

# INF102

## Algorithms, Data Structures and Programming

Marc Bezem<sup>1</sup>

<sup>1</sup>Department of Informatics  
University of Bergen

Fall 2016

## INF102, practical stuff

- ▶ Lecturer: Marc Bezem; Team: see homepage
- ▶ Homepage: [INF102](#) (requires login)
- ▶ Also: [INF102 on GitHub](#)
- ▶ Tentative [schedule](#)
- ▶ Textbook: [Algorithms, 4th edition](#)
- ▶ Prerequisites: INF100 + 101 ( $\approx$  Ch. 1.1 + 1.2)
- ▶ Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- ▶ Three compulsory exercises, must be passed
- ▶ Digital exam (Inspera) [02.12.2016](#)
- ▶ Old exams: [2004–2015](#)
- ▶ [Table of Contents of these slides](#)

# Resources

- ▶ Good textbook, USA-style: many pages, exercises etc.
- ▶ Average speed must be ca 50 pages p/w
- ▶ Lectures (ca 24) focus on the essentials
- ▶ Slides (ca 120, dense!) summarize the lectures
- ▶ Prepare yourself by reading in advance
- ▶ Workshops: selected exercises
- ▶ Test yourself by trying some exercises in advance
- ▶ If you can do the exercises (incl. compulsory), you are fine
- ▶ Review of exercises on Friday morning

## Generic Bags, Queues and Stacks

- ▶ Generic programming in Java, example: **PolyPair.java**
- ▶ Bag, Queue and Stack are generic, iterable collections
- ▶ Queue and Stack: Ch. 9 in textbook INF100/1 (!?)
- ▶ APIs include: `boolean isEmpty()` and `int size()`
- ▶ All three support adding an element
- ▶ Queue and Stack support removing an element (if any)
- ▶ FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- ▶ Dijkstra's Two-Stack Expression Evaluation **Movie**
- ▶ Example:  $( 1 + ( ( 2 + 3 ) * ( 4 * 5 ) ) )$

## Resizing Arrays

- ▶ Implementation problem: arrays have fixed size
- ▶ Solution: resize arrays, *wisely*
  - ▶ double the size when the array becomes overfull
  - ▶ halve the size when the array becomes quarter full
- ▶ Resizing takes time and space proportional to size
- ▶ Not too seldom (correctness), not too often (efficiency)
- ▶ Later: we retain *constant time direct access*
- ▶ Later: add operation in *constant time on average*
- ▶ Once we have understood resizing arrays: **ArrayList**

## Implementations

- ▶ `ResizingArray_Stack.java`
- ▶ Arrays give direct access, resizing at reasonable cost
- ▶ `LinkedList_Stack.java`
- ▶ No fixed size, but indirect access incurs a cost
- ▶ Pointers take space and dereferencing takes time
- ▶ Programming with pointers: make a picture
- ▶ `LinkedList_Queue.java`

## Computation time and memory space

- ▶ Two central questions:
  - ▶ How long will my program take?
  - ▶ Will there be enough memory?
- ▶ Example: **ThreeSum.java**
- ▶ Inner loop (here  $a[i] + a[j] + a[k] == 0$ ) is important
- ▶ Sorting helps: **ThreeSumOptimized.java**
- ▶ Run some experiments: `1Kints.txt`, `2Kints.txt`, ...
- ▶ Observe the 'doubling factor'

## Methods of Analysis

- ▶ Empirical:
  - ▶ Run program with randomized inputs, measuring time & space
  - ▶ Run program repeatedly, doubling the input size
  - ▶ Measuring time: **StopWatch**
  - ▶ Plot, or log-log plot and **linear regression**
- ▶ Theoretical:
  - ▶ Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
  - ▶ Try to count/estimate/average this cost as function of the input (size)
  - ▶ Use  $O(f(n))$  and  $f(n) \sim g(n)$



## ThreeSum, empirically

- ▶ Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4, 51.1 sec
- ▶ The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- ▶ Basis of the logarithm should be the same for both
- ▶ Linear regression gives  $y \approx 3x - 10$
- ▶  $\log(f(n)) = 3 \log(n) - 10$  iff

$$f(n) = 10^{\log(f(n))} = 10^{3 \log(n) - 10} = n^3 * 10^{-10}$$

- ▶ Conclusion: cubic in the input size, with constant  $\approx 10^{-10}$
- ▶ Strong dependence on input can be a problem
- ▶ Constant  $10^{-10}$  depends on computer, exponent 3 does not

## ThreeSum, theoretically

- ▶ Number of different picks of triples:  $g(n) = n(n-1)(n-2)/6$
- ▶ Inner loop  $a[i] + a[j] + a[k] == 0$  executed  $g(n)$  times
- ▶  $g(n) = n^3/6 - n^2/2 + n/3$
- ▶ Cubic term  $n^3/6$  wins for large  $n$
- ▶ Computational model # array accesses:  $3 * n^3/6 = n^3/2$
- ▶ Cost array access  $t$  sec: time  $t * n^3/2$  sec
- ▶ Cost models are (necessary) simplifications! (NB cache)

## Orders of Growth, Big Oh and $\sim$

- ▶ Q: 'wins for large  $n$ ' uhh???
- ▶ A: Big Oh, and  $\sim$  will clear this up
- ▶ Big Oh and  $\sim$  aim to capture 'order of growth'
- ▶ Costs are positive quantities, so  $f, g, \dots : \mathbb{N} \rightarrow \mathbb{R}^+$
- ▶ MNF130:  $f(n)$  is  $O(g(n))$  if there exist  $c \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$  such that  $f(n) \leq cg(n)$  for all  $n \geq N$  (that is, for  $n$  large enough)
- ▶ Example:  $n^2$  and even  $99n^3$  are  $O(n^3)$ , but  $n^3$  is not  $O(n^{2.9})$
- ▶ INF102:  $f(n) \sim g(n)$  if  $1 = \lim f(n)/g(n)$
- ▶ If  $f(n) \sim g(n)$ , then  $f(n)$  is  $O(g(n))$  and  $g(n)$  is  $O(f(n))$
- ▶ Not conversely: Big Oh disregards constant factors,  $\sim$  not
- ▶ Large constant factors are important!

## Important orders of growth

(we compare all for  $n = 20$  sec)

- ▶ constant:  $c$ , meaning  $f(n) = c$  for all  $n$
- ▶ linear:  $n$
- ▶ linearithmetic:  $n \log n$
- ▶ quadratic:  $n^2$
- ▶ cubic:  $n^3$
- ▶ exponential:  $2^n$
- ▶ general form:  $an^b(\log n)^c$

## Logarithms and Exponents

- ▶ Definition:  $\log_x z = y$  iff  $x^y = z$  for  $x > 0$
- ▶ Inverses:  $x^{\log_x y} = y$  and  $\log_x x^y = y$
- ▶ Exponent:  $x^{(y+z)} = x^y x^z$ ,  $x^{(yz)} = (x^y)^z$
- ▶ Logarithm:  $\log_x(yz) = \log_x y + \log_x z$ ,  $\log_x z = \log_x y \log_y z$
- ▶ Base of logarithm: the  $x$  in  $\log_x$
- ▶ Various bases:  $\log_2 = \lg$ ,  $\log_e = \ln$ ,  $\log_{10} = \log$
- ▶ Double exponent: e.g.  $2^{(2^n)}$  (not used in INF102)
- ▶ Double logarithm:  $\log(\log n)$  (not used in INF102)

## Worst case, average case, amortized cost

- ▶ Worst case: guaranteed, independent of input; Examples:
  - ▶ Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
  - ▶ Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- ▶ Average case: not guaranteed, dependent of input *distribution*
- ▶ Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost  $\leq 21$ , all others cost 1, amortized  $\leq 3$  p/o.
- ▶ Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- ▶ Special case of resizing array that is only growing:  
 $1(2)2(4)3(8)4(16)5(32)6(64)7(128)8(256)9(512) \dots 16(32768) \dots$ , with  $(n)$  the new size.  
 Resizing to  $(n)$  costs  $2n$  array accesses, so in total  
 $(1+4)+(1+8)+(2+16)+(4+32)+(8+64) \dots$ , so 9 p/push.

## Staying Connected

- ▶ We want efficient algorithms and datastructures for testing whether two objects are 'connected' (e.g., in networks)
- ▶ We assume connectedness to be an equivalence
- ▶ MNF130: relation  $E \subseteq V \times V$  is an *equivalence* if
  - ▶  $E$  is *reflexive*:  $\forall x \in V. E(x, x)$
  - ▶  $E$  is *symmetric*:  $\forall x, y \in V. E(x, y) \rightarrow E(y, x)$
  - ▶  $E$  is *transitive*:  $\forall x, y, z \in V. E(x, y) \wedge E(y, z) \rightarrow E(x, z)$
- ▶ Dynamic connectivity means (here) that  $E$  can grow
- ▶ Clear relationship with paths in graphs, (connected) components (MNF130)
- ▶ Input:  $N$  and pairs in  $V = \{0, \dots, N-1\}$  defining  $E$
- ▶ Challenge: efficient `boolean connected(int p, int q)`
- ▶ Example:  $N = 10$ , 4 3, 3 8, ... (`algs4-data/tinyUG.txt`)
- ▶ Picture on blackboard (don't print pairs that are already connected)

## Union-Find

- ▶ Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- ▶ Union, idea: for any new pair  $n\ m$  that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- ▶ API: **UF**; Cost model: number of array accesses
- ▶ Implementations:
  - ▶ **SlowUF.java**: `id[p]` identifier of  $p$   
`find()`  $\sim 1$ , `union()`  $\sim$  between  $n+3$  and  $2n+1$
  - ▶ **FastUF.java**: `int[] id` pointers, `id[p]==p`: identifier  
`find()`  $\sim 1+2d$ , `union()`  $\sim 1 + \text{two find()}'s$
  - ▶ **WeightedUF.java**: `int[] id` pointers, `int[] sz` subtree sizes  
`find()` and `union()` both  $\sim \lg n$
- ▶ WeightedUF: height of subtree of size  $k$  is at most  $\lg k$
- ▶ Path-compression: ultimate improvement of UF (almost  $O(1)$ , amortized)



# Sorting

- ▶ Sorting: putting objects in a certain order
- ▶ MNF130: relation  $R \subseteq V \times V$  is a *total order(ing)* if
  1.  $R$  is *reflexive*:  $\forall x \in V. R(x, x)$
  2.  $R$  is *transitive*:  $\forall x, y, z \in V. R(x, y) \wedge R(y, z) \rightarrow R(x, z)$
  3.  $R$  is *antisymmetric*:  $\forall x, y \in V. R(x, y) \wedge R(y, x) \rightarrow x = y$
  4.  $R$  is *total*:  $\forall x, y \in V. R(x, y) \vee R(y, x)$
- ▶ Natural orderings:
  - ▶ Numbers of any type: ordinary  $\leq$  and  $\geq$
  - ▶ Strings: lexicographic
  - ▶ Objects of a Comparable type: `v.compareTo(w) <= 0`

## Sorting (ctnd)

- ▶ Bubble sort: `ExampleSort.java`
- ▶ Certification: `assert isSorted(a)` in `main()`
- ▶ No guarantee against modifying the array (but `exch()` is safe)
- ▶ Costmodel 1: number of `exch()`'s and `less()`'s
- ▶ Costmodel 2: number of array accesses
- ▶ Pitfalls: cache misses, expensive `v.compareTo(w) < 0`
- ▶ Why studying sorting? (`java.util.Arrays.sort()`)
- ▶ Comparing sorting algorithms: `SortCompare.java`

## Selection Sort

- ▶ Bubble sort:  $\sim n^2/2$  compares,  $0 \leq \text{exchanges} \leq \sim n^2/2$
- ▶ Selection sort:
  - ▶ Find index of a minimum in  $a[0..n-1]$ , exchange with  $a[0]$
  - ▶ Find index of a minimum in  $a[1..n-1]$ , exchange with  $a[1]$
  - ▶ ... until  $n-2$
- ▶ Selection sort:  $\sim n^2/2$  compares,  $0 \leq \text{exchanges} \leq n-1$  (!)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=0; i<N-1; i++){  
        int min=i;  
        for (int j=i+1; j<N; j++) if (less(a[j],a[min])) min=j;  
        if (i != min) exch(a,i,min);  
    }  
}
```

## Insertion sort

- ▶ Insertion sort:
  - ▶ Insert  $a[1]$  on its correct place in (sorted)  $a[0..0]$
  - ▶ Insert  $a[2]$  on its correct place in (sorted)  $a[0..1]$
  - ▶ ... until  $a[n-1]$
- ▶ Very good for partially sorted arrays, costs:
  - ▶ Best case:  $n-1$  compares and 0 exchanges
  - ▶ Worst case:  $\sim n^2/2$  compares and exchanges
  - ▶ Average case:  $\sim n^2/4$  compares and exchanges (distinct keys)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=1; i<N; i++){  
        for (int j=i; j>0 && less(a[j],a[j-1]); j--)  
            exch(a,j,j-1);  
    }  
}
```

## Shell sort

- ▶ Insertion sort:
  - ▶ Very good for partially sorted arrays
  - ▶ Slow in transport: step by step `exch(a,j,j-1)`
- ▶ Idea: h-sort, `a[i], a[i+h], a[i+2h], ...` sorted (any `i`)

```
public static void hsort(int h, Comparable[] a) {  
    int N = a.length;  
    for (int i=h; i<N; i++)  
        for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)  
            exch(a,j,j-h);  
}
```

- ▶ Insertion sort: `hsort(1,a)`
- ▶ Shell sort: e.g., `hsort(10,a); hsort(1,a)`

## Shell sort (ctnd)

- ▶ `hsort(10,a); hsort(1,a)` faster than just `hsort(1,a)` !
- ▶ Q: How is this possible?
- ▶ A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- ▶ What about `hsort(100,a); hsort(10,a); hsort(1,a)`?
- ▶ To be expected: depends on the length  $N$  of the array
- ▶ Best practice:  $h = N/3, N/9, \dots, 364, 121, 40, 13, 4, 1$

# Mergesort

- ▶ Top-down (recursive) algorithm:
  - ▶ Mergesort left half, mergesort right half
  - ▶ Merge the results
- ▶ Using an auxiliary array: [TopDownMergeSort.java](#), [Movie](#)
- ▶ Bottom-up algorithm (16 elements):
  - ▶ Merge  $a[0], a[1]$ , so  $a[2], a[3]$ , so  $a[4], a[5]$ , so ...
  - ▶ Merge  $a[0..1], a[2..3]$ , so  $a[4..5], a[6..7]$ , so ...
  - ▶ Merge  $a[0..3], a[4..7]$ , so  $a[8..11], a[12..15]$
  - ▶ Merge  $a[0..7], a[8..15]$ , done!
- ▶ Also using an auxiliary array: [BottomUpMergeSort.java](#)

## Run-time and memory use of mergesort

- ▶ Mergesort uses between  $\sim (N/2) \lg N$  and  $\sim N \lg N$  compares. Proof on bb. Important formula ( $N = 2^n$ ):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- ▶ Mergesort uses at most  $\sim 6N \lg N$  array accesses
- ▶ Mergesort uses  $\sim 2N$  space (plus some var's)
- ▶ Q: How fast can compare-based sorting of  $N$  distinct keys be?
- ▶ A:  $\lg N! \sim N \lg N$ ; Proof in book and on bb. Keywords: binary *compare tree*, inner nodes for each `compare(a[i], a[j])`, permutations in the leaves,  
 $N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$



# Quicksort

- ▶ Top-down (recursive) algorithm:
  - ▶ Choose a (pivot) value  $v$  in the array
  - ▶ Partition the array in non-empty parts  $\leq v$  and  $\geq v$
  - ▶ Quicksort the two parts
- ▶ Pros: in-place, average computation time  $O(n \log n)$
- ▶ Cons: stack space for recursion, worst-case  $O(n^2)$ , not stable
- ▶ Implementation: **QuickSort.java**
- ▶ BTW: **Bug in java.util.Arrays.sort**

## Quicksort, details

- ▶ Subtleties in `sort()`: shuffling protects against worst-case behaviour
- ▶ Termination of recursive `quicksort()`
- ▶ Subtleties in `partition()`:
  - ▶ Invariants  $l \leq h$  in the two inner loops
  - ▶ Postcondition after the two inner loops
  - ▶ Invariant of the `for(;;)` loop
  - ▶ Termination of the `for(;;)` loop
  - ▶ There are some variations that are also correct

## Run-time and memory use of quicksort

- ▶ Compare Quicksort to other sorts ( $n = 10^2, 10^3, \dots$ )
- ▶ Quicksort: time  $O(n^2)$  if pivot is always smallest (or largest)
- ▶ Randomization: choose pivot randomly, or shuffle array
- ▶ If all keys are distinct and randomization is perfect, then quicksort uses on average  $\sim 2n \ln n$  compares and  $\sim (n/3) \ln n$  exchanges (proofs in book, complicated)
- ▶ Relevant improvements:
  - ▶ Cut-off to insertion sort for sizes  $\leq 15$  (ca.)
  - ▶ Median-of-three pivot
  - ▶ Taking advantage of duplicate keys (3-way partitioning)
- ▶ Quicksort is generally quite good
- ▶ In special situations other sorts are better (e.g., countsort)

## Priority Queues

- ▶ Assume collecting and processing items having keys
- ▶ Examples of keys: time-stamp, price-tag, priority-tag
- ▶ Assume: keys can be ordered
- ▶ Reasonable: processing currently highest (or lowest)
- ▶ Special cases: items time-stamped when added
  - ▶ Queue: dequeue currently oldest (lowest time-stamp)
  - ▶ Stack: pop currently newest (highest time-stamp)
- ▶ Priority queue generalizes this
- ▶ Examples: highest priority, largest transaction, lowest price
- ▶ Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

## Priority Queues

- ▶ Good info: [Wikipedia](#); API (the bare essentials):

```
public class  ArrayListPQ<Key extends Comparable<Key>>

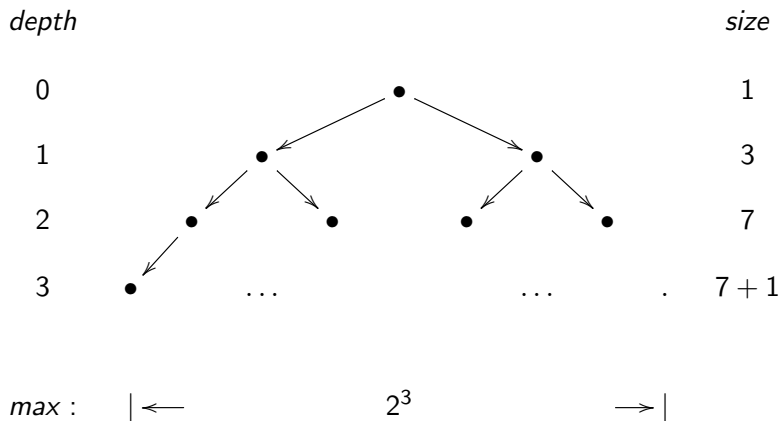
void          insert(Key v) // insert a key
Key           delMax() // delete a largest key, if any
boolean       isEmpty()
int           size()
```

- ▶ Aim: operations in logarithmic time, no extra space
- ▶ In case of duplicate keys: 'a' largest, not 'the'
- ▶ Typical application: the 1K largest keys of 1G unsorted keys
- ▶ Client: [BottomM.java](#) (Q: why is the output slowing down?)

## Binary Trees

- ▶ MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- ▶ MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height  $h$  has  $2^{h+1}-1$  nodes.
- ▶ INF102: A binary tree is (left-) *complete* if all levels  $< h$  are filled, level  $h$  may be partly empty from the right (picture bb). A (left-)complete binary tree of height  $h$  has between  $2^h$  and  $2^{h+1}-1$  nodes (from now on we leave out '(left-)').
- ▶ A complete binary tree of  $n$  nodes has height  $\lfloor \lg n \rfloor$

# Picture



## Heap-ordered Binary Trees

- ▶ A binary tree is *heap-ordered* if the key in each node is  $\geq$  the keys in its children (if any). Thus the root has a maximal key.
- ▶ Array representation of heap-ordered complete binary tree (bb)
- ▶ Methods `swim()` and `sink()`: picture on bb, code below
- ▶ Implementation: `ArrayListPQ.java`



## Run-time and memory use of heaps, applications

- ▶ In a heap of  $n$  elements (since height is  $\leq \lfloor \lg n \rfloor$ ):
  - ▶ `swim()`, and hence `insert()`, takes  $\leq 1 + \lfloor \lg n \rfloor$  compares and  $\leq \lfloor \lg n \rfloor$  exchanges
  - ▶ `sink()`, and hence `delMax()`, takes  $\leq 2\lfloor \lg n \rfloor$  compares
  - ▶ `sink()` takes  $\leq \lfloor \lg n \rfloor$  exchanges, and `delMax()`  $\leq 1 + \lfloor \lg n \rfloor$
- ▶ Heap construction by `insert()` can sometimes be improved
- ▶ Given an array of keys, right-to-left heap construction (bb) takes  $< 2n$  compares and  $< n$  exchanges
- ▶ Applications: **heapsort** and merging sorted streams (bb)
- ▶ Many variations with extended API (indexed priority queue)

## Purpose of Sorting

- ▶ Sorting makes the following easier and more efficient:
  - ▶ Searching (binary search, example: `ThreeSumOptimized`)
  - ▶ Searching and looking up, e.g., the `pagenumber` in an index
  - ▶ Removing duplicates
  - ▶ Finding the median, quartiles etc.
- ▶ Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- ▶ We do *pointer sorting*, manipulating refs to objects.
  - ▶ Pro: not moving full objects
  - ▶ Cons: pointer dereferencing, no `sort(int[] a)`
- ▶ More flexibility: pass a `Comparator` object to `sort()`

## Comparator object

- ▶ API: `void sort(Object[] a, Comparator c)`
- ▶ Call, e.g.: `sort(a, new Transaction.WhenOrder())`
- ▶ Call, e.g.: `sort(a, new Transaction.SizeOrder())`
- ▶ Obs: `import java.util.Comparator`
- ▶ Obs: `less(Object o1, Object o2, Comparator c)`
- ▶ Priority queues also with `Comparator`

```
public class Transaction {  
    ...  
    public static class MyOrder {  
        implements Comparator<Transaction>  
        public int compare(Transaction t, Transaction v){...}  
    } // End of Myorder  
    ...// similarly: WhenOrder, SizeOrder  
} // End of Transaction
```

## Applications of Sorting

- ▶ Consider sorting first to make other problems easier
- ▶ Commercial computing (sort on price, departure time, ...)
- ▶ Search for information: web-indexing, search engines
- ▶ Job scheduling heuristic: longest processing time first
- ▶ To come: Prim's, Dijkstra's and Kruskal's algorithms
- ▶ Huffman compression: a lossless compression based on using the shortest codes for the symbols that occur ofttest.  
Frequency counter: next chapter!
- ▶ Cryptology and genomics (e.g., longest repeated substring)

# Symbol Tables

- ▶ Symbol table associates *keys* with *values*: *key-value pairs*
- ▶ Examples: keyword-page number, ID number-personal data
- ▶ Important operations:
  - ▶ Insert a key-value pair in the symbol table: `void put(k,v)`
  - ▶ Search the value for a given key (if any): `Value get(k)`
- ▶ Important conventions:
  - ▶ Inserting key-value for existing key: overwriting the value
  - ▶ No duplicate keys, no null keys
  - ▶ Value null: no value for this key
  - ▶ Lazy deletion: insert key-null; Eager: really delete key
- ▶ **API** of unordered symbol table
- ▶ Aim: all operations in time  $\sim c \lg n$  with constant  $c$  small

## ST Basics

- ▶ Archetypical ST-client: frequency counter (code: `main`)
- ▶ Cost model: number of compares
- ▶ Naive ST: unordered linked list, linear search (INF101, Ch.9)
  - ▶ Search miss:  $\sim n$  compares
  - ▶ Search hit: between 1 and  $\sim n$  compares
  - ▶ Random search hit:  $(1 + \dots + n)/n \sim n/2$  compares
  - ▶ Inserting  $n$  distinct keys:  $(1 + \dots + (n - 1)) \sim n^2/2$  compares
- ▶ `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- ▶ Naive ST impracticable for genomics, internet
- ▶ Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- ▶ Better for unordered ST: hashing (in Ch. 3.4)

## Ordered Symbol Table

- ▶ Ordered ST: keys are ordered
- ▶ **API** of ordered symbol table
- ▶ Binary search: `get(Key k)` takes  $\sim \lg n$  comparisons
- ▶ What about `put(Key k, Value v)`? **ArrayListST**, good!
- ▶ TODO: test that `add(int i, E e)` is amortized  $O(1)$
- ▶ Implementation with binary search in **ArrayListST.java**
- ▶ Trace of inserts on bb: S E A R C H E X A M P L E
- ▶ Experiments with `tinyTale.txt`, `tale.txt`, ...

## Binary Search Trees

- ▶ *Binary search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- ▶ Search time: length of the path to the node where the key 'should' be
- ▶ Balanced binary tree with  $n$  keys has  $\lg n$  height
- ▶ Unbalanced binary trees can have height  $n$  (max depth)
- ▶ Search hits in a binary search tree, built without rebalancing, of  $n$  random keys take on average  $\sim 2 \ln n$  compares
- ▶ **UBST.java**: `put()`, `get()`, `size()`, `isEmpty()`
- ▶ Trace of inserts on bb: S E A R C H E X A M P L E



## Binary Search Trees (ctnd)

- ▶ Interrelated, increasing difficulty: `min(Node x)`, `deleteMin(Node x)`, `delete(Node x, Key k)`
- ▶ Node of minimum key: not `null`, and has left child `null`, and is root or left child of parent (picture on bb)

```
public Node min(Node x){ // x != null, subtree not empty
    while (x.left != null) x = x.left;
    return x;
} // cf. tail recursive min() in Alg. 3.3
```

- ▶ Delete minimum key, two cases:  
(1) both children `null`; (2) left child `null`
- ▶ Delete is really difficult: bb + **BST.java**
- ▶ Don't forget: update `x.N` along the path to the root!

## Balanced Search Trees: keep paths short!

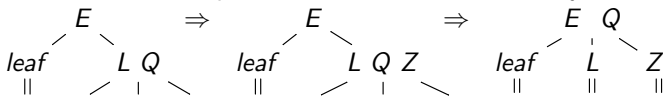
- ▶ NB tree balancing not as easy as in UF and Heap (4hrs!)
- ▶ A 2-3 search tree consists of 2-nodes and 3-nodes:
  - ▶ Each 2-node has two children and a key  $k$  such that all keys in the left subtree are  $< k$ , and all keys in the right subtree  $> k$
  - ▶ Each 3-node has three children and two keys  $k_1, k_2$  such that all keys in the left subtree are  $< k_1$ , all keys in the middle subtree  $> k_1$  and  $< k_2$ , and all keys in the right subtree  $> k_2$
- ▶ Examples and pictures on bb
- ▶ *Perfect* 2-3 search tree: paths from root to leaves equally long
- ▶ Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- ▶ Insert should keep tree perfect, rough idea:
  - ▶ into a 2-leaf: make it into a 3-leaf
  - ▶ into a 3-node: do something clever (explained next)

## Insert in Balanced Search Trees

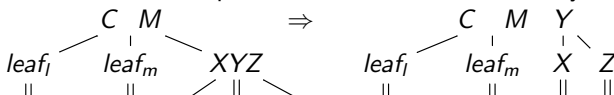
- ▶ Terminology: a *leaf* is a node all whose children are null
- ▶ Data invariant 1: 2-3 search tree
- ▶ Data invariant 2: paths from root to leaves equally long

- ▶ Insert into a 2-leaf  $L$  : either  $\begin{array}{c} A \ L \\ \diagdown \ \diagup \\ \parallel \end{array}$  or  $\begin{array}{c} L \ Z \\ \diagdown \ \diagup \\ \parallel \end{array}$

- ▶ into a 3-leaf whose parent is a 2-node: with new key  $Z$



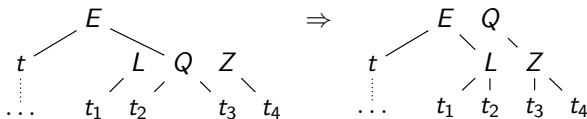
- ▶ into a 3-leaf whose parent is a 3-node: with new key  $Z$  added



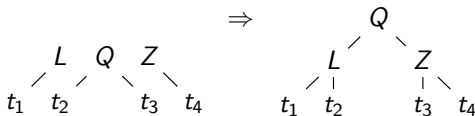
- ▶ into a 3-node whose parent is a 3-node: move up middle key!

## Insert (ctnd)

- ▶ Data invariant 1: 2-3 search tree
- ▶ Data invariant 2: paths from root to leaves equally long
- ▶ Insert works up from the leaf where the key 'should' be
  - ▶ if 2-node on path to root: make it into a 3-node (two cases)



- ▶ otherwise: split the root

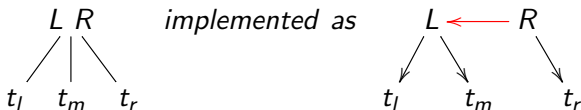


## Insert, summary and examples

- ▶ Six operations for eliminating 4-nodes:
  - ▶ if parent is 2-node: move middle key up (left and right case)
  - ▶ if parent is 3-node: move middle key up (left, middle, right)
  - ▶ if root: split root
- ▶ Search and insert visit at most  $\lfloor \lg n \rfloor$  nodes
- ▶ Proof: maximal path length is  $\geq \lfloor \log_3 n \rfloor$  and  $\leq \lfloor \log_2 n \rfloor$
- ▶ Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- ▶ Trace of inserts on bb: A C E H L M P R S X

## Red-black trees

- ▶ Red-black trees implement 2-3 trees
- ▶ Idea: one 3-node = two 2-nodes + extra info
- ▶ Extra info coded in color, picture:




- ▶ A *red-black tree* is a binary search tree with red and black links such that:
  - ▶ Only left links can be red (but need not be)
  - ▶ Never
  - ▶ Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- ▶ Equivalent: red-black tree and perfect 2-3 search tree

## Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

```
private class Node {  
    Key key;  
    Value value;  
    Node left, right;  
    boolean color; // true for red, false for black  
    int N;  
}  
private boolean isRed(Node n) {  
    if (n==null) {return false;} else {return x.color}
```

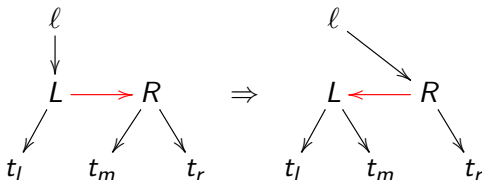
## Rotating and Color Flipping

- ▶ Aim: restoring the data invariants of red-black search trees
  1. Only left red links, but never two
  2. Search tree invariant
  3. Perfect black balance
- ▶ Invariants get violated by temporary 4-nodes, e.g.,
  - ▶ inserting  $Z$  in  $L \leftarrow R : L \leftarrow R \rightarrow Z$
  - ▶ inserting  $A$  in  $L \leftarrow R : A \leftarrow L \leftarrow R$
  - ▶ inserting  $M$  in  $L \leftarrow R : L \leftarrow R$   

- ▶ Restoring:
  - ▶ Color flip  $L \leftarrow R \rightarrow Z : L \leftarrow R \rightarrow Z$
  - ▶ Rotation right + color flip  $A \leftarrow L \leftarrow R : A \leftarrow L \rightarrow R$
  - ▶ Rotation left into  $M \leftarrow L \leftarrow R$ , then as previous



## Left Rotation

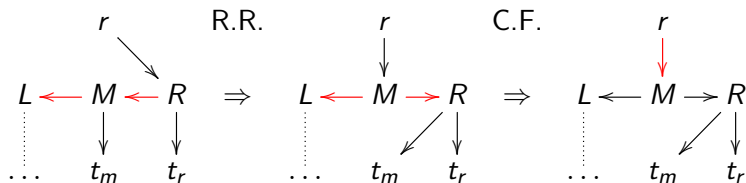
Call: `l = rotateLeft(l);`



```
private Node rotateLeft(Node l){
    Node r = l.right; l.right = r.left; r.left = l;
    r.color = l.color; l.color = true // == RED
    r.N = l.N; l.N -= 1+size(r.right); // Why?
    return r;
}
```

## Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- ▶ Code of `rotateRight()` like that of `rotateLeft()`
- ▶ NB1: operations are local (here only  $r$ ,  $M$ ,  $R$ )
- ▶ NB2: operations preserve data invariants
- ▶ NB3: root is a special case (always black)
- ▶ Deletions: complicated, but doable (Exc. 3.3.39–41)

## Run-time and memory use of Red-Black BSTs

- ▶ The height of a red-black BST with  $n$  nodes is  $\leq 2 \lg n$   
Proof: the worst-case is one 3-node path and the rest 2-nodes
- ▶ The average length of path from the root to a *node* (?) in a red-black BST with  $n$  nodes is  $\lg n$  ('empirical fact')
- ▶ In a red-black BST, search, insert, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node.
- ▶ For red-black BSTs, logarithmic time is guaranteed!

# Hashing

- ▶ Idea: if keys in  $[0..99]$  an array is the perfect symbol table
- ▶ A *hash function* maps a key to an array index
- ▶ Injectivity of the hash function is not guaranteed
- ▶ *Hash collision*: different keys are mapped to the same index
- ▶ In such a case we need *collision resolution*
- ▶ Symbol tables: hashing is fast, but unordered (no `max`, `min`)
- ▶ Aim: ST operations amortized  $O(1)$  time, extra space OK

## Space-Time Trade-Off

- ▶ Hashing is an example of a *space-time trade-off*
- ▶ Time: computation time required
- ▶ Space: memory space used
- ▶ Unlimited space: (1) use key as index (e.g., the bits)
- ▶ Unlimited time: (2) use linked list and linear search
- ▶ Hashing strikes a balance using (1) with some array of reasonable size, and (2) in case of collisions
- ▶ The balance between (1) and (2) can easily be tuned

## Hash functions

- ▶ Ideal (uniform hashing assumption, UHA): uniform and independent distribution of keys over integers from 0 to  $M - 1$
- ▶ Examples of **hash functions in Java**
- ▶ Horner:  $a_0 + x(a_1 + x(a_2 + \dots)) = a_0 + a_1x + a_2x^2 + \dots$
- ▶ Reasonably  $\approx$  UHA: modular hashing ( $M$  prime):  

```
private int hash(Key k){  
    return (key.hashCode() & 0x7fffffff) % M;}  

```
- ▶ Q: Why  $M$  prime?
- ▶ A: e.g.  $M = 32$  takes only into account the last five bits

## Collision Resolution

- ▶ Two methods of collision resolution:
  1. Hashing with separate chaining (picture on bb)
  2. Hashing with linear probing (picture on bb)
- ▶ Separate chaining: symbol table is an array of linked lists, linear search. If array has length  $M$ , then the linked lists have average length  $N/M$  with  $N$  keys.
- ▶ Linear probing: symbol table is an array of length  $M > N$ . Colliding keys are put at the first empty position. Linear search from the position where the key 'should have been'. Empty position: not found. Deletion tricky: reinsert all keys to the right of the deleted key, until the first empty position (picture on bb). Array must have length  $\geq N$  with  $N$  keys.

# Symbol Table with Hashing

- ▶ Implementation: `ArrayListHashST.java`
- ▶  $M = 1$ : measure overhead wrt. `ArrayListST.java`
- ▶ Tests with various values of  $M$ : 31, 997, 65521
- ▶ NB: *construction* versus *use* of ST (hashing better for *use*)
- ▶ Hashing can be combined with any other ST-implementation



## Quantitative analysis

- ▶ Throwing a dice 10 times, what is the probability of 3 fives?
- ▶ Under UHA, with  $N$  distinct keys, the probability that exactly  $k$  keys collide at some given hash value is

$$\binom{N}{k} \left(\frac{1}{M}\right)^k \left(\frac{M-1}{M}\right)^{N-k}, \text{ where e.g. } \binom{100}{10} \approx 1.7E13$$

- ▶ This is a small number for, say,  $N = M = 100$  and  $k = 10$
- ▶ For linear probing one typically takes  $M = 2N$
- ▶ For separate chaining one keeps  $N/8 \leq M \leq N/2$  (resizing  $M$ )
- ▶ Under UHA: search, insert, delete take amortized  $O(1)$  time
- ▶ Extra space used can be upto  $100N$  byte (objects, pointers); this on top of the space used by  $N$  key-value pairs

## Applications of Searching

- ▶ Synonyms: **associative array**, map, symbol table, or dictionary
- ▶ Origin of **symbol table**: compilers and interpreters
- ▶ Web-indexing, **search engines**
- ▶ Sparse matrices (many 0's): **dictionary**
  1. keys: (row, column)-pairs
  2. values: matrix entries
- ▶ Set API (no values, only keys, for deduplication, filtering):

```
public class SET<Key>
```

```
void          add(Key k)
void          delete(Key k)
boolean       contains(Key k)
boolean       isEmpty()
int           size()
```

## Applications of Searching (ctnd)

- ▶ Application (key, value)
- ▶ Phone book (name, phone number)
- ▶ Dictionary (word, meaning or translation)
- ▶ Account information (client ID, account information)
- ▶ Genomics (sequences of ACTG triplets, proteins)
- ▶ Experimental data of various kinds
- ▶ File systems (file name, address etc)
- ▶ Internet domain name system (domain name, IP address)
- ▶ Invertex index (value, key(s))

## Balanced Search Tree or Hash Table?

- ▶ Q: Which symbol table to use?
- ▶ A: The basic choice between BST and HT depends on ...
  1. Ordering of keys essential: BST
  2. Availability of good hash function (good = fast + UHA)
  3. Ordering of keys expensive (long strings): HT (or: Ch.5)
  4. Ordering of keys possible, but not essential: HT + BST
  5. Space considerations (ArrayListST uses the least extra space)
  6. Number of distinct keys and the space each key takes
  7. Distribution of insert/delete/search operations

## Odds and Ends Chapter 1–3

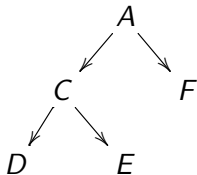
- ▶ Discuss methodological issues SortCompare
- ▶ Find out if `add(int i, E e)` in `ArrayList` is amortized  $O(1)$
- ▶ Discuss primitive types (objects are costly)
- ▶ **Distributed Hash Table**
- ▶ Double hashing: linear probing  
 $h_1(k), h_1(k) + h_2(k), h_1(k) + 2h_2(k), \dots$
- ▶ Indexed Priority Queues: next slide

## Indexed Priority Queues

- ▶ IPQ  $\approx$  array with direct access to minimum (maximum)
- ▶ API: `void insert(int i, Key k); void del(int i); int minKey();`  
`Key keyOf(int i);...` Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-
qp	1	0	2	4	3	-1	-1

heap



Do: `insert(6,G)`, `insert(5,B)`

## Graph classes

( MNF130: useful review of graph theory)

1. Undirected graphs: a set of *vertices* (or *nodes*)  $V$  and a set of *edges*  $E$  connecting the nodes
2. Directed graphs (*digraphs*): a set of nodes  $V$  and a set  $E$  of edges (or *arrows*) pointing from one node to another
3. *Edge-weighted graphs*: undirected graphs in which every edge has a number called the *weight* of the edge
4. *Edge-weighted digraphs*: digraphs in which every arrow has a weight

## Examples

1. Undirected graphs: social networks, communication networks (duplex communication), road maps
2. Directed graphs: logical circuits, job scheduling, flow graphs, hyperlinks, (class, module, package) dependencies
3. Edge-weighted graphs: roadmaps with geographical distance, communication networks with bandwidth
4. Edge-weighted digraphs: job scheduling with duration, transport of goods, financial transactions



## Undirected Graphs

- ▶ Undirected graph: a set of *vertices* (or *nodes*)  $V$  and a set of *edges*  $E$  connecting the nodes
- ▶ *Subgraph*: subset of  $E$  and subset of  $V$  forming a graph (!)
- ▶ *Path*: sequence of nodes connected by edges (!)
- ▶ *Simple path*: path with no node repeated
- ▶ *Length of path*: number of edges
- ▶ *Cycle*: path of length  $> 0$  with same start and end node
- ▶ *Simple cycle*: not repeating edges or nodes (apart from start and end node)
- ▶ *Acyclic graph*: graph without simple cycles
- ▶ *Connected graph*: having a path between every two nodes
- ▶ *Connected component*: a maximal connected subgraph

## Trees and Forests

- ▶ 'Anomalies' concerning edges:
  - ▶ Self-loop: edge connecting a node to itself
  - ▶ Parallel edges: two edges connecting the same node(s)
- ▶ When no anomalies,  $E \subseteq \{\{v, v'\} \mid v \in V, v' \in V, v \neq v'\}$
- ▶ *Tree*: connected acyclic graph (then: no anomalies)
- ▶ *Spanning tree*: maximal subgraph that is a tree
- ▶ Lemma: any spanning tree of a connected graph contains all nodes. Proof by contradiction (on bb).
- ▶ *Forest*: graph consisting of disjoint trees
- ▶ *Spanning Forest*: forest consisting of spanning trees of connected components of a graph
- ▶ Example: `tinyG.txt` on bb

## Undirected Graphs (ctnd)

- ▶ *Distance* between two nodes: length of a shortest connecting path if there is a path connecting these nodes, otherwise  $\infty$
- ▶ *Degree* of a node: number of edges connected to that node
- ▶ Graph  $G = (V, E)$ , the following are equivalent:
  - ▶  $G$  is a tree
  - ▶  $G$  has  $|V| - 1$  edges and no cycles
  - ▶  $G$  has  $|V| - 1$  edges and is connected
  - ▶  $G$  is acyclic and adding an edges creates a cycle
  - ▶ Any two nodes of  $G$  are connected by exactly one simple path
- ▶ Example: some subgraphs of `tinyG.txt` on bb

## Graph representation and implementation

- ▶ Impractical: **adjacency matrix**  $\sim V^2$ , **incidence matrix**  $\sim VE$
- ▶ Often practical: **adjacency lists**  $\sim (V+2E)$ , that is, `adj[v]` lists all nodes `w` connected to `v` by an edge
- ▶ Example: `tinyG.txt` by **LinkedListG.java**
- ▶ Graph API includes: `V()`, `E()`, `addEdge()`
- ▶ Basic algorithms: Depth-First Search (DFS) and Breadth-First Search (BFS)
- ▶ Both DFS and BFS 'walk through the graph', in different ways
- ▶ Both DFS and BFS can compute a spanning tree and forest

```
public void dfs(Integer v, boolean[] marked) {
    marked[v] = true;
    for (Integer w : adj[v])
        if (! marked[w]) dfs(w,marked);
} // dfs() is recursive, call: dfs(v,marked);

public void bfs(Queue<Integer> q, boolean[] marked) {
    while (!q.isEmpty()) {
        Integer v = q.dequeue();
        for (Integer w : adj[v])
            if (! marked[w]) {marked[w]=true; q.enqueue(w);}
    }
} // call: marked[v]=true; q.enqueue(v); bfs(q,marked);

// Example: 0-1, 0-3, 1-2, 1-3, 3-4
// Example: complete ternary tree of height 2
```

## Implementation and Properties of DFS/BFS

- ▶ **LinkedListG.java**: `pathdfs()`, `pathbfs()`
- ▶ DFS and BFS mark nodes connected to a given source node in time proportional to the sum of their degrees ( $\leq 2E$ ), and can return a path from a marked node to the given source in time proportional to the length of this path
- ▶ BFS always finds a shortest path (proof: queue only contains nodes at distance  $k$  followed by nodes at distance  $k + 1$ , while all nodes at distance  $\leq k$  not in queue have been processed)
- ▶ DFS finds a left-most path (long or short, example bb)
- ▶ BFS tends to use more space (but not always)
- ▶ UF tests connectivity, but finds no paths

## Applications

- ▶ `StringSTG.java`, flight connections, shortest path = minimum number of stop-overs
- ▶ Degrees of separation in social networks, e.g., Erdős number = length of shortest path to Paul Erdős in the co-author graph
- ▶ Connected components:
- ▶ Example: `tinyG.txt` on bb and by `LinkedListG.countcc()`

## Directed Graphs

- ▶ *Digraph*: a set of *vertices* (or *nodes*)  $V$  and a set of *directed edges* (or *arrows*)  $E$  pointing from one node to another
- ▶ *Subdigraph*, *directed (simple) path* (*dipath*), *directed (simple) cycle*, *acyclic*, *length*: as expected
- ▶ Often we leave out 'di' in digraph, dipath, etc.
- ▶ *DAG*: **D**irected **A**cyctic **G**raph;
- ▶ *Degree*: **in**-degree and **out**-degree
- ▶ Node  $v$  is *reachable* from  $w$ : a dipath from  $w$  to  $v$  exists
- ▶ *Strongly connected digraph*: dipath between every two nodes (for all  $v, w$ , there are dipaths from  $v$  to  $w$  and from  $w$  to  $v$ )
- ▶ *Strongly connected component*: maximal strongly connected subgraph ( $u \rightleftarrows v \rightarrow w$  has two scc's)
- ▶ Representation: adjacency lists even simpler!



## Reachability Problems

Assume we are given a directed graph  $G$ .

- ▶ Single-source: given a node  $s$ , the *source*, is a given node  $v$  reachable from  $s$ ? Example: `tinyCG.txt`
- ▶ Multiple-source: given a set of nodes  $S$ , is a given node  $v$  reachable from some node in  $S$ ?
- ▶ Solutions: same DFS and BFS algorithms as in Chapter 1
- ▶ Application (example): mark-and-sweep garbage collection
- ▶ Single-source path: given  $s, v$  such that  $v$  is reachable from  $s$ . Find a path from  $s$  to  $v$ .
- ▶ Single-source shortest path: given  $s, v$  such that  $v$  is reachable from  $s$ . Find a *shortest* path from  $s$  to  $v$ .
- ▶ Solutions: same DFS (path) and BFS (shortest path) algorithms as for undirected graphs

## Cycle Detection

- ▶ Recall: a *DAG* is a graph without a directed cycle
- ▶ Acyclicity test, cycle detection: easy extension of DFS. We keep track of the search path from the source. If there is an arrow from  $v$  to  $w$  and  $w$  is on the path from  $v$  to the source, then there is a cycle. (DFS finds the leftmost path to the leftmost cycle.) Two techniques (space-time trade-off!):
  - ▶ Go back the search path: `LinkedListG.slowCyclist()`
  - ▶ Memorize the search path: `LinkedListG.fastCyclist()`
- ▶ Application: precedence scheduling of jobs

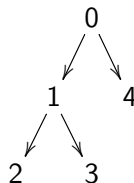
## Pre-order, post-order

- ▶ Graph walks based on DFS from a source node
- ▶ Pre-order: order in which DFS arrives at nodes
- ▶ Post-order: order in which DFS leaves nodes
- ▶ In-order *for binary trees*: e.g., in `UBST.show()`
- ▶ Example:

pre-order: 01234

post-order: 23140

in-order: 21304



- ▶ Example: `tinyCG.txt` on `bb` and by `LinkedListG`

## Topological order of acyclic digraph

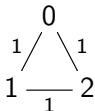
- ▶ Topological order: total order  $\prec$  compatible with the graph in the following sense: if there is an arrow from  $w$  to  $v$ , then  $v \preceq w$  (consequently:  $v \preceq w$  if  $v$  is reachable from  $w$ )
- ▶ Lemma: if a digraph has a topological order, then it is acyclic (proof: a cycle cannot be ordered compatibly)
- ▶ Lemma: if a digraph is acyclic, then it has a topological order (proof idea: if acyclic, the post-order is a topological order since, if there is an arrow from  $w$  to  $v$ , then  $w$  is not reachable from  $v$  and DFS will leave  $w$  after it has left  $v$ )
- ▶ Topological order is a job schedule respecting precedence

## Transitive closure

- ▶ Definition: given  $G$ , its transitive closure  $G^*$  is a graph with the same nodes and arrows from  $v$  to  $w$  for each  $w$  that is reachable from  $v$  in  $G$ .
- ▶ NB:  $G^*$  can have many more arrows than  $G$
- ▶ Implementation: adjacency matrix in case of high density of arrows (proof idea: if acyclic, the post-order is a topological order since, if there is an edge from  $w$  to  $v$ , then  $w$  is not reachable from  $v$  and dfs will leave  $w$  after it has left  $v$ )

## Minimum Spanning Tree

- ▶ Recall slide 66: *spanning tree* of a connected undirected graph is a maximal subgraph that is a tree (and thus contains all nodes and is acyclic)
- ▶ EWG = Edge-Weighted Graph, here always connected
- ▶ Example: `tinyEWG.txt` on bb
- ▶ Recall slide 67: all spanning trees have  $V - 1$  edges
- ▶ *Weight* of spanning tree: sum of the weights of its edges
- ▶ *Minimum Spanning Tree*: spanning tree with minimal weight
- ▶ Example: three MSTs of



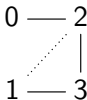
- ▶ Exc.4.3.3: if all weights different, then MST is unique
- ▶ From now on we assume all weights different!

## Minimum Spanning Tree (ctnd)

- ▶ Applications: power plants and electrical grid, airlines and flight routes, maps and distance
- ▶ Weights may be zero or negative (e.g., cost minus profit of a new network of roads between cities)
- ▶ Two important algorithms to find the MST: Prim's and Kruskal's

## Cuts and Crossing Edges

- ▶ Recall slide 67: deleting an edge from a tree creates two disjoint components, adding an edge creates a cycle
- ▶ *Cut*: a partition of  $V$  in two non-empty subsets of nodes
- ▶ *Crossing edge*: edge connecting two nodes in different subsets of a cut
- ▶ NB there can be more than one crossing edge:



- ▶ Lemma: for any cut in an EWG, the crossing edge of minimum weight is in the MST.
- ▶ Proof: given a cut, assume by contradiction there is a crossing edge  $e$  of weight smaller than the crossing edge(s) that is (are) in the MST (e.g., the dotted edge above). Adding  $e$  creates a simple cycle, which must contain one other crossing edge  $f$  in the MST. Replacing  $f$  by  $e$ : ✗



## Prim's Lazy Algorithm

- ▶ Datastructures:
  - ▶ EWG represented with adjacency lists  $\text{adj}[v]$
  - ▶ Minimum priority queue  $pq$  for edges
  - ▶ Array  $\text{marked}[v]$  for marking vertices
  - ▶ Queue  $mst$  for the minimum spanning tree
- ▶ Edge is *eligible* if not both endpoints marked (crossing!)
- ▶ Algorithm based on previous lemma, cut: un/marked
  1. mark 0 and add all eligible edges in  $\text{adj}[0]$  to  $pq$
  2. get and delete minimum edge  $e$  from  $pq$
  3. add  $e$  to  $mst$
  4. mark the unmarked endpoint of  $e$ , say  $k$ , and add all eligible edges in  $\text{adj}[k]$  to  $pq$
  5. delete ineligible minimum edges from  $pq$  and get new eligible minimum edge  $e$  from  $pq$
  6. continue at point 3 until  $pq$  is empty

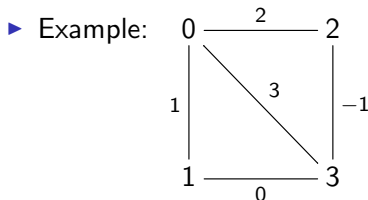
## Prim's Algorithm (ctnd)

- ▶ **LazyPrimMST.java**, methods `scan()` and `prim()`
- ▶ Invariant: at least one of the nodes of an edge in `pq` is marked
- ▶ NOT: all edges in `pq` are crossing edges wrt cut un/marked
- ▶ Lazy: ineligible edges are not eagerly deleted from `pq`
- ▶ Runtime: LazyPrimMST runs in  $O(E \log E)$  time (worst-case)
- ▶ Possible: edges in `pq` the crossing edges wrt cut un/marked
- ▶ Better: if  $v$  unmarked, the only crossing edge of interest is the *lightest* one connecting  $v$  to the marked edges (= MST so far)
- ▶ Runtime: **PrimMST.java** runs in  $O(E \log V)$  time (worst-case)

## Prim's Eager Algorithm

► Datastructures:

- EWG represented with adjacency lists `adj[v]`
- Boolean array `marked[v]` for marking vertices
- Array `distTo[v]`, minimum distances to MST so far
- Array `edgeTo[v]`, edges with minimum distance to MST so far
- Indexed minimum priority queue `pq`: `index=v`, `key=distTo[v]`
- Queue `mst` for the minimum spanning tree



- **PrimMST.java**, methods `scan()` and `prim()`

## Kruskal's Algorithm

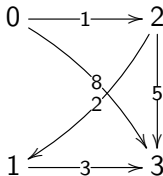
- ▶ Datastructures:
  - ▶ EWG represented with adjacency lists  $\text{adj}[v]$
  - ▶ Minimum priority queue  $\text{pq}$  for edges
  - ▶ Union-Find object  $\text{uf}$  testing connectivity
  - ▶ Queue  $\text{mst}$  for the minimum spanning tree
- ▶ Algorithm:
  1. delete the minimum edge  $e$  from  $\text{pq}$
  2. if the points connected by  $e$  are not connected, add  $e$  to  $\text{mst}$  and connect the points in  $\text{uf}$
  3. continue at point 1 until  $\text{pq}$  is empty or  $\text{uf}$  contains all nodes
- ▶ Examples: EWG on previous slide, `tinyEWG.txt`
- ▶ Correctness: same lemma about minimum-weight crossing edge of cut
- ▶ Implementation: `KruskalMST.java`, constructor method

## Memory-Use and Run-time Analysis

- ▶ Space, worst-case:
  - ▶ All methods use  $O(V + E)$  space for the graph, plus ...
  - ▶ Priority queue for edges (Lazy Prim and Kruskal):  $O(E)$  space
  - ▶ Priority queue for vertices (Eager Prim):  $O(V)$  space
  - ▶ Arrays indexed by vertices (all):  $O(V)$  space
- ▶ Time, worst-case:
  - ▶ Priority queue operations (Lazy Prim and Kruskal):  $O(E \log E)$  time
  - ▶ Priority queue operations (Eager Prim):  $O(E \log V)$  time

## Shortest paths

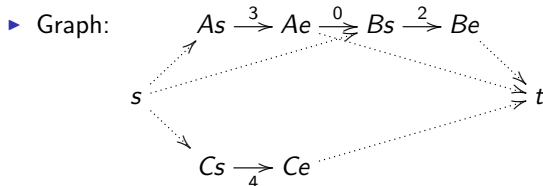
- ▶ Recall slide 63, *Edge-weighted digraphs*: digraphs in which every arrow has a weight
- ▶ *Weight* of a (di)path: sum of weights of the arrows
- ▶ *Shortest* path from node  $s$  to node  $t$ : minimum path weight
- ▶ EWD = Edge-Weighted Digraph, example: `tinyEWD.txt`
- ▶ Example: two SPs from 0 to 3:



- ▶ Shortest paths need not be unique, even if all weights are different!
- ▶ Shortest paths need not exist, for two independent reasons:
  - ▶ If  $t$  is not reachable from  $s$
  - ▶ If there is a negative cycle on the path to  $t$ , e.g.,  $1 \xrightarrow{-1} 1$

## Variations

- ▶ Single-source versus multiple sources
- ▶ Only non-negative weights versus all weights allowed
- ▶ Acyclic versus cycles, in particular negative cycles
- ▶ Important example: (parallel) scheduling of jobs A, B, and C
  - ▶ A (3 hrs), must be preceded by B (2 hr), independent C (4 hrs)



- ▶ Makespan, longest path, shortest negative path: **tinyJob.txt**
- ▶ Now add: A must start less than 2 hrs before B starts
- ▶ Negative cycle: no valid schedule!

## Dijkstra's Algorithm

- ▶ Single-source, only non-negative weights, cycles no problem
- ▶ Datastructures:
  - ▶ EWD with adjacency lists `adj[v]` of weighted out-edges
  - ▶ Boolean array `marked[v]` for marking vertices
  - ▶ Array `distToSource[v]`, minimum distances to source so far
  - ▶ Array `pathToSource[v]`, best arrow to source so far
- ▶ Algorithm: proceed with the unmarked node with least distance, until all marked; Simple example: slide 86
- ▶ Invariant:
  - ▶ Marked nodes: known shortest path to `s` (non-negativity!)
  - ▶ Unmarked nodes: known shortest path to `s` THROUGH marked nodes if such path exists
- ▶ Implementation `LinkedListEWD.slowEWD()`, examples `tinyJob.txt` and `tinyEWD.txt`



## Bellman-Ford

- ▶ Single-source, all weights, negative cycles detected
- ▶ Datastructures:
  - ▶ EWD with adjacency lists `adj[v]` of weighted out-edges
  - ▶ Array `distToSource[v]`, minimum distances to source so far
  - ▶ (Array `pathToSource[v]`, best arrow to source so far)
- ▶ Algorithm: do at most  $V$  rounds for every node  $v$  and every arrow  $e$  in `adj[v]`, if  $e$  shortens the distance to its endpoint  $w$ , update that distance (and path); stop after a round when no distances improve. If distances improve in the  $V$ -th round, a negative cycle is reachable from the source.
- ▶ Invariant: after  $n$  rounds the distances are less than or equal to the shortest path of length  $n$  from the source
- ▶ Implementation `LinkedListEWD.simpleBF()`, examples `tinyJob.txt` and `tiNoJob.txt`

## Memory-Use and Run-time Analysis

- ▶ Space, worst-case:
  - ▶ All methods use  $O(V + E)$  for the graph, plus  $O(V)$  extra
  - ▶ Still true for Dijkstra improved with an indexed priority queue
- ▶ Time, worst-case:
  - ▶  $V$  times finding minimum (original Dijkstra):  $O(V^2)$
  - ▶ Priority queue operations (improved Dijkstra):  $O(E \log V)$
  - ▶  $V$  rounds relaxing  $E$  edges (Bellman-Ford):  $O(EV)$

## Odds and Ends Chapter 4

- ▶ `StringSTG.java`, flight connections, shortest path = minimum number of stop-overs
- ▶ Degrees of separation in social networks
- ▶ Transitive closure

## ToC and topics of general interest

- ▶ Table of Contents on next slide (all items clickable)
- ▶ Practical stuff: slide 2

Introduction

Ch.1.3 Bags, Queues and Stacks

Ch.1.4 Analysis of Algorithms

Ch.1.5 Case Study: Union-Find

Ch.2.1 Elementary Sorts

Ch.2.2 Mergesort

Ch.2.3 Quicksort

Ch.2.4 Priority Queues

Ch.2.5 Applications

Ch.3.1 Symbol Tables

Ch.3.2 Binary Search Trees

Ch.3.3 Balanced Search Trees

Ch.3.4 Hash Tables

Ch.3.5 Applications of Searching

Odds and Ends Chapter 1–3

Ch.4.1 Undirected Graphs

Ch.4.2 Directed Graphs

Ch.4.3 Minimum Spanning Tree

Ch.4.4 Shortest Paths

Odds and Ends Chapter 4

Table of Contents