

INF102

Algorithms, Data Structures and Programming

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INF102, practical stuff

- ▶ Lecturer: Marc Bezem; Team: see homepage
- ▶ Homepage: [INF102](#) (hyperlinks in red)
- ▶ Also: [GitHub](#) (recommended); Dropbox: [slides](#), [schedule](#)
- ▶ Textbook: [Algorithms, 4th edition](#)
- ▶ Prerequisites: INF100 + 101 (\approx Ch. 1.1 + 1.2)
- ▶ Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- ▶ Exam: three compulsory exercises and a [written exam](#)
- ▶ Old exams: [2004–2013](#), [2014](#)
- ▶ [Table of Contents of these slides](#)

Resources

- ▶ Good textbook, USA-style: many pages, exercises etc.
- ▶ Average speed must be ca 50 pages p/w
- ▶ Lectures (ca 24) focus on the essentials
- ▶ Slides (ca 120, dense!) summarize the lectures
- ▶ Prepare yourself by reading in advance
- ▶ Workshops: selected exercises
- ▶ Test yourself by trying some exercises in advance
- ▶ If you can do the exercises (incl. compulsory), you are fine
- ▶ Review of exercises on Friday morning

Generic Bags, Queues and Stacks

- ▶ Generic programming in Java, example: **PolyPair.java**
- ▶ Bag, Queue and Stack are generic, iterable collections
- ▶ Queue and Stack: Ch. 9 in textbook INF100/1
- ▶ APIs include: `boolean isEmpty()` and `int size()`
- ▶ All three support adding an element
- ▶ Queue and Stack support removing an element (if any)
- ▶ FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- ▶ Dijkstra's Two-Stack Expression Evaluation **Movie**
- ▶ Example: $(1 + ((2 + 3) * (4 * 5)))$

Implementations

- ▶ `ResizingArray_Stack.java`
- ▶ Arrays give direct access, but have fixed size
- ▶ Resizing takes time and space proportional to size
- ▶ `LinkedList_Stack.java`
- ▶ No fixed size, but indirect access
- ▶ Pointers take space and dereferencing takes time
- ▶ Programming with pointers: make a picture
- ▶ `LinkedList_Queue.java`

Computation time and memory space

- ▶ Two central questions:
 - ▶ How long will my program take?
 - ▶ Will there be enough memory?
- ▶ Example: **ThreeSum.java**
- ▶ Inner loop (here $a[i] + a[j] + a[k] == 0$) is important
- ▶ Sorting helps: **ThreeSumOptimized.java**
- ▶ Run some experiments: `1Kints.txt`, `2Kints.txt`, ...

Methods of Analysis

- ▶ Empirical:
 - ▶ Run program with randomized inputs, measuring time & space
 - ▶ Run program repeatedly, doubling the input size
 - ▶ Measuring time: **StopWatch**
 - ▶ Plot, or log-log plot and **linear regression**
- ▶ Theoretical:
 - ▶ Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
 - ▶ Try to count/estimate/average this cost as function of the input (size)
 - ▶ Use $O(f(n))$ and $f(n) \sim g(n)$

ThreeSum, empirically

- ▶ Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4 ,51.1 sec
- ▶ The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- ▶ Basis of the logarithm should be the same for both
- ▶ Linear regression gives $y \approx 3x - 10$
- ▶ $\log(f(n)) = 3 \log(n) - 10$ iff

$$f(n) = 10^{\log(f(n))} = 10^{3 \log(n) - 10} = n^3 * 10^{-10}$$

- ▶ Conclusion: cubic in the input size, with constant $\approx 10^{-10}$
- ▶ Strong dependence on input can be a problem
- ▶ Constant 10^{-10} depends on computer, exponent 3 does not

ThreeSum, theoretically

- ▶ Number of different picks of triples: $g(n) = n(n-1)(n-2)/6$
- ▶ Inner loop $a[i] + a[j] + a[k] == 0$ executed $g(n)$ times
- ▶ $g(n) = n^3/6 - n^2/2 + n/3$
- ▶ Cubic term $n^3/6$ wins for large n
- ▶ Computational model # array accesses: $3 * n^3/6 = n^3/2$
- ▶ Cost array access t sec: time $t * n^3/2$ sec
- ▶ Cost models are abstractions! (NB cache)

Big Oh, and \sim

- ▶ Q: 'wins for large n ' uhh???
- ▶ A: Big Oh, and \sim will clear this up
- ▶ Costs are positive quantities, so $f, g, \dots : \mathbb{N} \rightarrow \mathbb{R}^+$
- ▶ MNF130: $f(n)$ is $O(g(n))$ if there exist $c \in \mathbb{R}^+$, $N \in \mathbb{N}$ such that $f(n) \leq cg(n)$ for all $n \geq N$ (that is,, for n large enough)
- ▶ Example: n^2 and even $99n^3$ are $O(n^3)$, but n^3 is not $O(n^{2.9})$
- ▶ INF102: $f(n) \sim g(n)$ if $1 = \lim f(n)/g(n)$
- ▶ If $f(n) \sim g(n)$, then $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$
- ▶ Big Oh and \sim aim to capture 'order of growth'
- ▶ Big Oh abstracts from constant factors, \sim does not
- ▶ Large constant factors are important!

Important orders of growth

- ▶ constant: c , $f(n) = c$ for all n
- ▶ linear: n (compare all for $n = 20$ sec)
- ▶ linearithmetic: $n \log n$
- ▶ quadratic: n^2
- ▶ cubic: n^3
- ▶ exponential: 2^n
- ▶ general form: $an^b(\log n)^c$

Logarithms and Exponents

- ▶ Definition: $\log_x z = y$ iff $x^y = z$ for $x > 0$
- ▶ Inverses: $x^{\log_x y} = y$ and $\log_x x^y = y$
- ▶ Exponent: $x^{(y+z)} = x^y x^z$, $x^{(yz)} = (x^y)^z$
- ▶ Logarithm: $\log_x(yz) = \log_x y + \log_x z$, $\log_x z = \log_x y \log_y z$
- ▶ Base of logarithm: the x in \log_x
- ▶ Various bases: $\log_2 = \lg$, $\log_e = \ln$, $\log_{10} = \log$
- ▶ Double exponent: e.g. $2^{(2^n)}$ (not used in INF102)
- ▶ Double logarithm: $\log(\log n)$ (not used in INF102)

Worst case, average case, amortized cost

- ▶ Worst case: guaranteed, independent of input; Examples:
 - ▶ Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
 - ▶ Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- ▶ Average case: not guaranteed, dependent of input *distribution*
- ▶ Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost ≤ 21 , all others cost 1, amortized ≤ 3 p/o.
- ▶ Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- ▶ Special case of resizing array that is only growing:
 $1(2)2(4)3(8)4(16)5(32)6(64)7(128)8(256)9(512) \dots 16(32768) \dots$, with (n) the new size.
 Resizing to (n) costs $2n$ array accesses, so in total
 $(1+4)+(1+8)+(2+16)+(4+32)+(8+64) \dots$, so 9 p/push.

Staying Connected

- ▶ We want efficient algorithms and datastructures for testing whether two objects are 'connected'
- ▶ MNF130: relation $E \subseteq V \times V$ is an *equivalence* if
 - ▶ E is *reflexive*: $\forall x \in V. E(x, x)$
 - ▶ E is *symmetric*: $\forall x, y \in V. E(x, y) \rightarrow E(y, x)$
 - ▶ E is *transitive*: $\forall x, y, z \in V. E(x, y) \wedge E(y, z) \rightarrow E(x, z)$
- ▶ We assume connectedness to be an equivalence
- ▶ Dynamic connectivity means (here) that E can grow
- ▶ Clear relationship with paths in graphs, (connected) components (MNF130)
- ▶ Input: N and pairs in $V = \{0, \dots, N-1\}$ defining E
- ▶ Challenge: efficient `boolean connected(int p, int q)`
- ▶ Example: $N = 10$, 4 3, 3 8, ... (`algs4-data/tinyUG.txt`)
- ▶ Picture on blackboard (don't print pairs that are already connected)

Union-Find

- ▶ Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- ▶ Union, idea: for any new pair $n\ m$ that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- ▶ API: **UF**; Cost model: number of array accesses
- ▶ Implementations:
 - ▶ **SlowUF.java**: `id[p]` identifier of p
`find()` ~ 1 , `union()` \sim between $n+3$ and $2n+1$
 - ▶ **FastUF.java**: `int[] id` pointers, `id[p]==p`: identifier
`find()` $\sim 1+2d$, `union()` $\sim 1 + \text{two find}()$'s
 - ▶ **WeightedUF.java**: `int[] id` pointers, `int[] sz` subtree sizes
`find()` and `union()` both $\sim \lg n$
- ▶ WeightedUF: height of subtree of size k is at most $\lg k$
- ▶ Path-compression: ultimate improvement of UF (almost $O(1)$, amortized)

Sorting

- ▶ Sorting: putting objects in a certain order
- ▶ MNF130: relation $R \subseteq V \times V$ is a *total order(ing)* if
 1. R is *reflexive*: $\forall x \in V. R(x, x)$
 2. R is *transitive*: $\forall x, y, z \in V. R(x, y) \wedge R(y, z) \rightarrow R(x, z)$
 3. R is *antisymmetric*: $\forall x, y \in V. R(x, y) \wedge R(y, x) \rightarrow x = y$
 4. R is *total*: $\forall x, y \in V. R(x, y) \vee R(y, x)$
- ▶ Natural orderings:
 - ▶ Numbers of any type: ordinary \leq and \geq
 - ▶ Strings: lexicographic
 - ▶ Objects of a Comparable type: `v.compareTo(w) <= 0`

Sorting (ctnd)

- ▶ Bubble sort: `ExampleSort.java`
- ▶ Certification: `assert isSorted(a)` in `main()`
- ▶ No guarantee against modifying the array (but `exch()` is safe)
- ▶ Costmodel 1: number of `exch()`'s and `less()`'s
- ▶ Costmodel 2: number of array accesses
- ▶ Pitfalls: cache misses, expensive `v.compareTo(w) < 0`
- ▶ Why studying sorting? (`java.util.Arrays.sort()`)
- ▶ Comparing sorting algorithms: `SortCompare.java`

Selection Sort

- ▶ Bubble sort: $\sim n^2/2$ compares, $0 \leq \text{exchanges} \leq \sim n^2/2$
- ▶ Selection sort:
 - ▶ Find index of a minimum in $a[0..n-1]$, exchange with $a[0]$
 - ▶ Find index of a minimum in $a[1..n-1]$, exchange with $a[1]$
 - ▶ ... until $n-2$
- ▶ Selection sort: $\sim n^2/2$ compares, $0 \leq \text{exchanges} \leq n-1$ (!)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=0; i<N-1; i++){  
        int min=i;  
        for (int j=i+1; j<N; j++) if (less(a[j],a[min])) min=j;  
        if (i != min) exch(a,i,min);  
    }  
}
```

Insertion sort

- ▶ Insertion sort:
 - ▶ Insert $a[1]$ on its correct place in (sorted) $a[0..0]$
 - ▶ Insert $a[2]$ on its correct place in (sorted) $a[0..1]$
 - ▶ ... until $a[n-1]$
- ▶ Very good for partially sorted arrays, costs:
 - ▶ Best case: $n-1$ compares and 0 exchanges
 - ▶ Worst case: $\sim n^2/2$ compares and exchanges
 - ▶ Average case: $\sim n^2/4$ compares and exchanges (distinct keys)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=1; i<N; i++){  
        for (int j=i; j>0 && less(a[j],a[j-1]); j--)  
            exch(a,j,j-1);  
    }  
}
```

Shell sort

- ▶ Insertion sort:
 - ▶ Very good for partially sorted arrays
 - ▶ Slow in transport: step by step `exch(a,j,j-1)`
- ▶ Idea: h-sort, `a[i], a[i+h], a[i+2h], ...` sorted (any `i`)

```
public static void hsort(int h, Comparable[] a) {  
    int N = a.length;  
    for (int i=h; i<N; i++)  
        for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)  
            exch(a,j,j-h);  
}
```

- ▶ Insertion sort: `hsort(1,a)`
- ▶ Shell sort: e.g., `hsort(10,a); hsort(1,a)`

Shell sort (ctnd)

- ▶ `hsort(10,a); hsort(1,a)` faster than just `hsort(1,a)` !
- ▶ Q: How is this possible?
- ▶ A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- ▶ What about `hsort(100,a); hsort(10,a); hsort(1,a)`?
- ▶ To be expected: depends on the length N of the array
- ▶ Best practice: $h = N/3, N/9, \dots, 364, 121, 40, 13, 4, 1$

Mergesort

- ▶ Top-down (recursive) algorithm:
 - ▶ Mergesort left half, mergesort right half
 - ▶ Merge the results
- ▶ Using an auxiliary array: [TopDownMergeSort.java](#), [Movie](#)
- ▶ Bottom-up algorithm (16 elements):
 - ▶ Merge $a[0], a[1]$, so $a[2], a[3]$, so $a[4], a[5]$, so ...
 - ▶ Merge $a[0..1], a[2..3]$, so $a[4..5], a[6..7]$, so ...
 - ▶ Merge $a[0..3], a[4..7]$, so $a[8..11], a[12..15]$
 - ▶ Merge $a[0..7], a[8..15]$, done!
- ▶ Also using an auxiliary array: [BottomUpMergeSort.java](#)

Run-time and memory use of mergesort

- ▶ Mergesort uses between $\sim (N/2) \lg N$ and $\sim N \lg N$ compares. Proof on bb. Important formula ($N = 2^n$):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- ▶ Mergesort uses at most $\sim 6N \lg N$ array accesses
- ▶ Mergesort uses $\sim 2N$ space (plus some var's)
- ▶ Q: How fast can compare-based sorting of N distinct keys be?
- ▶ A: $\lg N! \sim N \lg N$; Proof in book and on bb. Keywords: binary *compare tree*, inner nodes for each `compare(a[i], a[j])`, permutations in the leaves,
 $N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$

Quicksort

- ▶ Top-down (recursive) algorithm:
 - ▶ Choose a (pivot) value v in the array
 - ▶ Partition the array in non-empty parts $\leq v$ and $\geq v$
 - ▶ Quicksort the two parts
- ▶ Pros: in-place, average computation time $O(n \log n)$
- ▶ Cons: stack space for recursion, worst-case $O(n^2)$, not stable
- ▶ Implementation: **QuickSort.java**
- ▶ BTW: **Bug in java.util.Arrays.sort**

Quicksort, details

- ▶ Subtleties in `sort()`: shuffling protects against worst-case behaviour
- ▶ Termination of recursive `quicksort()`
- ▶ Subtleties in `partition()`:
 - ▶ Invariants $l \leq h$ in the two inner loops
 - ▶ Postcondition after the two inner loops
 - ▶ Invariant of the `for(;;)` loop
 - ▶ Termination of the `for(;;)` loop
 - ▶ There are some variations that are also correct

Run-time and memory use of quicksort

- ▶ Compare Quicksort to other sorts ($n = 10^2, 10^3, \dots$)
- ▶ Quicksort: time $O(n^2)$ if pivot is always smallest (or largest)
- ▶ Randomization: choose pivot randomly, or shuffle array
- ▶ If all keys are distinct and randomization is perfect, then quicksort uses on average $\sim 2n \ln n$ compares and $\sim (n/3) \ln n$ exchanges (proofs in book, complicated)
- ▶ Relevant improvements:
 - ▶ Cut-off to insertion sort for sizes ≤ 15 (ca.)
 - ▶ Median-of-three pivot
 - ▶ Taking advantage of duplicate keys (3-way partitioning)
- ▶ Quicksort is generally quite good
- ▶ In special situations other sorts are better (e.g., countsort)

Priority Queues

- ▶ Assume collecting and processing items having keys
- ▶ Examples of keys: time-stamp, price-tag, priority-tag
- ▶ Assume: keys can be ordered
- ▶ Reasonable: processing currently highest (or lowest)
- ▶ Special cases: items time-stamped when added
 - ▶ Queue: dequeue currently oldest (lowest time-stamp)
 - ▶ Stack: pop currently newest (highest time-stamp)
- ▶ Priority queue generalizes this
- ▶ Examples: highest priority, largest transaction, lowest price
- ▶ Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

Priority Queues

- ▶ Good info: [Wikipedia](#); API (the bare essentials):

```
public class  ArrayListPQ<Key extends Comparable<Key>>

void          insert(Key v) // insert a key
Key           delMax() // delete a largest key, if any
boolean       isEmpty()
int           size()
```

- ▶ Aim: operations in logarithmic time, no extra space
- ▶ In case of duplicate keys: 'a' largest, not 'the'
- ▶ Typical application: the 1K largest keys of 1G unsorted keys
- ▶ Client: [BottomM.java](#) (Q: why is the output slowing down?)

Binary Trees

- ▶ MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- ▶ MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height h has $2^{h+1}-1$ nodes.
- ▶ INF102: A binary tree is (left-) *complete* if all levels $< h$ are filled, level h may be partly empty from the right (picture bb). A (left-)complete binary tree of height h has between 2^h and $2^{h+1}-1$ nodes (from now on we leave out '(left-)').
- ▶ A complete binary tree of n nodes has height $\lfloor \lg n \rfloor$

Picture



Heap-ordered Binary Trees

- ▶ A binary tree is *heap-ordered* if the key in each node is \geq the keys in its children (if any). Thus the root has a maximal key.
- ▶ Array representation of heap-ordered complete binary tree (bb)
- ▶ Methods `swim()` and `sink()`: picture on bb, code below
- ▶ Implementation: `ArrayListPQ.java`

Run-time and memory use of heaps, applications

- ▶ In a heap of n elements (since height is $\leq \lfloor \lg n \rfloor$):
 - ▶ `swim()`, and hence `insert()`, takes $\leq 1 + \lfloor \lg n \rfloor$ compares and $\leq \lfloor \lg n \rfloor$ exchanges
 - ▶ `sink()`, and hence `delMax()`, takes $\leq 2\lfloor \lg n \rfloor$ compares
 - ▶ `sink()` takes $\leq \lfloor \lg n \rfloor$ exchanges, and `delMax()` $\leq 1 + \lfloor \lg n \rfloor$
- ▶ Heap construction by `insert()` can sometimes be improved
- ▶ Given an array of keys, right-to-left heap construction (bb) takes $< 2n$ compares and $< n$ exchanges
- ▶ Applications: **heapsort** and merging sorted streams (bb)
- ▶ Many variations with extended API (indexed priority queue)

Purpose of Sorting

- ▶ Sorting makes the following easier and more efficient:
 - ▶ Searching (binary search, example: `ThreeSumOptimized`)
 - ▶ Searching and looking up, e.g., the `pagenumber` in an index
 - ▶ Removing duplicates
 - ▶ Finding the median, quartiles etc.
- ▶ Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- ▶ We do *pointer sorting*, manipulating refs to objects.
 - ▶ Pro: not moving full objects
 - ▶ Cons: pointer dereferencing, no `sort(int[] a)`
- ▶ More flexibility: pass a `Comparator` object to `sort()`

Comparator object

- ▶ API: `void sort(Object[] a, Comparator c)`
- ▶ Call, e.g.: `sort(a, new Transaction.WhenOrder())`
- ▶ Call, e.g.: `sort(a, new Transaction.SizeOrder())`
- ▶ Obs: `import java.util.Comparator`
- ▶ Obs: `less(Object o1, Object o2, Comparator c)`
- ▶ Priority queues also with `Comparator`

```
public class Transaction {  
    ...  
    public static class MyOrder {  
        implements Comparator<Transaction>  
        public int compare(Transaction t, Transaction v){...}  
    } // End of Myorder  
    ...// similarly: WhenOrder, SizeOrder  
} // End of Transaction
```

Applications of Sorting

- ▶ Consider sorting first to make other problems easier
- ▶ Commercial computing (sort on price, departure time, ...)
- ▶ Search for information: web-indexing, search engines
- ▶ Job scheduling heuristic: longest processing time first
- ▶ To come: Prim's, Dijkstra's and Kruskal's algorithms
- ▶ Huffman compression: a lossless compression based on using the shortest codes for the symbols that occur often.
Frequency counter: next chapter!
- ▶ Cryptology and genomics (e.g., longest repeated substring)

Symbol Tables

- ▶ Symbol table associates *keys* with *values*: *key-value pairs*
- ▶ Examples: keyword-page number, ID number-personal data
- ▶ Important operations:
 - ▶ Insert a key-value pair in the symbol table: `void put(k,v)`
 - ▶ Search the value for a given key (if any): `Value get(k)`
- ▶ Important conventions:
 - ▶ Inserting key-value for existing key: overwriting the value
 - ▶ No duplicate keys, no null keys
 - ▶ Value null: no value for this key
 - ▶ Lazy deletion: insert key-null; Eager: really delete key
- ▶ **API** of unordered symbol table
- ▶ Aim: all operations in time $\sim c \lg n$ with constant c small

ST Basics

- ▶ Archetypical ST-client: frequency counter (code: `main`)
- ▶ Cost model: number of compares
- ▶ Naive ST: unordered linked list, linear search (INF101, Ch.9)
 - ▶ Search miss: $\sim n$ compares
 - ▶ Search hit: between 1 and $\sim n$ compares
 - ▶ Random search hit: $(1 + \dots + n)/n \sim n/2$ compares
 - ▶ Inserting n distinct keys: $(1 + \dots + (n-1)) \sim n^2/2$ compares
- ▶ `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- ▶ Naive ST impracticable for genomics, internet
- ▶ Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- ▶ Better for unordered ST: hashing (in Ch. 3.4)

Ordered Symbol Table

- ▶ Ordered ST: keys are ordered
- ▶ **API** of ordered symbol table
- ▶ Binary search: `get(Key k)` takes $\sim \lg n$ comparisons
- ▶ What about `put(Key k, Value v)`? **ArrayListST**, good!
- ▶ TODO: test that `add(int i, E e)` is amortized $O(1)$
- ▶ Implementation with binary search in **ArrayListST.java**
- ▶ Trace of inserts on bb: S E A R C H E X A M P L E
- ▶ Experiments with `tinyTale.txt`, `tale.txt`, ...

Binary Search Trees

- ▶ Binary *search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- ▶ Search time: length of the path to the node where the key 'should' be
- ▶ Balanced binary tree with n keys has $\lg n$ height
- ▶ Unbalanced binary trees can have height n (max depth)
- ▶ Search hits in a binary search tree, built without rebalancing, of n random keys take on average $\sim 2 \ln n$ compares
- ▶ **UBST.java**: `put()`, `get()`, `size()`, `isEmpty()`
- ▶ Trace of inserts on bb: S E A R C H E X A M P L E

Binary Search Trees (ctnd)

- ▶ Interrelated, increasing difficulty: `min(Node x)`, `deleteMin(Node x)`, `delete(Node x, Key k)`
- ▶ Node of minimum key: not `null`, and has left child `null`, and is root or left child of parent (picture on bb)

```
public Node min(Node x){// x != null, subtree not empty
    while (x.left!=null) x = x.left;
    return x;
} // cf. tail recursive min() in Alg. 3.3
```

- ▶ Delete minimum key, two cases:
(1) both children `null`; (2) left child `null`
- ▶ Delete is really difficult: bb + **BST.java**
- ▶ Don't forget: update `x.N` along the path to the root!

Balanced Search Trees: keep paths short!

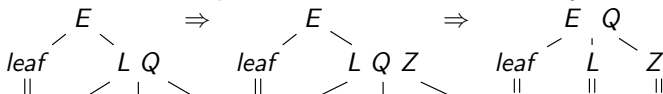
- ▶ NB tree balancing not as easy as in UF and Heap (4hrs!)
- ▶ A 2-3 search tree consists of 2-nodes and 3-nodes:
 - ▶ Each 2-node has two children and a key k such that all keys in the left subtree are $< k$, and all keys in the right subtree $> k$
 - ▶ Each 3-node has three children and two keys k_1, k_2 such that all keys in the left subtree are $< k_1$, all keys in the middle subtree $> k_1$ and $< k_2$, and all keys in the right subtree $> k_2$
- ▶ Examples and pictures on bb
- ▶ *Perfect* 2-3 search tree: paths from root to leaves equally long
- ▶ Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- ▶ Insert should keep tree perfect, rough idea:
 - ▶ into a 2-leaf: make it into a 3-leaf
 - ▶ into a 3-node: do something clever (explained next)

Insert in Balanced Search Trees

- ▶ Terminology: a *leaf* is a node all whose children are null
- ▶ Data invariant 1: 2-3 search tree
- ▶ Data invariant 2: paths from root to leaves equally long

- ▶ Insert into a 2-leaf L : either $\begin{array}{c} A \ L \\ \diagup \ \diagdown \\ \parallel \end{array}$ or $\begin{array}{c} L \ Z \\ \diagup \ \diagdown \\ \parallel \end{array}$

- ▶ into a 3-leaf whose parent is a 2-node: with new key Z



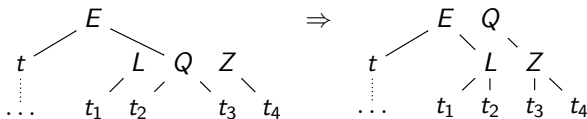
- ▶ into a 3-leaf whose parent is a 3-node: with new key Z added



- ▶ into a 3-node whose parent is a 3-node: move up middle key!

Insert (ctnd)

- ▶ Data invariant 1: 2-3 search tree
- ▶ Data invariant 2: paths from root to leaves equally long
- ▶ Insert works up from the leaf where the key 'should' be
 - ▶ if 2-node on path to root: make it into a 3-node (two cases)



- ▶ otherwise: split the root



Insert, summary and examples

- ▶ Six operations for eliminating 4-nodes:
 - ▶ if parent is 2-node: move middle key up (left and right case)
 - ▶ if parent is 3-node: move middle key up (left, middle, right)
 - ▶ if root: split root
- ▶ Search and insert visit at most $\lfloor \lg n \rfloor$ nodes
- ▶ Proof: maximal path length is $\geq \lfloor \log_3 n \rfloor$ and $\leq \lfloor \log_2 n \rfloor$
- ▶ Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- ▶ Trace of inserts on bb: A C E H L M P R S X

Red-black trees

- ▶ Red-black trees implement 2-3 trees
- ▶ Idea: one 3-node = two 2-nodes + extra info
- ▶ Extra info coded in color, picture:




- ▶ A *red-black tree* is a binary search tree with red and black links such that:
 - ▶ Only left links can be red (but need not be)
 - ▶ Never
 - ▶ Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- ▶ Equivalent: red-black tree and perfect 2-3 search tree

Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

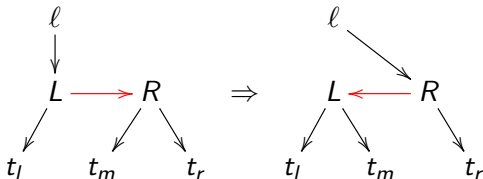
```
private class Node {  
    Key key;  
    Value value;  
    Node left, right;  
    boolean color; // true for red, false for black  
    int N;  
}  
private boolean isRed(Node n) {  
    if (n==null) {return false;} else {return x.color}
```

Rotating and Color Flipping

- ▶ Aim: restoring the data invariants of red-black search trees
 1. Only left red links, but never two
 2. Search tree invariant
 3. Perfect black balance
- ▶ Invariants get violated by temporary 4-nodes, e.g.,
 - ▶ inserting Z in $L \leftarrow R : L \leftarrow R \rightarrow Z$
 - ▶ inserting A in $L \leftarrow R : A \leftarrow L \leftarrow R$
 - ▶ inserting M in $L \leftarrow R : L \leftarrow R$

- ▶ Restoring:
 - ▶ Color flip $L \leftarrow R \rightarrow Z : L \leftarrow R \rightarrow Z$
 - ▶ Rotation right + color flip $A \leftarrow L \leftarrow R : A \leftarrow L \rightarrow R$
 - ▶ Rotation left into $M \leftarrow L \leftarrow R$, then as previous

Left Rotation

Call: `l = rotateLeft(l);`



```
private Node rotateLeft(Node l){
    Node r = l.right; l.right = r.left; r.left = l;
    r.color = l.color; l.color = true // == RED
    r.N = l.N; l.N -= 1+size(r.right); // Why?
    return r;
}
```


Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- ▶ Code of `rotateRight()` like that of `rotateLeft()`
- ▶ NB1: operations are local (here only r , M , R)
- ▶ NB2: operations preserve data invariants
- ▶ NB3: root is a special case (always black)
- ▶ Deletions: complicated, but doable (Exc. 3.3.39–41)

Run-time and memory use of Red-Black BSTs

- ▶ The height of a red-black BST with n nodes is $\leq 2 \lg n$
Proof: the worst-case is one 3-node path and the rest 2-nodes
- ▶ The average length of path from the root to a *node* (?) in a red-black BST with n nodes is $\lg n$ ('empirical fact')
- ▶ In a red-black BST, search, insert, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node.
- ▶ For red-black BSTs, logarithmic time is guaranteed!

Hashing

- ▶ Idea: if keys in $[0..99]$ an array is the perfect symbol table
- ▶ A *hash function* maps a key to an array index
- ▶ Injectivity of the hash function is not guaranteed
- ▶ *Hash collision*: different keys are mapped to the same index
- ▶ In such a case we need *collision resolution*
- ▶ Symbol tables: hashing is fast, but unordered (no `max`, `min`)
- ▶ Aim: ST operations amortized $O(1)$ time, extra space OK

Space-Time Trade-Off

- ▶ Hashing is an example of a *space-time trade-off*
- ▶ Time: computation time required
- ▶ Space: memory space used
- ▶ Unlimited space: (1) use key as index (e.g., the bits)
- ▶ Unlimited time: (2) use linked list and linear search
- ▶ Hashing strikes a balance using (1) with some array of reasonable size, and (2) in case of collisions
- ▶ The balance between (1) and (2) can easily be tuned

Hash functions

- ▶ Ideal (uniform hashing assumption, UHA): uniform and independent distribution of keys over integers from 0 to $M - 1$
- ▶ Examples of **hash functions in Java**
- ▶ Horner: $a_0 + x(a_1 + x(a_2 + \dots)) = a_0 + a_1x + a_2x^2 + \dots$
- ▶ Reasonably \approx UHA: modular hashing (M prime):

```
private int hash(Key k){  
    return (key.hashCode() & 0x7fffffff) % M;}
```
- ▶ Q: Why M prime?
- ▶ A: e.g. $M = 32$ takes only into account the last five bits

Collision Resolution

- ▶ Two methods of collision resolution:
 1. Hashing with separate chaining (picture on bb)
 2. Hashing with linear probing (picture on bb)
- ▶ Separate chaining: symbol table is an array of linked lists, linear search. If array has length M , then the linked lists have average length N/M with N keys.
- ▶ Linear probing: symbol table is an array of length $M > N$. Colliding keys are put at the first empty position. Linear search from the position where the key 'should have been'. Empty position: not found. Deletion tricky: reinsert all keys to the right of the deleted key, until the first empty position (picture on bb). Array must have length $\geq N$ with N keys.

Symbol Table with Hashing

- ▶ Implementation: `ArrayListHashST.java`
- ▶ $M = 1$: measure overhead wrt. `ArrayListST.java`
- ▶ Tests with various values of M : 31, 997, 65521
- ▶ NB: *construction* versus *use* of ST (hashing better for *use*)
- ▶ Hashing can be combined with any other ST-implementation

Quantitative analysis

- ▶ Throwing a dice 10 times, what is the probability of 3 fives?
- ▶ Under UHA, with N distinct keys, the probability that exactly k keys collide at some given hash value is

$$\binom{N}{k} \left(\frac{1}{M}\right)^k \left(\frac{M-1}{M}\right)^{N-k}, \text{ where e.g. } \binom{100}{10} \approx 1.7E13$$

- ▶ This is a small number for, say, $N = M = 100$ and $k = 10$
- ▶ For linear probing one typically takes $M = 2N$
- ▶ For separate chaining one keeps $N/8 \leq M \leq N/2$ (resizing M)
- ▶ Under UHA: search, insert, delete take amortized $O(1)$ time
- ▶ Extra space used can be upto $100N$ byte (objects, pointers); this on top of the space used by N key-value pairs

Applications of Searching

- ▶ Synonyms: **associative array**, map, symbol table, or dictionary
- ▶ Origin of **symbol table**: compilers and interpreters
- ▶ Web-indexing, **search engines**
- ▶ Sparse matrices (many 0's): **dictionary**
 1. keys: (row, column)-pairs
 2. values: matrix entries
- ▶ Set API (no values, only keys, for deduplication, filtering):

```
public class SET<Key>
```

```
void      add(Key k)
void      delete(Key k)
boolean   contains(Key k)
boolean   isEmpty()
int       size()
```

Applications of Searching (ctnd)

- ▶ Application (key, value)
- ▶ Phone book (name, phone number)
- ▶ Dictionary (word, meaning or translation)
- ▶ Account information (client ID, account information)
- ▶ Genomics (sequences of ACTG triplets, proteins)
- ▶ Experimental data of various kinds
- ▶ File systems (file name, address etc)
- ▶ Internet domain name system (domain name, IP address)
- ▶ Invertex index (value, key(s))

Balanced Search Tree or Hash Table?

- ▶ Q: Which symbol table to use?
- ▶ A: The basic choice between BST and HT depends on ...
 1. Ordering of keys essential: BST
 2. Availability of good hash function (good = fast + UHA)
 3. Ordering of keys expensive (long strings): HT (or: Ch.5)
 4. Ordering of keys possible, but not essential: HT + BST
 5. Space considerations (ArrayListST uses the least extra space)
 6. Number of distinct keys and the space each key takes
 7. Distribution of insert/delete/search operations

Odds and Ends Chapter 1–3

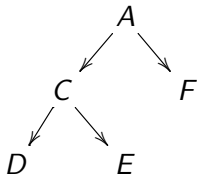
- ▶ Discuss methodological issues SortCompare
- ▶ Find out if `add(int i, E e)` in `ArrayList` is amortized $O(1)$
- ▶ Discuss primitive types (objects are costly)
- ▶ **Distributed Hash Table**
- ▶ Double hashing: linear probing
 $h_1(k), h_1(k) + h_2(k), h_1(k) + 2h_2(k), \dots$
- ▶ Indexed Priority Queues: next slide

Indexed Priority Queues

- ▶ IPQ \approx array with direct access to minimum (maximum)
- ▶ API: `void insert(int i, Key k); void del(int i); int minKey();`
`Key keyOf(int i);...` Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-
qp	1	0	2	4	3	-1	-1

heap



Do: `insert(6,G)`, `insert(5,B)`

Graph classes

(MNF130: useful review of graph theory)

1. Undirected graphs: a set of *vertices* (or *nodes*) V and a set of *edges* E connecting the nodes
2. Directed graphs (*digraphs*): a set of nodes V and a set E of edges (or *arrows*) pointing from one node to another
3. *Edge-weighted graphs*: undirected graphs in which every edge has a number called the *weight* of the edge
4. *Edge-weighted digraphs*: digraphs in which every arrow has a weight

Examples

1. Undirected graphs: social networks, communication networks (duplex communication), road maps
2. Directed graphs: logical circuits, job scheduling, flow graphs, hyperlinks, (class, module, package) dependencies
3. Edge-weighted graphs: roadmaps with geographical distance, communication networks with bandwidth
4. Edge-weighted digraphs: job scheduling with duration, transport of goods, financial transactions

Undirected Graphs

- ▶ Undirected graph: a set of *vertices* (or *nodes*) V and a set of *edges* E connecting the nodes
- ▶ *Subgraph*: subset of E and subset of V forming a graph (!)
- ▶ *Path*: sequence of nodes connected by edges (!)
- ▶ *Simple path*: path with no node repeated
- ▶ *Length of path*: number of edges
- ▶ *Cycle*: path of length > 0 with same start and end node
- ▶ *Simple cycle*: not repeating edges or nodes (apart from start and end node)
- ▶ *Acyclic graph*: graph without simple cycles
- ▶ *Connected graph*: having a path between every two nodes
- ▶ *Connected component*: a maximal connected subgraph

Trees and Forests

- ▶ 'Anomalies' concerning edges:
 - ▶ Self-loop: edge connecting a node to itself
 - ▶ Parallel edges: two edges connecting the same node(s)
- ▶ When no anomalies, $E \subseteq \{\{v, v'\} \mid v \in V, v' \in V, v \neq v'\}$
- ▶ *Tree*: connected acyclic graph (then: no anomalies)
- ▶ *Spanning tree*: maximal subgraph that is a tree
- ▶ Lemma: any spanning tree of a connected graph contains all nodes. Proof by contradiction (on bb).
- ▶ *Forest*: graph consisting of disjoint trees
- ▶ *Spanning Forest*: forest consisting of spanning trees of connected components of a graph
- ▶ Example: `tinyG.txt` on bb

Undirected Graphs (ctnd)

- ▶ *Distance* between two nodes: length of a shortest connecting path if there is a path connecting these nodes, otherwise ∞
- ▶ *Degree* of a node: number of edges connected to that node
- ▶ Graph $G = (V, E)$, the following are equivalent:
 - ▶ G is a tree
 - ▶ G has $|V| - 1$ edges and no cycles
 - ▶ G has $|V| - 1$ edges and is connected
 - ▶ G is acyclic and adding an edges creates a cycle
 - ▶ Any two nodes of G are connected by exactly one simple path
- ▶ Example: some subgraphs of `tinyG.txt` on bb

Graph representation and implementation

- ▶ Impractical: **adjacency matrix** $\sim V^2$, **incidence matrix** $\sim VE$
- ▶ Often practical: **adjacency lists** $\sim (V+2E)$, that is, `adj[v]` lists all nodes `w` connected to `v` by an edge
- ▶ Example: `tinyG.txt` by **LinkedListG.java**
- ▶ Graph API includes: `V()`, `E()`, `addEdge()`
- ▶ Basic algorithms: Depth-First Search (DFS) and Breadth-First Search (BFS)
- ▶ Both DFS and BFS 'walk through the graph', in different ways
- ▶ Both DFS and BFS can compute a spanning tree and forest

```
public void dfs(Integer v, boolean[] marked) {
    marked[v] = true;
    for (Integer w : adj[v])
        if (! marked[w]) dfs(w,marked);
} // dfs() is recursive, call: dfs(v,marked);

public void bfs(Queue<Integer> q, boolean[] marked) {
    while (!q.isEmpty()) {
        Integer v = q.dequeue();
        for (Integer w : adj[v])
            if (! marked[w]) {marked[w]=true; q.enqueue(w);}
    }
} // call: marked[v]=true; q.enqueue(v); bfs(q,marked);

// Example: 0-1, 0-3, 1-2, 1-3, 3-4
// Example: complete ternary tree of height 2
```

Implementation and Properties of DFS/BFS

- ▶ **LinkedListG.java**: `pathdfs()`, `pathbfs()`
- ▶ DFS and BFS mark nodes connected to a given source node in time proportional to the sum of their degrees ($\leq 2E$), and can return a path from a marked node to the given source in time proportional to the length of this path
- ▶ BFS always finds a shortest path (proof: queue only contains nodes at distance k followed by nodes at distance $k + 1$, while all nodes at distance $\leq k$ not in queue have been processed)
- ▶ DFS finds a left-most path (long or short, example bb)
- ▶ BFS tends to use more space (but not always)
- ▶ UF tests connectivity, but finds no paths

Applications

- ▶ `StringSTG.java`, flight connections, shortest path = minimum number of stop-overs
- ▶ Degrees of separation in social networks, e.g., Erdős number = length of shortest path to Paul Erdős in the co-author graph
- ▶ Connected components:
- ▶ Example: `tinyG.txt` on bb and by `LinkedListG.countcc()`

Directed Graphs

- ▶ *Digraph*: a set of *vertices* (or *nodes*) V and a set of *directed edges* (or *arrows*) E pointing from one node to another
- ▶ *Subdigraph*, *directed (simple) path* (*dipath*), *directed (simple) cycle*, *acyclic*, *length*: as expected
- ▶ Often we leave out 'di' in digraph, dipath, etc.
- ▶ DAG: **D**irected **A**cyctic **G**raph;
- ▶ *Degree*: **in**-degree and **out**-degree
- ▶ Node v is *reachable* from w : a dipath from w to v exists
- ▶ *Strongly connected digraph*: dipath between every two nodes (for all v, w , there are dipaths from v to w and from w to v)
- ▶ *Strongly connected component*: maximal strongly connected subgraph ($u \rightleftarrows v \rightarrow w$ has two scc's)
- ▶ Representation: adjacency lists even simpler!

Reachability Problems

Assume we are given a directed graph G .

- ▶ Single-source: given a node s , the *source*, is a given node v reachable from s ? Example: `tinyCG.txt`
- ▶ Multiple-source: given a set of nodes S , is a given node v reachable from some node in S ?
- ▶ Solutions: same DFS and BFS algorithms as in Chapter 1
- ▶ Application (example): mark-and-sweep garbage collection
- ▶ Single-source path: given s, v such that v is reachable from s . Find a path from s to v .
- ▶ Single-source shortest path: given s, v such that v is reachable from s . Find a *shortest* path from s to v .
- ▶ Solutions: same DFS (path) and BFS (shortest path) algorithms as for undirected graphs

Cycle Detection

- ▶ Recall: a *DAG* is a graph without a directed cycle
- ▶ Acyclicity test, cycle detection: easy extension of DFS. We keep track of the search path from the source. If there is an arrow from v to w and w is on the path from v to the source, then there is a cycle. (DFS finds the leftmost path to the leftmost cycle.) Two techniques (space-time trade-off!):
 - ▶ Go back the search path: `LinkedListG.slowCyclist()`
 - ▶ Memorize the search path: `LinkedListG.fastCyclist()`
- ▶ Application: precedence scheduling of jobs

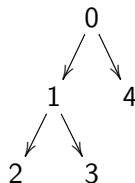
Pre-order, post-order

- ▶ Graph walks based on DFS from a source node
- ▶ Pre-order: order in which DFS arrives at nodes
- ▶ Post-order: order in which DFS leaves nodes
- ▶ In-order *for binary trees*: e.g., in `UBST.show()`
- ▶ Example:

pre-order: 01234

post-order: 23140

in-order: 21304



- ▶ Example: `tinyCG.txt` on `bb` and by `LinkedListG`

Topological order of acyclic digraph

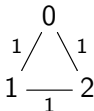
- ▶ Topological order: total order \prec compatible with the graph in the following sense: if there is an arrow from w to v , then $v \preceq w$ (consequently: $v \preceq w$ if v is reachable from w)
- ▶ Lemma: if a digraph has a topological order, then it is acyclic (proof: a cycle cannot be ordered compatibly)
- ▶ Lemma: if a digraph is acyclic, then it has a topological order (proof idea: if acyclic, the post-order is a topological order since, if there is an arrow from w to v , then w is not reachable from v and DFS will leave w after it has left v)
- ▶ Topological order is a job schedule respecting precedence

Transitive closure

- ▶ Definition: given G , its transitive closure G^* is a graph with the same nodes and arrows from v to w for each w that is reachable from v in G .
- ▶ NB: G^* can have many more arrows than G
- ▶ Implementation: adjacency matrix in case of high density of arrows (proof idea: if acyclic, the post-order is a topological order since, if there is an edge from w to v , then w is not reachable from v and dfs will leave w after it has left v)

Minimum Spanning Tree

- ▶ Recall slide 65: *spanning tree* of a connected undirected graph is a maximal subgraph that is a tree (and thus contains all nodes and is acyclic)
- ▶ EWG = Edge-Weighted Graph, here always connected
- ▶ Example: `tinyEWG.txt` on bb
- ▶ Recall slide 66: all spanning trees have $V - 1$ edges
- ▶ *Weight* of spanning tree: sum of the weights of its edges
- ▶ *Minimum Spanning Tree*: spanning tree with minimal weight
- ▶ Example: three MSTs of



- ▶ Exc.4.3.3: if all weights different, then MST is unique
- ▶ From now on we assume all weights different!

Minimum Spanning Tree (ctnd)

- ▶ Applications: power plants and electrical grid, airlines and flight routes, maps and distance
- ▶ Weights may be zero or negative (e.g., cost minus profit of a new network of roads between cities)
- ▶ Two important algorithms to find the MST: Prim's and Kruskal's

Cuts and Crossing Edges

- ▶ Recall slide 66: deleting an edge from a tree creates two disjoint components, adding an edge creates a cycle
- ▶ *Cut*: a partition of V in two non-empty subsets of nodes
- ▶ *Crossing edge*: edge connecting two nodes in different subsets of a cut
- ▶ NB there can be more than one crossing edge: 0 — 2



- ▶ Lemma: for any cut in an EWG, the crossing edge of minimum weight is in the MST.
- ▶ Proof: given a cut, assume by contradiction there is a crossing edge e of weight smaller than the crossing edge(s) that is (are) in the MST (e.g., the dotted edge above). Adding e creates a simple cycle, which must contain one other crossing edge f in the MST. Replacing f by e : ✗

Prim's Lazy Algorithm

- ▶ Datastructures:
 - ▶ EWG represented with adjacency lists $\text{adj}[v]$
 - ▶ Minimum priority queue pq for edges
 - ▶ Array $\text{marked}[v]$ for marking vertices
 - ▶ Queue mst for the minimum spanning tree
- ▶ Edge is *eligible* if not both endpoints marked (crossing!)
- ▶ Algorithm based on previous lemma, cut: un/marked
 1. mark 0 and add all eligible edges in $\text{adj}[0]$ to pq
 2. get and delete minimum edge e from pq
 3. add e to mst
 4. mark the unmarked endpoint of e , say k , and add all eligible edges in $\text{adj}[k]$ to pq
 5. delete ineligible minimum edges from pq and get new eligible minimum edge e from pq
 6. continue at point 3 until pq is empty

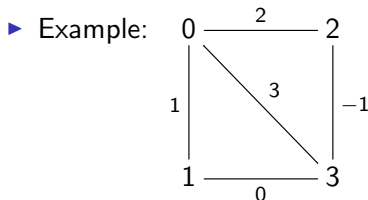
Prim's Algorithm (ctnd)

- ▶ **LazyPrimMST.java**, methods `scan()` and `prim()`
- ▶ Invariant: at least one of the nodes of an edge in `pq` is marked
- ▶ NOT: all edges in `pq` are crossing edges wrt cut un/marked
- ▶ Lazy: ineligible edges are not eagerly deleted from `pq`
- ▶ Runtime: LazyPrimMST runs in $O(E \log E)$ time (worst-case)
- ▶ Possible: edges in `pq` the crossing edges wrt cut un/marked
- ▶ Better: if v unmarked, the only crossing edge of interest is the *lightest* one connecting v to the marked edges (= MST so far)
- ▶ Runtime: **PrimMST.java** runs in $O(E \log V)$ time (worst-case)

Prim's Eager Algorithm

► Datastructures:

- EWG represented with adjacency lists `adj[v]`
- Boolean array `marked[v]` for marking vertices
- Array `distTo[v]`, minimum distances to MST so far
- Array `edgeTo[v]`, edges with minimum distance to MST so far
- Indexed minimum priority queue `pq`: `index=v`, `key=distTo[v]`
- Queue `mst` for the minimum spanning tree



- **PrimMST.java**, methods `scan()` and `prim()`

Kruskal's Algorithm

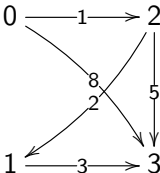
- ▶ Datastructures:
 - ▶ EWG represented with adjacency lists $\text{adj}[v]$
 - ▶ Minimum priority queue pq for edges
 - ▶ Union-Find object uf testing connectivity
 - ▶ Queue mst for the minimum spanning tree
- ▶ Algorithm:
 1. delete the minimum edge e from pq
 2. if the points connected by e are not connected, add e to mst and connect the points in uf
 3. continue at point 1 until pq is empty or uf contains all nodes
- ▶ Examples: EWG on previous slide, `tinyEWG.txt`
- ▶ Correctness: same lemma about minimum-weight crossing edge of cut
- ▶ Implementation: `KruskalMST.java`, constructor method

Memory-Use and Run-time Analysis

- ▶ Space, worst-case:
 - ▶ All methods use $O(V + E)$ space for the graph, plus ...
 - ▶ Priority queue for edges (Lazy Prim and Kruskal): $O(E)$ space
 - ▶ Priority queue for vertices (Eager Prim): $O(V)$ space
 - ▶ Arrays indexed by vertices (all): $O(V)$ space
- ▶ Time, worst-case:
 - ▶ Priority queue operations (Lazy Prim and Kruskal): $O(E \log E)$ time
 - ▶ Priority queue operations (Eager Prim): $O(E \log V)$ time

Shortest paths

- ▶ Recall slide 62, *Edge-weighted digraphs*: digraphs in which every arrow has a weight
- ▶ *Weight* of a (di)path: sum of weights of the (directed) edges
- ▶ *Shortest* path from node s to node t : minimum path weight
- ▶ EWD = Edge-Weighted Digraph

- ▶ Example: two SPs from 0 to 3: 

- ▶ Shortest paths need not be unique, even if all weights are different!
- ▶ Example: `tinyEWD.txt` on bb

Odds and Ends Chapter 4

- ▶ `StringSTG.java`, flight connections, shortest path = minimum number of stop-overs
- ▶ Degrees of separation in social networks
- ▶ Transitive closure

ToC and topics of general interest

- ▶ Table of Contents on next slide (all items clickable)
- ▶ Practical stuff: slide 2

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Ch.1.4 Analysis of Algorithms

Ch.1.5 Case Study: Union-Find

Ch.2.1 Elementary Sorts

Ch.2.2 Mergesort

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