INF102
Algorithms, Data Structures and Programming

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INF102, practical stuff

- Lecturer: Marc Bezem; Team: see homepage
- Homepage: INF102 (requires login)
- Also: INF102 on GitHub
- Tentative schedule
- Textbook: Algorithms, 4th edition
- Prerequisites: INF100 + 101 ($\approx$ Ch. 1.1 + 1.2)
- Syllabus (pensum): Ch. 1.3 – 1.5, Ch. 2 – 4
- Three compulsory exercises, must be passed
- Digital exam (Inspera) 06.12.2017
- Old exams: 2004–2017
- Table of Contents of these slides

Resources

- Good textbook, USA-style: many pages, exercises etc.
- Average speed must be ca 50 pages p/w
- Lectures (ca 24) focus on the essentials
- Slides (ca 120, dense!) summarize the lectures
- Prepare yourself by reading in advance
- Workshops: help with selected exercises
- Test yourself by trying some exercises in advance
- If you can do the exercises (incl. compulsory), you are fine
- Review of exercises on Tuesday morning

Generic Bags, Queues and Stacks

- Generic programming in Java, example: PolyPair.java
- Bag, Queue and Stack are generic, iterable collections
- All three support adding an element
- Queue and Stack support removing an element (if any)
- FIFO Queue (en/dequeue), LIFO Stack (push/pop)
- Stack: INF101; Bag is a Stack without pop
- APIs include: `boolean isEmpty()` and `int size()`

Implementations and one application

- Implementation of Stack: LinkedList_Stack.java
- Implementation of Queue: LinkedList_Queue.java
- Dijkstra's Two-Stack Expression Evaluation
- Example: (1 + ((2 + 3) * (4 * 5)))
- Now you learned something about compilers!
- Movie

Resizing Arrays

- Arrays have direct access, but have fixed size
- Linked lists have flexible size, but no direct access
- Best of both: use new, resized arrays, *wisely*:
  - double size when the array becomes overfull
  - halve size when the array becomes quarter full
- Resizing takes time and space proportional to size
- Not too seldom (correctness), not too often (efficiency)
- Later: we retain *constant time direct access*
- Later: add operation in *constant time on average*
- Once we have understood resizing arrays: ArrayList

Implementations

- ResizingArray_Stack.java
- Arrays give direct access, resizing at reasonable cost
- LinkedList_Stack.java
- No fixed size, but indirect access incurs a cost
- Pointers take space and dereferencing takes time
- Programming with pointers: make a picture

Computation time and memory space

- Two central questions:
  - How long will my program take?
  - Will there be enough memory?
- Example: ThreeSum.java
- Inner loop (here `a[i]+a[j]+a[k]==0`) is important
- Sorting helps: ThreeSumOptimized.java
- Run some experiments: `1Kints.txt, 2Kints.txt, ...`

Methods of Algorithm Analysis

- Empirical:
  - Run program with randomized inputs, measuring time & space
  - Run program repeatedly, varying (doubling) the input size
  - Measuring time: StopWatch
  - Plot, or log-log plot and linear regression
- Theoretical:
  - Define a cost model by abstraction (e.g., array accesses, comparisons, operations)
  - Try to count/estimate/average this cost as function of the input (size)
  - Use $O(f(n))$ (MNF130) and $f(n) \sim g(n)$ (see next slide)

Orders og Growth, Big Oh and $\sim$

- Big Oh and $\sim$ aim to capture 'order of growth'
- Costs are positive quantities, so $f, g, \ldots : \mathbb{N} \to \mathbb{R}^+$
- MNF130: $f(n)$ is $O(g(n))$ if there exist $c \in \mathbb{R}^+,\ N \in \mathbb{N}$ such that $f(n) \leq cg(n)$ for all $n \geq N$ (that is, for n large enough)
- Example: n^2 and even $99n^3$ are $O(n^3)$, but n^3 is not $O(n^{2.9})$
- INF102: $f(n) \sim g(n)$ if $1 = \lim_{n\to\infty}(f(n)/g(n))$
- If $f(n) \sim g(n)$, then $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$
- Not conversely: Big Oh disregards constant factors, $\sim$ not
- Factor c hidden in Big Oh is important in practice
- Bound N is important if it is large

Important orders of growth

order of growth as function of n	value for $n = 20$
constant: c, meaning $f(n) = c$ for all n	c sec
linear: n	20 sec
linearithmetic: $n \log n$	26 sec
quadratic: n^2	400 sec
cubic: n^3	8000 sec
exponential: 2^n	1048576 sec
general form: $an^b(\log n)^c$	$a \cdot 20^b \cdot (1.3)^c$ sec

ThreeSum, theoretically

- Number of different picks of triples: $g(n) = n(n-1)(n-2)/6$
- Inner loop `a[i]+a[j]+a[k]==0` executed $g(n)$ times
- $f(n) = n^3/6 - n^2/2 + n/3$
- Cubic term $n^3/6$ wins for large n: $f(n) \sim n^3/6$
- Cost model $\#$ array accesses: $\sim n^3/2$
- Cost array access t sec: total time $\sim t * n^3/2$ sec
- Cost models are (necessary) simplifications!

ThreeSum, empirically

- Input sizes 1K, 2K, 4K, 8K take time 0.1, 0.8, 6.4, 51.1 sec
- The log's are 3, 3.3, 3.6, 3.9 and -1, -0.1, 0.8, 1.71
- See the plots in plot sheet
- Linear regression gives $y \approx 3x - 10$
- $\log(f(n)) = 3\log(n) - 10$ iff

$$f(n) = 10^{\log(f(n))} = 10^{3\log(n)-10} = n^3 * 10^{-10}$$

- Conclusion: cubic in the input size, with constant $\approx 10^{-10}$
- No surprise: see the 3-nested loop in ThreeSum.java
- Strong dependence on input can be a problem
- Constant 10^{-10} depends on computer, exponent 3 does not

Worst case, average case, amortized cost

- Worst case: guaranteed, independent of input; Examples:
  - Linked list implementations of Stack, Queue and Bag: all operations take constant time in the worst case
  - Resizing array implementations of Stack, Queue and Bag: adding and deleting take linear time in the worst case (easy)
- Average case: not guaranteed, dependent of input *distribution*
- Amortized: worst-case cost *per operation*. E.g., each 10-th operation has cost ≤ 21, all others cost 1, amortized ≤ 3 p/o.
- Resizing arrays: adding and deleting take constant time *per operation* in the worst case (proof is difficult)
- Special case of resizing array that is only growing: 1(2)2(4)34(8)5678(16)9 ... 16(32) ..., with (n) the new size. Risizing to (n) costs $2n$ array accesses, so in total (1+4)+(1+8)+(2+16)+(4+32)+(8+64) ..., so 9 p/push.

Remarks and Pitfalls

- Theoretical approach:
  - Wrong cost model
  - JVM optimization can obscure the exponent
  - Caching can have large impact on memory access
  - Large constant factor in Big Oh
  - Worst case can be easy, average case difficult
- Empirical approach:
  - The focus is on run time (using space costs time)
  - Dependence on input, randomization does not always help
  - Machine/platform dependence
  - Linear regression not good for, e.g., $O(n^2 \log n)$

Exercise

Aim: better understand the empirical method.

1. Let input sizes 1, 2, 4, 6, 8K take 2, 7.9, 32, 72, 129 sec
2. Make a plot such as in plot sheet (download)
3. Compute the log's of the input sizes and of the run times and make the log-log plot such as in plot sheet (second plot)
4. Estimate a and b such that the log-log plot is $y \approx ax - b$
5. Estimate a and b through linear regression, compare with 4.
6. Find $f(n)$ given that $\log(f(n)) = a\log(n) - b$. Surprised?

In cases where the run time mostly depends on the size n of the input and not on the input itself, the function f is a reasonable (polynomial) estimation of the run time.

Logarithms and Exponents Cheat Sheet

- Definition: $\log_x z = y$ iff $x^y = z$ for $x > 0$
- Base of logarithm: the x in $\log_x$
- Inverses: $x^{\log_x y} = y$ and $\log_x x^y = y$
- Exponent, laws: $x^{(y+z)} = x^y x^z$, $x^{(yz)} = (x^y)^z$
- Logarithm, laws: $\log_x(yz) = \log_x y + \log_x z$, $\log_x z = \log_x y \log_y z$
- Various bases: $\log_2 = \lg$, $\log_e = \ln$, $\log_{10} = \log$
- Double exponent: e.g. $2^{(2^n)}$ (not used in INF102)
- Double logarithm: $\log(\log n)$ (not used in INF102)

Staying Connected

- We want efficient algorithms and datastructures for testing whether two objects are 'connected' (e.g., in networks)
- We assume connectedness to be an equivalence
- MNF130: relation $E \subseteq V \times V$ is an *equivalence* if
  - E is *reflexive*: $\forall x \in V.\ E(x,x)$
  - E is *symmetic*: $\forall x, y \in V.\ E(x,y) \rightarrow E(y,x)$
  - E is *transitive*: $\forall x, y, z \in V.\ E(x,y) \wedge E(y,z) \rightarrow E(x,z)$
- Dynamic connectivity means (here) that E can grow
- Relationship with paths in graphs, (connected) components (MNF130): nodes are connected if there is a path between them
- Input: N and pairs in $V = \{0, \ldots, N-1\}$ defining E
- Challenge: efficient `boolean connected(int p, int q)`

Example

- Example (`algs4-data/tinyUF.txt`) : $N = 10$
- Nodes $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$
- Edges: 4—3, 3—8, 6—5, 9—4, 2—1, 8—9, 5—0, ...
- Linear space: don't add pairs that are already connected!
- Q: what are the costs of storing all pairs that are connected, space and time?
- See: algoritmevisualisering by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke

Union-Find

- Find, idea: every component has one element as its identifier, `int find(int n)` computes this identifier
- Union, idea: for any new pair n m that are not already connected, `union(int n, int m)` takes the union of the two components, ensuring `find(n) == find(m)`
- API: UF; Cost model: number of array accesses
- Implementations:
  - SlowUF.java: `id[p]` identifier of `p`
    `find()`~ 1, `union()`~ 2 or between $n+3$ and $2n+1$ (!)
  - FastUF.java: `int[] id` pointers, `id[p]==p`: identifier
    `find()` $\sim 1+2d$, `union()` $\sim 1+$ two `find()`'s
  - WeightedUF.java: `int[] id` pointers, `int[] sz` subtree sizes
    `find()` and `union()` both $\sim \lg n$

Trees (cf. MNF130) and WeightedUF

A (rooted) *tree* consist of *nodes* (also called *vertices*) one of which is called the *root* r. Every node n is connected by an *edge* to zero or more other nodes, called the *children* of the *parent* n. Moreover, each node $n \neq r$ has a unique parent in a tree. Trees are naturally depicted in levels starting with the root at level 0, then the level 1 of the children of the root, level 2 of the children of the children of the root, and so on. The level of a node is also called its *depth*. In a finite tree there is always a highest level (maximum depth) and this is called the *heigth* of the tree.

- WeightedUF: height of subtree of size k is at most $\lg k$ (proof by induction on blackboard)
- Ultimate improvement of UF (almost $O(1)$, amortized): path-compression (sketch on bb)

Sorting

- Sorting: putting objects in a certain order
- MNF130: relation $R \subseteq V \times V$ is a *total order(ing)* if
  1. R is *reflexive*: $\forall x \in V.\ R(x,x)$
  2. R is *transitive*: $\forall x,y,z \in V.\ R(x,y) \wedge R(y,z) \rightarrow R(x,z)$
  3. R is *antisymmetic*: $\forall x,y \in V.\ R(x,y) \wedge R(y,x) \rightarrow x = y$
  4. R is *total*: $\forall x,y \in V.\ R(x,y) \vee R(y,x)$
- Natural orderings:
  - Numbers of any type: ordinary $\leq$ and $\geq$
  - Strings: lexicographic
  - Objects of a Comparable type: `v.compareTo(w) <= 0`

Sorting and searching: linear vs binary search

```
int linearSearch(Comparable key, Comparable[] a){
  for (int i=0; i < a.length; i++){
    if (key.compareTo(a[i])==0) {return i;}
  }
  return -1; // key not in array: O(a.length)
}

int binarySearch(Comparable key, Comparable[] a) {
  int lo=0; int hi=a.length-1; int mid; int test
  while (lo <= hi){
    mid = (lo+hi)/2; test = key.compareTo(a[mid]);
    if (test == 0) {return mid;}
    if (test < 0) { hi = mid-1;} else {lo = mid+1;}
  }
  return -1; // key not in SORTED array: O(lg(a.length))
}
```

Sorting (ctnd)

- Elementary sorts:
  1. Bubble sort (like gas bubbles in sparkling water)
  2. Selection sort (iterated selection of minima)
  3. Insertion sort (iterated insertion of elements)
  4. Shell sort (Shell's refinement of insertion sort)
- Bubble sort: ExampleSort.java
- Certification: `assert isSorted(a)` in `main()`
  (no guarantee against modifying the array, but `exch()` is safe)
- Costmodel(s): number of `less()`'s and of `exch()`'s
  (or array accesses; discuss pointer vs. object)
- Why studying sorting? (`java.util.Arrays.sort()`)
- Comparing sorting algorithms: SortCompare.java

Selection Sort

- Bubble sort: $\sim n^2/2$ compares, $0 \leq$ exchanges $\leq n^2/2$
- Selection sort:
  - Find index of a minimum in a[0..n-1], exchange with a[0]
  - Find index of a minimum in a[1..n-1], exchange with a[1]
  - ... until n-2
- Selection sort: $\sim n^2/2$ compares, $0 \leq$ exchanges $\leq n-1$ (!)

```
public  static void sort(Comparable[] a) {
 int N = a.length;
 for (int i=0; i<N-1; i++){
   int min=i;
   for (int j=i+1; j<N; j++) if less(a[j],a[min])) min=j;
   if (i != min) exch(a,i,min);
 }
}
```

Insertion sort

- Insertion sort:
  - Insert `a[1]` on its correct place in (sorted) `a[0..0]`
  - Insert `a[2]` on its correct place in (sorted) `a[0..1]`
  - ... until `a[n-1]`
- Very good for partially sorted arrays, costs:
  - Best case: $n-1$ compares and 0 exchanges
  - Worst case: $\sim n^2/2$ compares and exchanges
  - Average case: $\sim n^2/4$ compares and exchanges (distinct keys)

```
public  static void sort(Comparable[] a) {
 int N = a.length;
 for (int i=1; i<N; i++){
  for (int j=i; j>0 && less(a[j],a[j-1]); j--)
   exch(a,j,j-1);
 }
}
```

Shell sort

- Insertion sort:
  - Very good for partially sorted arrays
  - Slow due to one-step transport `exch(a,j,j-1)`
  - Why not larger steps `exch(a,j,j-h)` ?
- Idea: presort `a[i],a[i+h],a[i+2h],...` for `i = 0..h-1`

```
public static void hsort(int h, Comparable[] a) {
 int N = a.length;
 for (int i=h; i<N; i++)
  for (int j=i; j-h>=0 && less(a[j],a[j-h]); j-=h)
    exch(a,j,j-h);
}
```

- Insertion sort: `hsort(1,a)`
- Shell sort: e.g., `hsort(10,a); hsort(1,a)`

Shell sort (ctnd)

- `hsort(10,a); hsort(1,a)` faster than just `hsort(1,a)` !
- Q: How is this possible?
- A: `hsort(10,a)` transports items in steps of 10, which would be done by `hsort(1,a)` in 10 steps of 1.
- What about `hsort(100,a); hsort(10,a); hsort(1,a)`?
- To be expected: depends on the length N of the array
- The run-time analysis of Shell sort is very difficult
- Best practice: $h = N/3, N/9, \ldots, 364, 121, 40, 13, 4, 1$
- Example: algoritmevisualisering by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke

Mergesort

- Top-down (recursive) algorithm:
  - Mergesort left half, mergesort right half
  - Merge the results (example: 2468,1357)
- Using an auxiliary array: TopDownMergeSort.java, Movie
- Bottom-up algorithm (16 elements):
  - Merge `a[0],a[1]`, merge `a[2],a[3]`, merge `a[4],a[5]`, ...
  - Merge `a[0..1],a[2..3]`, merge `a[4..5],a[6..7]`, ...
  - Merge `a[0..3],a[4..7]`, merge `a[8..11],a[12..15]`
  - Merge `a[0..7],a[8..15]`, done!
- Also using an auxiliary array: BottomUpMergeSort.java

Run-time and memory use of mergesort

- Mergesort uses between $\sim (N/2) \lg N$ and $\sim N \lg N$ compares. Proof on bb. Important formula ($N = 2^n$):

$$2C(2^{n-1}) + 2^{n-1} \leq C(2^n) \leq 2C(2^{n-1}) + 2^n$$

- Mergesort uses at most $\sim 6N \lg N$ array accesses
- Mergesort uses $\sim 2N$ space (plus some var's)
- Q: How fast can compare-based sorting of N distinct keys be?
- A: $\lg N! \sim N \lg N$; Proof in book and on bb. Keywords: binary *compare tree*, inner nodes for each `compare(a[i],a[j])`, permutations in the leaves,

  $N! =$ number of permutations $\leq$ number of leaves $\leq 2^{\text{height of tree}}$

Quicksort

- Top-down (recursive) algorithm:
  - Choose a (pivot) value v in the array
  - Partition the array in non-empty parts $\leq v$ and $\geq v$
  - Quicksort the two parts
- Examples: algoritmevisualisering by Ragnhild Ålvik, Kristian Rosland, Knut Anders Stokke
- Pros: in-place, average computation time $O(n \log n)$
- Cons: stack space for recursion, worst-case $O(n^2)$, not stable
- Implementation: QuickSort.java
- BTW: Bug in java.util.Arrays.sort

Quicksort, details

- Subtleties in `Quicksort.sort()`: shuffling protects against worst-case behaviour
- Termination of recursive `quicksort()`
- Subtleties in `partition()`:
  - Invariants `l<=h` in the two inner loops
  - Postcondition after the two inner loops
  - Invariant of the `for(;;)` loop
  - Termination of the `for(;;)` loop
  - There are some variations that are also correct

Run-time and memory use of quicksort

- Compare Quicksort to other sorts ($n = 10^2, 10^3, \ldots$)
- Quicksort: time $O(n^2)$ if pivot is always smallest (or largest)
- Randomization: choose pivot randomly, or shuffle array
- If all keys are distinct and randomization is perfect, then quicksort uses on average $\sim 2n \ln n$ compares and $\sim (n/3) \ln n$ exchanges (proofs in book, complicated)
- Relevant improvements:
  - Cut-off to insertion sort for sizes ≤ 15 (ca.)
  - Median-of-three pivot
  - Taking advantage of duplicate keys (3-way partitioning)
- Quicksort is generally quite good
- In special situations other sorts are better (e.g., countsort)

Priority Queues

- Aim: collecting and processing items having keys
- Examples of keys: time-stamp, price-tag, priority-tag
- Assume: keys are ordered
- Reasonable: processing currently highest (or lowest)
- Special cases: items time-stamped when added
  - Queue: dequeue currently oldest (lowest time-stamp)
  - Stack: pop currently newest (highest time-stamp)
- Priority queue generalizes this
- Examples: highest priority, largest transaction, lowest price
- Abstract from 'item' and use only 'key' (in applications: use objects with fields `item` and `key` and compare on `key`)

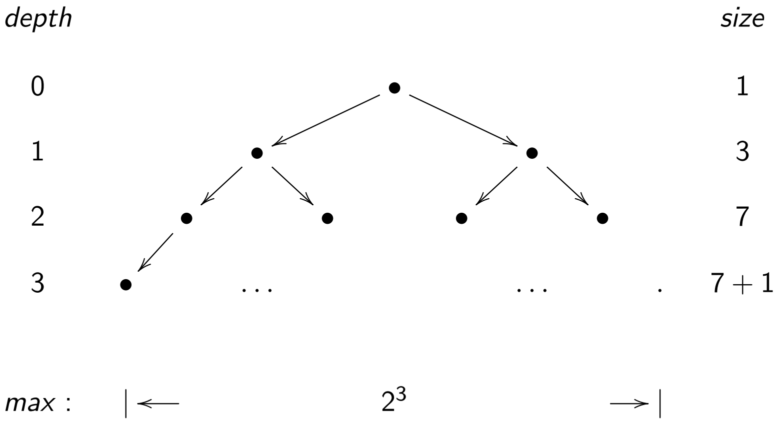
Priority Queues

- Good info: Wikipedia; API (the bare essentials):

```
public class  MaxPQ<Key extends Comparable<Key>>

void          insert(Key v) // insert a key
Key           delMax() // delete a largest key, if any
boolean       isEmpty()
int           size()
```

- In case of duplicate keys: ‘a’ largest, not ‘the’
- Typical application: the 1K smallest keys of 1G unsorted keys
- Client: BottomM.java (Q: why is the output slowing down?)

Example of left-complete binary tree



Binary Trees

- MNF130: Tree *size* is number of nodes, *depth* of a node is number of links to the root, tree *height* is maximum depth.
- MNF130: In a *binary* tree every node has at most two children.
- MNF130: A binary tree is *complete* if all levels are filled. So, a complete binary tree of height h has $2^{h+1}-1$ nodes.
- INF102: A binary tree is (left-)*complete* if all levels $< h$ are filled, level h may be partly empty from the right (picture bb). A (left-)complete binary tree of height h has between 2^h and $2^{h+1}-1$ nodes.
- A left-complete binary tree of n nodes has height $\lfloor \lg n \rfloor$ (from now on we leave out 'left-').

Heap-ordered Binary Trees

- Naive implementations:
  - Unsorted (resizing) array: fast `insert()`, linear `delMax()`
  - Sorted (resizing) array: linear `insert()`, fast `delMax()`
- Aim: operations in logarithmic time, no extra space
- A binary tree is *heap-ordered* if the key in each node is $\geq$ the keys in its children (if any). Thus the root has a maximal key.
- NB: a heap is NOT a search tree (different data invariants)!
- Array representation of heap-ordered complete binary tree (bb)
- Methods `swim()` and `sink()`: picture on bb, code below
- Implementation: ArrayListPQ.java

Run-time and memory use of heaps, applications

- In a heap of n elements (since height is $\leq \lfloor \lg n \rfloor$):
  - `insert()` takes $\leq 1 + \lfloor \lg n \rfloor$ compares and exchanges
  - `delMax()` takes $\leq 2\lfloor \lg n \rfloor$ compares and $\leq 1+\lfloor \lg n \rfloor$ exchanges
- Heap construction by `insert()` can sometimes be improved
- Example: maxheap from A B C D E F G H
- Given an array of n keys, right-to-left heap construction (bb) takes $< 2n$ compares and $< n$ exchanges
- Applications: heapsort and merging sorted streams
- Many variations with extended API (indexed priority queue)

Purpose of Sorting

- Sorting makes the following easier and more efficient:
  - Searching (binary search, example: `ThreeSumOptimized`
  - Searching and looking up, e.g., the pagenumber in an index
  - Finding and removing duplicates
  - Finding the median, quartiles etc.
- Our sorting algorithms are generic: `sort(Comparable[] a)`, for any user-defined data type with a `compareTo()` method
- We do *pointer sorting*, manipulating refs to objects.
  - Pro: not moving full objects
  - Cons: pointer dereferencing, no `sort(int[] a)`
- More flexibility: pass a `Comparator` object to `sort()`

Comparator object

- API: `void sort(Object[] a, Comparator c)`
- Call, e.g.: `sort(a, new Transaction.WhenOrder())`
- Call, e.g.: `sort(a, new Transaction.SizeOrder())`
- Obs: `import java.util.Comparator`
- Obs: `less(Object o1, Object o2, Comparator c)`
- Priority queues also with `Comparator`

```
public class Transaction {
 ...
 public static class MyOrder {
 implements Comparator<Transaction>
  public int compare(Transaction t, Transaction v){...}
 } // End of Myorder
...// similarly: WhenOrder, SizeOrder
} // End of Transaction
```

Applications of Sorting

- Consider sorting first to make other problems easier
- Commercial computing (sort on price, departure time, ...)
- Search for information: web-indexing, search engines
- Job scheduling heuristic: longest processing time first
- To come: Prim's, Dijkstra's and Kruskal's algorithms
- Huffman compression: a lossless compression based on using the shortest codes for the symbols that occur oftest. Frequency counter: next chapter!
- Cryptology and genomics (e.g., longest repeated substring)

Symbol Tables

- Symbol table associates *keys* with *values: key-value pairs*
- Examples: keyword-list of page nrs, ID number-personal data, word-frequency
- Important operations:
  - Insert a key-value pair in the symbol table: `void put(k,v)`
  - Search the value for a given key (if any): `Value get(k)`
- Important conventions:
  - Inserting key-value for existing key: overwriting the value
  - No duplicate keys, no null keys
  - Value null: no value for this key
  - Lazy deletion: insert key-null; Eager: really delete key-value
- API of unordered symbol table
- Aim: all operations in time $\sim c \lg n$ with constant c small

ST Basics

- Archetypical ST-client: frequency counter, ArrayListST.main
- Cost model: number of compares
- Naive ST: unordered linked list (or ArrayList), linear search
  - Search miss: $\sim n$ compares
  - Search hit: between 1 and $\sim n$ compares
  - Random search hit: $(1 + \cdots + n)/n \sim n/2$ compares
  - Inserting n distinct keys: $(1 + \cdots + (n-1)) \sim n^2/2$ compares
- `algs4-data/leipzig1M.txt`: 21M words, 500K distinct
- Naive ST impracticable for genomics, internet
- Scale: G-T keys, M-G distinct (Kilo,Mega,Giga,Tera)
- Better for unordered ST: hashing (in Ch. 3.4)

Ordered Symbol Table

- Ordered ST: keys are ordered, f.e., in an ArrayList
- API of ordered symbol table
- Binary search: `get(Key k)` takes $\sim \lg n$ comparisons
- What about `put(Key k, Value v)`? See ArrayList
- Pitfall: `add(int i, E e)` is linear, not amortized $O(1)$!
- Consequence: `put(Key k, Value v)` and `del(Key k)` *linear*
- Implementation with binary search in ArrayListST.java
- Trace of inserts on bb: S E A R C H E X A M P L E
- Experiments with `tinyTale.txt`, `tale.txt`, ...

Binary Search Trees

- Aim: `get`,`put`,`del` in logarithmic time, ST in linear space
- Binary *search* tree: for every node, all keys to the left of this node are smaller, and all keys to the right are larger
- Search time: lenght of the path to the node where the key 'should' be
- Balanced binary tree with n keys has $\lg n$ height
- Unbalanced binary trees can have height n (max depth)
- Search hits in a binary search tree, built without rebalancing, of n random keys take on average $\sim 2 \ln n$ compares
- UBST.java: `put()`, `get()`, `size()`, `isEmpty()`
- Trace of inserts on bb: S E A R C H E X A M P L E

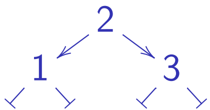
Binary Search Trees (`min()`, `delMin()`, `delete()`)

- Interrelated, increasing difficulty: `min(Node x)`, `deleteMin(Node x)`, `delete(Node x, Key k)`
- Node of minimum key: not `null`, and has left child `null`, and is root or left child of parent (picture on bb)

```
public Node min(Node x){// precondition x!=null
 while (x.left!=null) x = x.left; // inv x!=null
 return x;
} // cf. tail recursive min() in Alg. 3.3
```

- Delete minimum key, two cases:
  (1) both children `null`; (2) left child `null`
- Delete is really difficult: BST.java, cf. ArrayListST.java
- Don't forget: update `x.N` along the path to the root!

Delete from search tree, example:



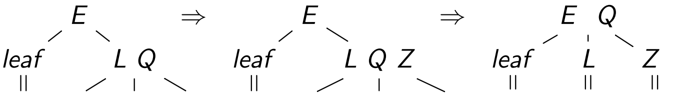

```
root=delete(root,3)                        (1st example)
| x=root; x.right=delete(x.right,3)
  | x'=x.right; return x'.left;            (x.right=null)
| update x.size;                           (x.size=2)
| return x;                                (root=x)

root=delete(root,2)                        (2nd example)
| x=root; t=x; x=min(t.right);             (x=t.right)
| x.right=deleteMin(t.right);              (x.right=null)
| x.left=t.left;
| update x.size;                           (x.size=2)
| return x;                                (root=x)
```

Balanced Search Trees: keep paths short!

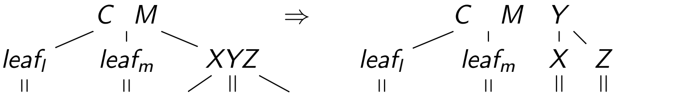
- NB tree balancing not as easy as in UF and Heap (4hrs!)
- A 2-3 search tree consists of 2-nodes and 3-nodes:
  - Each 2-node has two children and a key k such that all keys in the left subtree are $< k$, and all keys in the right subtree $> k$
  - Each 3-node has three children and two keys $k_1 < k_2$ such that all keys in the left subtree are $< k_1$, all keys in the middle subtree $> k_1$ and $< k_2$, and all keys in the right subtree $> k_2$
- Examples and pictures on bb
- *Perfect* 2-3 search tree: paths from root to leaves equally long
- Search: compare key with key(s) in node, if equal return corresponding value, else search in one of left, middle, right subtree where the key should be (if it occurs at all)
- Insert should keep tree perfect, rough idea:
  - into a 2-leaf: make it into a 3-leaf (easy)
  - any other case: (temporary) 4-nodes (see next, difficult)

Insert in Perfect 2-3 Search Trees

- Terminology: a *leaf* is a node all whose children are `null`
- Data invariant 1: tree is 2-3 search tree
- Data invariant 2: all paths from root to leaves equally long
  - Insert into a 2-leaf L : either $A\ L$ or $L\ Z$
  - into a 3-leaf whose parent is a 2-node: with new key Z (e.g.)



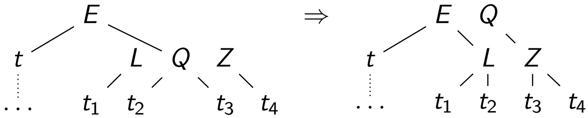
  - into a 3-leaf whose parent is a 3-node: with new key Z (e.g.)



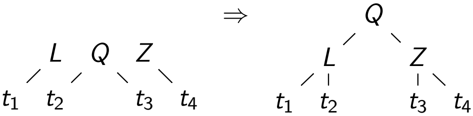
  - into a 3-node whose parent is a 3-node: move up middle key!

Insert (ctnd)

- Data invariant 1: tree is 2-3 search tree
- Data invariant 2: all paths from root to leaves equally long
- Insert works up from the leaf where the key 'should' be
  - if 2-node on path to root: make it into a 3-node; there are two cases, left and right, here is a picture of the latter



  - otherwise, work upwards and, finally, split the root:



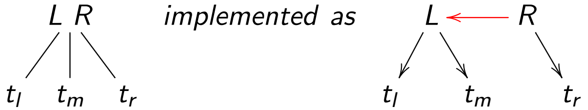
  - working upwards, there are three cases: left, middle, and right

Insert, summary and examples

- There exists a perfect 2-3 tree for any sequence of input keys
- Six operations for eliminating 4-nodes:
  - if parent is 2-node: move middle key up (left and right case)
  - if parent is 3-node: move middle key up (left, middle,right)
  - if root: split root
- Search and insert visit at most $\lfloor \lg n \rfloor$ nodes
- Proof: maximal path length is $\geq \lfloor \log_3 n \rfloor$ and $\leq \lfloor \log_2 n \rfloor$
- Trace of inserts on bb: S E A R C H (E) X (A) M P L (E)
- Trace of inserts on bb: A C E H L M P R S X (keep balance!)

Red-black trees

- Red-black trees implement 2-3 trees
- Idea: one 3-node = two 2-nodes + extra info
- Extra info coded in color, picture:



- A *red-black tree* is a binary search tree with red and black links such that:
  - Only left links can be red (but need not be)
  - Never $\leftarrow \; \leftarrow$
  - Perfect black balance (all paths from root to leaves same number of black links; this number is called the *black height*)
- Equivalent: red-black tree and perfect 2-3 search tree

Red-black trees (ctnd)

- Color is attribute of *incoming* link (why?)

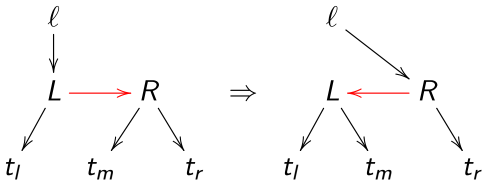
```
private class Node {
  Key key;
  Value value;
  Node left, right;
  boolean color; // true for red, false for black
  int N; // number of keys in subtree below this node
}
private boolean isRed(Node n) {
 if (n==null) {return false;} else {return x.color}
```

Rotating and Color Flipping

- Aim: restoring the data invariants of red-black search trees
  1. Only left links can be red, but never two successive
  2. Search tree invariant
  3. Perfect black balance
- Repairments use *rotations* and *color flips*, examples (leaves):
  - inserting Z in $L \longleftarrow R$: $L \longleftarrow R \longrightarrow Z$, violation (why?)
    repairment: color flip $L \longleftarrow R \longrightarrow Z$ (R must up)
  - inserting A in $L \longleftarrow R$: $A \longleftarrow L \longleftarrow R$, violation (why?)
    repairment: rotation right + color flip $A \longleftarrow L \longrightarrow R$
  - inserting M in $L \longleftarrow R$:
    $$\begin{tikzcd} L \arrow[dr] & & R \arrow[ll] \\ & M & \end{tikzcd}$$
    , violation
    repairment: rotation left into $L \longleftarrow M \longleftarrow R$, then as above
- Rotations change the root of the subtree, preserving invariants

Left Rotation

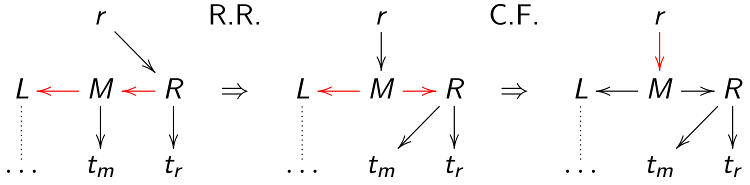
Call: `l = rotateLeft(l);`




```
private Node rotateLeft(Node l){
  Node r = l.right; l.right = r.left; r.left = l;
  r.color = l.color; l.color = true // == RED
  r.N = l.N; l.N -= 1+size(r.right); // Why?
  return r;
}
```

Right Rotation and Color Flip

Typically in the following situation (e.g., after insert(L) in a 3-leaf):



- Code of `rotateRight()` like that of `rotateLeft()`
- NB1: operations are local (here only `r`, `M` , `R`)
- NB2: operations preserve data invariants
- NB3: root is a special case (always black)
- Deletions: complicated, but doable (Exc. 3.3.39–41)

Run-time and memory use of Red-Black BSTs

- The height of a red-black BST with n nodes is $\leq 2\lg n$
  Proof: the worst-case is one 3-node path and the rest 2-nodes
- The average length of path (any color) from the root to a node in a red-black BST with n nodes is $\lg n$ ('empirical fact')
- In a red-black BST, search, insert, ..., and delete, take logarithmic time in the worst-case. Proof: a constant amount of work is done per visited node (book Prop. I, p. 447).
- For red-black BSTs, logarithmic time is guaranteed!

Hashing

- Idea: if keys in `[0..99]` an array is the perfect symbol table
- In fact: CountSort99.java counts frequencies like an ST client
- A *hash function* maps keys to array indices
- Injectivity of the hash function is not guaranteed
- *Hash collision*: different keys are mapped to the same index
- In such a case we need *collision resolution*
- Symbol tables: hashing is fast, but unordered (no `max`,`min`)
- Aim: ST operations in amortized $O(1)$ time, extra space OK

Space-Time Trade-Off

- Hashing is an example of a *space-time trade-off*
- Time: computation time required
- Space: memory space used
- Unlimited space: (1) use key as index (e.g., the bits)
- Unlimited time: (2) use linked list and linear search
- Hashing strikes a balance using (1) with some array of reasonable size, and (2) in case of collisions
- The balance between (1) and (2) can easily be tuned

Hash functions

- Ideal (uniform hashing assumption, UHA): uniform and independent distribution of keys over integers from 0 to $M-1$
- Examples of hash functions in Java
- Horner: $a_0 + x(a_1 + x(a_2 + \cdots)) = a_0 + a_1 x + a_2 x^2 + \cdots$
- Modular hashing (M prime), reasonably $\approx$ UHA:

```
private int hash(Key k){
 return (key.hashCode() & 0x7fffffff) % M;}
```

- Q: Why crazy `& 0x7fffffff` ???
- A: In Java, e.g., `(-5 % 3) == -2`) and not 1
- Q: Why M prime?
- A: E.g., $M = 32$ takes only into account the last five bits

Collision Resolution

- Two main methods of collision resolution:
  1. Hashing with separate chaining (picture on bb)
  2. Hashing with linear probing (picture on bb)
- Separate chaining: symbol table is an array of linked lists, linear search. If array has length M, then the linked lists have average length N/M with N keys.
- Linear probing: symbol table is an array of length $M \geq N$. Colliding keys are put at the first empty position. Linear search from the position where the key 'should have been'. Empty position: not found. Deletion tricky: reinsert all keys to the right of the deleted key, until the first empty position (picture on bb). Works better with $M >> N$.

Symbol Table with Hashing

- Implementation: `ArrayListHashST.java`
- $M = 1$: measure overhead wrt. `ArrayListST`
- Tests with various values of M: 31, 997, 65521
- NB1: `ArrayListST`, `UBST` are sorted (!)
- NB2: *construction* versus *use* of ST (hashing better for *use*)
- Hashing can be combined with any other ST-implementation
- UHA metaphor: for every key one throws a dice *once*, and remembers the value as the hash code of the key

Quantitative analysis

- Throwing a dice 10 times, what is the probability of 3 fives?
- Under UHA, with N distinct keys, the probability that exactly k keys collide at some given hash value is

$$\binom{N}{k}(\frac{1}{M})^k(\frac{M-1}{M})^{N-k}, \text{ where e.g. } \binom{100}{10} \approx 1.7E13$$

- This is a small number for, say, $N = M = 100$ and $k = 10$
- For linear probing one typically takes $M = 2N$
- For separate chaining one keeps $N/8 \leq M \leq N/2$ (resizing M)
- Under UHA: search, insert, delete take amortized $O(1)$ time
- Space used can be upto $100N$ byte (objects, pointers); this on top of the space used by N key-value pairs

Applications of Searching

- Synonyms: associative array, map, symbol table, or dictionary
- Origin of symbol table: compilers and interpreters
- Web-indexing, search engines
- Sparse matrices (many 0's): dictionary
  1. keys: (row, column)-pairs
  2. values: matrix entries
- Set API (no values, only keys, for deduplication, filtering):

```
public class  SET<Key>
 { void  add(Key k);
   void  delete(Key k);
boolean  contains(Key k);
boolean  isEmpty();
    int  size(); }
```

Indexes and Reverse Indexes

- Index (key, value)
- Reverse index (value, key(s))
- Phone book (name+address, phone number(s))
- Reverse phone book (phone number, name+address)
- Dictionary (word, meaning or translation)
- Account information (client ID, account information)
- Genomics (protein, sequences of ACTG triplets)
- File systems (file name, location on disk/ file attributes)
- Internet domain name system (domain name, IP address)

Balanced Search Tree or Hash Table?

- Q: Which symbol table to use?
- A: The basic choice between BST and HT depends on ...
  1. Ordering of keys essential: BST
  2. Availability of good hash function (good = fast + UHA)
  3. Ordering of keys expensive (long strings): HT (or: Ch.5)
  4. Ordering of keys possible, but not essential: HT + BST
  5. Space considerations (ArrayListST uses the least extra space)
  6. Number of distinct keys and the space each key takes
  7. Distribution of insert/delete/search operations

Overview Chapter 1–3

Chapter 1

- Stack and Queue, ThreeSum, Union-Find
- Theory: $\sim$ and O
- Experiments: loglog-plots, randomization

Chapter 2: Sorting

- Selection-, Insertion-, Shell-, Merge-, QuickSort
- Priority Queue, Binary Heap, HeapSort
- CountSort

Chapter 3

- Symbol Table
- Binary Search Tree, Perfect 2-3 Tree, Red-Black Tree
- Hashing: hash function and collision resolution

Odds and Ends Chapter 1–3

- Path-compression in UF (71)
- Randomizing the order of the tests
- Compare-based sorting requires $N \lg N$ comparisons (72)
- Double hashing: linear probing $h_1(k), h_1(k) + h_2(k), h_1(k) + 2h_2(k), \ldots$
- Indexed Priority Queues (74)

Randomizing the order of the tests

```
for (int i=0; i<N; i++) {
    a1[i] = StdRandom.uniform();
    a2[i] = a1[i]; }
if (StdRandom.uniform(2) == 0)
   {total1 += time(alg1, a1); // summing runtime of alg1
    total2 += time(alg2, a2); // summing runtime of alg2
    }
else
   {total2 += time(alg2, a2); // summing runtime of alg2
    total1 += time(alg1, a1); // summing runtime of alg1
    }
```

Path compression in UF

```
// Finding the "identifier" of the component of p in id:
public int find(int p) {
  while (p!=id[p]) { p=id[p]; }
  return p;
}
// now with path compression:

public int find(int p) {
  int q=p; // remember the starting point
  while (p!=id[p]) { p=id[p]; }
  // postcondition: p==id[p]==identifier of q
  while (id[q]!=p) { int aux=id[q]; id[q]=p; q=aux; }
  return p;
} // Example: int[] id={1,2,3,3}; find(0);
```

Compare-based sorting: worst-case $\geq N \lg N$

- Every compare-based sorting algorithm for N distinct keys in an array a leads to a *binary compare tree* with
  - nodes $(i{:}j)$ representing tests $a[i] < a[j]$
  - left subtree: $a[i] < a[j]$; right subtree: $a[i] > a[j]$
  - leaves: sorted permutations of the array
- Example with array of length 3 on bb
- Every permutation should occur at least once in a leaf!
- Binary tree of height h has at most 2^h leaves
- Length of path to leaf = number of comparisons
- Now $h \geq \lg N! \sim N \lg N$ by Stirling from this formula:

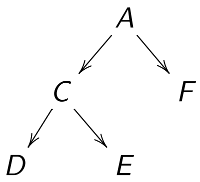
$$N! = \text{number of permutations} \leq \text{number of leaves} \leq 2^{\text{height of tree}}$$

Indexed Priority Queues (missing inverse qp of pq)

- IPQ ≈ array with direct access to minimum (maximum)
- API: void insert(int i, Key k); void del(int i); int minKey(); Key keyOf(int i);... Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-

heap



Do: insert(6,G), insert(5,B), insert(1,Z)

NB1: `keys[pq[i]]` on position `i` in heap

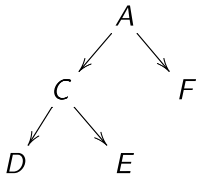
NB2: inverse index qp is needed

Indexed Priority Queues

- IPQ ≈ array with direct access to minimum (maximum)
- API: void insert(int i, Key k); void del(int i); int minKey(); Key keyOf(int i);... Example of implementation:

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	0	0
keys	C	A	F	E	D	-	-
qp	1	0	2	4	3	-1	-1

heap



Do: insert(6,G), insert(5,B)
NB qp is needed to find
the index of `key[i]` in pq
e.g., for insert(1,Z) (then: sink!)

Indexed Priority Queues (ctnd)

After insert(6,G):

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	6	0
keys	C	A	F	E	D	-	G
qp	1	0	2	4	3	-1	5

Step 1 insert(5,B):

index	0	1	2	3	4	5	6
pq	1	0	2	4	3	6	5
keys	C	A	F	E	D	B	G
qp	1	0	2	4	3	6	5

Step 2 (swaps)
pq[2] pq[6]
qp[pq[2]] qp[pq[6]]

index	0	1	2	3	4	5	6
pq	1	0	5	4	3	6	2
keys	C	A	F	E	D	B	G
qp	1	0	6	4	3	2	5

Graph classes

(MNF130: useful review of graph theory)

1. Undirected graphs: a set of *vertices* (or *nodes*) V and a set of *edges* E connecting the nodes
2. Directed graphs (*digraphs*): a set of nodes V and a set E of directed edges (or *arrows*) pointing from one node to another
3. *Edge-weighted graphs*: undirected graphs in which every edge comes with a number called the *weight* of the edge
4. *Edge-weighted digraphs*: digraphs in which every arrow has a weight, like the edges of an edge-weighted graph

Examples

1. Map (discuss: nodes, un/directed, un/weighted, multigraph)
2. Undirected graphs: social networks, communication networks (duplex communication)
3. Directed graphs: hyperlinks, (class, module, package) dependencies, logical circuits, job scheduling, flow graphs
4. Edge-weighted graphs: roadmaps with geographical distance, or with toll, communication networks with bandwidth
5. Edge-weighted digraphs: job scheduling with duration, flow with volume, financial transactions with size

Undirected Graphs

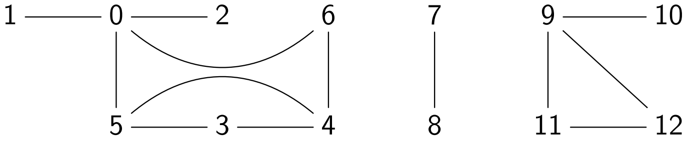
- Undirected graph: a set of *vertices* (or *nodes*) V and a set of *edges* E connecting the nodes
- *Subgraph*: subset of E and subset of V forming a graph (!)
- *Path*: sequence of nodes connected by edges (!)
- *Simple path*: path with no node repeated
- *Length of path*: number of edges
- *Cycle*: path of length > 0 with same start and end node
- *Simple* cycle: not repeating edges or nodes (apart from start and end node)
- *Acyclic graph*: graph without simple cycles
- *Connected graph*: having a path between every two nodes
- *Connected component*: a maximal connected subgraph

Trees and Forests

- Example: `tinyG.txt` on bb
- 'Anomalies' concerning edges:
  - Self-loop: edge connecting a node to itself
  - Parallel edges: two edges connecting the same node(s)
- When no anomalies, $E \subseteq \{\{v, v'\} \mid v \in V, v' \in V, v \neq v'\}$
- *Tree*: connected acyclic graph (then: no anomalies)
- *Spanning tree*: maximal subgraph that is a tree
- Lemma: any spanning tree of a connected graph contains all nodes. Proof on bb.
- *Forest*: graph consisting of disjoint trees
- *Spanning Forest*: forest consisting of spanning trees of connected components of a graph

Undirected Graphs (ctnd)

- *Distance* between two nodes: length of a shortest connecting path if there is a path connecting these nodes, otherwise ∞
- *Degree* of a node: number of edges connected to that node
- Graph $G = (V, E)$, the following are equivalent:
  - G is a tree (def: connected and acyclic)
  - G has $|V| - 1$ edges and no cycles
  - G has $|V| - 1$ edges and is connected
  - G is acyclic and adding an edge creates a cycle
  - Any two nodes of G are connected by exactly one simple path
- Example: some (connected) subgraphs of `tinyG.txt`



Graph representation and implementation

- Impractical: adjacency matrix $\sim V^2$, incidence matrix $\sim VE$
- Often practical: adjacency lists $\sim (V+2E)$, that is, `adj[v]` lists all nodes `w` connected to `v` by an edge
- Example: `tinyG.txt` by `LinkedListG.java`
- Graph API includes: `V()`, `E()`, `addEdge()`
- Basic algorithms: Depth-First Search (DFS) and Breadth-First Search (BFS)
- Both DFS and BFS 'walk through the graph', but differently
- Both DFS and BFS can compute a spanning tree and forest

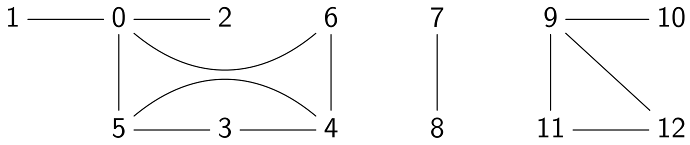
```
public void dfs(Integer v, boolean[] marked) {
  marked[v] = true;
  for (Integer w : adj[v])
    if (! marked[w]) dfs(w,marked);
} // dfs() is recursive, call: dfs(v,marked);

public void bfs(Queue<Integer> q, boolean[] marked) {
  while (!q.isEmpty()) {
     Integer v = q.dequeue();
     for (Integer w : adj[v])
       if (! marked[w]) {marked[w]=true; q.enqueue(w);}
  }
} // call: marked[v]=true; q.enqueue(v); bfs(q,marked);

// Example: 0-1, 0-3, 1-2, 1-3, 3-4
// Example: complete ternary tree of height 2
```

Implementation and Properties of DFS/BFS

- LinkedListG.java: `pathdfs(), pathbfs()`
- DFS and BFS mark nodes connected to a given source node in time proportional to the sum of their degrees ($\leq 2E$), and can return a path from a marked node to the given source in time proportional to the length of this path
- BFS always finds a shortest path (proof: queue only contains nodes at distance k followed by nodes at distance $k+1$, while all nodes at distance $\leq k$ not in queue have been processed)
- DFS finds a left-most path (long or short, example bb)
- BFS tends to use more space (but not always)
- UF (from Ch. 1) tests connectivity, but finds no paths

Applications

- StringSTG.java, flight connections, shortest path = minimum number of stop-overs
- Degrees of separation in social networks, e.g., Erdös number = length of shortest path to Paul Erdös in the co-author graph
- Connected components: `LinkedListG.countcc()`
- Example: `tinyG.txt` has three connected components

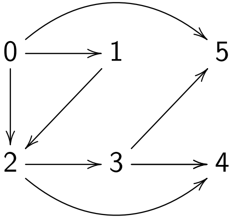


Directed Graphs

- *Digraph*: a set of *vertices* (or *nodes*) V and a set of *directed edges* (or *arrows*) E pointing from one node to another
- *Subdigraph, directed (simple) path (dipath), directed (simple) cycle, acyclic, length*: as expected
- Often we leave out 'di' in digraph, dipath, etc.
- *DAG*: Directed Acyclic Graph;
- *Degree*: in-degree and out-degree
- Node v is *reachable* from w: a dipath from w to v exists
- *Strongly connected digraph*: dipath between every two nodes (for all v, w, there are dipaths from v to w and from w to v)
- *Strongly connected component*: maximal strongly connected subgraph ($u \rightleftarrows v \rightarrow w$ has two scc's)
- Representation: adjacency lists even simpler!

Directed Graph, example

`tinyCG.txt`:



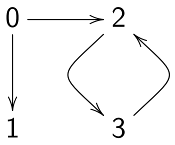
Reachability Problems

Assume we are given a directed graph G.

- Single-source: given a node s, the *source*, is a given node v reachable from s? Example: `tinyCG.txt`
- Multiple-source: given a set of nodes S, is a given node v reachable from some node in S?
- Solutions: same DFS and BFS algorithms as in Chapter 1
- Application (example): mark-and-sweep garbage collection
- Single-source path: given s, v such that v is reachable from s. Find a path from s to v.
- Single-source shortest path: given s, v such that v is reachable from s. Find a *shortest* path from s to v.
- Solutions: same DFS (path) and BFS (shortest path) algorithms as for undirected graphs

Cycle Detection

- Recall: a *DAG* is a graph without a directed cycle
- Acyclicity test, cycle detection: easy extension of DFS. We keep track of the search path from the source. If there is an arrow from v to w and w is on the path from the source to v, then there is a cycle. (DFS finds the leftmost path to the leftmost cycle.) Two techniques (space-time trade-off!):
  - Go back the search path: `LinkedListDiG.slowCyclist()`
  - Memorize the search path: `LinkedListDiG.fastCyclist()`
- Application: precedence scheduling of jobs
- Example: `cycleG.txt`



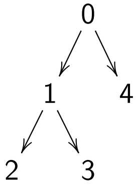
Pre-order, post-order

- Graph walks based on DFS from a source node
- Pre-order: order in which DFS arrives at nodes
- Post-order: order in which DFS leaves nodes
- In-order *for binary trees*: e.g., in `UBST.show()`
- Example:

pre-order: 01234

post-order: 23140

in-order: 21304



- Example: `tinyCG.txt` on bb and by `LinkedListDiG.java`

Topological order of acyclic digraph

- Topological order: total order $\prec$ compatible with the graph in the following sense: if there is an arrow from u to v, then $v \prec u$ (consequently: $v \preceq u$ if v is reachable from u)
- NB one can also take $\succ$, this is only a matter of definition
- Lemma: if a digraph has a topological order, then it is acyclic (proof: a cycle cannot be ordered compatibly)
- Lemma: if a digraph is acyclic, then it has a topological order (proof idea: if acyclic, the post-order is a topological order since, if there is an arrow from u to v, then u is not reachable from v and DFS will leave u after it has left v)
- Topological order is a job schedule respecting precedence
- Example: `1 <- 2 <- 4 -> 5 -> 3 <- 0`

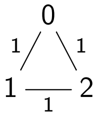
Transitive closure

- Definition: given G, its (reflexive!) transitive closure G^* is a graph with the same nodes, with arrows from u to v for each v that is reachable from u in G.
- NB: G^* can have many more arrows than G
- Implementation: adjacency matrix in case of many arrows

```
...
boolean[][] adjmat = new boolean[V][V];
for (int v=0; v<V; v++) {
   boolean[] marked = new boolean[V];
   dfs(v,marked);
   adjmat[v] = marked; // adjmat[v][v]==true: reflexive
...
 }
```

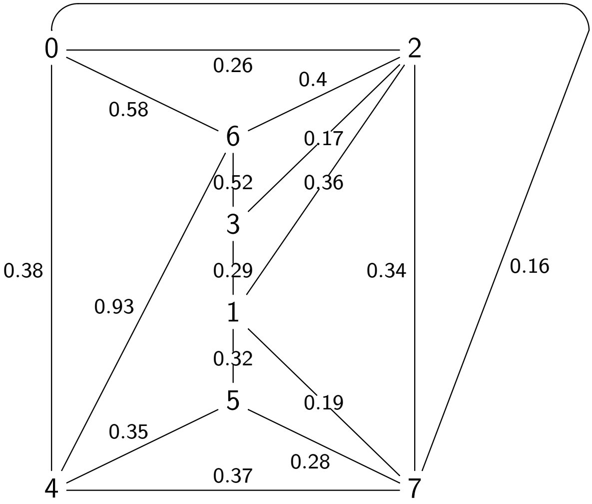
Minimum Spanning Tree

- Recall slide 79: *spanning tree* of a connected undirected graph is a maximal subgraph that is a tree (and thus contains all nodes and is acyclic)
- EWG = Edge-Weighted Graph, here always connected
- Recall slide 80: all spanning trees have $V - 1$ edges
- *Weight* of spanning tree: sum of the weights of its edges
- *Minimum Spanning Tree*: spanning tree with minimal weigth
- Example: three MSTs of



- Exc.4.3.3: if all weights different, then MST is unique
- From now on we assume all weights different!

MST Example (tinyEWG.txt)



Minimum Spanning Tree (ctnd)

- Applications: power plants and electrical grid, airlines and flight routes, maps and distance
- Weights may be zero or negative (e.g., cost minus profit of a new network of roads between cities)
- Two important algorithms to find the MST: Prim's and Kruskal's
- We explain Kruskal's algorithm first (unlike the book), since in previous years students found Prim's algorithm more difficult to understand

Cuts and Crossing Edges

- Recall slide 80: deleting an edge from a tree creates two disjoint components, adding an edge creates a cycle
- *Cut*: a partition of V in two non-empty subsets of nodes
- *Crossing edge*: edge connecting two nodes in different subsets of a cut
- NB there can be more than one crossing edge:

$$\begin{tikzcd} 0 \arrow[r, dash] & 2 \arrow[dl, dotted, dash] \arrow[d, dotted, dash] \\ 1 \arrow[r, dash] & 3 \end{tikzcd}$$

- Lemma: for any cut in an EWG, the crossing edge of minimum weight is in the MST.
- Proof: given a cut, assume by contradiction there is a crossing edge e of weight smaller than the crossing edge(s) that is (are) in the MST (e.g., the dotted edges above). Adding e creates a simple cycle, which must contain one other crossing edge f in the MST. Replacing f by e: $\lightning$

Kruskal's Algorithm

- Datastructures:
  - EWG represented with adjacency lists `adj[v]`
  - Minimum priority queue `pq` for edges
  - Union-Find object `uf` testing connectivity
  - Queue `mst` for the minimum spanning tree
- Algorithm:
  1. delete the minimum edge `e` from `pq`
  2. if the points connected by `e` are not connected, add `e` to `mst` and connect the points in `uf`
  3. continue at point 1 until `pq` is empty or `uf` contains all nodes
- Examples: EWG on slide 99, `tinyEWD.txt`
- Correctness: by lemma about minimum-weight crossing edge of cut (NB it is also correct to use a sorted array of edges)
- Implementation: KruskalMST.java, constructor method

Prim's Lazy Algorithm

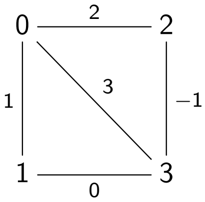
- Datastructures:
  - EWG represented with adjacency lists `adj[v]`
  - Minimum priority queue `pq` for edges
  - Array `marked[v]` for marking vertices
  - Queue `mst` for the minimum spanning tree
- Edge is *eligible* if not both endpoints marked (crossing!)
- Algorithm based on previous lemma, cut: un/marked nodes
  1. mark 0 and add all eligible edges in `adj[0]` to `pq`
  2. as long as `pq` is not empty, do:
     2.1 get and delete minimum edge `e` from `pq`
     2.2 add `e` to `mst`, say the unmarked endpoint of `e` is `k`
     2.3 mark `k` and add all eligible edges in `adj[k]` to `pq`
     2.4 delete ineligible minimum edges from `pq`
- After this algorithm, the queue `mst` contains the MST

Prim's Algorithm (ctnd)

- LazyPrimMST.java, methods `scan()` and `prim()`
- Invariant: at least one of the nodes of an edge in `pq` is marked
- NOT: all edges in `pq` are crossing edges wrt cut un/marked
- Lazy: ineligible edges are not eagerly deleted from `pq`
- Runtime: LazyPrimMST runs in $O(E \log E)$ time (worst-case)
- Possible: *only* crossing edges wrt cut un/marked in `pq`
- Eager: if `v` unmarked, the only crossing edge of interest is the *lightest* one connecting `v` to the marked edges (= MST so far)
- Runtime: eager Prim runs in $O(E \log V)$ time (worst-case)
- Max size `pq`: E edges for lazy; V nodes for eager

Prim's Eager Algorithm

- Datastructures:
  - EWG represented with adjacency lists `adj[v]`
  - Boolean array `marked[v]` for marking vertices
  - Array `distTo[v]`, minimum distances to MST so far
  - Array `edgeTo[v]`, edges with minimum distance to MST so far
  - Indexed minimum priority queue `pq`: index=`v`, key=`distTo[v]`
  - Queue `mst` for the MST based on `edgeTo`
- Example:



- PrimMST.java, methods `scan()` and `prim()`

Memory-Use and Run-time Analysis

- Space, worst-case:
  - All methods use $O(V + E)$ space for the graph, plus ...
  - Priority queue for edges (Lazy Prim and Kruskal): $O(E)$ space
  - Priority queue for vertices (Eager Prim): $O(V)$ space
  - Arrays indexed by vertices (all): $O(V)$ space
- Time, worst-case:
  - Priority queue operations (Lazy Prim and Kruskal): $O(E \log E)$ time
  - Priority queue operations (Eager Prim): $O(E \log V)$ time
- NB: $E \leq V^2$ implies $\log E \leq \log V^2 \leq 2 \log V$
- Example: complete graph on $0, \ldots, 9$, edge n-m weight $n+m$, MST consists of 0-m for $m = 1, \ldots, 9$

Shortest paths

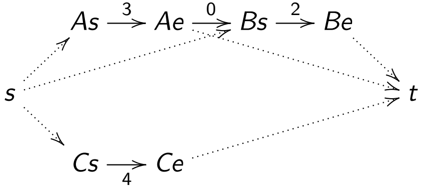
- Recall slide 76, *Edge-weighted digraphs*: digraphs in which every arrow has a weight
- *Weight* of a (di)path: sum of weights of the arrows
- *Shortest* path from node s to node t: minimum path weight
- EWD = Edge-Weighted Digraph, example: `tinyEWD.txt`
- Example: two SPs from 0 to 3:

$$\begin{tikzcd} 0 \arrow[r, "1"] \arrow[dr, "8"] & 2 \arrow[d, "5"] \arrow[dl, "2"] \\ 1 \arrow[r, "3"] & 3 \end{tikzcd}$$

- Shortest paths need not be unique, even if all weights are different!
- Shortest paths need not exist, for two independent reasons:
  - When target t is not reachable from source s
  - When there is a negative cycle on the path to t, e.g., $\begin{tikzcd} 1 \arrow[loop above, "-1"] \end{tikzcd}$

Variations

- Single-source versus multiple sources
- Only non-negative weights versus all weights allowed
- Acyclic versus cycles, in particular negative cycles
- Longest path: shortest path with weights negated
- Important example: (parallel) scheduling of jobs A, B, and C
  - A (3 hrs), must precede by B (2 hr), independent C (4 hrs)
  - Graph:



  - Schedule, longest paths, makespan: `tinyJob.txt`
  - Now add: A must start less than 2 hrs before B starts. Feasible? (No) And 4 hrs before? (Yes)

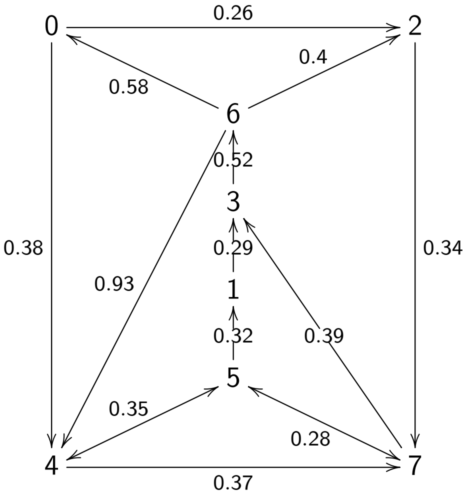
Dijkstra's Algorithm

- Single-source, only non-negative weights, cycles no problem
- Datastructures:
  - EWD with adjacency lists `adj[v]` of weighted out-edges
  - Boolean array `marked[v]` for marking vertices
  - Array `distToSource[v]`, minimum distances to source so far
  - Array `pathToSource[v]`, best arrow to source so far
- Algorithm: relax (see below) and mark the unmarked node with least distance, until all marked; Simple example: slide 101
- Invariants:
  - Marked nodes: known shortest path to source (non-negativity!)
  - Unmarked nodes: known shortest path to source THROUGH marked nodes (requires in-arrow from marked node)
- Implementation `LinkedListEWD.slowEWD()`, examples `tinyJob.txt` and `tinyEWD.txt`

Relaxation

- Assume an array `distToSource[v]` with minimum distances, so far, to a given source
- To *relax an edge e* from u to v with weight x means to update `distToSource[v]` to `distToSource[u]`$+x$ if the latter is smaller
- To *relax a node u* means to relax all edges in `adj[u]`, that is, to update `distToSource[v]` to `distToSource[u]`$+x$ if the latter is smaller, for every edge from u to v with weight x
- Dijkstra: relax and mark unmarked node v with minimal `distToSource[v]`, until all nodes marked
- Bellman-Ford: do max V rounds of relaxation of all edges (may stop after a round without updates)

EWD Example (tinyEWD.txt, NB 4 $\leftrightarrow$ 5 $\leftrightarrow$ 7!)



Bellman-Ford

- Single-source, also negative weights, negative cycles detected
- Datastructures:
  - EWD with adjacency lists `adj[v]` of weighted out-edges
  - Array `distToSource[v]`, minimum distances to source so far
  - (Array `pathToSource[v]`, best arrow to source so far)
- Algorithm: do at most V rounds for every node `v` and every arrow `e` in `adj[v]`, if `e` shortens the distance to its endpoint `w`, update that distance (and path); stop after a round when no distances improve. If distances improve in the V-th round, a negative cycle is reachable from the source.
- Invariant: after n rounds the distances are less than or equal to the shortest path of length n from the source
- Implementation `LinkedListEWD.simpleBF()`, examples `tinyJob.txt` and `tiNoJob.txt`

Memory-Use and Run-time Analysis

- Space
  - All methods use $O(V+E)$ for the graph, plus $O(V)$ extra
  - Still true for Dijkstra improved with an indexed priority queue, but indexed priority queue takes $\sim 3V$ space
- Time, worst-case:
  - V times finding a minimum (original Dijkstra): $O(V^2)$
  - Priority queue operations (improved Dijkstra): $O(E \log V)$
  - V rounds relaxing E edges (Bellman-Ford): $O(EV)$

Odds and Ends Chapter 4

- Bellman-Ford
- Indexed Priority Queue
- ...

ToC and topics of general interest

- Table of Contents on next slide (all items clickable)
- Practical stuff: slide 2