

# INF102

## Algorithms, Data Structures and Programming

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## INF102, practical stuff

- ▶ Lecturer: Marc Bezem; Team: see homepage
- ▶ Homepage: [INF102](#) (hyperlinks in red)
- ▶ Also: [GitHub](#) (recommended); Dropbox: [slides](#), [schedule](#)
- ▶ Textbook: [Algorithms, 4th edition](#)
- ▶ Prerequisites: INF100 + 101 ( $\approx$  Ch. 1.1 + 1.2)
- ▶ Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- ▶ Exam: three compulsory exercises and a [written exam](#)
- ▶ Old exams: [2004–2013](#), [2014](#)
- ▶ [Table of Contents of these slides](#)

# Resources

- ▶ Good textbook, USA-style: many pages, exercises etc.
- ▶ Average speed must be ca 50 pages p/w
- ▶ Lectures (ca 24) focus on the essentials
- ▶ Slides (ca 120, dense!) summarize the lectures
- ▶ Prepare yourself by reading in advance
- ▶ Workshops: selected exercises
- ▶ Test yourself by trying some exercises in advance
- ▶ If you can do the exercises (incl. compulsory), you are fine
- ▶ Review of exercises on Friday morning

## Generic Bags, Queues and Stacks

- ▶ Generic programming in Java, example: **PolyPair.java**
- ▶ Bag, Queue and Stack are generic, iterable collections
- ▶ Queue and Stack: Ch. 9 in textbook INF100/1
- ▶ APIs include: `boolean isEmpty()` and `int size()`
- ▶ All three support adding an element
- ▶ Queue and Stack support removing an element (if any)
- ▶ FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- ▶ Dijkstra's Two-Stack Expression Evaluation **Movie**
- ▶ Example:  $( 1 + ( ( 2 + 3 ) * ( 4 * 5 ) ) )$

## Implementations

- ▶ `ResizingArray_Stack.java`
- ▶ Arrays give direct access, but have fixed size
- ▶ Resizing takes time and space proportional to size
- ▶ `LinkedList_Stack.java`
- ▶ No fixed size, but indirect access
- ▶ Pointers take space and dereferencing takes time
- ▶ Programming with pointers: make a picture
- ▶ `LinkedList_Queue.java`

## Computation time and memory space

- ▶ Two central questions:
  - ▶ How long will my program take?
  - ▶ Will there be enough memory?
- ▶ Example: **ThreeSum.java**
- ▶ Inner loop (here  $a[i] + a[j] + a[k] == 0$ ) is important
- ▶ Sorting helps: **ThreeSumOptimized.java**
- ▶ Run some experiments: `1Kints.txt`, `2Kints.txt`, ...

# Methods of Analysis

- ▶ Empirical:
  - ▶ Run program with randomized inputs, measuring time & space
  - ▶ Run program repeatedly, doubling the input size
  - ▶ Measuring time: **StopWatch**
  - ▶ Plot, or log-log plot and **linear regression**
- ▶ Theoretical:
  - ▶ Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
  - ▶ Try to count/estimate/average this cost as function of the input (size)
  - ▶ Use  $O(f(n))$  and  $f(n) \sim g(n)$

## ThreeSum, empirically

- ▶ Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4 ,51.1 sec
- ▶ The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- ▶ Basis of the logarithm should be the same for both
- ▶ Linear regression gives  $y \approx 3x - 10$
- ▶  $\log(f(n)) = 3 \log(n) - 10$  iff

$$f(n) = 10^{\log(f(n))} = 10^{3 \log(n) - 10} = n^3 * 10^{-10}$$

- ▶ Conclusion: cubic in the input size, with constant  $\approx 10^{-10}$
- ▶ Strong dependence on input can be a problem
- ▶ Constant  $10^{-10}$  depends on computer, exponent 3 does not



## ThreeSum, theoretically

- ▶ Number of different picks of triples:  $g(n) = n(n-1)(n-2)/6$
- ▶ Inner loop  $a[i] + a[j] + a[k] == 0$  executed  $g(n)$  times
- ▶  $g(n) = n^3/6 - n^2/2 + n/3$
- ▶ Cubic term  $n^3/6$  wins for large  $n$
- ▶ Computational model # array accesses:  $3 * n^3/6 = n^3/2$
- ▶ Cost array access  $t$  sec: time  $t * n^3/2$  sec
- ▶ Cost models are abstractions! (NB cache)

## Big Oh, and $\sim$

- ▶ Q: 'wins for large  $n$ ' uhh???
- ▶ A: Big Oh, and  $\sim$  will clear this up
- ▶ Costs are positive quantities, so  $f, g, \dots : \mathbb{N} \rightarrow \mathbb{R}^+$
- ▶ MNF130:  $f(n)$  is  $O(g(n))$  if there exist  $c \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$  such that  $f(n) \leq cg(n)$  for all  $n \geq N$  (that is,, for  $n$  large enough)
- ▶ Example:  $n^2$  and even  $99n^3$  are  $O(n^3)$ , but  $n^3$  is not  $O(n^{2.9})$
- ▶ INF102:  $f(n) \sim g(n)$  if  $1 = \lim f(n)/g(n)$
- ▶ If  $f(n) \sim g(n)$ , then  $f(n)$  is  $O(g(n))$  and  $g(n)$  is  $O(f(n))$
- ▶ Big Oh and  $\sim$  aim to capture 'order of growth'
- ▶ Big Oh abstracts from constant factors,  $\sim$  does not
- ▶ Large constant factors are important!

## Important orders of growth

- ▶ constant:  $c$ ,  $f(n) = c$  for all  $n$
- ▶ linear:  $n$  (compare all for  $n = 20$  sec)
- ▶ linearithmetic:  $n \log n$
- ▶ quadratic:  $n^2$
- ▶ cubic:  $n^3$
- ▶ exponential:  $2^n$
- ▶ general form:  $an^b(\log n)^c$

## Logarithms and Exponents

- ▶ Definition:  $\log_x z = y$  iff  $x^y = z$  for  $x > 0$
- ▶ Inverses:  $x^{\log_x y} = y$  and  $\log_x x^y = y$
- ▶ Exponent:  $x^{(y+z)} = x^y x^z$ ,  $x^{(yz)} = (x^y)^z$
- ▶ Logarithm:  $\log_x(yz) = \log_x y + \log_x z$ ,  $\log_x z = \log_x y \log_y z$
- ▶ Base of logarithm: the  $x$  in  $\log_x$
- ▶ Various bases:  $\log_2 = \lg$ ,  $\log_e = \ln$ ,  $\log_{10} = \log$
- ▶ Double exponent: e.g.  $2^{(2^n)}$  (not used in INF102)
- ▶ Double logarithm:  $\log(\log n)$  (not used in INF102)

## Worst case, average case, amortized cost

- ▶ Worst case: guaranteed, independent of input; Examples:
  - ▶ Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
  - ▶ Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- ▶ Average case: not guaranteed, dependent of input *distribution*
- ▶ Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost  $\leq 21$ , all others cost 1, amortized  $\leq 3$  p/o.
- ▶ Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- ▶ Special case of resizing array that is only growing:  
 $1(2)2(4)3(8)4(16)5(32)6(64)7(128)8(256)9(512) \dots 16(32768) \dots$ , with  $(n)$  the new size.  
 Resizing to  $(n)$  costs  $2n$  array accesses, so in total  
 $(1+4)+(1+8)+(2+16)+(4+32)+(8+64) \dots$ , so 9 p/push.

## Staying Connected

- ▶ We want efficient algorithms and datastructures for testing whether two objects are 'connected'
- ▶ MNF130: relation  $E \subseteq V \times V$  is an *equivalence* if
  - ▶  $E$  is *reflexive*:  $\forall x \in V. E(x, x)$
  - ▶  $E$  is *symmetric*:  $\forall x, y \in V. E(x, y) \rightarrow E(y, x)$
  - ▶  $E$  is *transitive*:  $\forall x, y, z \in V. E(x, y) \wedge E(y, z) \rightarrow E(x, z)$
- ▶ We assume connectedness to be an equivalence
- ▶ Dynamic connectivity means (here) that  $E$  can grow
- ▶ Clear relationship with paths in graphs, (connected) components (MNF130)
- ▶ Input:  $N$  and pairs in  $V = \{0, \dots, N-1\}$  defining  $E$
- ▶ Challenge: efficient `boolean connected(int p, int q)`
- ▶ Example:  $N = 10$ , 4 3, 3 8, ... (`algs4-data/tinyUG.txt`)
- ▶ Picture on blackboard (don't print pairs that are already connected)

## Union-Find

- ▶ Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- ▶ Union, idea: for any new pair  $n\ m$  that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- ▶ API: **UF**; Cost model: number of array accesses
- ▶ Implementations:
  - ▶ **SlowUF.java**: `id[p]` identifier of  $p$   
`find()`  $\sim 1$ , `union()`  $\sim$  between  $n+3$  and  $2n+1$
  - ▶ **FastUF.java**: `int[] id` pointers, `id[p]==p`: identifier  
`find()`  $\sim 1+2d$ , `union()`  $\sim 1 + \text{two find()}'s$
  - ▶ **WeightedUF.java**: `int[] id` pointers, `int[] sz` subtree sizes  
`find()` and `union()` both  $\sim \lg n$
- ▶ WeightedUF: height of subtree of size  $k$  is at most  $\lg k$
- ▶ Path-compression: ultimate improvement of UF (almost  $O(1)$ , amortized)

# Sorting

- ▶ Sorting: putting objects in a certain order
- ▶ MNF130: relation  $R \subseteq V \times V$  is a *total order(ing)* if
  1.  $R$  is *reflexive*:  $\forall x \in V. R(x, x)$
  2.  $R$  is *transitive*:  $\forall x, y, z \in V. R(x, y) \wedge R(y, z) \rightarrow R(x, z)$
  3.  $R$  is *antisymmetric*:  $\forall x, y \in V. R(x, y) \wedge R(y, x) \rightarrow x = y$
  4.  $R$  is *total*:  $\forall x, y \in V. R(x, y) \vee R(y, x)$
- ▶ Natural orderings:
  - ▶ Numbers of any type: ordinary  $\leq$  and  $\geq$
  - ▶ Strings: lexicographic
  - ▶ Objects of a Comparable type: `v.compareTo(w) <= 0`



## Sorting (ctnd)

- ▶ Bubble sort: `ExampleSort.java`
- ▶ Certification: `assert isSorted(a)` in `main()`
- ▶ No guarantee against modifying the array (but `exch()` is safe)
- ▶ Costmodel 1: number of `exch()`'s and `less()`'s
- ▶ Costmodel 2: number of array accesses
- ▶ Pitfalls: cache misses, expensive `v.compareTo(w) < 0`
- ▶ Why studying sorting? (`java.util.Arrays.sort()`)
- ▶ Comparing sorting algorithms: `SortCompare.java`

## Selection Sort

- ▶ Bubble sort:  $\sim n^2/2$  compares,  $0 \leq \text{exchanges} \leq \sim n^2/2$
- ▶ Selection sort:
  - ▶ Find index of a minimum in  $a[0..n-1]$ , exchange with  $a[0]$
  - ▶ Find index of a minimum in  $a[1..n-1]$ , exchange with  $a[1]$
  - ▶ ... until  $n-2$
- ▶ Selection sort:  $\sim n^2/2$  compares,  $0 \leq \text{exchanges} \leq n-1$  (!)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=0; i<N-1; i++){  
        int min=i;  
        for (int j=i+1; j<N; j++) if (less(a[j],a[min])) min=j;  
        if (i != min) exch(a,i,min);  
    }  
}
```

## Insertion sort

- ▶ Insertion sort:
  - ▶ Insert  $a[1]$  on its correct place in (sorted)  $a[0..0]$
  - ▶ Insert  $a[2]$  on its correct place in (sorted)  $a[0..1]$
  - ▶ ... until  $a[n-1]$
- ▶ Very good for partially sorted arrays, costs:
  - ▶ Best case:  $n-1$  compares and 0 exchanges
  - ▶ Worst case:  $\sim n^2/2$  compares and exchanges
  - ▶ Average case:  $\sim n^2/4$  compares and exchanges (distinct keys)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=1; i<N; i++){  
        for (int j=i; j>0 && less(a[j],a[j-1]); j--)  
            exch(a,j,j-1);  
    }  
}
```

## Shell sort

- ▶ Insertion sort:
  - ▶ Very good for partially sorted arrays
  - ▶ Slow in transport: step by step `exch(a,j,j-1)`
- ▶ Idea: h-sort, `a[i], a[i+h], a[i+2h], ...` sorted (any `i`)

```
public static void hsort(int h, Comparable[] a) {  
    int N = a.length;  
    for (int i=h; i<N; i++)  
        for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)  
            exch(a,j,j-h);  
}
```

- ▶ Insertion sort: `hsort(1,a)`
- ▶ Shell sort: e.g., `hsort(10,a); hsort(1,a)`

## Shell sort (ctnd)

- ▶ `hsort(10,a); hsort(1,a)` faster than just `hsort(1,a)` !
- ▶ Q: How is this possible?
- ▶ A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- ▶ What about `hsort(100,a); hsort(10,a); hsort(1,a)`?
- ▶ To be expected: depends on the length  $N$  of the array
- ▶ Best practice:  $h = N/3, N/9, \dots, 364, 121, 40, 13, 4, 1$

# Mergesort

- ▶ Top-down (recursive) algorithm:
  - ▶ Mergesort left half, mergesort right half
  - ▶ Merge the results
- ▶ Using an auxiliary array: [TopDownMergeSort.java](#), [Movie](#)
- ▶ Bottom-up algorithm (16 elements):
  - ▶ Merge  $a[0], a[1]$ , so  $a[2], a[3]$ , so  $a[4], a[5]$ , so ...
  - ▶ Merge  $a[0..1], a[2..3]$ , so  $a[4..5], a[6..7]$ , so ...
  - ▶ Merge  $a[0..3], a[4..7]$ , so  $a[8..11], a[12..15]$
  - ▶ Merge  $a[0..7], a[8..15]$ , done!
- ▶ Also using an auxiliary array: [BottomUpMergeSort.java](#)

## Run-time and memory use of mergesort

- ▶ Mergesort uses between  $\sim (N/2) \lg N$  and  $\sim N \lg N$  compares. Proof on bb. Important formula ( $N = 2^n$ ):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- ▶ Mergesort uses at most  $\sim 6N \lg N$  array accesses
- ▶ Mergesort uses  $\sim 2N$  space (plus some var's)
- ▶ Q: How fast can compare-based sorting of  $N$  distinct keys be?
- ▶ A:  $\lg N! \sim N \lg N$ ; Proof in book and on bb. Keywords: binary *compare tree*, inner nodes for each `compare(a[i], a[j])`, permutations in the leaves,  
 $N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$

# Quicksort

- ▶ Top-down (recursive) algorithm:
  - ▶ Choose a (pivot) value  $v$  in the array
  - ▶ Partition the array in non-empty parts  $\leq v$  and  $\geq v$
  - ▶ Quicksort the two parts
- ▶ Pros: in-place, average computation time  $O(n \log n)$
- ▶ Cons: stack space for recursion, worst-case  $O(n^2)$ , not stable
- ▶ Implementation: **QuickSort.java**
- ▶ BTW: **Bug in java.util.Arrays.sort**



## Quicksort, details

- ▶ Subtleties in `sort()`: shuffling protects against worst-case behaviour
- ▶ Termination of recursive `quicksort()`
- ▶ Subtleties in `partition()`:
  - ▶ Invariants  $l \leq h$  in the two inner loops
  - ▶ Postcondition after the two inner loops
  - ▶ Invariant of the `for(;;)` loop
  - ▶ Termination of the `for(;;)` loop
  - ▶ There are some variations that are also correct

## Run-time and memory use of quicksort

- ▶ Compare Quicksort to other sorts ( $n = 10^2, 10^3, \dots$ )
- ▶ Quicksort: time  $O(n^2)$  if pivot is always smallest (or largest)
- ▶ Randomization: choose pivot randomly, or shuffle array
- ▶ If all keys are distinct and randomization is perfect, then quicksort uses on average  $\sim 2n \ln n$  compares and  $\sim (n/3) \ln n$  exchanges (proofs in book, complicated)
- ▶ Relevant improvements:
  - ▶ Cut-off to insertion sort for sizes  $\leq 15$  (ca.)
  - ▶ Median-of-three pivot
  - ▶ Taking advantage of duplicate keys (3-way partitioning)
- ▶ Quicksort is generally quite good
- ▶ In special situations other sorts are better (e.g., countsort)

## Priority Queues

- ▶ Assume collecting and processing items having keys
- ▶ Examples of keys: time-stamp, price-tag, priority-tag
- ▶ Assume: keys can be ordered
- ▶ Reasonable: processing currently highest (or lowest)
- ▶ Special cases: items time-stamped when added
  - ▶ Queue: dequeue currently oldest (lowest time-stamp)
  - ▶ Stack: pop currently newest (highest time-stamp)
- ▶ Priority queue generalizes this
- ▶ Examples: highest priority, largest transaction, lowest price
- ▶ Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

## Priority Queues

- ▶ Good info: [Wikipedia](#); API (the bare essentials):

```
public class  ArrayListPQ<Key extends Comparable<Key>>

void          insert(Key v) // insert a key
Key           delMax() // delete a largest key, if any
boolean       isEmpty()
int           size()
```

- ▶ Aim: operations in logarithmic time, no extra space
- ▶ In case of duplicate keys: 'a' largest, not 'the'
- ▶ Typical application: the 1K largest keys of 1G unsorted keys
- ▶ Client: [BottomM.java](#) (Q: why is the output slowing down?)

## Binary Trees

- ▶ MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- ▶ MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height  $h$  has  $2^{h+1}-1$  nodes.
- ▶ INF102: A binary tree is (left-) *complete* if all levels  $< h$  are filled, level  $h$  may be partly empty from the right (picture bb). A (left-)complete binary tree of height  $h$  has between  $2^h$  and  $2^{h+1}-1$  nodes (from now on we leave out '(left-)').
- ▶ A complete binary tree of  $n$  nodes has height  $\lfloor \lg n \rfloor$

## Picture



## Heap-ordered Binary Trees

- ▶ A binary tree is *heap-ordered* if the key in each node is  $\geq$  the keys in its children (if any). Thus the root has a maximal key.
- ▶ Array representation of heap-ordered complete binary tree (bb)
- ▶ Methods `swim()` and `sink()`: picture on bb, code below
- ▶ Implementation: `ArrayListPQ.java`

## Run-time and memory use of heaps, applications

- ▶ In a heap of  $n$  elements (since height is  $\leq \lfloor \lg n \rfloor$ ):
  - ▶ `swim()`, and hence `insert()`, takes  $\leq 1 + \lfloor \lg n \rfloor$  compares and  $\leq \lfloor \lg n \rfloor$  exchanges
  - ▶ `sink()`, and hence `delMax()`, takes  $\leq 2\lfloor \lg n \rfloor$  compares
  - ▶ `sink()` takes  $\leq \lfloor \lg n \rfloor$  exchanges, and `delMax()`  $\leq 1 + \lfloor \lg n \rfloor$
- ▶ Heap construction by `insert()` can sometimes be improved
- ▶ Given an array of keys, right-to-left heap construction (bb) takes  $< 2n$  compares and  $< n$  exchanges
- ▶ Applications: **heapsort** and merging sorted streams (bb)
- ▶ Many variations with extended API (indexed priority queue)



## Purpose of Sorting

- ▶ Sorting makes the following easier and more efficient:
  - ▶ Searching (binary search, example: `ThreeSumOptimized`)
  - ▶ Searching and looking up, e.g., the `pagenumber` in an index
  - ▶ Removing duplicates
  - ▶ Finding the median, quartiles etc.
- ▶ Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- ▶ We do *pointer sorting*, manipulating refs to objects.
  - ▶ Pro: not moving full objects
  - ▶ Cons: pointer dereferencing, no `sort(int[] a)`
- ▶ More flexibility: pass a `Comparator` object to `sort()`

## Comparator object

- ▶ API: `void sort(Object[] a, Comparator c)`
- ▶ Call, e.g.: `sort(a, new Transaction.WhenOrder())`
- ▶ Call, e.g.: `sort(a, new Transaction.SizeOrder())`
- ▶ Obs: `import java.util.Comparator`
- ▶ Obs: `less(Object o1, Object o2, Comparator c)`
- ▶ Priority queues also with `Comparator`

```
public class Transaction {  
    ...  
    public static class MyOrder {  
        implements Comparator<Transaction>  
        public int compare(Transaction t, Transaction v){...}  
    } // End of Myorder  
    ...// similarly: WhenOrder, SizeOrder  
} // End of Transaction
```

## Applications of Sorting

- ▶ Consider sorting first to make other problems easier
- ▶ Commercial computing (sort on price, departure time, ...)
- ▶ Search for information: web-indexing, search engines
- ▶ Job scheduling heuristic: longest processing time first
- ▶ To come: Prim's, Dijkstra's and Kruskal's algorithms
- ▶ Huffman compression: a lossless compression based on using the shortest codes for the symbols that occur often.  
Frequency counter: next chapter!
- ▶ Cryptology and genomics (e.g., longest repeated substring)

# Symbol Tables

- ▶ Symbol table associates *keys* with *values*: *key-value pairs*
- ▶ Examples: keyword-page number, ID number-personal data
- ▶ Important operations:
  - ▶ Insert a key-value pair in the symbol table: `void put(k,v)`
  - ▶ Search the value for a given key (if any): `Value get(k)`
- ▶ Important conventions:
  - ▶ Inserting key-value for existing key: overwriting the value
  - ▶ No duplicate keys, no null keys
  - ▶ Value null: no value for this key
  - ▶ Lazy deletion: insert key-null; Eager: really delete key
- ▶ **API** of unordered symbol table
- ▶ Aim: all operations in time  $\sim c \lg n$  with constant  $c$  small

# ST Basics

- ▶ Archetypical ST-client: frequency counter (code: `main`)
- ▶ Cost model: number of compares
- ▶ Naive ST: unordered linked list, linear search (INF101, Ch.9)
  - ▶ Search miss:  $\sim n$  compares
  - ▶ Search hit: between 1 and  $\sim n$  compares
  - ▶ Random search hit:  $(1 + \dots + n)/n \sim n/2$  compares
  - ▶ Inserting  $n$  distinct keys:  $(1 + \dots + (n-1)) \sim n^2/2$  compares
- ▶ `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- ▶ Naive ST impracticable for genomics, internet
- ▶ Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- ▶ Better for unordered ST: hashing (in Ch. 3.4)

## Ordered Symbol Table

- ▶ Ordered ST: keys are ordered
- ▶ **API** of ordered symbol table
- ▶ Binary search: `get(Key k)` takes  $\sim \lg n$  comparisons
- ▶ What about `put(Key k, Value v)`? **ArrayListST**, good!
- ▶ TODO: test that `add(int i, E e)` is amortized  $O(1)$
- ▶ Implementation with binary search in **ArrayListST.java**
- ▶ Trace of inserts on bb: **S E A R C H E X A M P L E**
- ▶ Experiments with `tinyTale.txt`, `tale.txt`, ...

## Binary Search Trees

- ▶ Binary *search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- ▶ Search time: length of the path to the node where the key 'should' be
- ▶ Balanced binary tree with  $n$  keys has  $\lg n$  height
- ▶ Unbalanced binary trees can have height  $n$  (max depth)
- ▶ Search hits in a binary search tree, built without rebalancing, of  $n$  random keys take on average  $\sim 2 \ln n$  compares
- ▶ **UBST.java**: `put()`, `get()`, `size()`, `isEmpty()`
- ▶ Trace of inserts on bb: S E A R C H E X A M P L E

## Binary Search Trees (ctnd)

- ▶ Interrelated, increasing difficulty: `min(Node x)`, `deleteMin(Node x)`, `delete(Node x, Key k)`
- ▶ Node of minimum key: not `null`, and has left child `null`, and is root or left child of parent (picture on bb)

```
public Node min(Node x){// x != null, subtree not empty
    while (x.left!=null) x = x.left;
    return x;
} // cf. tail recursive min() in Alg. 3.3
```

- ▶ Delete minimum key, two cases:  
(1) both children `null`; (2) left child `null`
- ▶ Delete is really difficult: bb + **BST.java**
- ▶ Don't forget: update `x.N` along the path to the root!



## Balanced Search Trees: keep paths short!

- ▶ NB tree balancing not as easy as in UF and Heap (4hrs!)
- ▶ A 2-3 search tree consists of 2-nodes and 3-nodes:
  - ▶ Each 2-node has two children and a key  $k$  such that all keys in the left subtree are  $< k$ , and all keys in the right subtree  $> k$
  - ▶ Each 3-node has three children and two keys  $k_1, k_2$  such that all keys in the left subtree are  $< k_1$ , all keys in the middle subtree  $> k_1$  and  $< k_2$ , and all keys in the right subtree  $> k_2$
- ▶ Examples and pictures on bb
- ▶ *Perfect* 2-3 search tree: paths from root to leaves equally long
- ▶ Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- ▶ Insert should keep tree perfect, rough idea:
  - ▶ into a 2-leaf: make it into a 3-leaf
  - ▶ into a 3-node: do something clever (explained next)

## Insert in Balanced Search Trees

- ▶ Terminology: a *leaf* is a node all whose children are null
- ▶ Data invariant 1: 2-3 search tree
- ▶ Data invariant 2: paths from root to leaves equally long

- ▶ Insert into a 2-leaf  $L$  : either  $\begin{array}{c} A \ L \\ | \quad | \end{array}$  or  $\begin{array}{c} L \ Z \\ | \quad | \end{array}$

- ▶ into a 3-leaf whose parent is a 2-node: with new key  $Z$



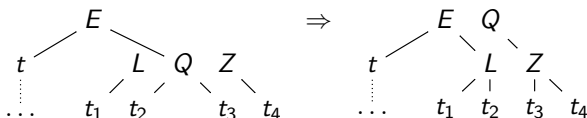
- ▶ into a 3-leaf whose parent is a 3-node: with new key  $Z$  added



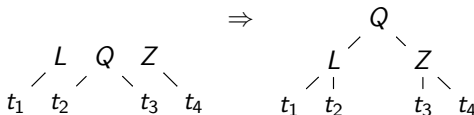
- ▶ into a 3-node whose parent is a 3-node: move up middle key!

## Insert (ctnd)

- ▶ Data invariant 1: 2-3 search tree
- ▶ Data invariant 2: paths from root to leaves equally long
- ▶ Insert works up from the leaf where the key 'should' be
  - ▶ if 2-node on path to root: make it into a 3-node (two cases)



- ▶ otherwise: split the root



## Insert, summary and examples

- ▶ Six operations for eliminating 4-nodes:
  - ▶ if parent is 2-node: move middle key up (left and right case)
  - ▶ if parent is 3-node: move middle key up (left, middle, right)
  - ▶ if root: split root
- ▶ Search and insert visit at most  $\lfloor \lg n \rfloor$  nodes
- ▶ Proof: maximal path length is  $\geq \lfloor \log_3 n \rfloor$  and  $\leq \lfloor \log_2 n \rfloor$
- ▶ Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- ▶ Trace of inserts on bb: A C E H L M P R S X

## Red-black trees

- ▶ Red-black trees implement 2-3 trees
- ▶ Idea: one 3-node = two 2-nodes + extra info
- ▶ Extra info coded in color, picture:




- ▶ A *red-black tree* is a binary search tree with red and black links such that:
  - ▶ Only left links can be red (but need not be)
  - ▶ Never
  - ▶ Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- ▶ Equivalent: red-black tree and perfect 2-3 search tree

## Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

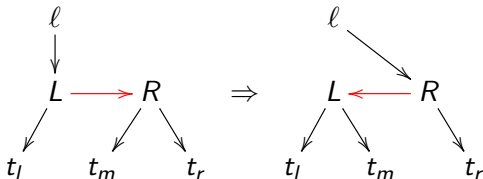
```
private class Node {  
    Key key;  
    Value value;  
    Node left, right;  
    boolean color; // true for red, false for black  
    int N;  
}  
private boolean isRed(Node n) {  
    if (n==null) {return false;} else {return x.color}
```

## Rotating and Color Flipping

- ▶ Aim: restoring the data invariants of red-black search trees
  1. Only left red links, but never two
  2. Search tree invariant
  3. Perfect black balance
- ▶ Invariants get violated by temporary 4-nodes, e.g.,
  - ▶ inserting  $Z$  in  $L \leftarrow R : L \leftarrow R \rightarrow Z$
  - ▶ inserting  $A$  in  $L \leftarrow R : A \leftarrow L \leftarrow R$
  - ▶ inserting  $M$  in  $L \leftarrow R : L \leftarrow R$   

- ▶ Restoring:
  - ▶ Color flip  $L \leftarrow R \rightarrow Z : L \leftarrow R \rightarrow Z$
  - ▶ Rotation right + color flip  $A \leftarrow L \leftarrow R : A \leftarrow L \rightarrow R$
  - ▶ Rotation left into  $M \leftarrow L \leftarrow R$ , then as previous

## Left Rotation

Call: `l = rotateLeft(l);`



```
private Node rotateLeft(Node l){  
    Node r = l.right; l.right = r.left; r.left = l;  
    r.color = l.color; l.color = true // == RED  
    r.N = l.N; l.N -= 1+size(r.right); // Why?  
    return r;  
}
```



## Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- ▶ Code of `rotateRight()` like that of `rotateLeft()`
- ▶ NB1: operations are local (here only  $r$ ,  $M$ ,  $R$ )
- ▶ NB2: operations preserve data invariants
- ▶ NB3: root is a special case (always black)
- ▶ Deletions: complicated, but doable (Exc. 3.3.39–41)

## Run-time and memory use of Red-Black BSTs

- ▶ The height of a red-black BST with  $n$  nodes is  $\leq 2 \lg n$   
Proof: the worst-case is one 3-node path and the rest 2-nodes
- ▶ The average length of path from the root to a *node* (?) in a red-black BST with  $n$  nodes is  $\lg n$  ('empirical fact')
- ▶ In a red-black BST, search, insert, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node.
- ▶ For red-black BSTs, logarithmic time is guaranteed!

# Hashing

- ▶ Idea: if keys in  $[0..99]$  an array is the perfect symbol table
- ▶ A *hash function* maps a key to an array index
- ▶ Injectivity of the hash function is not guaranteed
- ▶ *Hash collision*: different keys are mapped to the same index
- ▶ In such a case we need *collision resolution*
- ▶ Symbol tables: hashing is fast, but unordered (no `max`, `min`)
- ▶ Aim: ST operations amortized  $O(1)$  time, extra space OK

## Space-Time Trade-Off

- ▶ Hashing is an example of a *space-time trade-off*
- ▶ Time: computation time required
- ▶ Space: memory space used
- ▶ Unlimited space: (1) use key as index (e.g., the bits)
- ▶ Unlimited time: (2) use linked list and linear search
- ▶ Hashing strikes a balance using (1) with some array of reasonable size, and (2) in case of collisions
- ▶ The balance between (1) and (2) can easily be tuned

## Hash functions

- ▶ Ideal (uniform hashing assumption, UHA): uniform and independent distribution of keys over integers from 0 to  $M - 1$
- ▶ Examples of **hash functions in Java**
- ▶ Horner:  $a_0 + x(a_1 + x(a_2 + \dots)) = a_0 + a_1x + a_2x^2 + \dots$
- ▶ Reasonably  $\approx$  UHA: modular hashing ( $M$  prime):  

```
private int hash(Key k){  
    return (key.hashCode() & 0x7fffffff) % M;}  

```
- ▶ Q: Why  $M$  prime?
- ▶ A: e.g.  $M = 32$  takes only into account the last five bits

## Collision Resolution

- ▶ Two methods of collision resolution:
  1. Hashing with separate chaining (picture on bb)
  2. Hashing with linear probing (picture on bb)
- ▶ Separate chaining: symbol table is an array of linked lists, linear search. If array has length  $M$ , then the linked lists have average length  $N/M$  with  $N$  keys.
- ▶ Linear probing: symbol table is an array of length  $M > N$ . Colliding keys are put at the first empty position. Linear search from the position where the key 'should have been'. Empty position: not found. Deletion tricky: reinsert all keys to the right of the deleted key, until the first empty position (picture on bb). Array must have length  $\geq N$  with  $N$  keys.

## Symbol Table with Hashing

- ▶ Implementation: `ArrayListHashST.java`
- ▶  $M = 1$ : measure overhead wrt. `ArrayListST.java`
- ▶ Tests with various values of  $M$ : 31, 997, 65521
- ▶ NB: *construction* versus *use* of ST (hashing better for *use*)
- ▶ Hashing can be combined with any other ST-implementation

## Quantitative analysis

- ▶ Throwing a dice 10 times, what is the probability of 3 fives?
- ▶ Under UHA, with  $N$  distinct keys, the probability that exactly  $k$  keys collide at some given hash value is

$$\binom{N}{k} \left(\frac{1}{M}\right)^k \left(\frac{M-1}{M}\right)^{N-k}, \text{ where e.g. } \binom{100}{10} \approx 1.7E13$$

- ▶ This is a small number for, say,  $N = M = 100$  and  $k = 10$
- ▶ For linear probing one typically takes  $M = 2N$
- ▶ For separate chaining one keeps  $N/8 \leq M \leq N/2$  (resizing  $M$ )
- ▶ Under UHA: search, insert, delete take amortized  $O(1)$  time
- ▶ Extra space used can be upto  $100N$  byte (objects, pointers); this on top of the space used by  $N$  key-value pairs



## Applications of Searching

- ▶ Synonyms: **associative array**, map, symbol table, or dictionary
- ▶ Origin of **symbol table**: compilers and interpreters
- ▶ Web-indexing, **search engines**
- ▶ Sparse matrices (many 0's): **dictionary**
  1. keys: (row, column)-pairs
  2. values: matrix entries
- ▶ Set API (no values, only keys, for deduplication, filtering):

```
public class SET<Key>
```

```
void      add(Key k)
void      delete(Key k)
boolean   contains(Key k)
boolean   isEmpty()
int       size()
```

## Applications of Searching (ctnd)

- ▶ Application (key, value)
- ▶ Phone book (name, phone number)
- ▶ Dictionary (word, meaning or translation)
- ▶ Account information (client ID, account information)
- ▶ Genomics (sequences of ACTG triplets, proteins)
- ▶ Experimental data of various kinds
- ▶ File systems (file name, address etc)
- ▶ Internet domain name system (domain name, IP address)
- ▶ Invertex index (value, key(s))

## Balanced Search Tree or Hash Table?

- ▶ Q: Which symbol table to use?
- ▶ A: The basic choice between BST and HT depends on ...
  1. Ordering of keys essential: BST
  2. Availability of good hash function (good = fast + UHA)
  3. Ordering of keys expensive (long strings): HT (or: Ch.5)
  4. Ordering of keys possible, but not essential: HT + BST
  5. Space considerations (ArrayListST uses the least extra space)
  6. Number of distinct keys and the space each key takes
  7. Distribution of insert/delete/search operations

## Odds and Ends Chapter 1–3

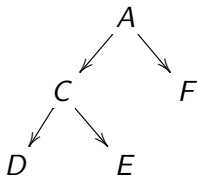
- ▶ Discuss methodological issues SortCompare
- ▶ Find out if `add(int i, E e)` in `ArrayList` is amortized  $O(1)$
- ▶ Discuss primitive types (objects are costly)
- ▶ **Distributed Hash Table**
- ▶ Double hashing: linear probing  
 $h_1(k), h_1(k) + h_2(k), h_1(k) + 2h_2(k), \dots$
- ▶ Indexed Priority Queues: next slide

## Indexed Priority Queues

- ▶ IPQ  $\approx$  array with direct access to minimum (maximum)
- ▶ API: `void insert(int i, Key k); void del(int i); int minKey();`  
`Key keyOf(int i);...` Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-
qp	1	0	2	4	3	-1	-1

heap



Do: `insert(6,G)`, `insert(5,B)`

## Graph classes

( MNF130: useful review of graph theory)

1. Undirected graphs: a set of *vertices* (or *nodes*)  $V$  and a set of *edges*  $E$  connecting the nodes
2. Directed graphs (*digraphs*): a set of nodes  $V$  and a set  $E$  of edges (or *arrows*) pointing from one node to another
3. *Edge-weighted graphs*: undirected graphs in which every edge has a number called the *weight* of the edge
4. *Edge-weighted digraphs*: digraphs in which every arrow has a weight

## Examples

1. Undirected graphs: social networks, communication networks (duplex communication), road maps
2. Directed graphs: logical circuits, job scheduling, flow graphs, hyperlinks, (class, module, package) dependencies
3. Edge-weighted graphs: roadmaps with geographical distance, communication networks with bandwidth
4. Edge-weighted digraphs: job scheduling with duration, transport of goods, financial transactions

## Undirected Graphs

- ▶ Undirected graph: a set of *vertices* (or *nodes*)  $V$  and a set of *edges*  $E$  connecting the nodes
- ▶ *Subgraph*: subset of  $E$  and subset of  $V$  forming a graph (!)
- ▶ *Path*: sequence of nodes connected by edges (!)
- ▶ *Simple path*: path with no node repeated
- ▶ *Length of path*: number of edges
- ▶ *Cycle*: path of length  $> 0$  with same start and end node
- ▶ *Simple cycle*: not repeating edges or nodes (apart from start and end node)
- ▶ *Acyclic graph*: graph without simple cycles
- ▶ *Connected graph*: having a path between every two nodes
- ▶ *Connected component*: a maximal connected subgraph



## Trees and Forests

- ▶ 'Anomalies' concerning edges:
  - ▶ Self-loop: edge connecting a node to itself
  - ▶ Parallel edges: two edges connecting the same node(s)
- ▶ When no anomalies,  $E \subseteq \{\{v, v'\} \mid v \in V, v' \in V, v \neq v'\}$
- ▶ *Tree*: connected acyclic graph (then: no anomalies)
- ▶ *Spanning tree*: maximal subgraph that is a tree
- ▶ Lemma: any spanning tree of a connected graph contains all nodes. Proof by contradiction (on bb).
- ▶ *Forest*: graph consisting of disjoint trees
- ▶ *Spanning Forest*: forest consisting of spanning trees of connected components of a graph
- ▶ Example: `tinyG.txt` on bb

## Undirected Graphs (ctnd)

- ▶ *Distance* between two nodes: length of a shortest connecting path if there is a path connecting these nodes, otherwise  $\infty$
- ▶ *Degree* of a node: number of edges connected to that node
- ▶ Graph  $G = (V, E)$ , the following are equivalent:
  - ▶  $G$  is a tree
  - ▶  $G$  has  $|V| - 1$  edges and no cycles
  - ▶  $G$  has  $|V| - 1$  edges and is connected
  - ▶  $G$  is acyclic and adding an edges creates a cycle
  - ▶ Any two nodes of  $G$  are connected by exactly one simple path
- ▶ Example: some subgraphs of `tinyG.txt` on bb

## Graph representation and implementation

- ▶ Impractical: **adjacency matrix**  $\sim V^2$ , **incidence matrix**  $\sim VE$
- ▶ Often practical: **adjacency lists**  $\sim (V+2E)$ , that is, `adj[v]` lists all nodes `w` connected to `v` by an edge
- ▶ Example: `tinyG.txt` by **LinkedListG.java**
- ▶ Graph API includes: `V()`, `E()`, `addEdge()`
- ▶ Basic algorithms: Depth-First Search (DFS) and Breadth-First Search (BFS)
- ▶ Both DFS and BFS 'walk through the graph', in different ways
- ▶ Both DFS and BFS can compute a spanning tree and forest

```
public void dfs(Integer v, boolean[] marked) {
    marked[v] = true;
    for (Integer w : adj[v])
        if (! marked[w]) dfs(w,marked);
} // dfs() is recursive, call: dfs(v,marked);

public void bfs(Queue<Integer> q, boolean[] marked) {
    while (!q.isEmpty()) {
        Integer v = q.dequeue();
        for (Integer w : adj[v])
            if (! marked[w]) {marked[w]=true; q.enqueue(w);}
    }
} // call: marked[v]=true; q.enqueue(v); bfs(q,marked);

// Example: 0-1, 0-3, 1-2, 1-3, 3-4
// Example: complete ternary tree of height 2
```

## Implementation and Properties of DFS/BFS

- ▶ **LinkedListG.java**: `pathdfs()`, `pathbfs()`
- ▶ DFS and BFS mark nodes connected to a given source node in time proportional to the sum of their degrees ( $\leq 2E$ ), and can return a path from a marked node to the given source in time proportional to the length of this path
- ▶ BFS always finds a shortest path (proof: queue only contains nodes at distance  $k$  followed by nodes at distance  $k + 1$ , while all nodes at distance  $\leq k$  not in queue have been processed)
- ▶ DFS finds a left-most path (long or short, example bb)
- ▶ BFS tends to use more space (but not always)
- ▶ UF tests connectivity, but finds no paths

## Applications

- ▶ `StringSTG.java`, flight connections, shortest path = minimum number of stop-overs
- ▶ Degrees of separation in social networks, e.g., Erdős number = length of shortest path to Paul Erdős in the co-author graph
- ▶ Connected components:
- ▶ Example: `tinyG.txt` on bb and by `LinkedListG.countcc()`

## Directed Graphs

- ▶ *Digraph*: a set of *vertices* (or *nodes*)  $V$  and a set of *directed edges* (or *arrows*)  $E$  pointing from one node to another
- ▶ *Subdigraph*, *directed (simple) path*, *directed (simple) cycle*, *acyclic*, *length*: as expected
- ▶ *Dag*: Directed **a**cyctic **g**raph; *dipath*: **d**irected **p**ath
- ▶ *Degree*: **i**n-degree and **o**ut-degree
- ▶ Node  $v$  is *reachable* from  $w$ : a dipath from  $w$  to  $v$  exists
- ▶ *Strongly connected digraph*: dipath between every two nodes (for all  $v, w$ , there are dipaths from  $v$  to  $w$  and from  $w$  to  $v$ )
- ▶ *Strongly connected component*: maximal strongly connected subgraph ( $u \rightleftarrows v \rightarrow w$  has two scc's)
- ▶ Often we leave out 'di' in digraph, dipath, etc.
- ▶ Representation: adjacency lists even simpler!

## Reachability Problems

Assume we are given a directed graph  $G$ .

- ▶ Single-source: given a node  $s$ , the *source*, is a given node  $v$  reachable from  $s$ ?
- ▶ Multiple-source: given a set of nodes  $S$ , is a given node  $v$  reachable from some node in  $S$ ?
- ▶ Solutions: same DFS and BFS algorithms as in Chapter 1
- ▶ Application (example): mark-and-sweep garbage collection
- ▶ Single-source path: given  $s, v$  such that  $v$  is reachable from  $s$ . Find a path from  $s$  to  $v$ .
- ▶ Single-source shortest path: given  $s, v$  such that  $v$  is reachable from  $s$ . Find a *shortest* path from  $s$  to  $v$ .
- ▶ Solutions: same DFS (path) and BFS (shortest path) algorithms as for undirected graphs



## Cycle Detection

- ▶ Recall: a *Dag* is a graph without a directed cycle
- ▶ Acyclicity test, cycle detection: easy extension of DFS. We keep track of the search path from the source. If there is an edge from  $v$  to  $w$  and  $w$  is on the path from  $v$  to the source, then there is a cycle. (DFS finds the leftmost path to the leftmost cycle.) Two techniques:
  - ▶ Go back the search path: `LinkedListG.slowCyclist()`
  - ▶ Memorize the search path: `LinkedListG.fastCyclist()`
- ▶ Application: precedence scheduling of jobs

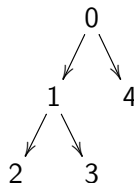
## Pre-order, post-order

- ▶ Graph walks based on dfs from a source node
- ▶ Pre-order: order in which dfs arrives at nodes
- ▶ Post-order: order in which dfs leaves nodes
- ▶ In-order *for binary trees*: e.g., in `UBST.show()`
- ▶ Example:

pre-order: 01234

post-order: 23140

in-order: 21304



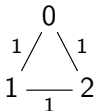
- ▶ Example: `tinyCG.txt` on `bb` and by `LinkedListG`

## Topological order of acyclic digraph

- ▶ Topological order: total order  $\prec$  compatible with the graph in the following sense: if there is an edge from  $w$  to  $v$ , then  $v \preceq w$  (consequently:  $v \preceq w$  if  $v$  is reachable from  $w$ )
- ▶ Lemma: if a digraph has a topological order, then it is acyclic (proof: a cycle cannot be ordered compatibly)
- ▶ Lemma: if a digraph is acyclic, then it has a topological order (proof idea: if acyclic, the post-order is a topological order since, if there is an edge from  $w$  to  $v$ , then  $w$  is not reachable from  $v$  and dfs will leave  $w$  after it has left  $v$ )

## Minimum Spanning Tree

- ▶ Recall 65: *spanning tree* of a connected undirected graph is a maximal subgraph that is a tree (and thus contains all nodes and is acyclic)
- ▶ Example: tinyEWG.txt on bb
- ▶ EWG = Edge-Weighted Graph, here always connected
- ▶ Recall 66: all spanning trees have  $V - 1$  edges
- ▶ *Weight* of spanning tree: sum of the weights of its edges
- ▶ *Minimum Spanning Tree*: spanning tree with minimal weight
- ▶ Example: three MSTs of




- ▶ Exc.4.3.3: if all weights different, then MST is unique
- ▶ From now on we assume all weights different!

## Minimum Spanning Tree (ctnd)

- ▶ Applications: power plants and electrical grid, airlines and flight routes, maps and distance
- ▶ Weights may be zero or negative (e.g., cost, profit)
- ▶ Two important algorithms to find the MST: Prim's and Kruskal's

## Cuts and Crossing Edges

- ▶ Recall **66**: deleting an edge from a tree creates two disjoint components, adding an edge creates a cycle
- ▶ *Cut*: a partition of  $V$  in two non-empty subsets of nodes
- ▶ *Crossing edge*: edge connecting two nodes in different subsets of a cut

- ▶ NB there can be more than one crossing edge:
 

- ▶ Lemma: for any cut in an EWG, the crossing edge of minimum weight is in the MST. Proof: given a cut, assume there is a crossing edge  $e$  of weight smaller than the crossing edge(s) that is (are) in the MST (e.g., the dotted edge above). Adding  $e$  creates a cycle, which must contain one other crossing edge  $f$  in the MST. Replacing  $f$  by  $e$  ...

## Prim's Algorithm

- ▶ Datastructures:
  - ▶ EWG represented with adjacency lists  $\text{adj}[v]$
  - ▶ Priority queue  $pq$  for edges
  - ▶ Array  $\text{marked}$  for marking vertices
  - ▶ Queue  $mst$  for the minimum spanning tree
- ▶ Edge is *eligible* if not both endpoints marked (crossing!)
- ▶ Algorithm based on previous lemma, cut: un/marked
  1. mark 0 and add all eligible edges in  $\text{adj}[0]$  to  $pq$
  2. get and delete minimum edge  $e$  from  $pq$
  3. add  $e$  to  $mst$
  4. mark the unmarked endpoint of  $e$ , say  $k$ , and add all eligible edges in  $\text{adj}[k]$  to  $pq$
  5. delete ineligible minimum edges from  $pq$  and get new eligible minimum edge  $e$  from  $pq$
  6. continue at point 3 until  $pq$  is empty

## Prim's Algorithm (ctnd)

- ▶ **LazyPrimMST.java**, methods `scan()` and `prim()`
- ▶ Invariant: at least one of the nodes of an edge in `pq` is marked
- ▶ NOT: all edges in `pq` are crossing edges wrt cut un/marked
- ▶ Lazy: ineligible edges are not eagerly deleted from `pq`
- ▶ Runtime: LazyPrimMST runs in  $O(E \log E)$  time (worst-case)
- ▶ Possible: edges in `pq` the crossing edges wrt cut un/marked
- ▶ Better: if  $v$  unmarked, the only crossing edge of interest is the *lightest* one connecting  $v$  to the marked edges (= `mst` so far)
- ▶ **PrimMST.java**, indexed `pq` + extra array containing lightest crossing edges to `mst` so far
- ▶ Runtime: PrimMST runs in  $O(E \log V)$  time (worst-case)



# Kruskal's Algorithm

- ▶ Datastructures:
  - ▶ EWG represented with adjacency lists  $\text{adj}[v]$
  - ▶ Priority queue  $\text{pq}$  for edges
  - ▶ Union-Find object  $\text{uf}$  testing connectivity
  - ▶ Queue  $\text{mst}$  for the minimum spanning tree
- ▶ Algorithm:
  1. delete the minimum edge  $e$  from  $\text{pq}$
  2. if the points connected by  $e$  are not connected, add  $e$  to  $\text{mst}$  and connect the points in  $\text{uf}$
  3. continue at point 1 until  $\text{pq}$  is empty or  $\text{uf}$  contains all nodes

## ToC and topics of general interest

- ▶ Table of Contents on next slide (all items clickable)
- ▶ Practical stuff: slide 2

Introduction

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