

INF102

Algorithms, Data Structures and Programming

Marc Bezem¹

¹Department of Informatics
University of Bergen

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INF102, practical stuff

- ▶ Lecturer: Marc Bezem; Team: see homepage
- ▶ Homepage: [INF102](#) (requires login)
- ▶ Also: [INF102 on GitHub](#)
- ▶ Tentative [schedule](#)
- ▶ Textbook: [Algorithms, 4th edition](#)
- ▶ Prerequisites: INF100 + 101 (\approx Ch. 1.1 + 1.2)
- ▶ Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- ▶ Three compulsory exercises, must be passed
- ▶ Digital exam (Inspira) [06.12.2017](#)
- ▶ Old exams: [2004–2017](#)
- ▶ [Table of Contents of these slides](#)

Resources

- ▶ Good textbook, USA-style: many pages, exercises etc.
- ▶ Average speed must be ca 50 pages p/w
- ▶ Lectures (ca 24) focus on the essentials
- ▶ Slides (ca 120, dense!) summarize the lectures
- ▶ Prepare yourself by reading in advance
- ▶ Workshops: help with **selected** exercises
- ▶ Test yourself by trying some exercises in advance
- ▶ If you can do the exercises (incl. compulsory), you are fine
- ▶ Review of exercises on Tuesday morning

Generic Bags, Queues and Stacks

- ▶ Generic programming in Java, example: **PolyPair.java**
- ▶ Bag, Queue and Stack are generic, iterable collections
- ▶ All three support adding an element
- ▶ Queue and Stack support removing an element (if any)
- ▶ FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- ▶ Stack: INF101; Bag is a Stack without pop
- ▶ APIs include: `boolean isEmpty()` and `int size()`

Implementations and one application

- ▶ Implementation of Stack: `LinkedList_Stack.java`
- ▶ Implementation of Queue: `LinkedList_Queue.java`
- ▶ Dijkstra's Two-Stack Expression Evaluation
- ▶ Example: $(1 + ((2 + 3) * (4 * 5)))$
- ▶ Now you learned something about compilers!
- ▶ **Movie**

Resizing Arrays

- ▶ Arrays have direct access, but have fixed size
- ▶ Linked lists have flexible size, but no direct access
- ▶ Best of both: use new, resized arrays, *wisely*:
 - ▶ double size when the array becomes overfull
 - ▶ halve size when the array becomes quarter full
- ▶ Resizing takes time and space proportional to size
- ▶ Not too seldom (correctness), not too often (efficiency)
- ▶ Later: we retain *constant time direct access*
- ▶ Later: add operation in *constant time on average*
- ▶ Once we have understood resizing arrays: **ArrayList**

Implementations

- ▶ `ResizingArray_Stack.java`
- ▶ Arrays give direct access, resizing at reasonable cost
- ▶ `LinkedList_Stack.java`
- ▶ No fixed size, but indirect access incurs a cost
- ▶ Pointers take space and dereferencing takes time
- ▶ Programming with pointers: make a picture

Computation time and memory space

- ▶ Two central questions:
 - ▶ How long will my program take?
 - ▶ Will there be enough memory?
- ▶ Example: **ThreeSum.java**
- ▶ Inner loop (here $a[i] + a[j] + a[k] == 0$) is important
- ▶ Sorting helps: **ThreeSumOptimized.java**
- ▶ Run some experiments: `1Kints.txt`, `2Kints.txt`, ...

Methods of Algorithm Analysis

- ▶ Empirical:
 - ▶ Run program with randomized inputs, measuring time & space
 - ▶ Run program repeatedly, varying (doubling) the input size
 - ▶ Measuring time: **StopWatch**
 - ▶ Plot, or log-log plot and **linear regression**
- ▶ Theoretical:
 - ▶ Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
 - ▶ Try to count/estimate/average this cost as function of the input (size)
 - ▶ Use $O(f(n))$ (MNF130) and $f(n) \sim g(n)$ (see next slide)

Orders of Growth, Big Oh and \sim

- ▶ Big Oh and \sim aim to capture 'order of growth'
- ▶ Costs are positive quantities, so $f, g, \dots : \mathbb{N} \rightarrow \mathbb{R}^+$
- ▶ MNF130: $f(n)$ is $O(g(n))$ if there exist $c \in \mathbb{R}^+$, $N \in \mathbb{N}$ such that $f(n) \leq cg(n)$ for all $n \geq N$ (that is, for n large enough)
- ▶ Example: n^2 and even $99n^3$ are $O(n^3)$, but n^3 is not $O(n^{2.9})$
- ▶ INF102: $f(n) \sim g(n)$ if $1 = \lim_{n \rightarrow \infty} (f(n)/g(n))$
- ▶ If $f(n) \sim g(n)$, then $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$
- ▶ Not conversely: Big Oh disregards constant factors, \sim not
- ▶ Factor c hidden in Big Oh is important in practice
- ▶ Bound N is important if it is large

Important orders of growth

order of growth as function of n	value for $n = 20$
constant: c , meaning $f(n) = c$ for all n	c sec
linear: n	20 sec
linearithmetic: $n \log n$	26 sec
quadratic: n^2	400 sec
cubic: n^3	8000 sec
exponential: 2^n	1048576 sec
general form: $an^b(\log n)^c$	$a \cdot 20^b \cdot (1.3)^c$ sec

ThreeSum, theoretically

- ▶ Number of different picks of triples: $g(n) = n(n-1)(n-2)/6$
- ▶ Inner loop $a[i] + a[j] + a[k] == 0$ executed $g(n)$ times
- ▶ $f(n) = n^3/6 - n^2/2 + n/3$
- ▶ Cubic term $n^3/6$ wins for large n : $f(n) \sim n^3/6$
- ▶ Cost model # array accesses: $\sim n^3/2$
- ▶ Cost array access t sec: total time $\sim t * n^3/2$ sec
- ▶ Cost models are (necessary) simplifications!

ThreeSum, empirically

- ▶ Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4, 51.1 sec
- ▶ The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- ▶ See the plots in [plot sheet](#)
- ▶ Linear regression gives $y \approx 3x - 10$
- ▶ $\log(f(n)) = 3 \log(n) - 10$ iff

$$f(n) = 10^{\log(f(n))} = 10^{3 \log(n) - 10} = n^3 * 10^{-10}$$

- ▶ Conclusion: cubic in the input size, with constant $\approx 10^{-10}$
- ▶ No surprise: see the 3-nested loop in [ThreeSum.java](#)
- ▶ Strong dependence on input can be a problem
- ▶ Constant 10^{-10} depends on computer, exponent 3 does not

Worst case, average case, amortized cost

- ▶ Worst case: guaranteed, independent of input; Examples:
 - ▶ Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
 - ▶ Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- ▶ Average case: not guaranteed, dependent of input *distribution*
- ▶ Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost ≤ 21 , all others cost 1, amortized ≤ 3 p/o.
- ▶ Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- ▶ Special case of resizing array that is only growing:
 $1(2)2(4)3(8)4(16)5(32)6(64)7(128)8(256)9(512) \dots 16(32768) \dots$, with (n) the new size.
 Resizing to (n) costs $2n$ array accesses, so in total
 $(1+4)+(1+8)+(2+16)+(4+32)+(8+64) \dots$, so 9 p/push.

Remarks and Pitfalls

- ▶ Theoretical approach:
 - ▶ Wrong cost model
 - ▶ JVM optimization can obscure the exponent
 - ▶ Caching can have large impact on memory access
 - ▶ Large constant factor in Big Oh
 - ▶ Worst case can be easy, average case difficult
- ▶ Empirical approach:
 - ▶ The focus is on run time (using space costs time)
 - ▶ Dependence on input, randomization does not always help
 - ▶ Machine/platform dependence
 - ▶ Linear regression not good for, e.g., $O(n^2 \log n)$

Exercise

Aim: better understand the empirical method.

1. Let input sizes 1, 2, 4, 6, 8K take 2, 7.9, 32, 72, 129 sec
2. Make a plot such as in [plot sheet](#) (download)
3. Compute the log's of the input sizes and of the run times and make the log-log plot such as in [plot sheet](#) (second plot)
4. Estimate a and b such that the log-log plot is $y \approx ax - b$
5. Estimate a and b through [linear regression](#), compare with 4.
6. Find $f(n)$ given that $\log(f(n)) = a \log(n) - b$. Surprised?

In cases where the run time mostly depends on the size n of the input and not on the input itself, the function f is a reasonable (polynomial) estimation of the run time.

Logarithms and Exponents Cheat Sheet

- ▶ Definition: $\log_x z = y$ iff $x^y = z$ for $x > 0$
- ▶ Base of logarithm: the x in \log_x
- ▶ Inverses: $x^{\log_x y} = y$ and $\log_x x^y = y$
- ▶ Exponent, laws: $x^{(y+z)} = x^y x^z$, $x^{(yz)} = (x^y)^z$
- ▶ Logarithm, laws: $\log_x(yz) = \log_x y + \log_x z$,
 $\log_x z = \log_x y \log_y z$
- ▶ Various bases: $\log_2 = \lg$, $\log_e = \ln$, $\log_{10} = \log$
- ▶ Double exponent: e.g. $2^{(2^n)}$ (not used in INF102)
- ▶ Double logarithm: $\log(\log n)$ (not used in INF102)

Staying Connected

- ▶ We want efficient algorithms and datastructures for testing whether two objects are 'connected' (e.g., in networks)
- ▶ We assume connectedness to be an equivalence
- ▶ MNF130: relation $E \subseteq V \times V$ is an *equivalence* if
 - ▶ E is *reflexive*: $\forall x \in V. E(x, x)$
 - ▶ E is *symmetric*: $\forall x, y \in V. E(x, y) \rightarrow E(y, x)$
 - ▶ E is *transitive*: $\forall x, y, z \in V. E(x, y) \wedge E(y, z) \rightarrow E(x, z)$
- ▶ Dynamic connectivity means (here) that E can grow
- ▶ Relationship with paths in graphs, (connected) components (MNF130): nodes are connected if there is a path between them
- ▶ Input: N and pairs in $V = \{0, \dots, N-1\}$ defining E
- ▶ Challenge: efficient boolean `connected(int p, int q)`

Example

- ▶ Example (`algs4-data/tinyUF.txt`) : $N = 10$
- ▶ Nodes 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- ▶ Edges: 4–3, 3–8, 6–5, 9–4, 2–1, 8–9, 5–0, ...
- ▶ Linear space: don't add pairs that are already connected!
- ▶ Q: what are the costs of storing all pairs that are connected, space and time?
- ▶ See: [algoritmevisualisering](#) by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke

Union-Find

- ▶ Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- ▶ Union, idea: for any new pair $n\ m$ that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- ▶ API: **UF**; Cost model: number of array accesses
- ▶ Implementations:
 - ▶ **SlowUF.java**: `id[p]` identifier of p
`find()` ~ 1 , `union()` ~ 2 or between $n+3$ and $2n+1$ (!)
 - ▶ **FastUF.java**: `int[] id` pointers, `id[p]==p`: identifier
`find()` $\sim 1+2d$, `union()` $\sim 1 + \text{two find()}'s$
 - ▶ **WeightedUF.java**: `int[] id` pointers, `int[] sz` subtree sizes
`find()` and `union()` both $\sim \lg n$

Trees (cf. MNF130) and WeightedUF

A (rooted) *tree* consist of *nodes* (also called *vertices*) one of which is called the *root* r . Every node n is connected by an *edge* to zero or more other nodes, called the *children* of the *parent* n . Moreover, each node $n \neq r$ has a unique parent in a tree. Trees are naturally depicted in levels starting with the root at level 0, then the level 1 of the children of the root, level 2 of the children of the children of the root, and so on. The level of a node is also called its *depth*. In a finite tree there is always a highest level (maximum depth) and this is called the *height* of the tree.

- ▶ WeightedUF: height of subtree of size k is at most $\lg k$ (proof by induction on blackboard)
- ▶ Ultimate improvement of UF (almost $O(1)$, amortized): path-compression (sketch on bb)

Sorting

- ▶ Sorting: putting objects in a certain order
- ▶ MNF130: relation $R \subseteq V \times V$ is a *total order(ing)* if
 1. R is *reflexive*: $\forall x \in V. R(x, x)$
 2. R is *transitive*: $\forall x, y, z \in V. R(x, y) \wedge R(y, z) \rightarrow R(x, z)$
 3. R is *antisymmetric*: $\forall x, y \in V. R(x, y) \wedge R(y, x) \rightarrow x = y$
 4. R is *total*: $\forall x, y \in V. R(x, y) \vee R(y, x)$
- ▶ Natural orderings:
 - ▶ Numbers of any type: ordinary \leq and \geq
 - ▶ Strings: lexicographic
 - ▶ Objects of a Comparable type: `v.compareTo(w) <= 0`

Sorting and searching: linear vs binary search

```
int linearSearch(Comparable key, Comparable[] a){  
    for (int i=0; i < a.length; i++){  
        if (key.compareTo(a[i])==0) {return i;}  
    }  
    return -1; // key not in array: O(a.length)  
}
```

```
int binarySearch(Comparable key, Comparable[] a) {  
    int lo=0; int hi=a.length-1; int mid; int test  
    while (lo <= hi){  
        mid = (lo+hi)/2; test = key.compareTo(a[mid]);  
        if (test == 0) {return mid;}  
        if (test < 0) { hi = mid-1;} else {lo = mid+1;}  
    }  
    return -1; // key not in SORTED array: O(lg(a.length))  
}
```

Sorting (ctnd)

- ▶ Elementary sorts:
 1. Bubble sort (like gas bubbles in sparkling water)
 2. Selection sort (iterated selection of minima)
 3. Insertion sort (iterated insertion of elements)
 4. Shell sort (Shell's refinement of insertion sort)
- ▶ Bubble sort: **ExampleSort.java**
- ▶ Certification: `assert isSorted(a)` in `main()`
(no guarantee against modifying the array, but `exch()` is safe)
- ▶ Costmodel(s): number of `less()`'s and of `exch()`'s
(or array accesses; discuss pointer vs. object)
- ▶ Why studying sorting? (`java.util.Arrays.sort()`)
- ▶ Comparing sorting algorithms: **SortCompare.java**

Selection Sort

- ▶ Bubble sort: $\sim n^2/2$ compares, $0 \leq \text{exchanges} \leq n^2/2$
- ▶ Selection sort:
 - ▶ Find index of a minimum in $a[0..n-1]$, exchange with $a[0]$
 - ▶ Find index of a minimum in $a[1..n-1]$, exchange with $a[1]$
 - ▶ ... until $n-2$
- ▶ Selection sort: $\sim n^2/2$ compares, $0 \leq \text{exchanges} \leq n-1$ (!)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=0; i<N-1; i++){  
        int min=i;  
        for (int j=i+1; j<N; j++) if (less(a[j],a[min])) min=j;  
        if (i != min) exch(a,i,min);  
    }  
}
```

Insertion sort

- ▶ Insertion sort:
 - ▶ Insert $a[1]$ on its correct place in (sorted) $a[0..0]$
 - ▶ Insert $a[2]$ on its correct place in (sorted) $a[0..1]$
 - ▶ ... until $a[n-1]$
- ▶ Very good for partially sorted arrays, costs:
 - ▶ Best case: $n-1$ compares and 0 exchanges
 - ▶ Worst case: $\sim n^2/2$ compares and exchanges
 - ▶ Average case: $\sim n^2/4$ compares and exchanges (distinct keys)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=1; i<N; i++){  
        for (int j=i; j>0 && less(a[j],a[j-1]); j--)  
            exch(a,j,j-1);  
    }  
}
```

Shell sort

- ▶ Insertion sort:
 - ▶ Very good for partially sorted arrays
 - ▶ Slow due to one-step transport `exch(a,j,j-1)`
 - ▶ Why not larger steps `exch(a,j,j-h)` ?
- ▶ Idea: presort `a[i], a[i+h], a[i+2h], ...` for `i = 0..h-1`

```
public static void hsort(int h, Comparable[] a) {  
    int N = a.length;  
    for (int i=h; i<N; i++)  
        for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)  
            exch(a,j,j-h);  
}
```

- ▶ Insertion sort: `hsort(1,a)`
- ▶ Shell sort: e.g., `hsort(10,a); hsort(1,a)`

Shell sort (ctnd)

- ▶ `hsort(10,a)`; `hsort(1,a)` faster than just `hsort(1,a)` !
- ▶ Q: How is this possible?
- ▶ A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- ▶ What about `hsort(100,a)`; `hsort(10,a)`; `hsort(1,a)`?
- ▶ To be expected: depends on the length N of the array
- ▶ The run-time analysis of Shell sort is very difficult
- ▶ Best practice: $h = N/3, N/9, \dots, 364, 121, 40, 13, 4, 1$
- ▶ Example: [algoritmevisualisering](#) by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke

Mergesort

- ▶ Top-down (recursive) algorithm:
 - ▶ Mergesort left half, mergesort right half
 - ▶ Merge the results (example: 2468,1357)
- ▶ Using an auxiliary array: **TopDownMergeSort.java**, **Movie**
- ▶ Bottom-up algorithm (16 elements):
 - ▶ Merge $a[0], a[1]$, merge $a[2], a[3]$, merge $a[4], a[5]$, ...
 - ▶ Merge $a[0..1], a[2..3]$, merge $a[4..5], a[6..7]$, ...
 - ▶ Merge $a[0..3], a[4..7]$, merge $a[8..11], a[12..15]$
 - ▶ Merge $a[0..7], a[8..15]$, done!
- ▶ Also using an auxiliary array: **BottomUpMergeSort.java**

Run-time and memory use of mergesort

- ▶ Mergesort uses between $\sim (N/2) \lg N$ and $\sim N \lg N$ compares. Proof on bb. Important formula ($N = 2^n$):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- ▶ Mergesort uses at most $\sim 6N \lg N$ array accesses
- ▶ Mergesort uses $\sim 2N$ space (plus some var's)
- ▶ Q: How fast can compare-based sorting of N distinct keys be?
- ▶ A: $\lg N! \sim N \lg N$; Proof in book and on bb. Keywords: binary *compare tree*, inner nodes for each `compare(a[i], a[j])`, permutations in the leaves,
 $N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$

Quicksort

- ▶ Top-down (recursive) algorithm:
 - ▶ Choose a (pivot) value v in the array
 - ▶ Partition the array in non-empty parts $\leq v$ and $\geq v$
 - ▶ Quicksort the two parts
- ▶ Examples: [algoritmevisualisering](#) by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke
- ▶ Pros: in-place, average computation time $O(n \log n)$
- ▶ Cons: stack space for recursion, worst-case $O(n^2)$, not stable
- ▶ Implementation: [QuickSort.java](#)
- ▶ BTW: [Bug in java.util.Arrays.sort](#)

Quicksort, details

- ▶ Subtleties in `Quicksort.sort()`: shuffling protects against worst-case behaviour
- ▶ Termination of recursive `quicksort()`
- ▶ Subtleties in `partition()`:
 - ▶ Invariants $l \leq h$ in the two inner loops
 - ▶ Postcondition after the two inner loops
 - ▶ Invariant of the `for(;;)` loop
 - ▶ Termination of the `for(;;)` loop
 - ▶ There are some variations that are also correct

Run-time and memory use of quicksort

- ▶ Compare Quicksort to other sorts ($n = 10^2, 10^3, \dots$)
- ▶ Quicksort: time $O(n^2)$ if pivot is always smallest (or largest)
- ▶ Randomization: choose pivot randomly, or shuffle array
- ▶ If all keys are distinct and randomization is perfect, then quicksort uses on average $\sim 2n \ln n$ compares and $\sim (n/3) \ln n$ exchanges (proofs in book, complicated)
- ▶ Relevant improvements:
 - ▶ Cut-off to insertion sort for sizes ≤ 15 (ca.)
 - ▶ Median-of-three pivot
 - ▶ Taking advantage of duplicate keys (3-way partitioning)
- ▶ Quicksort is generally quite good
- ▶ In special situations other sorts are better (e.g., countsort)

Priority Queues

- ▶ Aim: collecting and processing items having keys
- ▶ Examples of keys: time-stamp, price-tag, priority-tag
- ▶ Assume: keys are ordered
- ▶ Reasonable: processing currently highest (or lowest)
- ▶ Special cases: items time-stamped when added
 - ▶ Queue: dequeue currently oldest (lowest time-stamp)
 - ▶ Stack: pop currently newest (highest time-stamp)
- ▶ Priority queue generalizes this
- ▶ Examples: highest priority, largest transaction, lowest price
- ▶ Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

Priority Queues

- ▶ Good info: [Wikipedia](#); API (the bare essentials):

```
public class MaxPQ<Key extends Comparable<Key>>
```

```
void          insert(Key v) // insert a key
```

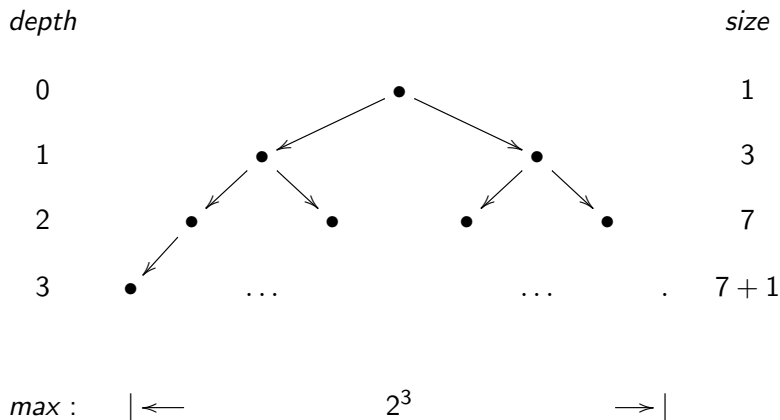
```
Key           delMax() // delete a largest key, if any
```

```
boolean       isEmpty()
```

```
int           size()
```

- ▶ In case of duplicate keys: 'a' largest, not 'the'
- ▶ Typical application: the 1K smallest keys of 1G unsorted keys
- ▶ Client: [BottomM.java](#) (Q: why is the output slowing down?)

Example of left-complete binary tree



Binary Trees

- ▶ MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- ▶ MNF130: In a *binary* tree every node has at most two children.
- ▶ MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height h has $2^{h+1}-1$ nodes.
- ▶ INF102: A binary tree is (left-) *complete* if all levels $< h$ are filled, level h may be partly empty from the right (picture bb). A (left-)complete binary tree of height h has between 2^h and $2^{h+1}-1$ nodes.
- ▶ A left-complete binary tree of n nodes has height $\lfloor \lg n \rfloor$ (from now on we leave out 'left-').

Heap-ordered Binary Trees

- ▶ Naive implementations:
 - ▶ Unsorted (resizing) array: fast `insert()`, linear `delMax()`
 - ▶ Sorted (resizing) array: linear `insert()`, fast `delMax()`
- ▶ Aim: operations in logarithmic time, no extra space
- ▶ A binary tree is *heap-ordered* if the key in each node is \geq the keys in its children (if any). Thus the root has a maximal key.
- ▶ NB: a heap is NOT a search tree (different data invariants)!
- ▶ Array representation of heap-ordered complete binary tree (bb)
- ▶ Methods `swim()` and `sink()`: picture on bb, code below
- ▶ Implementation: `ArrayListPQ.java`

Run-time and memory use of heaps, applications

- ▶ In a heap of n elements (since height is $\leq \lfloor \lg n \rfloor$):
 - ▶ `insert()` takes $\leq 1 + \lfloor \lg n \rfloor$ compares and exchanges
 - ▶ `delMax()` takes $\leq 2\lfloor \lg n \rfloor$ compares and $\leq 1 + \lfloor \lg n \rfloor$ exchanges
- ▶ Heap construction by `insert()` can sometimes be improved
- ▶ Example: maxheap from A B C D E F G H
- ▶ Given an array of n keys, right-to-left heap construction (bb) takes $< 2n$ compares and $< n$ exchanges
- ▶ Applications: **heapsort** and **merging sorted streams**
- ▶ Many variations with extended API (indexed priority queue)

Purpose of Sorting

- ▶ Sorting makes the following easier and more efficient:
 - ▶ Searching (binary search, example: `ThreeSumOptimized`)
 - ▶ Searching and looking up, e.g., the `pagenumber` in an index
 - ▶ Finding and removing duplicates
 - ▶ Finding the median, quartiles etc.
- ▶ Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- ▶ We do *pointer sorting*, manipulating refs to objects.
 - ▶ Pro: not moving full objects
 - ▶ Cons: pointer dereferencing, no `sort(int[] a)`
- ▶ More flexibility: pass a `Comparator` object to `sort()`

Comparator object

- ▶ API: `void sort(Object[] a, Comparator c)`
- ▶ Call, e.g.: `sort(a, new Transaction.WhenOrder())`
- ▶ Call, e.g.: `sort(a, new Transaction.SizeOrder())`
- ▶ Obs: `import java.util.Comparator`
- ▶ Obs: `less(Object o1, Object o2, Comparator c)`
- ▶ Priority queues also with `Comparator`

```
public class Transaction {  
    ...  
    public static class MyOrder {  
        implements Comparator<Transaction>  
        public int compare(Transaction t, Transaction v){...}  
    } // End of Myorder  
    ...// similarly: WhenOrder, SizeOrder  
} // End of Transaction
```

Applications of Sorting

- ▶ Consider sorting first to make other problems easier
- ▶ Commercial computing (sort on price, departure time, ...)
- ▶ Search for information: web-indexing, search engines
- ▶ Job scheduling heuristic: longest processing time first
- ▶ To come: Prim's, Dijkstra's and Kruskal's algorithms
- ▶ Huffman compression: a lossless compression based on using the shortest codes for the symbols that occur often.
Frequency counter: next chapter!
- ▶ Cryptology and genomics (e.g., longest repeated substring)

Symbol Tables

- ▶ Symbol table associates *keys* with *values*: *key-value pairs*
- ▶ Examples: keyword-list of page nrs, ID number-personal data, word-frequency
- ▶ Important operations:
 - ▶ Insert a key-value pair in the symbol table: `void put(k,v)`
 - ▶ Search the value for a given key (if any): `Value get(k)`
- ▶ Important conventions:
 - ▶ Inserting key-value for existing key: overwriting the value
 - ▶ No duplicate keys, no null keys
 - ▶ Value null: no value for this key
 - ▶ Lazy deletion: insert key-null; Eager: really delete key-value
- ▶ **API** of unordered symbol table
- ▶ Aim: all operations in time $\sim c \lg n$ with constant c small

ST Basics

- ▶ Archetypical ST-client: frequency counter, `ArrayListST.main`
- ▶ Cost model: number of compares
- ▶ Naive ST: unordered linked list (or `ArrayList`), linear search
 - ▶ Search miss: $\sim n$ compares
 - ▶ Search hit: between 1 and $\sim n$ compares
 - ▶ Random search hit: $(1 + \dots + n)/n \sim n/2$ compares
 - ▶ Inserting n distinct keys: $(1 + \dots + (n - 1)) \sim n^2/2$ compares
- ▶ `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- ▶ Naive ST impracticable for genomics, internet
- ▶ Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- ▶ Better for unordered ST: hashing (in Ch. 3.4)

Ordered Symbol Table

- ▶ Ordered ST: keys are ordered, f.e., in an `ArrayList`
- ▶ **API** of ordered symbol table
- ▶ Binary search: `get(Key k)` takes $\sim \lg n$ comparisons
- ▶ What about `put(Key k, Value v)`? See **`ArrayList`**
- ▶ Pitfall: `add(int i, E e)` is linear, not amortized $O(1)$!
- ▶ Consequence: `put(Key k, Value v)` and `del(Key k)` *linear*
- ▶ Implementation with binary search in **`ArrayListST.java`**
- ▶ Trace of inserts on bb: `SEARCHEXAMPLE`
- ▶ Experiments with `tinyTale.txt`, `tale.txt`, ...

Binary Search Trees

- ▶ Aim: `get`, `put`, `del` in logarithmic time, ST in linear space
- ▶ Binary *search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- ▶ Search time: length of the path to the node where the key 'should' be
- ▶ Balanced binary tree with n keys has $\lg n$ height
- ▶ Unbalanced binary trees can have height n (max depth)
- ▶ Search hits in a binary search tree, built without rebalancing, of n random keys take on average $\sim 2 \ln n$ compares
- ▶ **UBST.java**: `put()`, `get()`, `size()`, `isEmpty()`
- ▶ Trace of inserts on bb: S E A R C H E X A M P L E

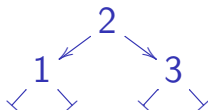
Binary Search Trees (`min()`, `delMin()`, `delete()`)

- ▶ Interrelated, increasing difficulty: `min(Node x)`, `deleteMin(Node x)`, `delete(Node x, Key k)`
- ▶ Node of minimum key: not null, and has left child null, and is root or left child of parent (picture on bb)

```
public Node min(Node x){// precondition x!=null
    while (x.left!=null) x = x.left; // inv x!=null
    return x;
} // cf. tail recursive min() in Alg. 3.3
```

- ▶ Delete minimum key, two cases:
(1) both children null; (2) left child null
- ▶ Delete is really difficult: `BST.java`, cf. `ArrayListST.java`
- ▶ Don't forget: update `x.N` along the path to the root!

Delete from search tree, example:



```
root=delete(root,3)
```

(1st example)

```
| x=root; x.right=delete(x.right,3)
```

(x.right=null)

```
| x'=x.right; return x'.left;
```

(x.size=2)

```
| update x.size;
```

(root=x)

```
| return x;
```

```
root=delete(root,2)
```

(2nd example)

```
| x=root; t=x; x=min(t.right);
```

(x=t.right)

```
| x.right=deleteMin(t.right);
```

(x.right=null)

```
| x.left=t.left;
```

(x.size=2)

```
| update x.size;
```

(root=x)

```
| return x;
```


Balanced Search Trees: keep paths short!

- ▶ NB tree balancing not as easy as in UF and Heap (4hrs!)
- ▶ A **2-3 search tree** consists of 2-nodes and 3-nodes:
 - ▶ Each 2-node has two children and a key k such that all keys in the left subtree are $< k$, and all keys in the right subtree $> k$
 - ▶ Each 3-node has three children and two keys $k_1 < k_2$ such that all keys in the left subtree are $< k_1$, all keys in the middle subtree $> k_1$ and $< k_2$, and all keys in the right subtree $> k_2$
- ▶ Examples and pictures on bb
- ▶ *Perfect* 2-3 search tree: paths from root to leaves equally long
- ▶ Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- ▶ Insert should keep tree perfect, rough idea:
 - ▶ into a 2-leaf: make it into a 3-leaf (easy)
 - ▶ any other case: (temporary) 4-nodes (see next, difficult)

Insert in Perfect 2-3 Search Trees

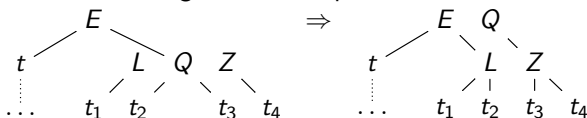
- Terminology: a *leaf* is a node all whose children are null
- Data invariant 1: tree is 2-3 search tree
- Data invariant 2: all paths from root to leaves equally long
 - Insert into a 2-leaf L : either $\begin{array}{c} A \ L \\ \diagdown \quad | \quad \diagup \\ \parallel \end{array}$ or $\begin{array}{c} L \ Z \\ \diagdown \quad | \quad \diagup \\ \parallel \end{array}$
 - into a 3-leaf whose parent is a 2-node: with new key Z (e.g.)

$\begin{array}{c} E \\ \diagdown \quad \diagup \\ leaf \quad L \ Q \\ \parallel \quad \diagdown \quad | \quad \diagup \end{array} \Rightarrow \begin{array}{c} E \\ \diagdown \quad \diagup \\ leaf \quad L \ Q \ Z \\ \parallel \quad \diagdown \quad | \quad \diagup \end{array} \Rightarrow \begin{array}{c} E \ Q \\ \diagdown \quad | \quad \diagup \\ leaf \quad L \ Z \\ \parallel \quad \parallel \quad \parallel \end{array}$
 - into a 3-leaf whose parent is a 3-node: with new key Z (e.g.)

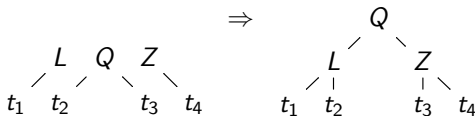
$\begin{array}{c} C \ M \\ \diagdown \quad | \quad \diagup \\ leaf_l \quad leaf_m \quad XYZ \\ \parallel \quad \parallel \quad \diagdown \quad \parallel \quad \diagup \end{array} \Rightarrow \begin{array}{c} C \ M \ Y \\ \diagdown \quad | \quad \diagup \\ leaf_l \quad leaf_m \quad X \ Z \\ \parallel \quad \parallel \quad \parallel \quad \parallel \end{array}$
 - into a 3-node whose parent is a 3-node: move up middle key!

Insert (ctnd)

- ▶ Data invariant 1: tree is 2-3 search tree
- ▶ Data invariant 2: all paths from root to leaves equally long
- ▶ Insert works up from the leaf where the key 'should' be
 - ▶ if 2-node on path to root: make it into a 3-node; there are two cases, left and right, here is a picture of the latter



- ▶ otherwise, work upwards and, finally, split the root:



- ▶ working upwards, there are three cases: left, middle, and right

Insert, summary and examples

- ▶ There exists a perfect 2-3 tree for any sequence of input keys
- ▶ Six operations for eliminating 4-nodes:
 - ▶ if parent is 2-node: move middle key up (left and right case)
 - ▶ if parent is 3-node: move middle key up (left, middle, right)
 - ▶ if root: split root
- ▶ Search and insert visit at most $\lfloor \lg n \rfloor$ nodes
- ▶ Proof: maximal path length is $\geq \lfloor \log_3 n \rfloor$ and $\leq \lfloor \log_2 n \rfloor$
- ▶ Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- ▶ Trace of inserts on bb: A C E H L M P R S X (keep balance!)

Red-black trees

- ▶ Red-black trees implement 2-3 trees
- ▶ Idea: one 3-node = two 2-nodes + extra info
- ▶ Extra info coded in color, picture:



- ▶ A *red-black tree* is a binary search tree with red and black links such that:
 - ▶ Only left links can be red (but need not be)
 - ▶ Never
 - ▶ Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- ▶ Equivalent: red-black tree and perfect 2-3 search tree


Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

```
private class Node {  
    Key key;  
    Value value;  
    Node left, right;  
    boolean color; // true for red, false for black  
    int N; // number of keys in subtree below this node  
}  
  
private boolean isRed(Node n) {  
    if (n==null) {return false;} else {return x.color}
```

Rotating and Color Flipping

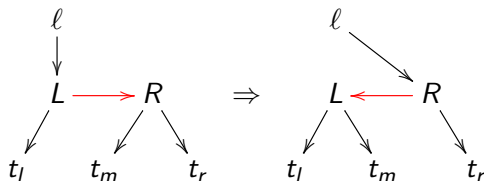
- ▶ Aim: restoring the data invariants of red-black search trees
 1. Only left links can be red, but never two successive
 2. Search tree invariant
 3. Perfect black balance
- ▶ Repairs use *rotations* and *color flips*, examples (leaves):
 - ▶ inserting Z in $L \leftarrow R : L \leftarrow R \rightarrow Z$, violation (why?)
 repairment: color flip $L \leftarrow R \rightarrow Z$ (R must up)
 - ▶ inserting A in $L \leftarrow R : A \leftarrow L \leftarrow R$, violation (why?)
 repairment: rotation right + color flip $A \leftarrow L \rightarrow R$
 - ▶ inserting M in $L \leftarrow R : L \leftarrow R$, violation



 repairment: rotation left into $L \leftarrow M \leftarrow R$, then as above
- ▶ Rotations change the root of the subtree, preserving invariants

Left Rotation

Call: `l = rotateLeft(l);`



```
private Node rotateLeft(Node l){
    Node r = l.right; l.right = r.left; r.left = l;
    r.color = l.color; l.color = true // == RED
    r.N = l.N; l.N -= 1+size(r.right); // Why?
    return r;
}
```


Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- ▶ Code of `rotateRight()` like that of `rotateLeft()`
- ▶ NB1: operations are local (here only r , M , R)
- ▶ NB2: operations preserve data invariants
- ▶ NB3: root is a special case (always black)
- ▶ Deletions: complicated, but doable (Exc. 3.3.39–41)

Run-time and memory use of Red-Black BSTs

- ▶ The height of a red-black BST with n nodes is $\leq 2 \lg n$
Proof: the worst-case is one 3-node path and the rest 2-nodes
- ▶ The average length of path (any color) from the root to a node in a red-black BST with n nodes is $\lg n$ ('empirical fact')
- ▶ In a red-black BST, search, insert, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node (book Prop. I, p. 447).
- ▶ For red-black BSTs, logarithmic time is guaranteed!

Hashing

- ▶ Idea: if keys in $[0..99]$ an array is the perfect symbol table
- ▶ In fact: `CountSort99.java` counts frequencies like an ST client
- ▶ A *hash function* maps keys to array indices
- ▶ Injectivity of the hash function is not guaranteed
- ▶ *Hash collision*: different keys are mapped to the same index
- ▶ In such a case we need *collision resolution*
- ▶ Symbol tables: hashing is fast, but unordered (no `max,min`)
- ▶ Aim: ST operations in amortized $O(1)$ time, extra space OK

Space-Time Trade-Off

- ▶ Hashing is an example of a *space-time trade-off*
- ▶ Time: computation time required
- ▶ Space: memory space used
- ▶ Unlimited space: (1) use key as index (e.g., the bits)
- ▶ Unlimited time: (2) use linked list and linear search
- ▶ Hashing strikes a balance using (1) with some array of reasonable size, and (2) in case of collisions
- ▶ The balance between (1) and (2) can easily be tuned

Hash functions

- ▶ Ideal (uniform hashing assumption, UHA): uniform and independent distribution of keys over integers from 0 to $M - 1$
- ▶ Examples of **hash functions in Java**
- ▶ Horner: $a_0 + x(a_1 + x(a_2 + \dots)) = a_0 + a_1x + a_2x^2 + \dots$
- ▶ Modular hashing (M prime), reasonably \approx UHA:

```
private int hash(Key k){  
    return (key.hashCode() & 0x7fffffff) % M;}  

```
- ▶ Q: Why crazy $\& 0x7fffffff$???
- ▶ A: In Java, e.g., $(-5 \% 3) == -2$ and not 1
- ▶ Q: Why M prime?
- ▶ A: E.g., $M = 32$ takes only into account the last five bits

Collision Resolution

- ▶ Two main methods of collision resolution:
 1. Hashing with separate chaining (picture on bb)
 2. Hashing with linear probing (picture on bb)
- ▶ Separate chaining: symbol table is an array of linked lists, linear search. If array has length M , then the linked lists have average length N/M with N keys.
- ▶ Linear probing: symbol table is an array of length $M \geq N$. Colliding keys are put at the first empty position. Linear search from the position where the key 'should have been'. Empty position: not found. Deletion tricky: reinsert all keys to the right of the deleted key, until the first empty position (picture on bb). Works better with $M \gg N$.

Symbol Table with Hashing

- ▶ Implementation: `ArrayListHashST.java`
- ▶ $M = 1$: measure overhead wrt. `ArrayListST`
- ▶ Tests with various values of M : 31, 997, 65521
- ▶ NB1: `ArrayListST`, `UBST` are sorted (!)
- ▶ NB2: *construction* versus *use* of ST (hashing better for *use*)
- ▶ Hashing can be combined with any other ST-implementation
- ▶ UHA metaphor: for every key one throws a dice *once*, and remembers the value as the hash code of the key

Quantitative analysis

- ▶ Throwing a dice 10 times, what is the probability of 3 fives?
- ▶ Under UHA, with N distinct keys, the probability that exactly k keys collide at some given hash value is

$$\binom{N}{k} \left(\frac{1}{M}\right)^k \left(\frac{M-1}{M}\right)^{N-k}, \text{ where e.g. } \binom{100}{10} \approx 1.7E13$$

- ▶ This is a small number for, say, $N = M = 100$ and $k = 10$
- ▶ For linear probing one typically takes $M = 2N$
- ▶ For separate chaining one keeps $N/8 \leq M \leq N/2$ (resizing M)
- ▶ Under UHA: search, insert, delete take amortized $O(1)$ time
- ▶ Space used can be upto $100N$ byte (objects, pointers); this on top of the space used by N key-value pairs

Applications of Searching

- ▶ Synonyms: **associative array**, map, symbol table, or dictionary
- ▶ Origin of **symbol table**: compilers and interpreters
- ▶ Web-indexing, **search engines**
- ▶ Sparse matrices (many 0's): **dictionary**
 1. keys: (row, column)-pairs
 2. values: matrix entries
- ▶ Set API (no values, only keys, for deduplication, filtering):

```
public class SET<Key>
{ void add(Key k);
  void delete(Key k);
boolean contains(Key k);
boolean isEmpty();
  int size(); }
```

Indexes and Reverse Indexes

- ▶ Index (key, value)
- ▶ Reverse index (value, key(s))
- ▶ Phone book (name+address, phone number(s))
- ▶ Reverse phone book (phone number, name+address)
- ▶ Dictionary (word, meaning or translation)
- ▶ Account information (client ID, account information)
- ▶ Genomics (protein, sequences of ACTG triplets)
- ▶ File systems (file name, location on disk/ file attributes)
- ▶ Internet domain name system (domain name, IP address)

Balanced Search Tree or Hash Table?

- ▶ Q: Which symbol table to use?
- ▶ A: The basic choice between BST and HT depends on ...
 1. Ordering of keys essential: BST
 2. Availability of good hash function (good = fast + UHA)
 3. Ordering of keys expensive (long strings): HT (or: Ch.5)
 4. Ordering of keys possible, but not essential: HT + BST
 5. Space considerations (ArrayListST uses the least extra space)
 6. Number of distinct keys and the space each key takes
 7. Distribution of insert/delete/search operations

Overview Chapter 1–3

Chapter 1

- ▶ Stack and Queue, ThreeSum, Union-Find
- ▶ Theory: \sim and O
- ▶ Experiments: loglog-plots, randomization

Chapter 2: Sorting

- ▶ Selection-, Insertion-, Shell-, Merge-, QuickSort
- ▶ Priority Queue, Binary Heap, HeapSort
- ▶ CountSort

Chapter 3

- ▶ Symbol Table
- ▶ Binary Search Tree, Perfect 2-3 Tree, Red-Black Tree
- ▶ Hashing: hash function and collision resolution

Odds and Ends Chapter 1–3

- ▶ Path-compression in UF (71)
- ▶ Randomizing the order of the tests
- ▶ Compare-based sorting requires $N \lg N$ comparisons (72)
- ▶ Double hashing: linear probing
 $h_1(k), h_1(k) + h_2(k), h_1(k) + 2h_2(k), \dots$
- ▶ Indexed Priority Queues (74)

Randomizing the order of the tests

```
for (int i=0; i<N; i++) {  
    a1[i] = StdRandom.uniform();  
    a2[i] = a1[i]; }  
if (StdRandom.uniform(2) == 0)  
    {total1 += time(alg1, a1); // summing runtime of alg1  
    total2 += time(alg2, a2); // summing runtime of alg2  
    }  
else  
    {total2 += time(alg2, a2); // summing runtime of alg2  
    total1 += time(alg1, a1); // summing runtime of alg1  
    }
```

Path compression in UF

```
// Finding the "identifier" of the component of p in id:  
public int find(int p) {  
    while (p!=id[p]) { p=id[p]; }  
    return p;  
}
```

// now with path compression:

```
public int find(int p) {  
    int q=p; // remember the starting point  
    while (p!=id[p]) { p=id[p]; }  
    // postcondition: p==id[p]==identifier of q  
    while (q!=p) { int aux=id[q]; id[q]=p; q=aux; }  
    return p;  
} // Example: int[] id={1,2,3,3}; find(0);
```

Compare-based sorting: worst-case $\geq N \lg N$

- ▶ Every compare-based sorting algorithm for N distinct keys in an array a leads to a *binary compare tree* with
 - ▶ nodes $(i:j)$ representing tests $a[i] < a[j]$
 - ▶ left subtree: $a[i] < a[j]$; right subtree: $a[i] > a[j]$
 - ▶ leaves: sorted permutations of the array
- ▶ Example with array of length 3 on bb
- ▶ Every permutation should occur at least once in a leaf!
- ▶ Binary tree of height h has at most 2^h leaves
- ▶ Length of path to leaf = number of comparisons
- ▶ Now $h \geq \lg N! \sim N \lg N$ by Stirling from this formula:

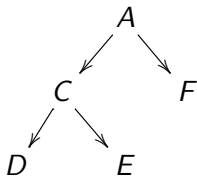
$$N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$$

Indexed Priority Queues (missing inverse qp of pq)

- ▶ IPQ \approx array with direct access to minimum (maximum)
- ▶ API: `void insert(int i, Key k); void del(int i); int minKey();`
`Key keyOf(int i);...` Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-

heap



Do: `insert(6,G)`, `insert(5,B)`, `insert(1,Z)`

NB1: `keys[pq[i]]` on position `i` in heap

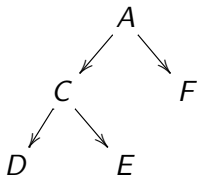
NB2: inverse index qp is needed

Indexed Priority Queues

- ▶ IPQ \approx array with direct access to minimum (maximum)
- ▶ API: `void insert(int i, Key k); void del(int i); int minKey();`
`Key keyOf(int i);...` Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-
qp	1	0	2	4	3	-1	-1

heap



Do: `insert(6,G)`, `insert(5,B)`
 NB qp is needed to find
 the index of `key[i]` in pq
 e.g., for `insert(1,Z)` (then: sink!)

Indexed Priority Queues (ctnd)

After insert(6,G):

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	6	0
keys	C	A	F	E	D	-	G
qp	1	0	2	4	3	-1	5

Step 1 insert(5,B):

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	6	5
keys	C	A	F	E	D	B	G
qp	1	0	2	4	3	6	5

Step 2 (swaps)
 pq[2] pq[6]
 qp[pq[2]] qp[pq[6]]

index	0	1	2	3	4	5	6
pq	1	0	5	4	3	6	2
keys	C	A	F	E	D	B	G
qp	1	0	6	4	3	2	5

Graph classes

(MNF130: useful review of graph theory)

1. Undirected graphs: a set of *vertices* (or *nodes*) V and a set of *edges* E connecting the nodes
2. Directed graphs (*digraphs*): a set of nodes V and a set E of edges (or *arrows*) pointing from one node to another
3. *Edge-weighted graphs*: undirected graphs in which every edge has a number called the *weight* of the edge
4. *Edge-weighted digraphs*: digraphs in which every arrow has a weight

Examples

1. **Map** (discuss: un/directed, un/weighted, multigraph)
2. Undirected graphs: social networks, communication networks (duplex communication)
3. Directed graphs: hyperlinks, (class, module, package) dependencies, logical circuits, job scheduling, flow graphs
4. Edge-weighted graphs: roadmaps with geographical distance, or with toll, communication networks with bandwidth
5. Edge-weighted digraphs: job scheduling with duration, transport of goods, financial transactions

Undirected Graphs

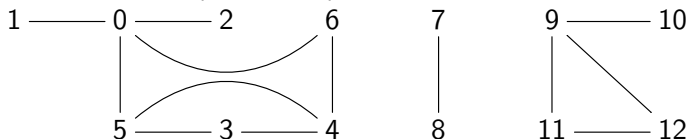
- ▶ Undirected graph: a set of *vertices* (or *nodes*) V and a set of *edges* E connecting the nodes
- ▶ *Subgraph*: subset of E and subset of V forming a graph (!)
- ▶ *Path*: sequence of nodes connected by edges (!)
- ▶ *Simple path*: path with no node repeated
- ▶ *Length of path*: number of edges
- ▶ *Cycle*: path of length > 0 with same start and end node
- ▶ *Simple cycle*: not repeating edges or nodes (apart from start and end node)
- ▶ *Acyclic graph*: graph without simple cycles
- ▶ *Connected graph*: having a path between every two nodes
- ▶ *Connected component*: a maximal connected subgraph

Trees and Forests

- ▶ 'Anomalies' concerning edges:
 - ▶ Self-loop: edge connecting a node to itself
 - ▶ Parallel edges: two edges connecting the same node(s)
- ▶ When no anomalies, $E \subseteq \{\{v, v'\} \mid v \in V, v' \in V, v \neq v'\}$
- ▶ *Tree*: connected acyclic graph (then: no anomalies)
- ▶ *Spanning tree*: maximal subgraph that is a tree
- ▶ Lemma: any spanning tree of a connected graph contains all nodes. Proof by contradiction (on bb).
- ▶ *Forest*: graph consisting of disjoint trees
- ▶ *Spanning Forest*: forest consisting of spanning trees of connected components of a graph
- ▶ Example: `tinyG.txt` on bb

Undirected Graphs (ctnd)

- ▶ *Distance* between two nodes: length of a shortest connecting path if there is a path connecting these nodes, otherwise ∞
- ▶ *Degree* of a node: number of edges connected to that node
- ▶ Graph $G = (V, E)$, the following are equivalent:
 - ▶ G is a tree (def: connected and acyclic)
 - ▶ G has $|V| - 1$ edges and no cycles
 - ▶ G has $|V| - 1$ edges and is connected
 - ▶ G is acyclic and adding an edge creates a cycle
 - ▶ Any two nodes of G are connected by exactly one simple path
- ▶ Example: some (connected) subgraphs of `tinyG.txt`



Graph representation and implementation

- ▶ Impractical: **adjacency matrix** $\sim V^2$, **incidence matrix** $\sim VE$
- ▶ Often practical: **adjacency lists** $\sim (V+2E)$, that is, `adj[v]` lists all nodes `w` connected to `v` by an edge
- ▶ Example: `tinyG.txt` by **LinkedListG.java**
- ▶ Graph API includes: `V()`, `E()`, `addEdge()`
- ▶ Basic algorithms: Depth-First Search (DFS) and Breadth-First Search (BFS)
- ▶ Both DFS and BFS 'walk through the graph', in different ways
- ▶ Both DFS and BFS can compute a spanning tree and forest

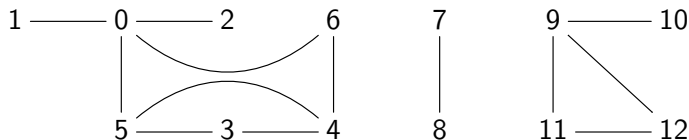
```
public void dfs(Integer v, boolean[] marked) {  
    marked[v] = true;  
    for (Integer w : adj[v])  
        if (! marked[w]) dfs(w,marked);  
} // dfs() is recursive, call: dfs(v,marked);  
  
public void bfs(Queue<Integer> q, boolean[] marked) {  
    while (!q.isEmpty()) {  
        Integer v = q.dequeue();  
        for (Integer w : adj[v])  
            if (! marked[w]) {marked[w]=true; q.enqueue(w);}  
    }  
} // call: marked[v]=true; q.enqueue(v); bfs(q,marked);  
  
// Example: 0-1, 0-3, 1-2, 1-3, 3-4  
// Example: complete ternary tree of height 2
```

Implementation and Properties of DFS/BFS

- ▶ **LinkedListG.java**: `pathdfs()`, `pathbfs()`
- ▶ DFS and BFS mark nodes connected to a given source node in time proportional to the sum of their degrees ($\leq 2E$), and can return a path from a marked node to the given source in time proportional to the length of this path
- ▶ BFS always finds a shortest path (proof: queue only contains nodes at distance k followed by nodes at distance $k + 1$, while all nodes at distance $\leq k$ not in queue have been processed)
- ▶ DFS finds a left-most path (long or short, example bb)
- ▶ BFS tends to use more space (but not always)
- ▶ UF tests connectivity, but finds no paths

Applications

- ▶ `StringSTG.java`, flight connections, shortest path = minimum number of stop-overs
- ▶ Degrees of separation in social networks, e.g., Erdős number = length of shortest path to Paul Erdős in the co-author graph
- ▶ Connected components: `LinkedListG.countcc()`
- ▶ Example: `tinyG.txt` has three connected components

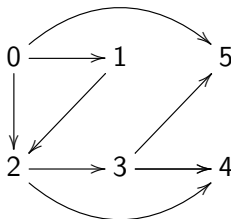


Directed Graphs

- ▶ *Digraph*: a set of *vertices* (or *nodes*) V and a set of *directed edges* (or *arrows*) E pointing from one node to another
- ▶ *Subdigraph*, *directed (simple) path* (*dipath*), *directed (simple) cycle*, *acyclic*, *length*: as expected
- ▶ Often we leave out 'di' in digraph, dipath, etc.
- ▶ *DAG*: **D**irected **A**cyctic **G**raph;
- ▶ *Degree*: **in**-degree and **out**-degree
- ▶ Node v is *reachable* from w : a dipath from w to v exists
- ▶ *Strongly connected digraph*: dipath between every two nodes (for all v, w , there are dipaths from v to w and from w to v)
- ▶ *Strongly connected component*: maximal strongly connected subgraph ($u \rightleftarrows v \rightarrow w$ has two scc's)
- ▶ Representation: adjacency lists even simpler!

Directed Graph, example

tinyCG.txt:



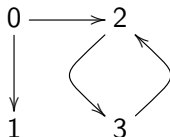
Reachability Problems

Assume we are given a directed graph G .

- ▶ Single-source: given a node s , the *source*, is a given node v reachable from s ? Example: `tinyCG.txt`
- ▶ Multiple-source: given a set of nodes S , is a given node v reachable from some node in S ?
- ▶ Solutions: same DFS and BFS algorithms as in Chapter 1
- ▶ Application (example): **mark-and-sweep garbage collection**
- ▶ Single-source path: given s, v such that v is reachable from s . Find a path from s to v .
- ▶ Single-source shortest path: given s, v such that v is reachable from s . Find a *shortest* path from s to v .
- ▶ Solutions: same DFS (path) and BFS (shortest path) algorithms as for undirected graphs

Cycle Detection

- ▶ Recall: a *DAG* is a graph without a directed cycle
- ▶ Acyclicity test, cycle detection: easy extension of DFS. We keep track of the search path from the source. If there is an arrow from v to w and w is on the path from the source to v , then there is a cycle. (DFS finds the leftmost path to the leftmost cycle.) Two techniques (space-time trade-off!):
 - ▶ Go back the search path: `LinkedListDiG.slowCyclist()`
 - ▶ Memorize the search path: `LinkedListDiG.fastCyclist()`
- ▶ Application: precedence scheduling of jobs
- ▶ Example: `cycleG.txt`



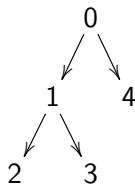
Pre-order, post-order

- ▶ Graph walks based on DFS from a source node
- ▶ Pre-order: order in which DFS arrives at nodes
- ▶ Post-order: order in which DFS leaves nodes
- ▶ In-order *for binary trees*: e.g., in `UBST.show()`
- ▶ Example:

pre-order: 01234

post-order: 23140

in-order: 21304



- ▶ Example: `tinyCG.txt` on `bb` and by `LinkedListDiG.java`

Topological order of acyclic digraph

- ▶ Topological order: total order \prec compatible with the graph in the following sense: if there is an arrow from u to v , then $v \prec u$ (consequently: $v \preceq u$ if v is reachable from u)
- ▶ NB one can also take \succ , this is only a matter of definition
- ▶ Lemma: if a digraph has a topological order, then it is acyclic (proof: a cycle cannot be ordered compatibly)
- ▶ Lemma: if a digraph is acyclic, then it has a topological order (proof idea: if acyclic, the post-order is a topological order since, if there is an arrow from u to v , then u is not reachable from v and DFS will leave u after it has left v)
- ▶ Topological order is a job schedule respecting precedence
- ▶ Example: $1 \leftarrow 2 \leftarrow 4 \rightarrow 5 \rightarrow 3 \leftarrow 0$

Transitive closure

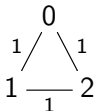
- ▶ Definition: given G , its (reflexive!) transitive closure G^* is a graph with the same nodes, with arrows from u to v for each v that is reachable from u in G .
- ▶ NB: G^* can have many more arrows than G
- ▶ Implementation: adjacency matrix in case of many arrows

...

```
boolean[][] adjmat = new boolean[V][V];  
for (int v=0; v<V; v++) {  
    boolean[] marked = new boolean[V];  
    dfs(v,marked);  
    adjmat[v] = marked; // adjmat[v][v]==true: reflexive  
    ...  
}
```

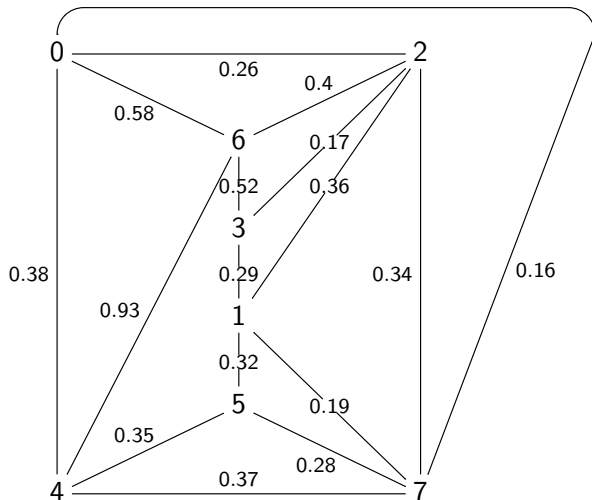
Minimum Spanning Tree

- ▶ Recall slide 79: *spanning tree* of a connected undirected graph is a maximal subgraph that is a tree (and thus contains all nodes and is acyclic)
- ▶ EWG = Edge-Weighted Graph, here always connected
- ▶ Example: `tinyEWG.txt` on bb
- ▶ Recall slide 80: all spanning trees have $V - 1$ edges
- ▶ *Weight* of spanning tree: sum of the weights of its edges
- ▶ *Minimum Spanning Tree*: spanning tree with minimal weight
- ▶ Example: three MSTs of



- ▶ Exc.4.3.3: if all weights different, then MST is unique
- ▶ From now on we assume all weights different!

MST Example (tinyEWG.txt)



Minimum Spanning Tree (ctnd)

- ▶ Applications: power plants and electrical grid, airlines and flight routes, maps and distance
- ▶ Weights may be zero or negative (e.g., cost minus profit of a new network of roads between cities)
- ▶ Two important algorithms to find the MST: Prim's and Kruskal's

Cuts and Crossing Edges

- ▶ Recall slide 80: deleting an edge from a tree creates two disjoint components, adding an edge creates a cycle
- ▶ *Cut*: a partition of V in two non-empty subsets of nodes
- ▶ *Crossing edge*: edge connecting two nodes in different subsets of a cut
- ▶ NB there can be more than one crossing edge: 0 — 2



- ▶ Lemma: for any cut in an EWG, the crossing edge of minimum weight is in the MST.
- ▶ Proof: given a cut, assume by contradiction there is a crossing edge e of weight smaller than the crossing edge(s) that is (are) in the MST (e.g., the dotted edge above). Adding e creates a simple cycle, which must contain one other crossing edge f in the MST. Replacing f by e : ✗

Prim's Lazy Algorithm

- ▶ Datastructures:
 - ▶ EWG represented with adjacency lists $\text{adj}[v]$
 - ▶ Minimum priority queue pq for edges
 - ▶ Array $\text{marked}[v]$ for marking vertices
 - ▶ Queue mst for the minimum spanning tree
- ▶ Edge is *eligible* if not both endpoints marked (crossing!)
- ▶ Algorithm based on previous lemma, cut: un/marked nodes
 1. mark 0 and add all eligible edges in $\text{adj}[0]$ to pq
 2. as long as pq is not empty, do:
 - 2.1 get and delete minimum edge e from pq
 - 2.2 add e to mst , say the unmarked endpoint of e is k
 - 2.3 mark k and add all eligible edges in $\text{adj}[k]$ to pq
 - 2.4 delete ineligible minimum edges from pq
- ▶ After this algorithm, the queue mst contains the MST

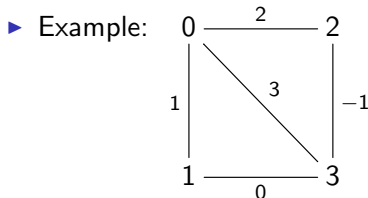
Prim's Algorithm (ctnd)

- ▶ **LazyPrimMST.java**, methods `scan()` and `prim()`
- ▶ Invariant: at least one of the nodes of an edge in `pq` is marked
- ▶ NOT: all edges in `pq` are crossing edges wrt cut un/marked
- ▶ Lazy: ineligible edges are not eagerly deleted from `pq`
- ▶ Runtime: LazyPrimMST runs in $O(E \log E)$ time (worst-case)
- ▶ Possible: *only* crossing edges wrt cut un/marked in `pq`
- ▶ Eager: if v unmarked, the only crossing edge of interest is the *lightest* one connecting v to the marked edges (= MST so far)
- ▶ Runtime: eager Prim runs in $O(E \log V)$ time (worst-case)
- ▶ Max size `pq`: E edges for lazy; V nodes for eager

Prim's Eager Algorithm

- ▶ Datastructures:

- ▶ EWG represented with adjacency lists `adj[v]`
- ▶ Boolean array `marked[v]` for marking vertices
- ▶ Array `distTo[v]`, minimum distances to MST so far
- ▶ Array `edgeTo[v]`, edges with minimum distance to MST so far
- ▶ Indexed minimum priority queue `pq`: `index=v`, `key=distTo[v]`
- ▶ Queue `mst` for the MST based on `edgeTo`



- ▶ **PrimMST.java**, methods `scan()` and `prim()`

Kruskal's Algorithm

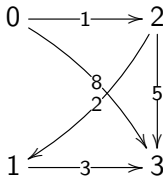
- ▶ Datastructures:
 - ▶ EWG represented with adjacency lists $\text{adj}[v]$
 - ▶ Minimum priority queue pq for edges
 - ▶ Union-Find object uf testing connectivity
 - ▶ Queue mst for the minimum spanning tree
- ▶ Algorithm:
 1. delete the minimum edge e from pq
 2. if the points connected by e are not connected, add e to mst and connect the points in uf
 3. continue at point 1 until pq is empty or uf contains all nodes
- ▶ Examples: EWG on previous slide, `tinyEWG.txt`
- ▶ Correctness: same lemma about minimum-weight crossing edge of cut
- ▶ Implementation: `KruskalMST.java`, constructor method

Memory-Use and Run-time Analysis

- ▶ Space, worst-case:
 - ▶ All methods use $O(V + E)$ space for the graph, plus ...
 - ▶ Priority queue for edges (Lazy Prim and Kruskal): $O(E)$ space
 - ▶ Priority queue for vertices (Eager Prim): $O(V)$ space
 - ▶ Arrays indexed by vertices (all): $O(V)$ space
- ▶ Time, worst-case:
 - ▶ Priority queue operations (Lazy Prim and Kruskal): $O(E \log E)$ time
 - ▶ Priority queue operations (Eager Prim): $O(E \log V)$ time
- ▶ NB: $E \leq V^2$ implies $\log E \leq \log V^2 \leq 2 \log V$
- ▶ Example: complete graph on $0, \dots, 9$, edge $n-m$ weight $n+m$, MST consists of $0-m$ for $m = 1, \dots, 9$

Shortest paths

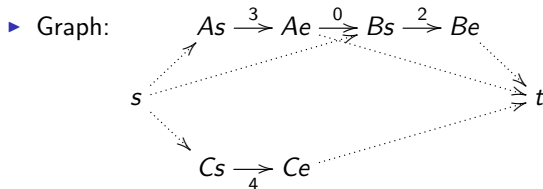
- ▶ Recall slide **76**, *Edge-weighted digraphs*: digraphs in which every arrow has a weight
- ▶ *Weight* of a (di)path: sum of weights of the arrows
- ▶ *Shortest* path from node s to node t : minimum path weight
- ▶ EWD = Edge-Weighted Digraph, example: `tinyEWD.txt`
- ▶ Example: two SPs from 0 to 3:



- ▶ Shortest paths need not be unique, even if all weights are different!
- ▶ Shortest paths need not exist, for two independent reasons:
 - ▶ When target t is not reachable from source s
 - ▶ When there is a negative cycle on the path to t , e.g., $1 \xrightarrow{-1} 1$

Variations

- ▶ Single-source versus multiple sources
- ▶ Only non-negative weights versus all weights allowed
- ▶ Acyclic versus cycles, in particular negative cycles
- ▶ Longest path: shortest path with weights negated
- ▶ Important example: (parallel) scheduling of jobs A, B, and C
 - ▶ A (3 hrs), must precede by B (2 hr), independent C (4 hrs)



- ▶ Schedule, longest paths, makespan: `tinyJob.txt`
- ▶ Now add: A must start less than 2 hrs before B starts.
Feasible? (No) And 4 hrs before? (Yes)

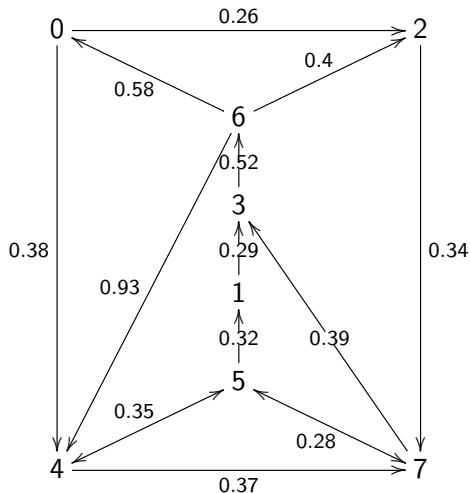
Dijkstra's Algorithm

- ▶ Single-source, only non-negative weights, cycles no problem
- ▶ Datastructures:
 - ▶ EWD with adjacency lists `adj[v]` of weighted out-edges
 - ▶ Boolean array `marked[v]` for marking vertices
 - ▶ Array `distToSource[v]`, minimum distances to source so far
 - ▶ Array `pathToSource[v]`, best arrow to source so far
- ▶ Algorithm: relax (see below) and mark the unmarked node with least distance, until all marked; Simple example: slide 101
- ▶ Invariants:
 - ▶ Marked nodes: known shortest path to source (non-negativity!)
 - ▶ Unmarked nodes: known shortest path to source THROUGH marked nodes (requires in-arrow from marked node)
- ▶ Implementation `LinkedListEWD.slowEWD()`, examples `tinyJob.txt` and `tinyEWD.txt`

Relaxation

- ▶ Assume an array `distToSource[v]` with minimum distances, so far, to a given source
- ▶ To *relax an edge* e from u to v with weight x means to update `distToSource[v]` to `distToSource[u] + x` if the latter is smaller
- ▶ To *relax a node* u means to relax all edges in `adj[u]`, that is, to update `distToSource[v]` to `distToSource[u] + x` if the latter is smaller, for every edge from u to v with weight x
- ▶ Dijkstra: relax and mark unmarked node v with minimal `distToSource[v]`, until all nodes marked
- ▶ Bellman-Ford: do max V rounds of relaxation of all edges (may stop after a round without updates)

EWD Example (tinyEWD.txt, NB $4 \leftrightarrow 5 \leftrightarrow 7$!)



Bellman-Ford

- ▶ Single-source, also negative weights, negative cycles detected
- ▶ Datastructures:
 - ▶ EWD with adjacency lists `adj[v]` of weighted out-edges
 - ▶ Array `distToSource[v]`, minimum distances to source so far
 - ▶ (Array `pathToSource[v]`, best arrow to source so far)
- ▶ Algorithm: do at most V rounds for every node v and every arrow e in `adj[v]`, if e shortens the distance to its endpoint w , update that distance (and path); stop after a round when no distances improve. If distances improve in the V -th round, a negative cycle is reachable from the source.
- ▶ Invariant: after n rounds the distances are less than or equal to the shortest path of length n from the source
- ▶ Implementation `LinkedListEWD.simpleBF()`, examples `tinyJob.txt` and `tiNoJob.txt`

Memory-Use and Run-time Analysis

- ▶ Space
 - ▶ All methods use $O(V + E)$ for the graph, plus $O(V)$ extra
 - ▶ Still true for Dijkstra improved with an indexed priority queue, but indexed priority queue takes $\sim 3V$ space
- ▶ Time, worst-case:
 - ▶ V times finding a minimum (original Dijkstra): $O(V^2)$
 - ▶ Priority queue operations (improved Dijkstra): $O(E \log V)$
 - ▶ V rounds relaxing E edges (Bellman-Ford): $O(EV)$

Odds and Ends Chapter 4

- ▶ Bellman-Ford
- ▶ Indexed Priority Queue
- ▶ ...

ToC and topics of general interest

- ▶ Table of Contents on next slide (all items clickable)
- ▶ Practical stuff: slide 2

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Ch.1.5 Case Study: Union-Find

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Ch.2.2 Mergesort

Ch.2.3 Quicksort

Ch.2.4 Priority Queues

Ch.2.5 Applications

Ch.3.1 Symbol Tables

Ch.3.2 Binary Search Trees

Ch.3.3 Balanced Search Trees

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Ch.3.5 Applications of Searching

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