

# INF102

## Algorithms, Data Structures and Programming

Marc Bezem<sup>1</sup>

<sup>1</sup>Department of Informatics  
University of Bergen

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## INF102, practical stuff

- ▶ Lecturer: Marc Bezem; Team: see homepage
- ▶ Homepage: [INF102](#) (requires login)
- ▶ Also: [INF102 on GitHub](#)
- ▶ Tentative [schedule](#)
- ▶ Textbook: [Algorithms, 4th edition](#)
- ▶ Prerequisites: INF100 + 101 ( $\approx$  Ch. 1.1 + 1.2)
- ▶ Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- ▶ Three compulsory exercises, must be passed
- ▶ Digital exam (Inspera) [06.12.2017](#)
- ▶ Old exams: [2004–2017](#)
- ▶ [Table of Contents of these slides](#)

# Resources

- ▶ Good textbook, USA-style: many pages, exercises etc.
- ▶ Average speed must be ca 50 pages p/w
- ▶ Lectures (ca 24) focus on the essentials
- ▶ Slides (ca 120, dense!) summarize the lectures
- ▶ Prepare yourself by reading in advance
- ▶ Workshops: help with **selected** exercises
- ▶ Test yourself by trying some exercises in advance
- ▶ If you can do the exercises (incl. compulsory), you are fine
- ▶ Review of exercises on Tuesday morning

## Generic Bags, Queues and Stacks

- ▶ Generic programming in Java, example: **PolyPair.java**
- ▶ Bag, Queue and Stack are generic, iterable collections
- ▶ All three support adding an element
- ▶ Queue and Stack support removing an element (if any)
- ▶ FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- ▶ Stack: INF101; Bag is a Stack without pop
- ▶ APIs include: `boolean isEmpty()` and `int size()`

## Implementations and one application

- ▶ Implementation of Stack: `LinkedList_Stack.java`
- ▶ Implementation of Queue: `LinkedList_Queue.java`
- ▶ Dijkstra's Two-Stack Expression Evaluation
- ▶ Example:  $( 1 + ( ( 2 + 3 ) * ( 4 * 5 ) ) )$
- ▶ Now you learned something about compilers!
- ▶ `Movie`

## Resizing Arrays

- ▶ Arrays have direct access, but have fixed size
- ▶ Linked lists have flexible size, but no direct access
- ▶ Best of both: use new, resized arrays, *wisely*:
  - ▶ double size when the array becomes overfull
  - ▶ halve size when the array becomes quarter full
- ▶ Resizing takes time and space proportional to size
- ▶ Not too seldom (correctness), not too often (efficiency)
- ▶ Later: we retain *constant time direct access*
- ▶ Later: add operation in *constant time on average*
- ▶ Once we have understood resizing arrays: **ArrayList**

## Implementations

- ▶ `ResizingArray_Stack.java`
- ▶ Arrays give direct access, resizing at reasonable cost
- ▶ `LinkedList_Stack.java`
- ▶ No fixed size, but indirect access incurs a cost
- ▶ Pointers take space and dereferencing takes time
- ▶ Programming with pointers: make a picture

## Computation time and memory space

- ▶ Two central questions:
  - ▶ How long will my program take?
  - ▶ Will there be enough memory?
- ▶ Example: **ThreeSum.java**
- ▶ Inner loop (here  $a[i] + a[j] + a[k] == 0$ ) is important
- ▶ Sorting helps: **ThreeSumOptimized.java**
- ▶ Run some experiments: `1Kints.txt`, `2Kints.txt`, ...



# Methods of Algorithm Analysis

- ▶ Empirical:
  - ▶ Run program with randomized inputs, measuring time & space
  - ▶ Run program repeatedly, varying (doubling) the input size
  - ▶ Measuring time: **StopWatch**
  - ▶ Plot, or log-log plot and **linear regression**
- ▶ Theoretical:
  - ▶ Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
  - ▶ Try to count/estimate/average this cost as function of the input (size)
  - ▶ Use  $O(f(n))$  (MNF130) and  $f(n) \sim g(n)$  (see next slide)

## Orders of Growth, Big Oh and $\sim$

- ▶ Big Oh and  $\sim$  aim to capture 'order of growth'
- ▶ Costs are positive quantities, so  $f, g, \dots : \mathbb{N} \rightarrow \mathbb{R}^+$
- ▶ MNF130:  $f(n)$  is  $O(g(n))$  if there exist  $c \in \mathbb{R}^+$ ,  $N \in \mathbb{N}$  such that  $f(n) \leq cg(n)$  for all  $n \geq N$  (that is, for  $n$  large enough)
- ▶ Example:  $n^2$  and even  $99n^3$  are  $O(n^3)$ , but  $n^3$  is not  $O(n^{2.9})$
- ▶ INF102:  $f(n) \sim g(n)$  if  $1 = \lim_{n \rightarrow \infty} (f(n)/g(n))$
- ▶ If  $f(n) \sim g(n)$ , then  $f(n)$  is  $O(g(n))$  and  $g(n)$  is  $O(f(n))$
- ▶ Not conversely: Big Oh disregards constant factors,  $\sim$  not
- ▶ Factor  $c$  hidden in Big Oh is important in practice
- ▶ Bound  $N$  is important if it is large

## Important orders of growth

order of growth as function of $n$	value for $n = 20$
constant: $c$ , meaning $f(n) = c$ for all $n$	$c$ sec
linear: $n$	20 sec
linearithmetic: $n \log n$	26 sec
quadratic: $n^2$	400 sec
cubic: $n^3$	8000 sec
exponential: $2^n$	1048576 sec
general form: $an^b(\log n)^c$	$a \cdot 20^b \cdot (1.3)^c$ sec

## ThreeSum, theoretically

- ▶ Number of different picks of triples:  $g(n) = n(n-1)(n-2)/6$
- ▶ Inner loop  $a[i] + a[j] + a[k] == 0$  executed  $g(n)$  times
- ▶  $f(n) = n^3/6 - n^2/2 + n/3$
- ▶ Cubic term  $n^3/6$  wins for large  $n$ :  $f(n) \sim n^3/6$
- ▶ Cost model # array accesses:  $\sim n^3/2$
- ▶ Cost array access  $t$  sec: total time  $\sim t * n^3/2$  sec
- ▶ Cost models are (necessary) simplifications!

## ThreeSum, empirically

- ▶ Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4, 51.1 sec
- ▶ The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- ▶ See the plots in [plot sheet](#)
- ▶ Linear regression gives  $y \approx 3x - 10$
- ▶  $\log(f(n)) = 3 \log(n) - 10$  iff

$$f(n) = 10^{\log(f(n))} = 10^{3 \log(n) - 10} = n^3 * 10^{-10}$$

- ▶ Conclusion: cubic in the input size, with constant  $\approx 10^{-10}$
- ▶ No surprise: see the 3-nested loop in [ThreeSum.java](#)
- ▶ Strong dependence on input can be a problem
- ▶ Constant  $10^{-10}$  depends on computer, exponent 3 does not

## Worst case, average case, amortized cost

- ▶ Worst case: guaranteed, independent of input; Examples:
  - ▶ Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
  - ▶ Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- ▶ Average case: not guaranteed, dependent of input *distribution*
- ▶ Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost  $\leq 21$ , all others cost 1, amortized  $\leq 3$  p/o.
- ▶ Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- ▶ Special case of resizing array that is only growing:  
 $1(2)2(4)3(8)4(16)5(32)6(64)7(128)8(256)9(512) \dots 16(32768) \dots$ , with  $(n)$  the new size.  
 Resizing to  $(n)$  costs  $2n$  array accesses, so in total  
 $(1+4)+(1+8)+(2+16)+(4+32)+(8+64) \dots$ , so 9 p/push.

## Remarks and Pitfalls

- ▶ Theoretical approach:
  - ▶ Wrong cost model
  - ▶ JVM optimization can obscure the exponent
  - ▶ Caching can have large impact on memory access
  - ▶ Large constant factor in Big Oh
  - ▶ Worst case can be easy, average case difficult
- ▶ Empirical approach:
  - ▶ The focus is on run time (using space costs time)
  - ▶ Dependence on input, randomization does not always help
  - ▶ Machine/platform dependence
  - ▶ Linear regression not good for, e.g.,  $O(n^2 \log n)$

## Exercise

Aim: better understand the empirical method.

1. Let input sizes 1, 2, 4, 6, 8K take 2, 7.9, 32, 72, 129 sec
2. Make a plot such as in [plot sheet](#) (download)
3. Compute the log's of the input sizes and of the run times and make the log-log plot such as in [plot sheet](#) (second plot)
4. Estimate  $a$  and  $b$  such that the log-log plot is  $y \approx ax - b$
5. Estimate  $a$  and  $b$  through [linear regression](#), compare with 4.
6. Find  $f(n)$  given that  $\log(f(n)) = a \log(n) - b$ . Surprised?

In cases where the run time mostly depends on the size  $n$  of the input and not on the input itself, the function  $f$  is a reasonable (polynomial) estimation of the run time.



## Logarithms and Exponents Cheat Sheet

- ▶ Definition:  $\log_x z = y$  iff  $x^y = z$  for  $x > 0$
- ▶ Base of logarithm: the  $x$  in  $\log_x$
- ▶ Inverses:  $x^{\log_x y} = y$  and  $\log_x x^y = y$
- ▶ Exponent, laws:  $x^{(y+z)} = x^y x^z$ ,  $x^{(yz)} = (x^y)^z$
- ▶ Logarithm, laws:  $\log_x(yz) = \log_x y + \log_x z$ ,  
 $\log_x z = \log_x y \log_y z$
- ▶ Various bases:  $\log_2 = \lg$ ,  $\log_e = \ln$ ,  $\log_{10} = \log$
- ▶ Double exponent: e.g.  $2^{(2^n)}$  (not used in INF102)
- ▶ Double logarithm:  $\log(\log n)$  (not used in INF102)

## Staying Connected

- ▶ We want efficient algorithms and datastructures for testing whether two objects are 'connected' (e.g., in networks)
- ▶ We assume connectedness to be an equivalence
- ▶ MNF130: relation  $E \subseteq V \times V$  is an *equivalence* if
  - ▶  $E$  is *reflexive*:  $\forall x \in V. E(x, x)$
  - ▶  $E$  is *symmetric*:  $\forall x, y \in V. E(x, y) \rightarrow E(y, x)$
  - ▶  $E$  is *transitive*:  $\forall x, y, z \in V. E(x, y) \wedge E(y, z) \rightarrow E(x, z)$
- ▶ Dynamic connectivity means (here) that  $E$  can grow
- ▶ Relationship with paths in graphs, (connected) components (MNF130): nodes are connected if there is a path between them
- ▶ Input:  $N$  and pairs in  $V = \{0, \dots, N-1\}$  defining  $E$
- ▶ Challenge: efficient boolean `connected(int p, int q)`

## Example

- ▶ Example (`algs4-data/tinyUF.txt`) :  $N = 10$
- ▶ Nodes 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- ▶ Edges: 4–3, 3–8, 6–5, 9–4, 2–1, 8–9, 5–0, ...
- ▶ Linear space: don't add pairs that are already connected!
- ▶ Q: what are the costs of storing all pairs that are connected, space and time?
- ▶ See: [algoritmevisualisering](#) by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke

## Union-Find

- ▶ Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- ▶ Union, idea: for any new pair  $n\ m$  that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- ▶ API: **UF**; Cost model: number of array accesses
- ▶ Implementations:
  - ▶ **SlowUF.java**: `id[p]` identifier of  $p$   
`find()`  $\sim 1$ , `union()`  $\sim 2$  or between  $n+3$  and  $2n+1$  (!)
  - ▶ **FastUF.java**: `int[] id` pointers, `id[p]==p`: identifier  
`find()`  $\sim 1+2d$ , `union()`  $\sim 1 + \text{two find()}'s$
  - ▶ **WeightedUF.java**: `int[] id` pointers, `int[] sz` subtree sizes  
`find()` and `union()` both  $\sim \lg n$

## Trees (cf. MNF130) and WeightedUF

A (rooted) *tree* consist of *nodes* (also called *vertices*) one of which is called the *root*  $r$ . Every node  $n$  is connected by an *edge* to zero or more other nodes, called the *children* of the *parent*  $n$ . Moreover, each node  $n \neq r$  has a unique parent in a tree. Trees are naturally depicted in levels starting with the root at level 0, then the level 1 of the children of the root, level 2 of the children of the children of the root, and so on. The level of a node is also called its *depth*. In a finite tree there is always a highest level (maximum depth) and this is called the *height* of the tree.

- ▶ WeightedUF: height of subtree of size  $k$  is at most  $\lg k$  (proof by induction on blackboard)
- ▶ Ultimate improvement of UF (almost  $O(1)$ , amortized): path-compression (sketch on bb)

# Sorting

- ▶ Sorting: putting objects in a certain order
- ▶ MNF130: relation  $R \subseteq V \times V$  is a *total order(ing)* if
  1.  $R$  is *reflexive*:  $\forall x \in V. R(x, x)$
  2.  $R$  is *transitive*:  $\forall x, y, z \in V. R(x, y) \wedge R(y, z) \rightarrow R(x, z)$
  3.  $R$  is *antisymmetric*:  $\forall x, y \in V. R(x, y) \wedge R(y, x) \rightarrow x = y$
  4.  $R$  is *total*:  $\forall x, y \in V. R(x, y) \vee R(y, x)$
- ▶ Natural orderings:
  - ▶ Numbers of any type: ordinary  $\leq$  and  $\geq$
  - ▶ Strings: lexicographic
  - ▶ Objects of a Comparable type: `v.compareTo(w) <= 0`

## Sorting and searching: linear vs binary search

```
int linearSearch(Comparable key, Comparable[] a){  
    for (int i=0; i < a.length; i++){  
        if (key.compareTo(a[i])==0) {return i;}  
    }  
    return -1; // key not in array: O(a.length)  
}
```

```
int binarySearch(Comparable key, Comparable[] a) {  
    int lo=0; int hi=a.length-1; int mid; int test  
    while (lo <= hi){  
        mid = (lo+hi)/2; test = key.compareTo(a[mid]);  
        if (test == 0) {return mid;}  
        if (test < 0) { hi = mid-1;} else {lo = mid+1;}  
    }  
    return -1; // key not in SORTED array: O(lg(a.length))  
}
```

## Sorting (ctnd)

- ▶ Elementary sorts:
  1. Bubble sort (like gas bubbles in sparkling water)
  2. Selection sort (iterated selection of minima)
  3. Insertion sort (iterated insertion of elements)
  4. Shell sort (Shell's refinement of insertion sort)
- ▶ Bubble sort: **ExampleSort.java**
- ▶ Certification: `assert isSorted(a)` in `main()`  
(no guarantee against modifying the array, but `exch()` is safe)
- ▶ Costmodel(s): number of `less()`'s and of `exch()`'s  
(or array accesses; discuss pointer vs. object)
- ▶ Why studying sorting? (`java.util.Arrays.sort()`)
- ▶ Comparing sorting algorithms: **SortCompare.java**



## Selection Sort

- ▶ Bubble sort:  $\sim n^2/2$  compares,  $0 \leq \text{exchanges} \leq n^2/2$
- ▶ Selection sort:
  - ▶ Find index of a minimum in  $a[0..n-1]$ , exchange with  $a[0]$
  - ▶ Find index of a minimum in  $a[1..n-1]$ , exchange with  $a[1]$
  - ▶ ... until  $n-2$
- ▶ Selection sort:  $\sim n^2/2$  compares,  $0 \leq \text{exchanges} \leq n-1$  (!)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=0; i<N-1; i++){  
        int min=i;  
        for (int j=i+1; j<N; j++) if (less(a[j],a[min])) min=j;  
        if (i != min) exch(a,i,min);  
    }  
}
```

## Insertion sort

- ▶ Insertion sort:
  - ▶ Insert  $a[1]$  on its correct place in (sorted)  $a[0..0]$
  - ▶ Insert  $a[2]$  on its correct place in (sorted)  $a[0..1]$
  - ▶ ... until  $a[n-1]$
- ▶ Very good for partially sorted arrays, costs:
  - ▶ Best case:  $n-1$  compares and 0 exchanges
  - ▶ Worst case:  $\sim n^2/2$  compares and exchanges
  - ▶ Average case:  $\sim n^2/4$  compares and exchanges (distinct keys)

```
public static void sort(Comparable[] a) {  
    int N = a.length;  
    for (int i=1; i<N; i++){  
        for (int j=i; j>0 && less(a[j],a[j-1]); j--)  
            exch(a,j,j-1);  
    }  
}
```

## Shell sort

- ▶ Insertion sort:
  - ▶ Very good for partially sorted arrays
  - ▶ Slow due to one-step transport `exch(a,j,j-1)`
  - ▶ Why not larger steps `exch(a,j,j-h)` ?
- ▶ Idea: presort `a[i], a[i+h], a[i+2h], ...` for `i = 0..h-1`

```
public static void hsort(int h, Comparable[] a) {  
    int N = a.length;  
    for (int i=h; i<N; i++)  
        for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)  
            exch(a,j,j-h);  
}
```

- ▶ Insertion sort: `hsort(1,a)`
- ▶ Shell sort: e.g., `hsort(10,a); hsort(1,a)`

## Shell sort (ctnd)

- ▶ `hsort(10,a)`; `hsort(1,a)` faster than just `hsort(1,a)` !
- ▶ Q: How is this possible?
- ▶ A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- ▶ What about `hsort(100,a)`; `hsort(10,a)`; `hsort(1,a)`?
- ▶ To be expected: depends on the length  $N$  of the array
- ▶ The run-time analysis of Shell sort is very difficult
- ▶ Best practice:  $h = N/3, N/9, \dots, 364, 121, 40, 13, 4, 1$
- ▶ Example: [algoritmevisualisering](#) by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke

# Mergesort

- ▶ Top-down (recursive) algorithm:
  - ▶ Mergesort left half, mergesort right half
  - ▶ Merge the results (example: 2468,1357)
- ▶ Using an auxiliary array: [TopDownMergeSort.java](#), [Movie](#)
- ▶ Bottom-up algorithm (16 elements):
  - ▶ Merge  $a[0], a[1]$ , merge  $a[2], a[3]$ , merge  $a[4], a[5]$ , ...
  - ▶ Merge  $a[0..1], a[2..3]$ , merge  $a[4..5], a[6..7]$ , ...
  - ▶ Merge  $a[0..3], a[4..7]$ , merge  $a[8..11], a[12..15]$
  - ▶ Merge  $a[0..7], a[8..15]$ , done!
- ▶ Also using an auxiliary array: [BottomUpMergeSort.java](#)

## Run-time and memory use of mergesort

- ▶ Mergesort uses between  $\sim (N/2) \lg N$  and  $\sim N \lg N$  compares.  
Proof on bb. Important formula ( $N = 2^n$ ):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- ▶ Mergesort uses at most  $\sim 6N \lg N$  array accesses
- ▶ Mergesort uses  $\sim 2N$  space (plus some var's)
- ▶ Q: How fast can compare-based sorting of  $N$  distinct keys be?
- ▶ A:  $\lg N! \sim N \lg N$ ; Proof in book and on bb. Keywords:  
binary *compare tree*, inner nodes for each  
`compare(a[i], a[j])`, permutations in the leaves,  
 $N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$

# Quicksort

- ▶ Top-down (recursive) algorithm:
  - ▶ Choose a (pivot) value  $v$  in the array
  - ▶ Partition the array in non-empty parts  $\leq v$  and  $\geq v$
  - ▶ Quicksort the two parts
- ▶ Examples: [algoritmevisualisering](#) by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke
- ▶ Pros: in-place, average computation time  $O(n \log n)$
- ▶ Cons: stack space for recursion, worst-case  $O(n^2)$ , not stable
- ▶ Implementation: [QuickSort.java](#)
- ▶ BTW: [Bug in java.util.Arrays.sort](#)

## Quicksort, details

- ▶ Subtleties in `Quicksort.sort()`: shuffling protects against worst-case behaviour
- ▶ Termination of recursive `quicksort()`
- ▶ Subtleties in `partition()`:
  - ▶ Invariants  $l \leq h$  in the two inner loops
  - ▶ Postcondition after the two inner loops
  - ▶ Invariant of the `for(;;)` loop
  - ▶ Termination of the `for(;;)` loop
  - ▶ There are some variations that are also correct



## Run-time and memory use of quicksort

- ▶ Compare Quicksort to other sorts ( $n = 10^2, 10^3, \dots$ )
- ▶ Quicksort: time  $O(n^2)$  if pivot is always smallest (or largest)
- ▶ Randomization: choose pivot randomly, or shuffle array
- ▶ If all keys are distinct and randomization is perfect, then quicksort uses on average  $\sim 2n \ln n$  compares and  $\sim (n/3) \ln n$  exchanges (proofs in book, complicated)
- ▶ Relevant improvements:
  - ▶ Cut-off to insertion sort for sizes  $\leq 15$  (ca.)
  - ▶ Median-of-three pivot
  - ▶ Taking advantage of duplicate keys (3-way partitioning)
- ▶ Quicksort is generally quite good
- ▶ In special situations other sorts are better (e.g., countsort)

# Priority Queues

- ▶ Aim: collecting and processing items having keys
- ▶ Examples of keys: time-stamp, price-tag, priority-tag
- ▶ Assume: keys are ordered
- ▶ Reasonable: processing currently highest (or lowest)
- ▶ Special cases: items time-stamped when added
  - ▶ Queue: dequeue currently oldest (lowest time-stamp)
  - ▶ Stack: pop currently newest (highest time-stamp)
- ▶ Priority queue generalizes this
- ▶ Examples: highest priority, largest transaction, lowest price
- ▶ Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

## Priority Queues

- ▶ Good info: [Wikipedia](#); API (the bare essentials):

```
public class MaxPQ<Key extends Comparable<Key>>
```

```
void          insert(Key v) // insert a key
```

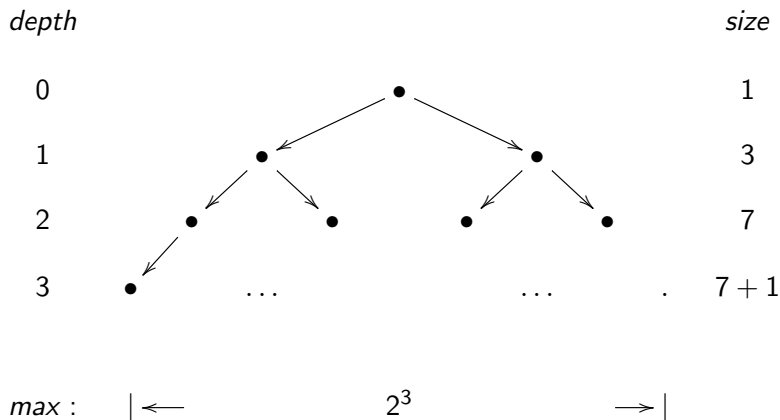
```
Key           delMax() // delete a largest key, if any
```

```
boolean       isEmpty()
```

```
int           size()
```

- ▶ In case of duplicate keys: 'a' largest, not 'the'
- ▶ Typical application: the 1K smallest keys of 1G unsorted keys
- ▶ Client: [BottomM.java](#) (Q: why is the output slowing down?)

## Example of left-complete binary tree



## Binary Trees

- ▶ MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- ▶ MNF130: In a *binary* tree every node has at most two children.
- ▶ MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height  $h$  has  $2^{h+1}-1$  nodes.
- ▶ INF102: A binary tree is (left-) *complete* if all levels  $< h$  are filled, level  $h$  may be partly empty from the right (picture bb). A (left-)complete binary tree of height  $h$  has between  $2^h$  and  $2^{h+1}-1$  nodes.
- ▶ A left-complete binary tree of  $n$  nodes has height  $\lfloor \lg n \rfloor$  (from now on we leave out 'left-').

## Heap-ordered Binary Trees

- ▶ Naive implementations:
  - ▶ Unsorted (resizing) array: fast `insert()`, linear `delMax()`
  - ▶ Sorted (resizing) array: linear `insert()`, fast `delMax()`
- ▶ Aim: operations in logarithmic time, no extra space
- ▶ A binary tree is *heap-ordered* if the key in each node is  $\geq$  the keys in its children (if any). Thus the root has a maximal key.
- ▶ NB: a heap is NOT a search tree (different data invariants)!
- ▶ Array representation of heap-ordered complete binary tree (bb)
- ▶ Methods `swim()` and `sink()`: picture on bb, code below
- ▶ Implementation: `ArrayListPQ.java`

## Run-time and memory use of heaps, applications

- ▶ In a heap of  $n$  elements (since height is  $\leq \lfloor \lg n \rfloor$ ):
  - ▶ `insert()` takes  $\leq 1 + \lfloor \lg n \rfloor$  compares and exchanges
  - ▶ `delMax()` takes  $\leq 2\lfloor \lg n \rfloor$  compares and  $\leq 1 + \lfloor \lg n \rfloor$  exchanges
- ▶ Heap construction by `insert()` can sometimes be improved
- ▶ Example: maxheap from A B C D E F G H
- ▶ Given an array of  $n$  keys, right-to-left heap construction (bb) takes  $< 2n$  compares and  $< n$  exchanges
- ▶ Applications: **heapsort** and **merging sorted streams**
- ▶ Many variations with extended API (indexed priority queue)

## Purpose of Sorting

- ▶ Sorting makes the following easier and more efficient:
  - ▶ Searching (binary search, example: `ThreeSumOptimized`)
  - ▶ Searching and looking up, e.g., the `pagenumber` in an index
  - ▶ Finding and removing duplicates
  - ▶ Finding the median, quartiles etc.
- ▶ Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- ▶ We do *pointer sorting*, manipulating refs to objects.
  - ▶ Pro: not moving full objects
  - ▶ Cons: pointer dereferencing, no `sort(int[] a)`
- ▶ More flexibility: pass a `Comparator` object to `sort()`



## Comparator object

- ▶ API: `void sort(Object[] a, Comparator c)`
- ▶ Call, e.g.: `sort(a, new Transaction.WhenOrder())`
- ▶ Call, e.g.: `sort(a, new Transaction.SizeOrder())`
- ▶ Obs: `import java.util.Comparator`
- ▶ Obs: `less(Object o1, Object o2, Comparator c)`
- ▶ Priority queues also with `Comparator`

```
public class Transaction {  
    ...  
    public static class MyOrder {  
        implements Comparator<Transaction>  
        public int compare(Transaction t, Transaction v){...}  
    } // End of Myorder  
    ...// similarly: WhenOrder, SizeOrder  
} // End of Transaction
```

## Applications of Sorting

- ▶ Consider sorting first to make other problems easier
- ▶ Commercial computing (sort on price, departure time, ...)
- ▶ Search for information: web-indexing, search engines
- ▶ Job scheduling heuristic: longest processing time first
- ▶ To come: Prim's, Dijkstra's and Kruskal's algorithms
- ▶ Huffman compression: a lossless compression based on using the shortest codes for the symbols that occur often.  
Frequency counter: next chapter!
- ▶ Cryptology and genomics (e.g., longest repeated substring)

# Symbol Tables

- ▶ Symbol table associates *keys* with *values*: *key-value pairs*
- ▶ Examples: keyword-list of page nrs, ID number-personal data, word-frequency
- ▶ Important operations:
  - ▶ Insert a key-value pair in the symbol table: `void put(k,v)`
  - ▶ Search the value for a given key (if any): `Value get(k)`
- ▶ Important conventions:
  - ▶ Inserting key-value for existing key: overwriting the value
  - ▶ No duplicate keys, no null keys
  - ▶ Value null: no value for this key
  - ▶ Lazy deletion: insert key-null; Eager: really delete key-value
- ▶ **API** of unordered symbol table
- ▶ Aim: all operations in time  $\sim c \lg n$  with constant  $c$  small

## ST Basics

- ▶ Archetypical ST-client: frequency counter, `ArrayListST.main`
- ▶ Cost model: number of compares
- ▶ Naive ST: unordered linked list (or `ArrayList`), linear search
  - ▶ Search miss:  $\sim n$  compares
  - ▶ Search hit: between 1 and  $\sim n$  compares
  - ▶ Random search hit:  $(1 + \dots + n)/n \sim n/2$  compares
  - ▶ Inserting  $n$  distinct keys:  $(1 + \dots + (n - 1)) \sim n^2/2$  compares
- ▶ `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- ▶ Naive ST impracticable for genomics, internet
- ▶ Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- ▶ Better for unordered ST: hashing (in Ch. 3.4)

## Ordered Symbol Table

- ▶ Ordered ST: keys are ordered, f.e., in an `ArrayList`
- ▶ **API** of ordered symbol table
- ▶ Binary search: `get(Key k)` takes  $\sim \lg n$  comparisons
- ▶ What about `put(Key k, Value v)`? See **`ArrayList`**
- ▶ Pitfall: `add(int i, E e)` is linear, not amortized  $O(1)$ !
- ▶ Consequence: `put(Key k, Value v)` and `del(Key k)` *linear*
- ▶ Implementation with binary search in **`ArrayListST.java`**
- ▶ Trace of inserts on bb: `SEARCHEXAMPLE`
- ▶ Experiments with `tinyTale.txt`, `tale.txt`, ...

## Binary Search Trees

- ▶ Aim: `get`, `put`, `del` in logarithmic time, ST in linear space
- ▶ Binary *search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- ▶ Search time: length of the path to the node where the key 'should' be
- ▶ Balanced binary tree with  $n$  keys has  $\lg n$  height
- ▶ Unbalanced binary trees can have height  $n$  (max depth)
- ▶ Search hits in a binary search tree, built without rebalancing, of  $n$  random keys take on average  $\sim 2 \ln n$  compares
- ▶ **UBST.java**: `put()`, `get()`, `size()`, `isEmpty()`
- ▶ Trace of inserts on bb: S E A R C H E X A M P L E

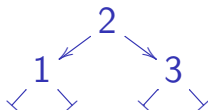
## Binary Search Trees (`min()`, `delMin()`, `delete()`)

- ▶ Interrelated, increasing difficulty: `min(Node x)`, `deleteMin(Node x)`, `delete(Node x, Key k)`
- ▶ Node of minimum key: not null, and has left child null, and is root or left child of parent (picture on bb)

```
public Node min(Node x){// precondition x!=null
    while (x.left!=null) x = x.left; // inv x!=null
    return x;
} // cf. tail recursive min() in Alg. 3.3
```

- ▶ Delete minimum key, two cases:  
(1) both children null; (2) left child null
- ▶ Delete is really difficult: `BST.java`, cf. `ArrayListST.java`
- ▶ Don't forget: update `x.N` along the path to the root!

## Delete from search tree, example:



```
root=delete(root,3)
```

(1st example)

```
| x=root; x.right=delete(x.right,3)
```

(x.right=null)

```
| | x'=x.right; return x'.left;
```

(x.size=2)

```
| | update x.size;
```

(root=x)

```
| | return x;
```

```
root=delete(root,2)
```

(2nd example)

```
| x=root; t=x; x=min(t.right);
```

(x=t.right)

```
| x.right=deleteMin(t.right);
```

(x.right=null)

```
| x.left=t.left;
```

(x.size=2)

```
| | update x.size;
```

(root=x)

```
| | return x;
```



## Balanced Search Trees: keep paths short!

- ▶ NB tree balancing not as easy as in UF and Heap (4hrs!)
- ▶ A **2-3 search tree** consists of 2-nodes and 3-nodes:
  - ▶ Each 2-node has two children and a key  $k$  such that all keys in the left subtree are  $< k$ , and all keys in the right subtree  $> k$
  - ▶ Each 3-node has three children and two keys  $k_1 < k_2$  such that all keys in the left subtree are  $< k_1$ , all keys in the middle subtree  $> k_1$  and  $< k_2$ , and all keys in the right subtree  $> k_2$
- ▶ Examples and pictures on bb
- ▶ *Perfect* 2-3 search tree: paths from root to leaves equally long
- ▶ Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- ▶ Insert should keep tree perfect, rough idea:
  - ▶ into a 2-leaf: make it into a 3-leaf (easy)
  - ▶ any other case: (temporary) 4-nodes (see next, difficult)

## Insert in Perfect 2-3 Search Trees

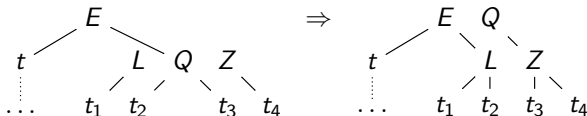
- ▶ Terminology: a *leaf* is a node all whose children are null
- ▶ Data invariant 1: tree is 2-3 search tree
- ▶ Data invariant 2: all paths from root to leaves equally long
  - ▶ Insert into a 2-leaf  $L$  : either  $\begin{array}{c} A \ L \\ \diagup \quad \diagdown \\ | \end{array}$  or  $\begin{array}{c} L \ Z \\ \diagup \quad \diagdown \\ | \end{array}$
  - ▶ into a 3-leaf whose parent is a 2-node: with new key  $Z$  (e.g.)
 

$$\begin{array}{c} E \\ \diagup \quad \diagdown \\ leaf \quad L \ Q \\ || \quad | \quad \diagdown \end{array} \Rightarrow \begin{array}{c} E \\ \diagup \quad \diagdown \\ leaf \quad L \ Q \ Z \\ || \quad | \quad \diagdown \end{array} \Rightarrow \begin{array}{c} E \ Q \\ \diagup \quad \diagdown \\ leaf \quad L \ Z \\ || \quad || \quad || \end{array}$$
  - ▶ into a 3-leaf whose parent is a 3-node: with new key  $Z$  (e.g.)
 

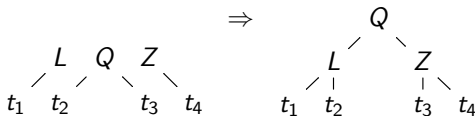
$$\begin{array}{c} C \ M \\ \diagup \quad \diagdown \\ leaf_l \quad leaf_m \quad XYZ \\ || \quad || \quad \diagdown \end{array} \Rightarrow \begin{array}{c} C \ M \ Y \\ \diagup \quad \diagdown \quad \diagdown \\ leaf_l \quad leaf_m \quad X \ Z \\ || \quad || \quad || \quad || \end{array}$$
  - ▶ into a 3-node whose parent is a 3-node: move up middle key!

## Insert (ctnd)

- ▶ Data invariant 1: tree is 2-3 search tree
- ▶ Data invariant 2: all paths from root to leaves equally long
- ▶ Insert works up from the leaf where the key 'should' be
  - ▶ if 2-node on path to root: make it into a 3-node; there are two cases, left and right, here is a picture of the latter



- ▶ otherwise, work upwards and, finally, split the root:



- ▶ working upwards, there are three cases: left, middle, and right

## Insert, summary and examples

- ▶ There exists a perfect 2-3 tree for any sequence of input keys
- ▶ Six operations for eliminating 4-nodes:
  - ▶ if parent is 2-node: move middle key up (left and right case)
  - ▶ if parent is 3-node: move middle key up (left, middle, right)
  - ▶ if root: split root
- ▶ Search and insert visit at most  $\lfloor \lg n \rfloor$  nodes
- ▶ Proof: maximal path length is  $\geq \lfloor \log_3 n \rfloor$  and  $\leq \lfloor \log_2 n \rfloor$
- ▶ Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- ▶ Trace of inserts on bb: A C E H L M P R S X (keep balance!)

## Red-black trees

- ▶ Red-black trees implement 2-3 trees
- ▶ Idea: one 3-node = two 2-nodes + extra info
- ▶ Extra info coded in color, picture:



- ▶ A *red-black tree* is a binary search tree with red and black links such that:
  - ▶ Only left links can be red (but need not be)
  - ▶ Never
  - ▶ Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- ▶ Equivalent: red-black tree and perfect 2-3 search tree


## Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

```
private class Node {  
    Key key;  
    Value value;  
    Node left, right;  
    boolean color; // true for red, false for black  
    int N; // number of keys in subtree below this node  
}  
  
private boolean isRed(Node n) {  
    if (n==null) {return false;} else {return x.color}
```

## Rotating and Color Flipping

- ▶ Aim: restoring the data invariants of red-black search trees
  1. Only left links can be red, but never two successive
  2. Search tree invariant
  3. Perfect black balance
- ▶ Repairs use *rotations* and *color flips*, examples (leaves):
  - ▶ inserting  $Z$  in  $L \leftarrow R : L \leftarrow R \rightarrow Z$ , violation (why?)  
 repairment: color flip  $L \leftarrow R \rightarrow Z$  ( $R$  must up)
  - ▶ inserting  $A$  in  $L \leftarrow R : A \leftarrow L \leftarrow R$ , violation (why?)  
 repairment: rotation right + color flip  $A \leftarrow L \rightarrow R$
  - ▶ inserting  $M$  in  $L \leftarrow R : L \leftarrow R$ , violation  

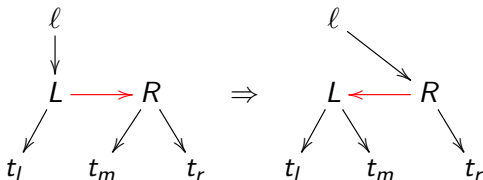


$M$

 repairment: rotation left into  $L \leftarrow M \leftarrow R$ , then as above
- ▶ Rotations change the root of the subtree, preserving invariants

## Left Rotation

Call: `l = rotateLeft(l);`



```
private Node rotateLeft(Node l){
    Node r = l.right; l.right = r.left; r.left = l;
    r.color = l.color; l.color = true // == RED
    r.N = l.N; l.N -= 1+size(r.right); // Why?
    return r;
}
```



## Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- ▶ Code of `rotateRight()` like that of `rotateLeft()`
- ▶ NB1: operations are local (here only  $r$ ,  $M$ ,  $R$ )
- ▶ NB2: operations preserve data invariants
- ▶ NB3: root is a special case (always black)
- ▶ Deletions: complicated, but doable (Exc. 3.3.39–41)

## Run-time and memory use of Red-Black BSTs

- ▶ The height of a red-black BST with  $n$  nodes is  $\leq 2 \lg n$   
Proof: the worst-case is one 3-node path and the rest 2-nodes
- ▶ The average length of path (any color) from the root to a node in a red-black BST with  $n$  nodes is  $\lg n$  ('empirical fact')
- ▶ In a red-black BST, search, insert, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node (book Prop. I, p. 447).
- ▶ For red-black BSTs, logarithmic time is guaranteed!

# Hashing

- ▶ Idea: if keys in  $[0..99]$  an array is the perfect symbol table
- ▶ In fact: `CountSort99.java` counts frequencies like an ST client
- ▶ A *hash function* maps keys to array indices
- ▶ Injectivity of the hash function is not guaranteed
- ▶ *Hash collision*: different keys are mapped to the same index
- ▶ In such a case we need *collision resolution*
- ▶ Symbol tables: hashing is fast, but unordered (no `max,min`)
- ▶ Aim: ST operations in amortized  $O(1)$  time, extra space OK

# Space-Time Trade-Off

- ▶ Hashing is an example of a *space-time trade-off*
- ▶ Time: computation time required
- ▶ Space: memory space used
- ▶ Unlimited space: (1) use key as index (e.g., the bits)
- ▶ Unlimited time: (2) use linked list and linear search
- ▶ Hashing strikes a balance using (1) with some array of reasonable size, and (2) in case of collisions
- ▶ The balance between (1) and (2) can easily be tuned

## Hash functions

- ▶ Ideal (uniform hashing assumption, UHA): uniform and independent distribution of keys over integers from 0 to  $M - 1$
- ▶ Examples of **hash functions in Java**
- ▶ Horner:  $a_0 + x(a_1 + x(a_2 + \dots)) = a_0 + a_1x + a_2x^2 + \dots$
- ▶ Modular hashing ( $M$  prime), reasonably  $\approx$  UHA:  

```
private int hash(Key k){  
    return (key.hashCode() & 0x7fffffff) % M;}  

```
- ▶ Q: Why crazy  $\& 0x7fffffff$  ???
- ▶ A: In Java, e.g.,  $(-5 \% 3) == -2$  and not 1
- ▶ Q: Why  $M$  prime?
- ▶ A: E.g.,  $M = 32$  takes only into account the last five bits

## Collision Resolution

- ▶ Two main methods of collision resolution:
  1. Hashing with separate chaining (picture on bb)
  2. Hashing with linear probing (picture on bb)
- ▶ Separate chaining: symbol table is an array of linked lists, linear search. If array has length  $M$ , then the linked lists have average length  $N/M$  with  $N$  keys.
- ▶ Linear probing: symbol table is an array of length  $M \geq N$ . Colliding keys are put at the first empty position. Linear search from the position where the key 'should have been'. Empty position: not found. Deletion tricky: reinsert all keys to the right of the deleted key, until the first empty position (picture on bb). Works better with  $M \gg N$ .

## Symbol Table with Hashing

- ▶ Implementation: `ArrayListHashST.java`
- ▶  $M = 1$ : measure overhead wrt. `ArrayListST`
- ▶ Tests with various values of  $M$ : 31, 997, 65521
- ▶ NB1: `ArrayListST`, `UBST` are sorted (!)
- ▶ NB2: *construction* versus *use* of ST (hashing better for *use*)
- ▶ Hashing can be combined with any other ST-implementation
- ▶ UHA metaphor: for every key one throws a dice *once*, and remembers the value as the hash code of the key

## Quantitative analysis

- ▶ Throwing a dice 10 times, what is the probability of 3 fives?
- ▶ Under UHA, with  $N$  distinct keys, the probability that exactly  $k$  keys collide at some given hash value is

$$\binom{N}{k} \left(\frac{1}{M}\right)^k \left(\frac{M-1}{M}\right)^{N-k}, \text{ where e.g. } \binom{100}{10} \approx 1.7E13$$

- ▶ This is a small number for, say,  $N = M = 100$  and  $k = 10$
- ▶ For linear probing one typically takes  $M = 2N$
- ▶ For separate chaining one keeps  $N/8 \leq M \leq N/2$  (resizing  $M$ )
- ▶ Under UHA: search, insert, delete take amortized  $O(1)$  time
- ▶ Space used can be upto  $100N$  byte (objects, pointers); this on top of the space used by  $N$  key-value pairs



## Applications of Searching

- ▶ Synonyms: **associative array**, map, symbol table, or dictionary
- ▶ Origin of **symbol table**: compilers and interpreters
- ▶ Web-indexing, **search engines**
- ▶ Sparse matrices (many 0's): **dictionary**
  1. keys: (row, column)-pairs
  2. values: matrix entries
- ▶ Set API (no values, only keys, for deduplication, filtering):

```
public class SET<Key>
{ void add(Key k);
  void delete(Key k);
boolean contains(Key k);
boolean isEmpty();
  int size(); }
```

## Indexes and Reverse Indexes

- ▶ Index (key, value)
- ▶ Reverse index (value, key(s))
- ▶ Phone book (name+address, phone number(s))
- ▶ Reverse phone book (phone number, name+address)
- ▶ Dictionary (word, meaning or translation)
- ▶ Account information (client ID, account information)
- ▶ Genomics (protein, sequences of ACTG triplets)
- ▶ File systems (file name, location on disk/ file attributes)
- ▶ Internet domain name system (domain name, IP address)

## Balanced Search Tree or Hash Table?

- ▶ Q: Which symbol table to use?
- ▶ A: The basic choice between BST and HT depends on ...
  1. Ordering of keys essential: BST
  2. Availability of good hash function (good = fast + UHA)
  3. Ordering of keys expensive (long strings): HT (or: Ch.5)
  4. Ordering of keys possible, but not essential: HT + BST
  5. Space considerations (ArrayListST uses the least extra space)
  6. Number of distinct keys and the space each key takes
  7. Distribution of insert/delete/search operations

# Overview Chapter 1–3

## Chapter 1

- ▶ Stack and Queue, ThreeSum, Union-Find
- ▶ Theory:  $\sim$  and  $O$
- ▶ Experiments: loglog-plots, randomization

## Chapter 2: Sorting

- ▶ Selection-, Insertion-, Shell-, Merge-, QuickSort
- ▶ Priority Queue, Binary Heap, HeapSort
- ▶ CountSort

## Chapter 3

- ▶ Symbol Table
- ▶ Binary Search Tree, Perfect 2-3 Tree, Red-Black Tree
- ▶ Hashing: hash function and collision resolution

## Odds and Ends Chapter 1–3

- ▶ Path-compression in UF (71)
- ▶ Randomizing the order of the tests
- ▶ Compare-based sorting requires  $N \lg N$  comparisons (72)
- ▶ Double hashing: linear probing  
 $h_1(k), h_1(k) + h_2(k), h_1(k) + 2h_2(k), \dots$
- ▶ Indexed Priority Queues (74)

## Randomizing the order of the tests

```
for (int i=0; i<N; i++) {  
    a1[i] = StdRandom.uniform();  
    a2[i] = a1[i]; }  
if (StdRandom.uniform(2) == 0)  
    {total1 += time(alg1, a1); // summing runtime of alg1  
    total2 += time(alg2, a2); // summing runtime of alg2  
    }  
else  
    {total2 += time(alg2, a2); // summing runtime of alg2  
    total1 += time(alg1, a1); // summing runtime of alg1  
    }
```

## Path compression in UF

```
// Finding the "identifier" of the component of p in id:  
public int find(int p) {  
    while (p!=id[p]) { p=id[p]; }  
    return p;  
}
```

// now with path compression:

```
public int find(int p) {  
    int q=p; // remember the starting point  
    while (p!=id[p]) { p=id[p]; }  
    // postcondition: p==id[p]==identifier of q  
    while (id[q]!=p) { int aux=id[q]; id[q]=p; q=aux; }  
    return p;  
} // Example: int[] id={1,2,3,3}; find(0);
```

## Compare-based sorting: worst-case $\geq N \lg N$

- ▶ Every compare-based sorting algorithm for  $N$  distinct keys in an array  $a$  leads to a *binary compare tree* with
  - ▶ nodes  $(i:j)$  representing tests  $a[i] < a[j]$
  - ▶ left subtree:  $a[i] < a[j]$ ; right subtree:  $a[i] > a[j]$
  - ▶ leaves: sorted permutations of the array
- ▶ Example with array of length 3 on bb
- ▶ Every permutation should occur at least once in a leaf!
- ▶ Binary tree of height  $h$  has at most  $2^h$  leaves
- ▶ Length of path to leaf = number of comparisons
- ▶ Now  $h \geq \lg N! \sim N \lg N$  by Stirling from this formula:

$$N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$$

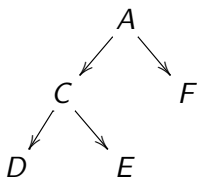


## Indexed Priority Queues (missing inverse qp of pq)

- ▶ IPQ  $\approx$  array with direct access to minimum (maximum)
- ▶ API: `void insert(int i, Key k); void del(int i); int minKey();`  
`Key keyOf(int i);...` Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-

heap



Do: `insert(6,G)`, `insert(5,B)`, `insert(1,Z)`

NB1: `keys[pq[i]]` on position `i` in heap

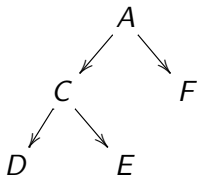
NB2: inverse index qp is needed

## Indexed Priority Queues

- ▶ IPQ  $\approx$  array with direct access to minimum (maximum)
- ▶ API: `void insert(int i, Key k); void del(int i); int minKey();`  
`Key keyOf(int i);...` Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-
qp	1	0	2	4	3	-1	-1

heap



Do: `insert(6,G)`, `insert(5,B)`  
 NB qp is needed to find  
 the index of `key[i]` in pq  
 e.g., for `insert(1,Z)` (then: sink!)

# Indexed Priority Queues (ctnd)

After insert(6,G):

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	6	0
keys	C	A	F	E	D	-	G
qp	1	0	2	4	3	-1	5

Step 1 insert(5,B):

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	6	5
keys	C	A	F	E	D	B	G
qp	1	0	2	4	3	6	5

Step 2 (swaps)  
 pq[2]      pq[6]  
 qp[pq[2]] qp[pq[6]]

index	0	1	2	3	4	5	6
pq	1	0	5	4	3	6	2
keys	C	A	F	E	D	B	G
qp	1	0	6	4	3	2	5

## Graph classes

( MNF130: useful review of graph theory)

1. Undirected graphs: a set of *vertices* (or *nodes*)  $V$  and a set of *edges*  $E$  connecting the nodes
2. Directed graphs (*digraphs*): a set of nodes  $V$  and a set  $E$  of directed edges (or *arrows*) pointing from one node to another
3. *Edge-weighted graphs*: undirected graphs in which every edge comes with a number called the *weight* of the edge
4. *Edge-weighted digraphs*: digraphs in which every arrow has a weight, like the edges of an edge-weighted graph

## Examples

1. **Map** (discuss: nodes, un/directed, un/weighted, multigraph)
2. Undirected graphs: social networks, communication networks (duplex communication)
3. Directed graphs: hyperlinks, (class, module, package) dependencies, logical circuits, job scheduling, flow graphs
4. Edge-weighted graphs: roadmaps with geographical distance, or with toll, communication networks with bandwidth
5. Edge-weighted digraphs: job scheduling with duration, flow with volume, financial transactions with size

## Undirected Graphs

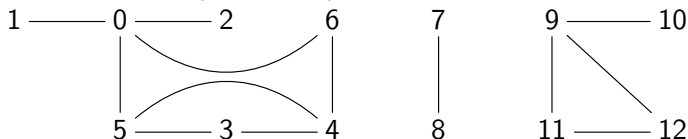
- ▶ Undirected graph: a set of *vertices* (or *nodes*)  $V$  and a set of *edges*  $E$  connecting the nodes
- ▶ *Subgraph*: subset of  $E$  and subset of  $V$  forming a graph (!)
- ▶ *Path*: sequence of nodes connected by edges (!)
- ▶ *Simple path*: path with no node repeated
- ▶ *Length of path*: number of edges
- ▶ *Cycle*: path of length  $> 0$  with same start and end node
- ▶ *Simple cycle*: not repeating edges or nodes (apart from start and end node)
- ▶ *Acyclic graph*: graph without simple cycles
- ▶ *Connected graph*: having a path between every two nodes
- ▶ *Connected component*: a maximal connected subgraph

## Trees and Forests

- ▶ Example: `tinyG.txt` on bb
- ▶ 'Anomalies' concerning edges:
  - ▶ Self-loop: edge connecting a node to itself
  - ▶ Parallel edges: two edges connecting the same node(s)
- ▶ When no anomalies,  $E \subseteq \{\{v, v'\} \mid v \in V, v' \in V, v \neq v'\}$
- ▶ *Tree*: connected acyclic graph (then: no anomalies)
- ▶ *Spanning tree*: maximal subgraph that is a tree
- ▶ Lemma: any spanning tree of a connected graph contains all nodes. Proof on bb.
- ▶ *Forest*: graph consisting of disjoint trees
- ▶ *Spanning Forest*: forest consisting of spanning trees of connected components of a graph

## Undirected Graphs (ctnd)

- ▶ *Distance* between two nodes: length of a shortest connecting path if there is a path connecting these nodes, otherwise  $\infty$
- ▶ *Degree* of a node: number of edges connected to that node
- ▶ Graph  $G = (V, E)$ , the following are equivalent:
  - ▶  $G$  is a tree (def: connected and acyclic)
  - ▶  $G$  has  $|V| - 1$  edges and no cycles
  - ▶  $G$  has  $|V| - 1$  edges and is connected
  - ▶  $G$  is acyclic and adding an edge creates a cycle
  - ▶ Any two nodes of  $G$  are connected by exactly one simple path
- ▶ Example: some (connected) subgraphs of `tinyG.txt`





## Graph representation and implementation

- ▶ Impractical: **adjacency matrix**  $\sim V^2$ , **incidence matrix**  $\sim VE$
- ▶ Often practical: **adjacency lists**  $\sim (V+2E)$ , that is, `adj[v]` lists all nodes `w` connected to `v` by an edge
- ▶ Example: `tinyG.txt` by **LinkedListG.java**
- ▶ Graph API includes: `V()`, `E()`, `addEdge()`
- ▶ Basic algorithms: Depth-First Search (DFS) and Breadth-First Search (BFS)
- ▶ Both DFS and BFS 'walk through the graph', but differently
- ▶ Both DFS and BFS can compute a spanning tree and forest

```
public void dfs(Integer v, boolean[] marked) {
    marked[v] = true;
    for (Integer w : adj[v])
        if (! marked[w]) dfs(w,marked);
} // dfs() is recursive, call: dfs(v,marked);

public void bfs(Queue<Integer> q, boolean[] marked) {
    while (!q.isEmpty()) {
        Integer v = q.dequeue();
        for (Integer w : adj[v])
            if (! marked[w]) {marked[w]=true; q.enqueue(w);}
    }
} // call: marked[v]=true; q.enqueue(v); bfs(q,marked);

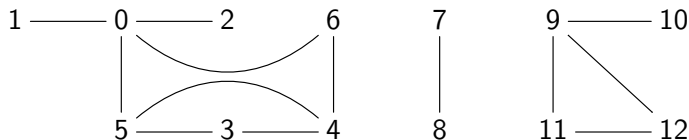
// Example: 0-1, 0-3, 1-2, 1-3, 3-4
// Example: complete ternary tree of height 2
```

## Implementation and Properties of DFS/BFS

- ▶ **LinkedListG.java**: `pathdfs()`, `pathbfs()`
- ▶ DFS and BFS mark nodes connected to a given source node in time proportional to the sum of their degrees ( $\leq 2E$ ), and can return a path from a marked node to the given source in time proportional to the length of this path
- ▶ BFS always finds a shortest path (proof: queue only contains nodes at distance  $k$  followed by nodes at distance  $k + 1$ , while all nodes at distance  $\leq k$  not in queue have been processed)
- ▶ DFS finds a left-most path (long or short, example bb)
- ▶ BFS tends to use more space (but not always)
- ▶ UF (from Ch. 1) tests connectivity, but finds no paths

## Applications

- ▶ `StringSTG.java`, flight connections, shortest path = minimum number of stop-overs
- ▶ Degrees of separation in social networks, e.g., Erdős number = length of shortest path to Paul Erdős in the co-author graph
- ▶ Connected components: `LinkedListG.countcc()`
- ▶ Example: `tinyG.txt` has three connected components

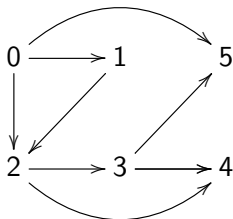


## Directed Graphs

- ▶ *Digraph*: a set of *vertices* (or *nodes*)  $V$  and a set of *directed edges* (or *arrows*)  $E$  pointing from one node to another
- ▶ *Subdigraph*, *directed (simple) path* (*dipath*), *directed (simple) cycle*, *acyclic*, *length*: as expected
- ▶ Often we leave out 'di' in digraph, dipath, etc.
- ▶ *DAG*: **D**irected **A**cyctic **G**raph;
- ▶ *Degree*: **in**-degree and **out**-degree
- ▶ Node  $v$  is *reachable* from  $w$ : a dipath from  $w$  to  $v$  exists
- ▶ *Strongly connected digraph*: dipath between every two nodes (for all  $v, w$ , there are dipaths from  $v$  to  $w$  and from  $w$  to  $v$ )
- ▶ *Strongly connected component*: maximal strongly connected subgraph ( $u \rightleftarrows v \rightarrow w$  has two scc's)
- ▶ Representation: adjacency lists even simpler!

## Directed Graph, example

tinyCG.txt:



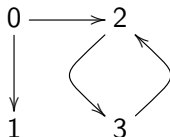
## Reachability Problems

Assume we are given a directed graph  $G$ .

- ▶ Single-source: given a node  $s$ , the *source*, is a given node  $v$  reachable from  $s$ ? Example: `tinyCG.txt`
- ▶ Multiple-source: given a set of nodes  $S$ , is a given node  $v$  reachable from some node in  $S$ ?
- ▶ Solutions: same DFS and BFS algorithms as in Chapter 1
- ▶ Application (example): **mark-and-sweep garbage collection**
- ▶ Single-source path: given  $s, v$  such that  $v$  is reachable from  $s$ . Find a path from  $s$  to  $v$ .
- ▶ Single-source shortest path: given  $s, v$  such that  $v$  is reachable from  $s$ . Find a *shortest* path from  $s$  to  $v$ .
- ▶ Solutions: same DFS (path) and BFS (shortest path) algorithms as for undirected graphs

## Cycle Detection

- ▶ Recall: a *DAG* is a graph without a directed cycle
- ▶ Acyclicity test, cycle detection: easy extension of DFS. We keep track of the search path from the source. If there is an arrow from  $v$  to  $w$  and  $w$  is on the path from the source to  $v$ , then there is a cycle. (DFS finds the leftmost path to the leftmost cycle.) Two techniques (space-time trade-off!):
  - ▶ Go back the search path: `LinkedListDiG.slowCyclist()`
  - ▶ Memorize the search path: `LinkedListDiG.fastCyclist()`
- ▶ Application: precedence scheduling of jobs
- ▶ Example: `cycleG.txt`





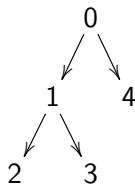
## Pre-order, post-order

- ▶ Graph walks based on DFS from a source node
- ▶ Pre-order: order in which DFS arrives at nodes
- ▶ Post-order: order in which DFS leaves nodes
- ▶ In-order *for binary trees*: e.g., in `UBST.show()`
- ▶ Example:

pre-order: 01234

post-order: 23140

in-order: 21304



- ▶ Example: `tinyCG.txt` on `bb` and by `LinkedListDiG.java`

## Topological order of acyclic digraph

- ▶ Topological order: total order  $\prec$  compatible with the graph in the following sense: if there is an arrow from  $u$  to  $v$ , then  $v \prec u$  (consequently:  $v \preceq u$  if  $v$  is reachable from  $u$ )
- ▶ NB one can also take  $\succ$ , this is only a matter of definition
- ▶ Lemma: if a digraph has a topological order, then it is acyclic (proof: a cycle cannot be ordered compatibly)
- ▶ Lemma: if a digraph is acyclic, then it has a topological order (proof idea: if acyclic, the post-order is a topological order since, if there is an arrow from  $u$  to  $v$ , then  $u$  is not reachable from  $v$  and DFS will leave  $u$  after it has left  $v$ )
- ▶ Topological order is a job schedule respecting precedence
- ▶ Example:  $1 \leftarrow 2 \leftarrow 4 \rightarrow 5 \rightarrow 3 \leftarrow 0$

## Transitive closure

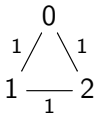
- ▶ Definition: given  $G$ , its (reflexive!) transitive closure  $G^*$  is a graph with the same nodes, with arrows from  $u$  to  $v$  for each  $v$  that is reachable from  $u$  in  $G$ .
- ▶ NB:  $G^*$  can have many more arrows than  $G$
- ▶ Implementation: adjacency matrix in case of many arrows

...

```
boolean[][] adjmat = new boolean[V][V];  
for (int v=0; v<V; v++) {  
    boolean[] marked = new boolean[V];  
    dfs(v,marked);  
    adjmat[v] = marked; // adjmat[v][v]==true: reflexive  
...  
}
```

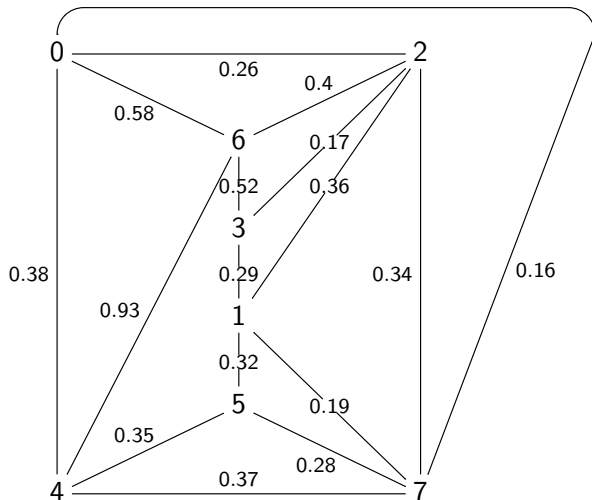
## Minimum Spanning Tree

- ▶ Recall slide 79: *spanning tree* of a connected undirected graph is a maximal subgraph that is a tree (and thus contains all nodes and is acyclic)
- ▶ EWG = Edge-Weighted Graph, here always connected
- ▶ Recall slide 80: all spanning trees have  $V - 1$  edges
- ▶ *Weight* of spanning tree: sum of the weights of its edges
- ▶ *Minimum Spanning Tree*: spanning tree with minimal weight
- ▶ Example: three MSTs of



- ▶ Exc.4.3.3: if all weights different, then MST is unique
- ▶ From now on we assume all weights different!

## MST Example (tinyEWG.txt)



## Minimum Spanning Tree (ctnd)

- ▶ Applications: power plants and electrical grid, airlines and flight routes, maps and distance
- ▶ Weights may be zero or negative (e.g., cost minus profit of a new network of roads between cities)
- ▶ Two important algorithms to find the MST: Prim's and Kruskal's
- ▶ We explain Kruskal's algorithm first (unlike the book), since in previous years students found Prim's algorithm more difficult to understand

## Cuts and Crossing Edges

- ▶ Recall slide 80: deleting an edge from a tree creates two disjoint components, adding an edge creates a cycle
- ▶ *Cut*: a partition of  $V$  in two non-empty subsets of nodes
- ▶ *Crossing edge*: edge connecting two nodes in different subsets of a cut
- ▶ NB there can be more than one crossing edge: 0 — 2



- ▶ Lemma: for any cut in an EWG, the crossing edge of minimum weight is in the MST.
- ▶ Proof: given a cut, assume by contradiction there is a crossing edge  $e$  of weight smaller than the crossing edge(s) that is (are) in the MST (e.g., the dotted edges above). Adding  $e$  creates a simple cycle, which must contain one other crossing edge  $f$  in the MST. Replacing  $f$  by  $e$ : ✗

## Kruskal's Algorithm

- ▶ Datastructures:
  - ▶ EWG represented with adjacency lists  $\text{adj}[v]$
  - ▶ Minimum priority queue  $\text{pq}$  for edges
  - ▶ Union-Find object  $\text{uf}$  testing connectivity
  - ▶ Queue  $\text{mst}$  for the minimum spanning tree
- ▶ Algorithm:
  1. delete the minimum edge  $e$  from  $\text{pq}$
  2. if the points connected by  $e$  are not connected, add  $e$  to  $\text{mst}$  and connect the points in  $\text{uf}$
  3. continue at point 1 until  $\text{pq}$  is empty or  $\text{uf}$  contains all nodes
- ▶ Examples: EWG on slide 99, `tinyEWD.txt`
- ▶ Correctness: by lemma about minimum-weight crossing edge of cut (NB it is also correct to use a sorted array of edges)
- ▶ Implementation: `KruskalMST.java`, constructor method



## Prim's Lazy Algorithm

- ▶ Datastructures:
  - ▶ EWG represented with adjacency lists  $\text{adj}[v]$
  - ▶ Minimum priority queue  $\text{pq}$  for edges
  - ▶ Array  $\text{marked}[v]$  for marking vertices
  - ▶ Queue  $\text{mst}$  for the minimum spanning tree
- ▶ Edge is *eligible* if not both endpoints marked (crossing!)
- ▶ Algorithm based on previous lemma, cut: un/marked nodes
  1. mark 0 and add all eligible edges in  $\text{adj}[0]$  to  $\text{pq}$
  2. as long as  $\text{pq}$  is not empty, do:
    - 2.1 get and delete minimum edge  $e$  from  $\text{pq}$
    - 2.2 add  $e$  to  $\text{mst}$ , say the unmarked endpoint of  $e$  is  $k$
    - 2.3 mark  $k$  and add all eligible edges in  $\text{adj}[k]$  to  $\text{pq}$
    - 2.4 delete ineligible minimum edges from  $\text{pq}$
- ▶ After this algorithm, the queue  $\text{mst}$  contains the MST

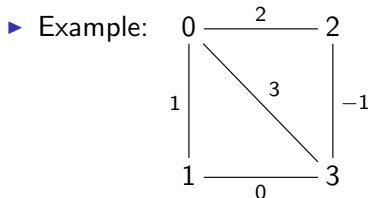
## Prim's Algorithm (ctnd)

- ▶ **LazyPrimMST.java**, methods `scan()` and `prim()`
- ▶ Invariant: at least one of the nodes of an edge in `pq` is marked
- ▶ NOT: all edges in `pq` are crossing edges wrt cut un/marked
- ▶ Lazy: ineligible edges are not eagerly deleted from `pq`
- ▶ Runtime: LazyPrimMST runs in  $O(E \log E)$  time (worst-case)
- ▶ Possible: *only* crossing edges wrt cut un/marked in `pq`
- ▶ Eager: if  $v$  unmarked, the only crossing edge of interest is the *lightest* one connecting  $v$  to the marked edges (= MST so far)
- ▶ Runtime: eager Prim runs in  $O(E \log V)$  time (worst-case)
- ▶ Max size `pq`:  $E$  edges for lazy;  $V$  nodes for eager

## Prim's Eager Algorithm

► Datastructures:

- EWG represented with adjacency lists `adj[v]`
- Boolean array `marked[v]` for marking vertices
- Array `distTo[v]`, minimum distances to MST so far
- Array `edgeTo[v]`, edges with minimum distance to MST so far
- Indexed minimum priority queue `pq`: `index=v`, `key=distTo[v]`
- Queue `mst` for the MST based on `edgeTo`



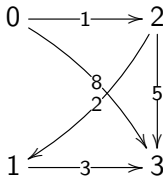
- **PrimMST.java**, methods `scan()` and `prim()`

## Memory-Use and Run-time Analysis

- ▶ Space, worst-case:
  - ▶ All methods use  $O(V + E)$  space for the graph, plus ...
  - ▶ Priority queue for edges (Lazy Prim and Kruskal):  $O(E)$  space
  - ▶ Priority queue for vertices (Eager Prim):  $O(V)$  space
  - ▶ Arrays indexed by vertices (all):  $O(V)$  space
- ▶ Time, worst-case:
  - ▶ Priority queue operations (Lazy Prim and Kruskal):  $O(E \log E)$  time
  - ▶ Priority queue operations (Eager Prim):  $O(E \log V)$  time
- ▶ NB:  $E \leq V^2$  implies  $\log E \leq \log V^2 \leq 2 \log V$
- ▶ Example: complete graph on  $0, \dots, 9$ , edge  $n-m$  weight  $n+m$ , MST consists of  $0-m$  for  $m = 1, \dots, 9$

## Shortest paths

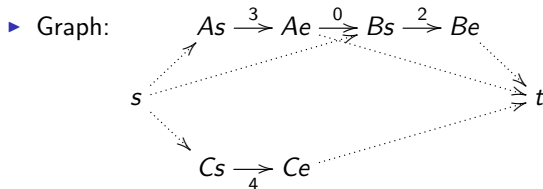
- ▶ Recall slide 76, *Edge-weighted digraphs*: digraphs in which every arrow has a weight
- ▶ *Weight* of a (di)path: sum of weights of the arrows
- ▶ *Shortest* path from node  $s$  to node  $t$ : minimum path weight
- ▶ EWD = Edge-Weighted Digraph, example: `tinyEWD.txt`
- ▶ Example: two SPs from 0 to 3:



- ▶ Shortest paths need not be unique, even if all weights are different!
- ▶ Shortest paths need not exist, for two independent reasons:
  - ▶ When target  $t$  is not reachable from source  $s$
  - ▶ When there is a negative cycle on the path to  $t$ , e.g.,  $1 \xrightarrow{-1} 1$

## Variations

- ▶ Single-source versus multiple sources
- ▶ Only non-negative weights versus all weights allowed
- ▶ Acyclic versus cycles, in particular negative cycles
- ▶ Longest path: shortest path with weights negated
- ▶ Important example: (parallel) scheduling of jobs A, B, and C
  - ▶ A (3 hrs), must precede by B (2 hr), independent C (4 hrs)



- ▶ Schedule, longest paths, makespan: `tinyJob.txt`
- ▶ Now add: A must start less than 2 hrs before B starts.  
Feasible? (No) And 4 hrs before? (Yes)

## Dijkstra's Algorithm

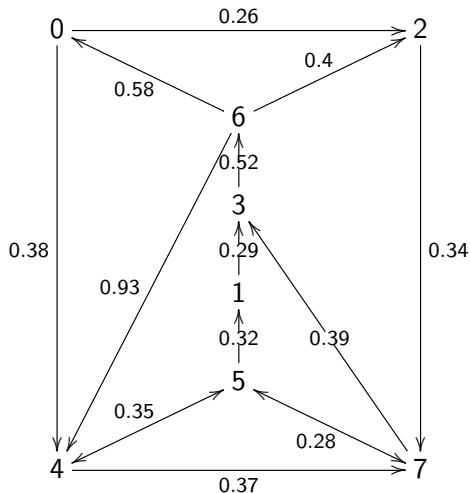
- ▶ Single-source, only non-negative weights, cycles no problem
- ▶ Datastructures:
  - ▶ EWD with adjacency lists `adj[v]` of weighted out-edges
  - ▶ Boolean array `marked[v]` for marking vertices
  - ▶ Array `distToSource[v]`, minimum distances to source so far
  - ▶ Array `pathToSource[v]`, best arrow to source so far
- ▶ Algorithm: relax (see below) and mark the unmarked node with least distance, until all marked; Simple example: slide 101
- ▶ Invariants:
  - ▶ Marked nodes: known shortest path to source (non-negativity!)
  - ▶ Unmarked nodes: known shortest path to source THROUGH marked nodes (requires in-arrow from marked node)
- ▶ Implementation `LinkedListEWD.slowEWD()`, examples `tinyJob.txt` and `tinyEWD.txt`

## Relaxation

- ▶ Assume an array `distToSource[v]` with minimum distances, so far, to a given source
- ▶ To *relax an edge*  $e$  from  $u$  to  $v$  with weight  $x$  means to update `distToSource[v]` to `distToSource[u] + x` if the latter is smaller
- ▶ To *relax a node*  $u$  means to relax all edges in `adj[u]`, that is, to update `distToSource[v]` to `distToSource[u] + x` if the latter is smaller, for every edge from  $u$  to  $v$  with weight  $x$
- ▶ Dijkstra: relax and mark unmarked node  $v$  with minimal `distToSource[v]`, until all nodes marked
- ▶ Bellman-Ford: do max  $V$  rounds of relaxation of all edges (may stop after a round without updates)



## EWD Example (tinyEWD.txt, NB $4 \leftrightarrow 5 \leftrightarrow 7$ !)



## Bellman-Ford

- ▶ Single-source, also negative weights, negative cycles detected
- ▶ Datastructures:
  - ▶ EWD with adjacency lists `adj[v]` of weighted out-edges
  - ▶ Array `distToSource[v]`, minimum distances to source so far
  - ▶ (Array `pathToSource[v]`, best arrow to source so far)
- ▶ Algorithm: do at most  $V$  rounds for every node  $v$  and every arrow  $e$  in `adj[v]`, if  $e$  shortens the distance to its endpoint  $w$ , update that distance (and path); stop after a round when no distances improve. If distances improve in the  $V$ -th round, a negative cycle is reachable from the source.
- ▶ Invariant: after  $n$  rounds the distances are less than or equal to the shortest path of length  $n$  from the source
- ▶ Implementation `LinkedListEWD.simpleBF()`, examples `tinyJob.txt` and `tiNoJob.txt`

## Memory-Use and Run-time Analysis

- ▶ Space
  - ▶ All methods use  $O(V + E)$  for the graph, plus  $O(V)$  extra
  - ▶ Still true for Dijkstra improved with an indexed priority queue, but indexed priority queue takes  $\sim 3V$  space
- ▶ Time, worst-case:
  - ▶  $V$  times finding a minimum (original Dijkstra):  $O(V^2)$
  - ▶ Priority queue operations (improved Dijkstra):  $O(E \log V)$
  - ▶  $V$  rounds relaxing  $E$  edges (Bellman-Ford):  $O(EV)$

## Odds and Ends Chapter 4

- ▶ Bellman-Ford
- ▶ Indexed Priority Queue
- ▶ ...

## ToC and topics of general interest

- ▶ Table of Contents on next slide (all items clickable)
- ▶ Practical stuff: slide 2

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Ch.1.5 Case Study: Union-Find

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Ch.2.2 Mergesort

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