x)$.
Corollary. Generalization of B.M. exit time. $\mathbb{P}(X^x_{T_{a,b}} = a) = \frac{p(b)-p(x)}{p(b)-p(a)}$, and $\mathbb{P}(X^x_{T_{a,b}} = b) = \frac{p(x)-p(a)}{p(b)-p(a)}$
Proof. Since $p(x) = \mathbb{E}[p(X_{T_{a,b}})] = p(a)\mathbb{P}[X_{T_{a,b}} = a] + p(b)\mathbb{P}[X_{T_{a,b}} = b]$. Since $\mathbb{P}[X_{T_{a,b}} = a] = 1 - \mathbb{P}[X_{T_{a,b}} = b]$, so we can derive the result. ■

9.3 Feller Test of Explosion

9.3.1 Scale Function and None-Explosion at Finite

Prop. Sufficient condition for None-Explosion in Finite Time. Consider setups: (1) SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$, (non-degenerate) $\sigma^2(X_t) > 0, \forall X_t \in I$, (local integrability) $\forall x \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\int_{x-\varepsilon}^{x+\varepsilon} \frac{|b(y)|}{\sigma^2(y)} dy < \infty$, constant initial condition $X_0 = x \in I = (l, r)$, (2) scale function is $p(x) := \int_c^x \exp\left(-2\int_c^y \frac{b(z)}{\sigma^2(z)} dz\right) dy$, (3) stopping time is $S = \inf\{t \geq 0 | X_t \notin (l, r)\}$, then
(a) if $p(l+) = -\infty$, $p(r-) = \infty$, then $\mathbb{P}(S = +\infty) = 1 = \mathbb{P}(\sup\limits_{0\leq t<\infty} X_t = r) = \mathbb{P}(\inf\limits_{0\leq t<\infty} X_t = l)$, so the process will not explode in finite time since $p(x)$ is strict increasing. Further, X is recurrent, so for $\forall y \in I$, $\mathbb{P}(X_t = y, \exists 0 \leq t < \infty) = 1$ because of recurrent on $I = (l, r)$
(b) If $p(l+) > -\infty$, $p(r-) = \infty$, then $\mathbb{P}(\lim\limits_{t\nearrow S} X_t = l) = 1 = \mathbb{P}(\sup\limits_{0\leq t<S} X_t < r)$, strict less than r
(c) If $p(l+) = -\infty$, $p(r-) < \infty$, then $\mathbb{P}(\lim\limits_{t\nearrow S} X_t = r) = 1 = \mathbb{P}(\inf\limits_{0\leq t<S} X_t > l)$
(d) If $p(l+) > -\infty$, $p(r+) < \infty$, then $\mathbb{P}[\lim\limits_{t\nearrow S} X_t = l] = 1 - \mathbb{P}[\lim\limits_{t\nearrow S} X_t = r] = \frac{p(r-)-p(x)}{p(r-)-p(l+)}$, generalized from sec 2-2.8
Proof.(a). if $p(l+) = -\infty$, $p(r-) = \infty$, then $\mathbb{P}(S = +\infty) = 1 = \mathbb{P}(\sup\limits_{0\leq t<\infty} X_t = r) = \mathbb{P}(\inf\limits_{0\leq t<\infty} X_t = l)$
(part-i) for $l < a < x < b < r$, to find probability of inf, sandwich it, no need to find distribution,

$$\begin{aligned}
1 &\geq \mathbb{P}(\inf_{0\leq t<s} X^x_t \leq a), \text{ corss b } \infty \text{ time before cross a} \\
&\geq \mathbb{P}(X^x_{T_{a,b}} = a), \text{only cross a, not cross b} \\
&= \frac{p(b)-p(x)}{p(b)-p(a)}, X_0 = x \\
&= \frac{1-p(x)/p(b)}{1-p(a)/p(b)}, \text{if } b \nearrow r, p(b) \nearrow p(r-) = \infty \\
&\overset{b\nearrow r}{\to} 1
\end{aligned}$$

so, $\mathbb{P}(\inf\limits_{0\leq t<s} X^x_t \leq a) = 1$, also true when $a \searrow l$, so set $\{a_n\}_{n=1}^{\infty}, a_n \searrow l$, by continuous from above ($A_n \searrow A$, $\mathbb{P}(A_n) \searrow \mathbb{P}(A)$), $1 = \lim\limits_{n\to\infty} \mathbb{P}(\inf\limits_{0\leq t} X_t \leq a_n) = \mathbb{P}(\inf\limits_{0\leq t} X_t \leq \lim\limits_{n\to\infty} a_n) = \mathbb{P}(\inf\limits_{0\leq t} X_t \leq l)$ and it is of course $1 = \mathbb{P}(\inf\limits_{0\leq t} X_t = l)$ since $X_t \in I$.
(part ii) Similarly,

$$\begin{aligned}
1 &\geq \mathbb{P}(\inf_{0\leq t<s} X^x_t \geq b), \text{ corss a } \infty \text{ time before cross b} \\
&\geq \mathbb{P}(X^x_{T_{a,b}} = b), \text{only cross b once, not cross a} \\
&= \frac{p(x)-p(a)}{p(b)-p(a)}, X_0 = x \\
&= \frac{p(x)/p(a)-1}{p(b)/p(a)-1}, \text{if } a \searrow l, p(a) \searrow p(l+) = -\infty \\
&\overset{a\searrow r}{\to} 1
\end{aligned}$$

same by taking $b_n \nearrow r$ as continuity from below , $\mathbb{P}(\inf\limits_{0\leq t<s} X^x_t = r) = 1$
(part iii) Suppose $\mathbb{P}(S < \infty) > 0$, then the limit $\lim\limits_{s\nearrow S} X_t$ exists, and can either be $X_t = l$ or $X_t = r$, then only $\mathbb{P}[\sup\limits_{0\leq t<\infty} X_t = r] < 1$ or $\mathbb{P}[\inf\limits_{0\leq t<\infty} X_t = l] < 1$, contradiction. $\mathbb{P}(S < \infty) = 0$

$\underline{\textit{Proof.}(b).}$ If $p(l+) > -\infty$, $p(r-) = \infty$, then $\mathbb{P}(\lim_{t\nearrow S} X_t = l) = 1 = \mathbb{P}(\sup_{0\le t<S} X_t < r)$, strict less than r.
(part-i) for $X_t = r$, for $l < a < x < b < r$,

$$
\begin{aligned}
\mathbb{P}[\sup_{0\le t<S} X_t = r] &\le \mathbb{P}[X_t = b, \exists 0 < t < S], \text{ corss b } \infty \text{ time before cross r} \\
&= \frac{p(x) - p(l)}{p(b) - p(l)}, \text{Sec2.8-Corollary} \\
&\overset{b\nearrow r}{\to} 0
\end{aligned}
$$

Thus, $\mathbb{P}[\sup_{0\le t<S} X_t = r] = 0$, thus $\mathbb{P}[\sup_{0\le t<S} X_t < r] = 1$
(part-ii) $\mathbb{P}(\lim_{t\nearrow S} X_t = l) = 1$ by (a)
(part-iii) To show limit exists, i.e. $\lim_{t\nearrow S} X_t = \inf_{0\le t<S} X_t$ or $\lim_{t\to\infty} X_{t\wedge S}$ limit exists a.e.. And limits has to l.
Recall $S_n = \inf\{t \ge 0 : X_t \notin (l_n, r_n)\}$, take $l_n \searrow l$, $r_n \nearrow r$, and define $S = \lim_{n\to\infty} S$.
Define process $Y_t^{(n)} := p(X_{t\wedge S_n}) - p(l+) \ge 0, 0 \le t < \infty$, since p is strict increasing.
Since $p(\cdot)$ is a local martingale proved in Section 2.4., $p(l+)$ is constant, then $Y_t^{(n)}$ is also a local martingale. Since $Y_t^{(n)}$ is nonnegative, then $Y_t^{(n)}$ is a supermartingale. Then $\lim_{n\to\infty} Y_t^{(n)} =: Y_t = p(X_{t\wedge S}) - p(l+) \ge 0$ also supermartingale.
By Fatou Lemma, $t > u$, $\mathbb{E}[\liminf Y_t^{(n)}|\mathcal{F}_u] \le \liminf_n \mathbb{E}[Y_t^{(n)}|\mathcal{F}_u] = \liminf Y_u^{(n)} = Y_u$ by local martingality. [Corollary: limit of super martingale sequence is also supermartingale]. So $Y_t^{(n)}$ converges to Y_t, $\forall t$.
Since $p \circ X_t$ is consitnuous and increasing, then $Y_t \in L^1$, and $p : [l, r] \to \mathbb{R}$ has continuous inverse, as $X_t = p^{-1}(Y_t)$, then $\lim_{t\to\infty} X_{t\wedge S}$ exists (finite).
$\underline{\textit{Proof.}(c).}$ Similar proof to (b).
$\underline{\textit{Proof.}(d).}$ If $p(l+) > -\infty$, $p(r+) < \infty$, then $\mathbb{P}[\lim_{t\nearrow S} X_t = l] = 1 - \mathbb{P}[\lim_{t\nearrow S} X_t = r] = \frac{p(r-)-p(x)}{p(r-)-p(l+)}$, just use what we have proved in Section 2.8.

9.3.2 Value Function

<u>Def. Value Function.</u> Define a recursion sequence $\{u_n\}_{n=0}^{\infty}$ on I, u_n are real-valued function, $u_0 \equiv 1$, $u_n(x) = \int_c^x p'(y) \left(\int_c^y u_{n-1}(z)m(dz)\right) dy, x \in I, n \ge 1$, and c is fixed number in I, then the value function $v(x)$ is, $v(x) := u_1(x) = \int_c^x p'(y) \int_c^y u_0(x)m(dz)dy = \int_c^x p'(y) \int_c^y m(dz)dy$.
<u>Remark.</u> $v(x) = \int_c^x (p(x) - p(y))dm(y)$. Proof below.

$$
\begin{aligned}
v(x) := u_1(x) &= \int_c^x p'(y) \int_c^y u_0(x)m(dz)dy \\
&= \int_c^x p'(y) \underbrace{\int_c^y \frac{2dz}{p'(z)\sigma^2(z)}}_{:=m(c,y)} dy \\
&= p(y)m(c,y)|_c^x - \int_c^x p(y)dm(c,y), \text{integral by parts} \\
&= p(x)m(c,x) - \int_c^x p(y)dm(c,y), m(c,c) = 0 \\
&= p(x)\int_c^x dm(c,y) - \int_c^x p(y)dm(c,y) = \int_c^x (p(x) - p(y))dm(y)
\end{aligned}
$$

<u>Lemma. Convergence and ODE solution for $u(x)$.</u> [GTM113, Chap 5.5, Lemma 5.26] Assume (non-degenerate) $\sigma^2(X_t) > 0, \forall X_t \in I = (l, r)$, (local integrability) $\forall x \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\int_{x-\varepsilon}^{x+\varepsilon} \frac{|b(y)|}{\sigma^2(y)} dy < \infty$, define $u(x) = \sum_{n=0}^{\infty} u_n(x), x \in I$. Then
(1) $u(x)$ converges uniformly on any compact subsets on I,
(2) $u(x)$ is differentiable and has absolutely continuous derivative on I,

(3) fix $c \in I$, $u(x)$ strictly increasing on (c, r), and strictly decreasing on (l, c), $u(c) = u_0(c) = 1$, so $u(x) \geq 1$

(4) $u(x) \geq 0$, and $u(x)$ os a solution to ODE $\begin{cases} \frac{1}{2}\sigma^2(x)u''(x) + b(x)u'(x) = u(x) \\ u(c) = 1, u'(c) = 0 \end{cases}$, $u(x)$ is autonomous, $x \in I$,

(5) $1 + v(x) \leq u(x) \leq e^{v(x)}$, $x \in I$

Proof.(3). Easy, just $\int_c^x$, and $-\int_x^c$

Proof.(4). Since,

$$
\begin{aligned}
b(x)u_n'(x) &= b(x)p'(x)\int_c^x u_{n-1}(z)m(dz) \\
\frac{1}{2}\sigma^2(x)u_n''(x) &= \frac{1}{2}\sigma^2(x)p''(x)\int_c^x u_{n-1}(z)m(dz) + \frac{1}{2}\sigma^2(x)p'(x)u_{n-1}(x)\frac{2}{p'(x)\sigma^2(x)} \\
&= \frac{1}{2}\sigma^2(x)p''(x)\int_c^x u_{n-1}(z)m(dz) + u_{n-1}(x) \\
&= -b(x)p'(x)\int_c^x u_{n-1}(z)m(dz) + u_{n-1}(x), p''(x) = -2\frac{b(x)}{\sigma^2(x)}p'(x)
\end{aligned}
$$

So, $b(x)u_n'(x) + \frac{1}{2}\sigma^2(x)u_n''(x) = u_{n-1}(x)$. Then,

$$
b(x)\sum_{n=1}^{\infty} u_n'(x) + \frac{1}{2}\sigma^2\sum_{n=1}^{\infty} u_n''(x) = \sum_{n=1}^{\infty} u_{n-1}(x) b(x)u'(x) + \frac{1}{2}\sigma^2 u''(x) = u(x)
$$

assume $u'(x)$ is summable and we can use Fubini Theorem. Here u_n is not autonomous, but u is.

Proof.(5). To show $1 + v(x) \leq u(x)$, for $c \leq x < r$, $u_n'(x) \geq 0$, $u_n'(c) = 0$, $u = \sum_{n=1}^{\infty} u_n$, so $u \geq u_0 + u_1$. Same for $l < x \leq c$.

To show $u(x) \leq e^{v(x)}$, STS $u_n(x) \leq \frac{v^n(x)}{n!}$, $n = 0, 1, 2, ...$ Proof by induction,

For $n = 0$, $u_0(x) \equiv 1$, $1 \leq 1$, True. Assume it is true for $n = k - 1$, $u_{k-1}(x) \leq \frac{v^{k-1}(x)}{n!}$, then for $n = k$,

$$
u_k'(x) = p'(x)\int_c^x u_{k-1}(z)m(dz)
$$

Then assume $c \leq x < r$, then,

$$
\begin{aligned}
u_k(x) &= \int_c^x p'(y)\left(\int_c^y u_{k-1}(z)m(dz)\right)dy \\
&\leq \int_c^x p'(y)\left(\int_c^y \frac{v^{k-1}(z)}{n!}m(dz)\right)dy \\
&\leq \frac{1}{(k-1)!}\int_c^x p'(y)v^{k-1}(y)\int_c^y m(dz)dy \\
&, \text{by } v'(x) = p'(x)\int_c^x m(dz) > 0 \\
&= \frac{1}{(k-1)!}\int_c^x v^{k-1}(y)dv(y) \\
&= \frac{v^k(x)}{k!}
\end{aligned}
$$

Same for $l < x \leq c$. ■.

9.3.3 Scale Function Implies Value Function

Lemma. Sufficient Conditon for Infiniteness Behavior. [GTM113, Chap.5.5, Prob.5.27] For scale function p and value funciton v, (1) $p(r-) = \infty \Rightarrow v(r-) = \infty$, (2) $p(l+) = -\infty \Rightarrow v(l+) = \infty$.

Lemma. Change integrate interval. For integral from c, denote $v_c(x) := \int_c^x p'(y)\int_c^y m(dz)dy$, then by above lemma, $v_a(x) = v_a(c) + v_a'(c)p_c(x) + v_c(x)$, $x, a, c \in I$. And finiteness or nonfiniteness of $v_c(r-), v_c(l+)$ does not depend on c.

9.3.4 Feller Test of None-Explosion

Theorem. Feller Test for None-Exlposion. [GTM113, Chap5.5, Theorem 5.29, Sufficient and necessary condition for explosion at infinite time [5]] Assume the SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ satisfies (non-degenerate) $\sigma^2(X_t) > 0, \forall X_t \in I = (l, r)$, (local integrability) $\forall x \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\int_{x-\varepsilon}^{x+\varepsilon} \frac{|b(y)|}{\sigma^2(y)} dy < \infty$. $(X, W), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ is a weak solution in I, initial condition is fixed $X_0 = x \in I$, then $\mathbb{P}(S = +\infty) = 1 \iff v(r-) = v(l+) = \infty$

Proof. Interpret: process not explode, or process only exlpode at $\pm\infty$, iff, v(both end) $= \infty$.

$\boxed{\Leftarrow}$ By contradiction, $\inf\{\emptyset\} = 1$. First, we construct a martingale in L^1 that has same finiteness of $u(\cdot)$. Define a localizing sequence $\tau_n = \inf\{t \geq 0 : \int_0^t \sigma^2(X_s)ds \geq n\}$. Stopped process $Z_t^n = u(X_{t\wedge S_n \wedge \tau_n}) \geq 0$. Apply Ito's Formula,

$$\begin{aligned}
du(X_t) &= u'(X_t)dX_t + \frac{1}{2}u''(X_t)(dX_t)^2 \\
&= (u'(X_t)b(X_t) + \frac{1}{2}u''(X_t)\sigma^2(X_t))dt + \sigma(X_t)u'(X_t)dW_t \\
&\text{recall },\begin{cases} \frac{1}{2}\sigma^2(x)u''(x) + b(x)u'(x) = u(x) \\ u(c) = 1, u'(c) = 0 \end{cases} \\
&= u(X_t)dt + \sigma(X_t)u'(X_t)dW_t
\end{aligned}$$

Then, $Z_t^n = Z_0^n + \int_0^{t\wedge S_n\wedge\tau_n} u(X_s)ds + \int_0^{t\wedge S_n\wedge\tau_n} u'(X_s)\sigma(X_s)dW_s$. Define sequence $M_t^n := e^{-t\wedge S_n\wedge\tau_n} Z_t^n$, then take Ito's formula,

$$\begin{aligned}
dM_t^n = de^{-t\wedge S_n\wedge\tau_n} Z_t^n &= -e^{-t\wedge S_n\wedge\tau_n} Z_t^n dt + e^{-t\wedge S_n\wedge\tau_n} dZ_t^n \\
&= -e^{-t\wedge S_n\wedge\tau_n} u(X_{t\wedge S_n\wedge\tau_n})dt + e^{-t\wedge S_n\wedge\tau_n} du(X_{t\wedge S_n\wedge\tau_n}) \\
&= -e^{-t\wedge S_n\wedge\tau_n} u(X_{t\wedge S_n\wedge\tau_n})dt + e^{-t\wedge S_n\wedge\tau_n} u(X_{t\wedge S_n\wedge\tau_n})dt \\
&\quad + e^{-t\wedge S_n\wedge\tau_n} \sigma(X_{t\wedge S_n\wedge\tau_n})u'(X_{t\wedge S_n\wedge\tau_n})dW_t \\
&= e^{-t\wedge S_n\wedge\tau_n} \sigma(X_{t\wedge S_n\wedge\tau_n})u'(X_{t\wedge S_n\wedge\tau_n})dW_t \\
\Rightarrow M_t^n = M_0^n &+ \int_0^{t\wedge S_n\wedge\tau_n} e^{-s}\sigma(X_s)u'(X_s)dW_s
\end{aligned}$$

Then clearly, M_t^n is a local martingale, and it's nonnegative since $\min u(X_{t\wedge S_n\wedge\tau_n}) \geq u(c) = 1 \geq 0$, so $Z_t^n \geq 0$, so $M_t^n \geq 0$, then M_t^n is a supermartingale. Then its limit $M_t = \lim\limits_{n\to\infty} M_t^n = e^{-t\wedge S} u(X_{t\wedge S})$ is also supermartingale, $\forall 0 \leq t < \infty$, since $M_t \geq 0$bounded below, then by Martingale Central Limit Theorem, $M_t \in L^1$, $\lim\limits_{t\to\infty} M_t := M_\infty = \lim\limits_{t\to\infty} e^{-t\wedge S} u(X_{t\wedge S})$ eixsts in L^1 thus finite a.e..

Then. Back to value function, if $v(r-) = v(l+) = \infty$, then by $1 + v(x) \leq u(x)$, $u(r-) = u(l+) = \infty$. Suppose $\mathbb{P}(S < \infty) > 0$, then for some S, $M_\infty = e^{-S}u(r - \wedge S) = e^{-S}u(r-) = \infty$ by $S = \inf\{t : X_t \notin (l, r)\}$. Contradiction, since $M_t \in L^1, 0 \leq t \leq \infty$, so $S = \infty$ a.e., aka, $\mathbb{P}(S = \infty) = 1$.

$\boxed{\Rightarrow}$ Prove by contradiction.

Suppose $\mathbb{P}(S = \infty) = 1$ and $v(r-) < \infty$ (or $v(l+) = \infty$), then $u(r-) < \infty$ by $u(x) < e^{v(x)}$.

Since fitiness doesn't depende on c, assume $c < x < r$, then set stopping time $T_c = \inf\{t \geq 0 : X_t = c\}$, define continuous process $M_{t\wedge T_c} := e^{-(t\wedge S\wedge T_c)} u(X_{t\wedge S\wedge T_c})$, $\forall t \in [0, \infty)$ is again local martingale (1).

Since v is increasing, v is bounded here by $v(r-) < \infty$, then u is bounded above, and $u \geq 0$ bounded below. So, $M_{t\wedge T_c}$ is a bounded local martingale (2),

thus $M_{t\wedge T_c}$ is a bounded martingale (3), thus converge a.s. in L^1 to $M_{T_c} := \lim\limits_{t\to\infty} M_{t\wedge T_c} \in L^1$ (4). Back to value function, since $M_0 = u(X_0) = u(x), X_0 = x \in I, c < x < r-$, and M_{T_c} is a martingale,

$$\begin{aligned}
u(x) &= \mathbb{E}[e^{-S\wedge T_c} u(X_{S\wedge T_c})] \\
&= \mathbb{E}[e^{-S\wedge T_c} u(X_{S\wedge T_c})\mathbb{I}_{S<T_c}] + \mathbb{E}[e^{-S\wedge T_c} u(X_{S\wedge T_c})\mathbb{I}_{S\geq T_c}] \\
&= \mathbb{E}[e^{-S} u(X_S)\mathbb{I}_{S<T_c}] + \mathbb{E}[e^{-T_c} u(X_{T_c})\mathbb{I}_{S\geq T_c}] \\
&= u(r-)\mathbb{E}[e^{-S}\mathbb{I}_{S<T_c}] + u(c)\mathbb{E}[e^{-T_c}\mathbb{I}_{S\geq T_c}], c \leq x < r-
\end{aligned}$$

Here we have $\mathbb{P}(S=\infty)=1$, then, $S>T_c$ since by OST $T_c \le \infty$ a.s. So, $u(x)=u(c)\mathbb{E}[e^{-T_c}\mathbb{I}_{S\ge T_c}] \le u(c)$, by $e^{-T_c}\mathbb{I}_{S\ge T_c}>0$. But we have show that $u(x)$ strict increases on $(c,r+)$, if $x>c$, then $u(x)>u(c)$, contradiction! So if $\mathbb{P}(S=\infty)=1$ and $v(r-)=\infty$.
Similarly, for $\mathbb{P}(S=\infty)=1$ and $v(l+)<\infty$, we only need to consider $l+<x<c$, same we will get $u(x)<u(c)$, but actually $u(x)>u(c)$ since c is the U-shaped curve's bottom. So contradiction! So $\mathbb{P}(S=\infty)=1$ and $v(l+)=\infty$.
So, if $\mathbb{P}(S=\infty)=1$, then $v(l+)=\infty$, $v(r-)=\infty$.

9.3.5 Feller Test of Explosion II

__Theorem. Feller Test for Exlposion at Finite Time.__ [GTM113, Chap5.5, Prop5.32] Assume the SDE $dX_t=b(X_t)dt+\sigma(X_t)dW_t$ satisfies (non-degenerate) $\sigma^2(X_t)>0, \forall X_t \in I=(l,r)$, (local integrability) $\forall x \in \mathbb{R}, \exists \varepsilon>0$ s.t. $\int_{x-\varepsilon}^{x+\varepsilon} \frac{|b(y)|}{\sigma^2(y)}dy<\infty$. $(X,W),(\Omega,\mathcal{F},\mathbb{P}),\{\mathcal{F}_t\}$ is a weak solution in I, initial condition is fixed $X_0=x\in I$, then $\mathbb{P}(S<+\infty)=1 \iff$ one of following holds (i)

$$\begin{cases} (i)\ v(r-)<\infty, v(l+)<\infty, \text{both finite} \\ (ii)\ v(r-)<\infty, p(l+)=-\infty \\ (iii)\ v(l+)<\infty, p(r-)=\infty \end{cases}.$$

__*Corollary.*__ If (i) is true, then $\mathbb{E}[S]<\infty$.
__*Proof.*__ By 3.3. Lemma, then $p(r-)<\infty, p(r+)>-\infty$. Recall Green's Function $G(x,y):=\frac{(p(x\wedge y)-p(l+))(p(r-)-p(x\vee y))}{p(r-)-p(l+)}$, $(x,y)\in I^2=(l,r)\times(l,r)$, and $M(x)=\int_l^r G(x,y)m(dy)$, $m(dy)=\frac{2}{p'(y)\sigma^2(y)}dy$, then $M(l+)=0=M(r-)$. Recall 2.8 Corollary, $\mathbb{E}[T_{l,r}]=\mathbb{E}[S]=M(x)$. Since p is increasing, and finite, then G is finite, and $m(dy)$ is thus finite, so $\mathbb{E}[S]\le M(x)<\infty$.
__Proof Theorem.__
$\boxed{\Leftarrow}$
(i) $v(r-)<\infty, v(r+)<\infty \Rightarrow \mathbb{P}(S=+\infty)<1$
By corollary, then $\mathbb{E}[S]=\mathbb{E}_x[S]<\infty$, starting from $X_0=x\in I$, so we cannot have $\mathbb{P}(S=+\infty)>0$, otherwise expectation cannot be finite. So, $\mathbb{P}(S<+\infty)=1$.
(ii) $v(r-)<\infty, p(l+)=-\infty \Rightarrow \mathbb{P}(S=+\infty)<1$
Consider intervals, $l_n \searrow l$, and $r_n \nearrow r$, then define stopping time $R_n=\inf\{t\ge 0: X_t=l_n\}$, $T_r=\inf\{t\ge 0: X_t=r\}$, $T_l=\inf\{t\ge 0: X_t=l\}=\lim_{n\to\infty} R_n$. Since $v(r-)<\infty$ by assumption, and $v(l_n)<\infty$ since $p(l_n)<\infty$ since p is strictly increasing and only $p(l)=p(\lim_{n\to\infty} l_n)=\infty$. By corollary, $\mathbb{E}[R_n\wedge T_{l,r}]<\infty$, and $\mathbb{E}[R_n\wedge T_{l,r}]=\mathbb{E}[R_n\wedge T_r]$, so $R_n\wedge T_r<\infty$ a.s., $\forall n\ge 1$.
By $v(r-)<\infty$, then $p(r-)<\infty$. By Section 3.1. Theorem-(c), since $1=\mathbb{P}(\inf_{0\le t<S} X_t>l)$, thus $\exists N\ge 1$ large enough, $n>N$, s.t. $X_t>l_n$, thus $R_n=\infty$ or at least $R_n>S$. Thus, $S(\omega)=\lim_{n\to\infty} R_n\wedge T_r=T_r<\infty$.
(iii) $v(r-)=\infty, v(l+)=\infty \Rightarrow \mathbb{P}(S=+\infty)<1$. Similar to (ii)
$\boxed{\Rightarrow}$
If $\mathbb{P}[S<\infty]=1$, the by Feller I, either $v(l+)<\infty$ (or $v(r-)<\infty$). Proof by contradiction, suppose none of (i), (ii), (iii), hold, aka, $v(r-)=\infty>p(r-), p(l+)>-\infty, p(r-)<\infty$. This is Section 3.1. Theorem-(d) type. Define event $A_r:=\{\lim_{t\nearrow S} X_t=r\}$. Then, $\mathbb{P}(A_r)=\frac{p(x)-p(l+)}{p(r-)-p(l+)}>0$.
By Feller I, $\boxed{\Leftarrow}$, $M_t=\lim_{n\to\infty} M_t^n=e^{-t\wedge S}u(X_{t\wedge S})$ is a nonegative supermartingale, and $\lim_{t\to\infty} M_t:=M_\infty=\lim_{t\to\infty} e^{-t\wedge S}u(X_{t\wedge S})\in L^1$,
By $u(x)>1+v(x)$, then $u(r-)=\infty$, then on A_r, $M_\infty=e^{-S}u(r+)=\infty$ for $\omega\in A_r$, then we can conclude $\mathbb{P}[S=\infty]>0$, contradiction. So we must have one of (i), (ii), (iii) holds.

9.4 Case Analysis of Simple Geometric BM

9.4.1 Explosive behavior of Geometric BM

__Prop.__ Consider dynamic $X_t=x+\mu\int_0^t X_s ds+\nu\int_0^t X_s dW_s$, $x>0$, then
(1) If $\mu<\frac{\nu^2}{2}$, $\lim_{n\to\infty} X_t=0$, $\sup_{t\ge 0} X_t<+\infty$

(2) If $\mu > \frac{\nu^2}{2}$, $\lim_{n\to\infty} X_t = +\infty$, $\inf_{t\geq 0} X_t > 0$

(3) If $\mu = \frac{\nu^2}{2}$, $\inf_t X_t = 0$, $\sup_{t\geq 0} X_t = +\infty$. This is a good example to check sufficient condition for Feller II.

Proof. The scale function is $p(x) = \int_c^x \exp(-2\int_c^y \frac{b(z)}{\sigma^2(z)}dz)dy = \frac{c}{1-\frac{2\mu}{\nu^2}}((\frac{x}{c})^{1-\frac{2\mu}{\nu^2}} - 1)$. And value function is, $v(x) = \int_c^x p'(y)\int_c^y \frac{2dz}{p'(z)\sigma^2(z)}dy = \frac{2\log(x)}{2\mu-\nu^2} - \frac{2c^{2\mu/\nu^2}}{2\mu-\nu^2}x^{1-2\mu/\nu^2} + K$.
If $\mu < \nu^2/2$, then $1 - \frac{2\mu}{\nu^2} > 1$, then $p(0+) > -\infty$, and $p(+\infty) = +\infty$. But $2\mu - \nu^2 < 0$ then, $v(0) = +\infty$, $v(+\infty) = +\infty$. By Feller I, $\mathbb{P}(S = +\infty) = 1$.

References

[1] solutions stack exchange.

[2] Carr, P., & Nadtochiy, S. (2011). Static hedging under time-homogeneous diffusions. *SIAM Journal on Financial Mathematics*, *2*(1), 794-838.

[3] Bayraktar, E., & Nadtochiy, S. (2015). Weak reflection principle for Lévy processes.

[4] Carr, P., & Zhang, Z. (2018). Generalizing Geometric Brownian Motion. *arXiv preprint arXiv:1809.02245*.

[5] Feller, W. (1952). The parabolic differential equations and the associated semi-groups of transformations. *Annals of Mathematics*, 468-519.

Stochastic Differential, Oh What a Miracle

Amidst the vastness of the unknown,
Where chaos reigns and chance is sown,
A humble equation seeks to show
The paths that randomness may go.

Stochastic differential, its name,
A fusion of two worlds, it came,
Where physics meets the dice's game,
And probability finds its aim.

Through drift and diffusion, it can trace,
The steps that lead to time and space,
Each moment born of chance and fate,
Yet governed by its rules innate.

The Brownian motion it can tame,
And model systems with no name,
From finance to the brain's terrain,
Its power shines, its might sustains.

Oh, stochastic differential,
Thy beauty lies in the improbable,
In the ways that chance can ripple,
And bring forth order from the crumble.

May thy equations guide our way,
As we explore the realm of stray,
And find the patterns in the fray,
That govern our world day by day.