INF102
Algorithms, Data Structures and Programming

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INF102, practical stuff

- Lecturer: Marc Bezem; Team: see homepage
- Homepage: INF102 (hyperlinks in red)
- Also: GitHub (recommended); Dropbox: slides, schedule
- Textbook: Algorithms, 4th edition
- Prerequisites: INF100 + 101 ($\approx$ Ch. 1.1 + 1.2)
- Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- Exam: three compulsory exercises and a written exam
- Old exams: 2004–2013, 2014
- Table of Contents of these slides

Resources

- Good textbook, USA-style: many pages, exercises etc.
- Average speed must be ca 50 pages p/w
- Lectures (ca 24) focus on the essentials
- Slides (ca 120, dense!) summarize the lectures
- Prepare yourself by reading in advance
- Workshops: selected exercises
- Test yourself by trying some exercises in advance
- If you can do the exercises (incl. compulsory), you are fine
- Review of exercises on Friday morning

Generic Bags, Queues and Stacks

- Generic programming in Java, example: PolyPair.java
- Bag, Queue and Stack are generic, iterable collections
- Queue and Stack: Ch. 9 in textbook INF100/1
- APIs include: `boolean isEmpty()` and `int size()`
- All three support adding an element
- Queue and Stack support removing an element (if any)
- FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- Dijkstra's Two-Stack Expression Evaluation Movie
- Example: (1 + ((2 + 3) * (4 * 5)))

Implementations

- ResizingArray_Stack.java
- Arrays give direct access, but have fixed size
- Resizing takes time and space proportional to size
- LinkedList_Stack.java
- No fixed size, but indirect access
- Pointers take space and dereferencing takes time
- Programming with pointers: make a picture
- LinkedList_Queue.java

Computation time and memory space

- Two central questions:
  - How long will my program take?
  - Will there be enough memory?
- Example: ThreeSum.java
- Inner loop (here `a[i]+a[j]+a[k]==0`) is important
- Sorting helps: ThreeSumOptimized.java
- Run some experiments: `1Kints.txt, 2Kints.txt, ...`

Methods of Analysis

- Empirical:
  - Run program with randomized inputs, measuring time & space
  - Run program repeatedly, doubling the input size
  - Measuring time: StopWatch
  - Plot, or log-log plot and linear regression
- Theoretical:
  - Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
  - Try to count/estimate/average this cost as function of the input (size)
  - Use $O(f(n))$ and $f(n) \sim g(n)$

ThreeSum, empirically

- Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4 ,51.1 sec
- The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- Basis of the logarithm should be the same for both
- Linear regression gives $y \approx 3x - 10$
- $\log(f(n)) = 3\log(n) - 10$ iff

$$f(n) = 10^{\log(f(n))} = 10^{3\log(n)-10} = n^3 * 10^{-10}$$

- Conclusion: cubic in the input size, with constant $\approx 10^{-10}$
- Strong dependence on input can be a problem
- Constant 10^{-10} depends on computer, exponent 3 does not

ThreeSum, theoretically

- Number of different picks of triples: $g(n) = n(n-1)(n-2)/6$
- Inner loop `a[i]+a[j]+a[k]==0` executed $g(n)$ times
- $g(n) = n^3/6 - n^2/2 + n/3$
- Cubic term $n^3/6$ wins for large n
- Computational model # array accesses: $3 * n^3/6 = n^3/2$
- Cost array access t sec: time $t * n^3/2$ sec
- Cost models are abstractions! (NB cache)

Big Oh, and $\sim$

- Q: ‘wins for large n’ uhh???
- A: Big Oh, and $\sim$ will clear this up
- Costs are positive quantities, so $f, g, \ldots : \mathbb{N} \to \mathbb{R}^+$
- MNF130: $f(n)$ is $O(g(n))$ if there exist $c \in \mathbb{R}^+,\ N \in \mathbb{N}$ such that $f(n) \leq cg(n)$ for all $n \geq N$ (that is,, for n large enough)
- Example: n^2 and even $99n^3$ are $O(n^3)$, but n^3 is not $O(n^{2.9})$
- INF102: $f(n) \sim g(n))$ if $1 = \lim f(n)/g(n)$
- If $f(n) \sim g(n))$, then $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$
- Big Oh and $\sim$ aim to capture ‘order of growth’
- Big Oh abstracts from constant factors, $\sim$ does not
- Large constant factors are important!

Important orders of growth

- constant: c, $f(n) = c$ for all n
- linear: n (compare all for $n = 20$ sec)
- linearithmetic: $n \log n$
- quadratic: n^2
- cubic: n^3
- exponential: 2^n
- general form: $an^b(\log n)^c$

Logarithms and Exponents

- Definition: $\log_x z = y$ iff $x^y = z$ for $x > 0$
- Inverses: $x^{\log_x y} = y$ and $\log_x x^y = y$
- Exponent: $x^{(y+z)} = x^y x^z$, $x^{(yz)} = (x^y)^z$
- Logarithm: $\log_x(yz) = \log_x y + \log_x z$, $\log_x z = \log_x y \log_y z$
- Base of logarithm: the x in $\log_x$
- Various bases: $\log_2 = \lg$, $\log_e = \ln$, $\log_{10} = \log$
- Double exponent: e.g. $2^{(2^n)}$ (not used in INF102)
- Double logarithm: $\log(\log n)$ (not used in INF102)

Worst case, average case, amortized cost

- Worst case: guaranteed, independent of input; Examples:
  - Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
  - Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- Average case: not guaranteed, dependent of input *distribution*
- Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost ≤ 21, all others cost 1, amortized ≤ 3 p/o.
- Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- Special case of resizing array that is only growing: 1(2)2(4)34(8)5678(16)9 ... 16(32) ..., with (n) the new size. Risizing to (n) costs $2n$ array accesses, so in total (1+4)+(1+8)+(2+16)+(4+32)+(8+64) ..., so 9 p/push.

Staying Connected

- We want efficient algorithms and datastructures for testing whether two objects are 'connected'
- MNF130: relation $E \subseteq V \times V$ is an *equivalence* if
  - E is *reflexive*: $\forall x \in V.\ E(x, x)$
  - E is *symmetic*: $\forall x, y \in V.\ E(x, y) \to E(y, x)$
  - E is *transitive*: $\forall x, y, z \in V.\ E(x, y) \wedge E(y, z) \to E(x, z)$
- We assume connectedness to be an equivalence
- Dynamic connectivity means (here) that E can grow
- Clear relationship with paths in graphs, (connected) components (MNF130)
- Input: N and pairs in $V = \{0, \ldots, N-1\}$ defining E
- Challenge: efficient `boolean connected(int p, int q)`
- Example: $N = 10$, 4 3, 3 8, ... (`algs4-data/tinyUG.txt`)
- Picture on blackboard (don't print pairs that are already connected)

Union-Find

- Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- Union, idea: for any new pair n m that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- API: UF; Cost model: number of array accesses
- Implementations:
  - SlowUF.java: `id[p]` identifier of `p`
    `find()`$\sim$ 1, `union()`$\sim$ between n+3 and 2n+1
  - FastUF.java: `int[] id` pointers, `id[p]==p`: identifier
    `find()` $\sim$ 1+2d, `union()` $\sim$ 1+ two `find()`'s
  - WeightedUF.java: `int[] id` pointers, `int[] sz` subtree sizes
    `find()` and `union()` both $\sim \lg n$
- WeightedUF: height of subtree of size k is at most $\lg k$
- Path-compression: ultimate improvement of UF (almost $O(1)$, amortized)

Sorting

- Sorting: putting objects in a certain order
- MNF130: relation $R \subseteq V \times V$ is a *total order(ing)* if
  1. R is *reflexive*: $\forall x \in V.\ R(x,x)$
  2. R is *transitive*: $\forall x,y,z \in V.\ R(x,y) \wedge R(y,z) \rightarrow R(x,z)$
  3. R is *antisymmetic*: $\forall x,y \in V.\ R(x,y) \wedge R(y,x) \rightarrow x = y$
  4. R is *total*: $\forall x,y \in V.\ R(x,y) \vee R(y,x)$
- Natural orderings:
  - Numbers of any type: ordinary $\leq$ and $\geq$
  - Strings: lexicographic
  - Objects of a Comparable type: `v.compareTo(w) <= 0`

Sorting (ctnd)

- Bubble sort: ExampleSort.java
- Certification: `assert isSorted(a)` in `main()`
- No guarantee against modifying the array (but `exch()` is safe)
- Costmodel 1: number of `exch()`'s and `less()`'s
- Costmodel 2: number of array accesses
- Pitfalls: cache misses, expensive `v.compareTo(w) < 0`
- Why studying sorting? (`java.util.Arrays.sort()`)
- Comparing sorting algorithms: SortCompare.java

Selection Sort

- Bubble sort: $\sim n^2/2$ compares, $0 \leq$ exchanges $\leq\sim n^2/2$
- Selection sort:
  - Find index of a minimum in `a[0..n-1]`, exchange with `a[0]`
  - Find index of a minimum in `a[1..n-1]`, exchange with `a[1]`
  - ... until `n-2`
- Selection sort: $\sim n^2/2$ compares, $0 \leq$ exchanges $\leq n-1$ (!)

```
public  static void sort(Comparable[] a) {
 int N = a.length;
 for (int i=0; i<N-1; i++){
   int min=i;
   for (int j=i+1; j<N; j++) if less(a[j],a[min])) min=j;
   if (i != min) exch(a,i,min);
 }
}
```

Insertion sort

- Insertion sort:
  - Insert `a[1]` on its correct place in (sorted) `a[0..0]`
  - Insert `a[2]` on its correct place in (sorted) `a[0..1]`
  - ... until `a[n-1]`
- Very good for partially sorted arrays, costs:
  - Best case: $n-1$ compares and 0 exchanges
  - Worst case: $\sim n^2/2$ compares and exchanges
  - Average case: $\sim n^2/4$ compares and exchanges (distinct keys)

```
public  static void sort(Comparable[] a) {
 int N = a.length;
 for (int i=1; i<N; i++){
  for (int j=i; j>0 && less(a[j],a[j-1]); j--)
   exch(a,j,j-1);
 }
}
```

Shell sort

- Insertion sort:
  - Very good for partially sorted arrays
  - Slow in transport: step by step `exch(a,j,j-1)`
- Idea: h-sort, `a[i],a[i+h],a[i+2h],...` sorted (any `i`)

```
public static void hsort(int h, Comparable[] a) {
 int N = a.length;
 for (int i=h; i<N; i++)
  for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)
    exch(a,j,j-h);
}
```

- Insertion sort: `hsort(1,a)`
- Shell sort: e.g., `hsort(10,a); hsort(1,a)`

Shell sort (ctnd)

- `hsort(10,a); hsort(1,a)` faster than just `hsort(1,a)` !
- Q: How is this possible?
- A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- What about `hsort(100,a); hsort(10,a); hsort(1,a)`?
- To be expected: depends on the length N of the array
- Best practice: $h = N/3, N/9, \ldots, 364, 121, 40, 13, 4, 1$

Mergesort

- Top-down (recursive) algorithm:
  - Mergesort left half, mergesort right half
  - Merge the results
- Using an auxiliary array: TopDownMergeSort.java, Movie
- Bottom-up algorithm (16 elements):
  - Merge `a[0]`,`a[1]`, so `a[2]`,`a[3]`, so `a[4]`,`a[5]`, so ...
  - Merge `a[0..1]`,`a[2..3]`, so `a[4..5]`,`a[6..7]`, so ...
  - Merge `a[0..3]`,`a[4..7]`, so `a[8..11]`,`a[12..15]`
  - Merge `a[0..7]`,`a[8..15]`, done!
- Also using an auxiliary array: BottomUpMergeSort.java

Run-time and memory use of mergesort

- Mergesort uses between $\sim (N/2) \lg N$ and $\sim N \lg N$ compares. Proof on bb. Important formula ($N = 2^n$):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- Mergesort uses at most $\sim 6N \lg N$ array accesses
- Mergesort uses $\sim 2N$ space (plus some var's)
- Q: How fast can compare-based sorting of N distinct keys be?
- A: $\lg N! \sim N \lg N$; Proof in book and on bb. Keywords: binary *compare tree*, inner nodes for each `compare(a[i],a[j])`, permutations in the leaves,

  $N! =$ number of permutations $\leq$ number of leaves $\leq 2^{\text{height of tree}}$

Quicksort

- Top-down (recursive) algorithm:
  - Choose a (pivot) value v in the array
  - Partition the array in non-empty parts $\leq v$ and $\geq v$
  - Quicksort the two parts
- Pros: in-place, average computation time $O(n \log n)$
- Cons: stack space for recursion, worst-case $O(n^2)$, not stable
- Implementation: QuickSort.java
- BTW: Bug in java.util.Arrays.sort

Quicksort, details

- Subtleties in `sort()`: shuffling protects against worst-case behaviour
- Termination of recursive `quicksort()`
- Subtleties in `partition()`:
  - Invariants `l<=h` in the two inner loops
  - Postcondition after the two inner loops
  - Invariant of the `for(;;)` loop
  - Termination of the `for(;;)` loop
  - There are some variations that are also correct

Run-time and memory use of quicksort

- Compare Quicksort to other sorts ($n = 10^2, 10^3, \dots$)
- Quicksort: time $O(n^2)$ if pivot is always smallest (or largest)
- Randomization: choose pivot randomly, or shuffle array
- If all keys are distinct and randomization is perfect, then quicksort uses on average $\sim 2n \ln n$ compares and $\sim (n/3) \ln n$ exchanges (proofs in book, complicated)
- Relevant improvements:
  - Cut-off to insertion sort for sizes ≤ 15 (ca.)
  - Median-of-three pivot
  - Taking advantage of duplicate keys (3-way partitioning)
- Quicksort is generally quite good
- In special situations other sorts are better (e.g., countsort)

Priority Queues

- Assume collecting and processing items having keys
- Examples of keys: time-stamp, price-tag, priority-tag
- Assume: keys can be ordered
- Reasonable: processing currently highest (or lowest)
- Special cases: items time-stamped when added
  - Queue: dequeue currently oldest (lowest time-stamp)
  - Stack: pop currently newest (highest time-stamp)
- Priority queue generalizes this
- Examples: highest priority, largest transaction, lowest price
- Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

Priority Queues

- Good info: Wikipedia; API (the bare essentials):

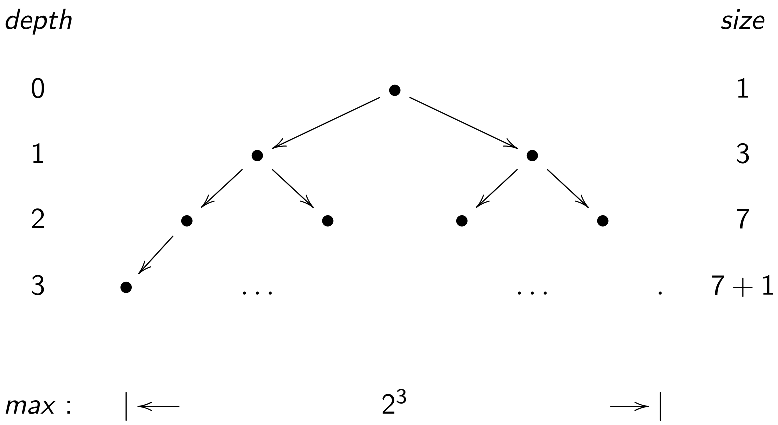
```
public class  ArrayListPQ<Key extends Comparable<Key>>

void          insert(Key v) // insert a key
Key           delMax() // delete a largest key, if any
boolean       isEmpty()
int           size()
```

- Aim: operations in logarithmic time, no extra space
- In case of duplicate keys: ‘a’ largest, not ‘the’
- Typical application: the 1K largest keys of 1G unsorted keys
- Client: BottomM.java (Q: why is the output slowing down?)

Binary Trees

- MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height h has $2^{h+1}-1$ nodes.
- INF102: A binary tree is (left-)*complete* if all levels $< h$ are filled, level h may be partly empty from the right (picture bb). A (left-)complete binary tree of height h has between 2^h and $2^{h+1}-1$ nodes (from now on we leave out '(left-)').
- A complete binary tree of n nodes has height $\lfloor \lg n \rfloor$

Picture



Heap-ordered Binary Trees

- A binary tree is *heap-ordered* if the key in each node is $\geq$ the keys in its children (if any). Thus the root has a maximal key.
- Array representation of heap-ordered complete binary tree (bb)
- Methods `swim()` and `sink()`: picture on bb, code below
- Implementation: ArrayListPQ.java

Run-time and memory use of heaps, applications

- In a heap of n elements (since height is $\leq \lfloor \lg n \rfloor$):
  - `swim()`, and hence `insert()`, takes $\leq 1 + \lfloor \lg n \rfloor$ compares and $\leq \lfloor \lg n \rfloor$ exchanges
  - `sink()`, and hence `delMax()`, takes $\leq 2\lfloor \lg n \rfloor$ compares
  - `sink()` takes $\leq \lfloor \lg n \rfloor$ exchanges, and `delMax()` $\leq 1+\lfloor \lg n \rfloor$
- Heap construction by `insert()` can sometimes be improved
- Given an array of keys, right-to-left heap construction (bb) takes $< 2n$ compares and $< n$ exchanges
- Applications: heapsort and merging sorted streams (bb)
- Many variations with extended API (indexed priority queue)

Purpose of Sorting

- Sorting makes the following easier and more efficient:
  - Searching (binary search, example: `ThreeSumOptimized`
  - Searching and looking up, e.g., the pagenumber in an index
  - Removing duplicates
  - Finding the median, quartiles etc.
- Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- We do *pointer sorting*, manipulating refs to objects.
  - Pro: not moving full objects
  - Cons: pointer dereferencing, no `sort(int[] a)`
- More flexibility: pass a `Comparator` object to `sort()`

Comparator object

- API: `void sort(Object[] a, Comparator c)`
- Call, e.g.: `sort(a, new Transaction.WhenOrder())`
- Call, e.g.: `sort(a, new Transaction.SizeOrder())`
- Obs: `import java.util.Comparator`
- Obs: `less(Object o1, Object o2, Comparator c)`
- Priority queues also with `Comparator`

```
public class Transaction {
 ...
 public static class MyOrder {
 implements Comparator<Transaction>
  public int compare(Transaction t, Transaction v){...}
 } // End of Myorder
...// similarly: WhenOrder, SizeOrder
} // End of Transaction
```

Applications of Sorting

- Consider sorting first to make other problems easier
- Commercial computing (sort on price, departure time, ...)
- Search for information: web-indexing, search engines
- Job scheduling heuristic: longest processing time first
- To come: Prim's, Dijkstra's and Kruskal's algorithms
- Huffman compression: a lossless compression based on using the shortest codes for the symbols that occur oftest. Frequency counter: next chapter!
- Cryptology and genomics (e.g., longest repeated substring)

Symbol Tables

- Symbol table associates *keys* with *values: key-value pairs*
- Examples: keyword-page number, ID number-personal data
- Important operations:
  - Insert a key-value pair in the symbol table: `void put(k,v)`
  - Search the value for a given key (if any): `Value get(k)`
- Important conventions:
  - Inserting key-value for existing key: overwriting the value
  - No duplicate keys, no null keys
  - Value null: no value for this key
  - Lazy deletion: insert key-null; Eager: really delete key
- API of unordered symbol table
- Aim: all operations in time $\sim c \lg n$ with constant c small

ST Basics

- Archetypical ST-client: frequency counter (code: `main`)
- Cost model: number of compares
- Naive ST: unordered linked list, linear search (INF101, Ch.9)
  - Search miss: $\sim n$ compares
  - Search hit: between 1 and $\sim n$ compares
  - Random search hit: $(1 + \cdots + n)/n \sim n/2$ compares
  - Inserting n distinct keys: $(1 + \cdots + (n - 1)) \sim n^2/2$ compares
- `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- Naive ST impracticable for genomics, internet
- Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- Better for unordered ST: hashing (in Ch. 3.4)

Ordered Symbol Table

- Ordered ST: keys are ordered
- API of ordered symbol table
- Binary search: `get(Key k)` takes $\sim \lg n$ comparisons
- What about `put(Key k, Value v)`? ArrayListST, good!
- TODO: test that `add(int i, E e)` is amortized $O(1)$
- Implementation with binary search in ArrayListST.java
- Trace of inserts on bb: S E A R C H E X A M P L E
- Experiments with `tinyTale.txt`, `tale.txt`, ...

Binary Search Trees

- Binary *search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- Search time: lenght of the path to the node where the key 'should' be
- Balanced binary tree with n keys has $\lg n$ height
- Unbalanced binary trees can have height n (max depth)
- Search hits in a binary search tree, built without rebalancing, of n random keys take on average $\sim 2 \ln n$ compares
- UBST.java: `put()`, `get()`, `size()`, `isEmpty()`
- Trace of inserts on bb: S E A R C H E X A M P L E

Binary Search Trees (ctnd)

- Interrelated, increasing difficulty: `min(Node x)`, `deleteMin(Node x)`, `delete(Node x, Key k)`
- Node of minimum key: not `null`, and has left child `null`, and is root or left child of parent (picture on bb)

```
public Node min(Node x){// x != null, subtree not empty
 while (x.left!=null) x = x.left;
 return x;
} // cf. tail recursive min() in Alg. 3.3
```

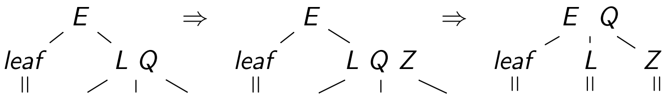
- Delete minimum key, two cases:
  (1) both children `null`; (2) left child `null`
- Delete is really difficult: bb + BST.java
- Don't forget: update `x.N` along the path to the root!

Balanced Search Trees: keep paths short!

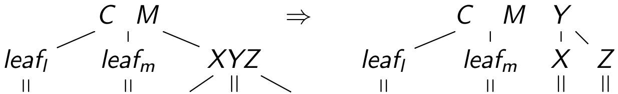
- NB tree balancing not as easy as in UF and Heap (4hrs!)
- A *2-3 search* tree consists of 2-nodes and 3-nodes:
  - Each 2-node has two children and a key k such that all keys in the left subtree are $< k$, and all keys in the right subtree $> k$
  - Each 3-node has three children and two keys k_1, k_2 such that all keys in the left subtree are $< k_1$, all keys in the middle subtree $> k_1$ and $< k_2$, and all keys in the right subtree $> k_2$
- Examples and pictures on bb
- *Perfect* 2-3 search tree: paths from root to leaves equally long
- Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- Insert should keep tree perfect, rough idea:
  - into a 2-leaf: make it into a 3-leaf
  - into a 3-node: do something clever (explained next)

Insert in Balanced Search Trees

- Terminology: a *leaf* is a node all whose children are `null`
- Data invariant 1: 2-3 search tree
- Data invariant 2: paths from root to leaves equally long
  - Insert into a 2-leaf *L* : either *A L* or *L Z*
  - into a 3-leaf whose parent is a 2-node: with new key *Z*



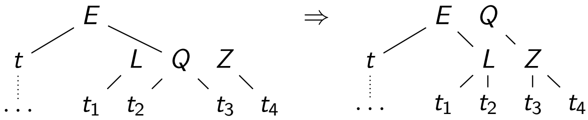
  - into a 3-leaf whose parent is a 3-node: with new key *Z* added



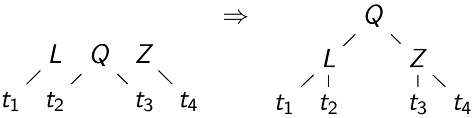
  - into a 3-node whose parent is a 3-node: move up middle key!

Insert (ctnd)

- Data invariant 1: 2-3 search tree
- Data invariant 2: paths from root to leaves equally long
- Insert works up from the leaf where the key ‘should’ be
  - if 2-node on path to root: make it into a 3-node (two cases)



  - otherwise: split the root

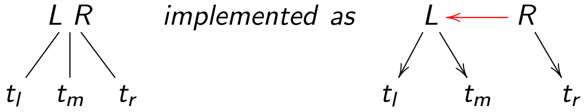


Insert, summary and examples

- Six operations for eliminating 4-nodes:
  - if parent is 2-node: move middle key up (left and right case)
  - if parent is 3-node: move middle key up (left, middle,right)
  - if root: split root
- Search and insert visit at most $\lfloor \lg n \rfloor$ nodes
- Proof: maximal path length is $\geq \lfloor \log_3 n \rfloor$ and $\leq \lfloor \log_2 n \rfloor$
- Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- Trace of inserts on bb: A C E H L M P R S X

Red-black trees

- Red-black trees implement 2-3 trees
- Idea: one 3-node = two 2-nodes + extra info
- Extra info coded in color, picture:



- A *red-black tree* is a binary search tree with red and black links such that:
  - Only left links can be red (but need not be)
  - Never $\leftarrow \leftarrow$
  - Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- Equivalent: red-black tree and perfect 2-3 search tree

Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

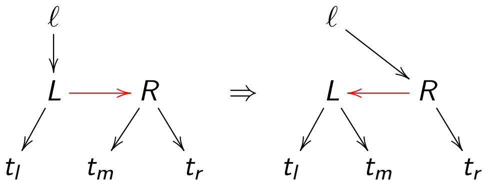
```
private class Node {
  Key key;
  Value value;
  Node left, right;
  boolean color; // true for red, false for black
  int N;
}
private boolean isRed(Node n) {
 if (n==null) {return false;} else {return x.color}
```

Rotating and Color Flipping

- Aim: restoring the data invariants of red-black search trees
  1. Only left red links, but never two
  2. Search tree invariant
  3. Perfect black balance
- Invariants get violated by temporary 4-nodes, e.g.,
  - inserting Z in $L \longleftarrow R$: $L \longleftarrow R \longrightarrow Z$
  - inserting A in $L \longleftarrow R$: $A \longleftarrow L \longleftarrow R$
  - inserting M in $L \longleftarrow\!\!\!-\!\!\!-\!\!\!-\, R$:
$$\begin{tikzcd} L \arrow[dr] & & R \arrow[ll] \\ & M & \end{tikzcd}$$
- Restoring:
  - Color flip $L \longleftarrow R \longrightarrow Z$: $L \longleftarrow R \longrightarrow Z$
  - Rotation right + color flip $A \longleftarrow L \longleftarrow R$: $A \longleftarrow L \longrightarrow R$
  - Rotation left into $M \longleftarrow L \longleftarrow R$, then as previous

Left Rotation

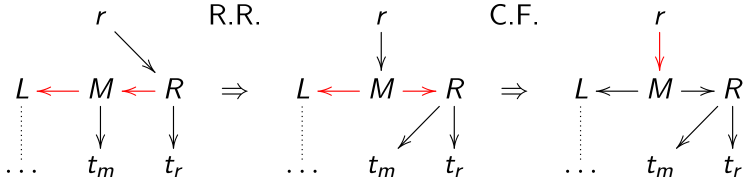
Call: `l = rotateLeft(l);`




```
private Node rotateLeft(Node l){
  Node r = l.right; l.right = r.left; r.left = l;
  r.color = l.color; l.color = true // == RED
  r.N = l.N; l.N -= 1+size(r.right); // Why?
  return r;
}
```

Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- Code of `rotateRight()` like that of `rotateLeft()`
- NB1: operations are local (here only `r`, `M` , `R`)
- NB2: operations preserve data invariants
- NB3: root is a special case (always black)
- Deletions: complicated, but doable (Exc. 3.3.39–41)

Run-time and memory use of Red-Black BSTs

- The height of a red-black BST with n nodes is $\leq 2 \lg n$
  Proof: the worst-case is one 3-node path and the rest 2-nodes
- The average length of path from the root to a *node (?)* in a red-black BST with n nodes is $\lg n$ ('empirical fact')
- In a red-black BST, search, insert, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node.
- For red-black BSTs, logarithmic time is guaranteed!

Hashing

- Idea: if keys in `[0..99]` an array is the perfect symbol table
- A *hash function* maps a key to an array index
- Injectivity of the hash function is not guaranteed
- *Hash collision*: different keys are mapped to the same index
- In such a case we need *collision resolution*
- Symbol tables: hashing is fast, but unordered (no `max`,`min`)
- Aim: ST operations amortized $O(1)$ time, extra space OK

Space-Time Trade-Off

- Hashing is an example of a *space-time trade-off*
- Time: computation time required
- Space: memory space used
- Unlimited space: (1) use key as index (e.g., the bits)
- Unlimited time: (2) use linked list and linear search
- Hashing strikes a balance using (1) with some array of reasonable size, and (2) in case of collisions
- The balance between (1) and (2) can easily be tuned

Hash functions

- Ideal (uniform hashing assumption, UHA): uniform and independent distribution of keys over integers from 0 to $M-1$
- Examples of hash functions in Java
- Horner: $a_0 + x(a_1 + x(a_2 + \cdots)) = a_0 + a_1 x + a_2 x^2 + \cdots$
- Reasonably $\approx$ UHA: modular hashing (M prime):

```
private int hash(Key k){
 return (key.hashCode() & 0x7fffffff) % M;}
```

- Q: Why M prime?
- A: e.g. $M = 32$ takes only into account the last five bits

Collision Resolution

- Two methods of collision resolution:
  1. Hashing with separate chaining (picture on bb)
  2. Hashing with linear probing (picture on bb)
- Separate chaining: symbol table is an array of linked lists, linear search. If array has length M, then the linked lists have average length N/M with N keys.
- Linear probing: symbol table is an array of length $M > N$. Colliding keys are put at the first empty position. Linear search from the position where the key 'should have been'. Empty position: not found. Deletion tricky: reinsert all keys to the right of the deleted key, until the first empty position (picture on bb). Array must have length $\geq N$ with N keys.

Symbol Table with Hashing

- Implementation: `ArrayListHashST.java`
- $M = 1$: measure overhead wrt. `ArrayListST.java`
- Tests with various values of M: 31, 997, 65521
- NB: *construction* versus *use* of ST (hashing better for *use*)
- Hashing can be combined with any other ST-implementation

Quantitative analysis

- Throwing a dice 10 times, what is the probability of 3 fives?
- Under UHA, with N distinct keys, the probability that exactly k keys collide at some given hash value is

$$\binom{N}{k}(\frac{1}{M})^k(\frac{M-1}{M})^{N-k}, \text{ where e.g. } \binom{100}{10} \approx 1.7E13$$

- This is a small number for, say, $N = M = 100$ and $k = 10$
- For linear probing one typically takes $M = 2N$
- For separate chaining one keeps $N/8 \leq M \leq N/2$ (resizing M)
- Under UHA: search, insert, delete take amortized $O(1)$ time
- Extra space used can be upto $100N$ byte (objects, pointers); this on top of the space used by N key-value pairs

Applications of Searching

- Synonyms: associative array, map, symbol table, or dictionary
- Origin of symbol table: compilers and interpreters
- Web-indexing, search engines
- Sparse matrices (many 0's): dictionary
  1. keys: (row, column)-pairs
  2. values: matrix entries
- Set API (no values, only keys, for deduplication, filtering):

```
public class  SET<Key>

void          add(Key k)
void          delete(Key k)
boolean       contains(Key k)
boolean       isEmpty()
int           size()
```

Applications of Searching (ctnd)

- Application (key, value)
- Phone book (name, phone number)
- Dictionary (word, meaning or translation)
- Account information (client ID, account information)
- Genomics (sequences of ACTG triplets, proteins)
- Experimental data of various kinds
- File systems (file name, address etc)
- Internet domain name system (domain name, IP address)
- Invertex index (value, key(s))

Balanced Search Tree or Hash Table?

- Q: Which symbol table to use?
- A: The basic choice between BST and HT depends on ...
  1. Ordering of keys essential: BST
  2. Availability of good hash function (good = fast + UHA)
  3. Ordering of keys expensive (long strings): HT (or: Ch.5)
  4. Ordering of keys possible, but not essential: HT + BST
  5. Space considerations (ArrayListST uses the least extra space)
  6. Number of distinct keys and the space each key takes
  7. Distribution of insert/delete/search operations

Odds and Ends Chapter 1–3

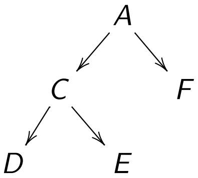
- Discuss methodological issues `SortCompare`
- Find out if `add(int i, E e)` in ArrayList is amortized $O(1)$
- Discuss primitive types (objects are costly)
- Distributed Hash Table
- Double hashing: linear probing
  $h_1(k), h_1(k) + h_2(k), h_1(k) + 2h_2(k), \ldots$
- Indexed Priority Queues: next slide

Indexed Priority Queues

- IPQ ≈ array with direct access to minimum (maximum)
- API: void insert(int i, Key k); void del(int i); int minKey(); Key keyOf(int i);... Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-
qp	1	0	2	4	3	-1	-1

heap



Do: insert(6,G), insert(5,B)

Graph classes

(MNF130: useful review of graph theory)

1. Undirected graphs: a set of *vertices* (or *nodes*) V and a set of *edges* E connecting the nodes
2. Directed graphs (*digraphs*): a set of nodes V and a set E of edges (or *arrows*) pointing from one node to another
3. *Edge-weighted graphs*: undirected graphs in which every edge has a number called the *weight* of the edge
4. *Edge-weighted digraphs*: digraphs in which every arrow has a weight

Examples

1. Undirected graphs: social networks, communication networks (duplex communication), road maps
2. Directed graphs: logical circuits, job scheduling, flow graphs, hyperlinks, (class, module, package) dependencies
3. Edge-weighted graphs: roadmaps with geographical distance, communication networks with bandwidth
4. Edge-weighted digraphs: job scheduling with duration, transport of goods, financial transactions

Undirected Graphs

- Undirected graph: a set of *vertices* (or *nodes*) V and a set of *edges* E connecting the nodes
- *Subgraph*: subset of E and subset of V forming a graph (!)
- *Path*: sequence of nodes connected by edges (!)
- *Simple path*: path with no node repeated
- *Length of path*: number of edges
- *Cycle*: path of length > 0 with same start and end node
- *Simple* cycle: not repeating edges or nodes (apart from start and end node)
- *Acyclic graph*: graph without simple cycles
- *Connected graph*: having a path between every two nodes
- *Connected component*: a maximal connected subgraph

Trees and Forests

- 'Anomalies' concerning edges:
  - Self-loop: edge connecting a node to itself
  - Parallel edges: two edges connecting the same node(s)
- When no anomalies, $E \subseteq \{\{v, v'\} \mid v \in V, v' \in V, v \neq v'\}$
- *Tree*: connected acyclic graph (then: no anomalies)
- *Spanning tree*: maximal subgraph that is a tree
- Lemma: any spanning tree of a connected graph contains all nodes. Proof by contradiction (on bb).
- *Forest*: graph consisting of disjoint trees
- *Spanning Forest*: forest consisting of spanning trees of connected components of a graph
- Example: `tinyG.txt` on bb

Undirected Graphs (ctnd)

- *Distance* between two nodes: length of a shortest connecting path if there is a path connecting these nodes, otherwise ∞
- *Degree* of a node: number of edges connected to that node
- Graph $G = (V, E)$, the following are equivalent:
  - G is a tree
  - G has $|V| - 1$ edges and no cycles
  - G has $|V| - 1$ edges and is connected
  - G is acyclic and adding an edges creates a cycle
  - Any two nodes of G are connected by exactly one simple path
- Example: some subgraphs of `tinyG.txt` on bb

Graph representation and implementation

- Impractical: adjacency matrix $\sim V^2$, incidence matrix $\sim VE$
- Often practical: adjacency lists $\sim (V+2E)$, that is, `adj[v]` lists all nodes `w` connected to `v` by an edge
- Example: `tinyG.txt` by `LinkedListG.java`
- Graph API includes: `V()`, `E()`, `addEdge()`
- Basic algorithms: Depth-First Search (DFS) and Breadth-First Search (BFS)
- Both DFS and BFS ‘walk through the graph’, in different ways
- Both DFS and BFS can compute a spanning tree and forest

```
public void dfs(Integer v, boolean[] marked) {
  marked[v] = true;
  for (Integer w : adj[v])
    if (! marked[w]) dfs(w,marked);
} // dfs() is recursive, call: dfs(v,marked);

public void bfs(Queue<Integer> q, boolean[] marked) {
  while (!q.isEmpty()) {
     Integer v = q.dequeue();
     for (Integer w : adj[v])
       if (! marked[w]) {marked[w]=true; q.enqueue(w);}
  }
} // call: marked[v]=true; q.enqueue(v); bfs(q,marked);

// Example: 0-1, 0-3, 1-2, 1-3, 3-4
// Example: complete ternary tree of height 2
```

Implementation and Properties of DFS/BFS

- LinkedListG.java: `pathdfs(), pathbfs()`
- DFS and BFS mark nodes connected to a given source node in time proportional to the sum of their degrees ($\leq 2E$), and can return a path from a marked node to the given source in time proportional to the length of this path
- BFS always finds a shortest path (proof: queue only contains nodes at distance k followed by nodes at distance $k+1$, while all nodes at distance $\leq k$ not in queue have been processed)
- DFS finds a left-most path (long or short, example bb)
- BFS tends to use more space (but not always)
- UF tests connectivity, but finds no paths

Applications

- StringSTG.java, flight connections, shortest path = minimum number of stop-overs
- Degrees of separation in social networks, e.g., Erdös number = length of shortest path to Paul Erdös in the co-author graph
- Connected components:
- Example: `tinyG.txt` on bb and by `LinkedListG.countcc()`

Directed Graphs

- *Digraph*: a set of *vertices* (or *nodes*) V and a set of *directed edges* (or *arrows*) E pointing from one node to another
- *Subdigraph, directed (simple) path, directed (simple) cycle, acyclic, length*: as expected
- *Dag*: Directed acyclic graph; *dipath*: directed path
- *Degree*: in-degree and out-degree
- Node v is *reachable* from w: a dipath from w to v exists
- *Strongly connected digraph*: dipath between every two nodes (for all v, w, there are dipaths from v to w and from w to v)
- *Strongly connected component*: maximal strongly connected subgraph ($u \rightleftarrows v \to w$ has two scc's)
- Often we leave out 'di' in digraph, dipath, etc.
- Representation: adjacency lists even simpler!

Reachability Problems

Assume we are given a directed graph G.

- Single-source: given a node s, the *source*, is a given node v reachable from s?
- Multiple-source: given a set of nodes S, is a given node v reachable from some node in S?
- Solutions: same DFS and BFS algorithms as in Chapter 1
- Application (example): mark-and-sweep garbage collection
- Single-source path: given s, v such that v is reachable from s. Find a path from s to v.
- Single-source shortest path: given s, v such that v is reachable from s. Find a *shortest* path from s to v.
- Solutions: same DFS (path) and BFS (shortest path) algorithms as for undirected graphs

Cycle Detection

- Recall: a *Dag* is a graph without a directed cycle
- Acyclicity test, cycle detection: easy extension of DFS. We keep track of the search path from the source. If there is an edge from v to w and w is on the path from v to the source, then there is a cycle. (DFS finds the leftmost path to the leftmost cycle.) Two techniques:
  - Go back the search path: `LinkedListG.slowCyclist()`
  - memorize the search path: `LinkedListG.fastCyclist()`
- Application: precedence scheduling of jobs

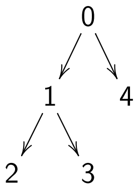
Pre-order, post-order

- Graph walks based on dfs from a source node
- Pre-order: order in which dfs arrives at nodes
- Post-order: order in which dfs leaves nodes
- In-order *for binary trees*: e.g., in UBST.java
- Example:

pre-order: 01234

post-order: 23140

in-order: 21304



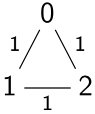
- Example: `tinyCG.txt` on bb and by LinkedListG.java

Topological order of acyclic digraph

- Topological order: total order $\prec$ compatible with the graph in the following sense: if there is an edge from w to v, then $v \preceq w$ (consequently: $v \preceq w$ if v is reachable from w)
- Lemma: if a digraph has a topological order, then it is acyclic (proof: a cycle cannot be ordered compatibly)
- Lemma: if a digraph is acyclic, then it has a topological order (proof idea: if acyclic, the post-order is a topological order since, if there is an edge from w to v, then w is not reachable from v and dfs will leave w after it has left v)

Minimum Spanning Tree

- Recall 65: *spanning tree* of a connected undirected graph is a maximal subgraph that is a tree (and thus contains all nodes and is acyclic)
- Example: `tinyEWG.txt` on bb
- EWG = Edge-Weighted Graph, here always connected
- Recall 66: all spanning trees have $V - 1$ edges
- *Weight* of spanning tree: sum of the weights of its edges
- *Minimum Spanning Tree*: spanning tree with minimal weigth
- Example: three MSTs of



- Exc.4.3.3: if all weights different, then MST is unique
- From now on we assume all weights different!

Minimum Spanning Tree (ctnd)

- Applications: power plants and electrical grid, airlines and flight routes, maps and distance
- Weights may be zero or negative (e.g., cost, profit)
- Two important algorithms to find the MST: Prim's and Kruskal's

Cuts and Crossing Edges

- Recall 66: deleting an edge from a tree creates two disjoint components, adding an edge creates a cycle
- *Cut*: a partition of V in two non-empty subsets of nodes
- *Crossing edge*: edge connecting two nodes in different subsets of a cut
- NB there can be more than one crossing edge:

$$\begin{tikzcd}[arrows=-] 0 \arrow[r] & 2 \arrow[d] \arrow[dl, dotted] \\ 1 \arrow[r] & 3 \end{tikzcd}$$

- Lemma: for any cut in an EWG, the crossing edge of minimum weight is in the MST. Proof: given a cut, assume there is a crossing edge e of weight smaller than the crossing edge(s) that is (are) in the MST (e.g., the dotted edge above). Adding e creates a cycle, which must contain one other crossing edge f in the MST. Replacing f by e ...

Prim's Algorithm

- Datastructures:
  - EWG represented with adjacency lists `adj[v]`
  - Priority queue `pq` for edges
  - Array `marked` for marking vertices
  - Queue `mst` for the minimum spanning tree
- Edge is *eligible* if not both endpoints marked (crossing!)
- Algorithm based on previous lemma, cut: un/marked
  1. mark 0 and add all eligible edges in `adj[0]` to `pq`
  2. get and delete minimum edge `e` from `pq`
  3. add `e` to `mst`
  4. mark the unmarked endpoint of `e`, say `k`, and add all eligible edges in `adj[k]` to `pq`
  5. delete ineligible minimum edges from `pq` and get new eligible minimum edge `e` from `pq`
  6. continue at point 3 until `pq` is empty

Prim's Algorithm (ctnd)

- LazyPrimMST.java, methods `scan()` and `prim()`
- Invariant: at least one of the nodes of an edge in `pq` is marked
- NOT: all edges in `pq` are crossing edges wrt cut un/marked
- Lazy: ineligible edges are not eagerly deleted from `pq`
- Runtime: LazyPrimMST runs in $O(E \log E)$ time (worst-case)
- Possible: edges in `pq` the crossing edges wrt cut un/marked
- Better: if `v` unmarked, the only crossing edge of interest is the *lightest* one connecting `v` to the marked edges ($=$ `mst` so far)
- PrimMST.java, indexed `pq` + extra array containing lightest crossing edges to `mst` so far
- Runtime: PrimMST runs in $O(E \log V)$ time (worst-case)

Kruskal's Algorithm

- Datastructures:
  - EWG represented with adjacency lists `adj[v]`
  - Priority queue `pq` for edges
  - Union-Find object `uf` testing connectivity
  - Queue `mst` for the minimum spanning tree
- Algorithm:
  1. delete the minimum edge `e` from `pq`
  2. if the points connected by `e` are not connected, add `e` to `mst` and connect the points in `uf`
  3. continue at point 1 until `pq` is empty or `uf` contains all nodes

ToC and topics of general interest

- Table of Contents on next slide (all items clickable)
- Practical stuff: slide 2