Numerical Analysis

Mathematics of Scientific Computing

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Solution of Nonlinear Equations

Contents

Fixed Points and Functional Iteration

- A sequence of points is computed by a formula of the form

$$x_{n+1} = F(x_n) \quad (n \geqslant 0) \tag{1}$$

The algorithm defined by such an equation is called functional iteration.
In Newton's method, the function F is given by

$$F(x) = x - \frac{f(x)}{f'(x)}$$

whereas in Steffensen's method, we have

$$F(x) = x - \frac{[f(x)]^2}{f(x + f(x)) - f(x)}$$

Fixed Points and Functional Iteration

- Suppose that
$$\lim_{n\to\infty} x_n = s$$
If F is continuous, then
$$F(s) = F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = s$$
Thus, $F(s) = s$, and we call s a fixed point of the function F.

- The theorem to be proved concerns contractive mapping. A mapping(or function) F is said to be contractive if there exists a number λ less than 1 such that
$$|F(x) - F(y)| \leqslant \lambda |x - y| \tag{2}$$
for all points x and y in the domain of F.

Contractive Mapping Theorem

THEOREM 1 (Contractive Mapping Theorem)

Let C be a closed subset of the real line. If F is a contractive mapping of C into C, then F has a unique fixed point. Moreover, this fixed point is the limit of every sequence obtained from Equation(1) with a starting point $x_0 \in C$.

Proof.

We use the contractive property(2) together with Equation(1) to write

$$|x_n - x_{n-1}| = |F(x_{n-1}) - F(x_{n-2})| \leqslant \lambda |x_{n-1} - x_{n-2}|$$

This argument can be repeated to get

$$|x_n - x_{n-1}| \leqslant \lambda |x_{n-1} - x_{n-2}| \leqslant \lambda^2 |x_{n-2} - x_{n-3}| \leqslant \ldots \leqslant \lambda^{n-1} |x_1 - x_0|$$

Contractive Mapping Theorem

Proof.

Since x_n can be written in the form

$$x_n = x_0 + (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})$$

We see that the sequence $[x_n]$ converges if and only if the series

$$\sum_{n=1}^{\infty} (x_n - x_{n-1})$$

converges. It suffices to prove that the series

$$\sum_{n=1}^{\infty} |x_n - x_{n-1}|$$

Contractive Mapping Theorem

Proof.

This is easy because we can use the comparision test and the previous work

$$\sum_{n=1}^{\infty} |x_n - x_{n-1}| \leqslant \sum_{n=1}^{\infty} \lambda^{n-1}|x_1 - x_0| = \frac{1}{1-\lambda}|x_1 - x_0|$$

Let $s = \lim_{n\to\infty} x_n$. Then $F(s) = s$, as noted previously.
As for the unicity of the fixed point, if x and y are fixed points, then

$$|x - y| = |F(x) - F(y)| \leqslant \lambda|x - y|$$

Since $\lambda < 1, |x - y| = 0$. Finally, note that the point s that is obtained belongs to C since s is the limit of a sequence lying in C. □

Fixed Points and Functional Iteration

EXAMPLE Use the Contractive Mapping Theorem to compute a fixed point of the function

$$F(x) = 4 + \frac{1}{3}\sin x$$

Solution By the Mean-Value Theorem, we have

$$|F(x)-F(y)| = \frac{1}{3}|\sin 2x-\sin 2y| = \frac{2}{3}|\cos 2\zeta||x-y| \leqslant \frac{2}{3}|x-y|$$

for some ζ between x and y. This shows that F is contractive, with $\lambda = 2/3$. By Theorem 1, F has a fixed point. A computer program to compute this fixed point can be based on the following algorithm.

Contractive Mapping Theorem

```
x=4;M=20
for k=1:M
    x=4+1/3*sin(2*x);
    x=vpa(x,8)
end
```

Error Analysis

- We suppose F has a fixed point, s, and that a sequence $[x_n]$ has been defined by the formula $x_{n+1} = F(x_n)$. Let

$$e_n = x_n - s$$

If F' exists and is continuous, then by the Mean-Value Theorem,

$$x_{n+1} - s = F(x_n) - F(s) = F'(\zeta_n)(x_n - s) \quad or \quad e_{n+1} = F'(\zeta_n)e_n$$

where ζ_n is a point between x_n and s. The condition $|F'(x)| < 1$ for all x ensures that errors decrease in magnitude. If e_n is small, then ζ_n is near s, and $F'(\zeta_n) \approx F'(s)$. One would expect rapid convergence if $F'(s)$ is small.

Contents

Computing Roots of Polynomials

- We write a polynomial in the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0 \tag{3}$$

in which the coefficients a_k and the variable z may be complex numbers. If $a_n \neq 0$, then p has **degree** n. We are interested in finding the roots of p.

THEOREM 2 (Fundamental Theorem of Algebra)

Every nonconstant polynomial has at least one root in the complex field.

Fundamental Theorem of Algebra

Proof.

We want to show that $p(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Since p is not constant, $|p(z)| \to \infty$ when $|z| \to \infty$. Let D be a disk centered at 0 outside of which $|p(z)| \geqslant |p(0)|$. Let z_0 be a point where $inf_{z\in D}|P(z)|$ is attained. Since $0 \in D$, $|P(z_0)| \leqslant |p(0)|$. Thus, $|p(z_0)| \leqslant |p(z)|$ for all $z \in \mathbb{C}$. Put $q(z) = p(z + z_0)$. We want to prove that $q(0) = 0$, so that $p(z_0) = 0$. Write $q(z) = c_0 + c_j z^j + \ldots + c_n z^n = c_0 + c_j z^j + z^{j+1} r(z)$, in which $c_j \neq 0$ and r is a polynomial. Now we want to prove that $c_0 = 0$. □

Fundamental Theorem of Algebra

Proof.

Suppose $c_0 \neq 0$. Let w be any complex number such that $c_j w^j = -c_0$. Define $N = sup_{0<\varepsilon<1} |r(\varepsilon w)|$. Select ε in $(0,1)$ so small that $\varepsilon |w|^{j+1} N < |c_0|$. Then we obtain a contradiction as follows:

$$
\begin{aligned}
|q(\varepsilon w)| &\leqslant |c_0 + c_j \varepsilon^j w^j| + \varepsilon^{j+1} |w^{j+1}||r(\varepsilon w)| \\
&= |c_0 - c_0 \varepsilon^j| + \varepsilon^j \varepsilon |w|^{j+1} N \\
&< |c_0|(1-\varepsilon^j) + \varepsilon^j |c_0| = |c_0| = |q(0)| \\
&= |p(z_0)| \leqslant |p(z_0 + \varepsilon w)| = |q(\varepsilon w)|
\end{aligned}
$$

□

Computing Roots of Polynomials

- If the polynomial p, having degree n at least 1, is divided by a linear factor $z - c$, the result ia a quotient q and a remainder r. The latter is a complex number, and the former is a polynomial of degree $n - 1$. We can represent the process by the equation

$$p(z) = (z - c)\,q(z) + r$$

- From this we see (by letting $z = c$) that $p(c) = r$. This fact is known as the Remainder Theorem.
- If c is a root of p, then $r = 0$. This implication is known as the Factor Theorem.

Computing Roots of Polynomials

- Let us write $p(z) = (z - r_1)q_1(z)$, where r_1 is any root of p. The equation can be

$$p(z) = (z - r_1)(z - r_2)\dots(z - r_n)q_n$$

This proves that a polynomial of degree n has a factorization into a product of linear factors, each corresponding to a root of p. It is clear that p can have no other roots. Since some of the roots r_k may be equal to each other, we see that a polynomial of degree n can have at most n roots.

Computing Roots of Polynomials

THEOREM 3 (Theorem on Complex Roots of Polynomials)

A polynomial of degree n has exactly n roots in the complex plane, it being agreed that each root shall be counted a number of times equal to its multiplicity.

THEOREM 4 (Localization Theorem)

All roots of the polynomial in Equation(3) lie in the open disk whose center is at the origin of the complex plane and whose radius is

$$\rho = 1 + |a_n|^{-1} \max_{0 \leqslant k < n} |a_k|$$

Computing Roots of Polynomials

Proof.

Put $c = \max_{0\leqslant k<n}|a_k|$ so that $c|a_n|^{-1} = \rho - 1$. If $c = 0$, our result is trivially true. Hence, assume $c > 0$. Then $\rho > 1$. If $|z| \geqslant \rho$, then (because $\rho > 1$)

$$
\begin{aligned}
|p(z)| &\geqslant |a_n z^n| - |a_{n-1}z^{n-1} + \ldots + a_0| \\
&\geqslant |a_n z^n| - c\sum_{k=0}^{n-1}|z|^k \\
&> |a_n z^n| - c|z|^n(|z|-1)^{-1} \\
&= |a_n z^n|\{1 - c|a_n|^{-1}(|z|-1)^{-1}\} \\
&\geqslant |a_n z^n|\{1 - c|a_n|^{-1}(\rho-1)^{-1}\} = 0
\end{aligned}
$$

□

Computing Roots of Polynomials

Take the polynomial p of Equation(3), and consider the function $s(z) = z^n p(1/z)$. Then

$$\begin{aligned} s(z) &= z^n[a_n\left(\frac{1}{z}\right)^n + a_{n-1}\left(\frac{1}{z}\right)^{n-1} + \ldots + a_0] \\ &= a_n + a_{n-1}z + a_{n-2}z^2 + \ldots + a_0 z^n \end{aligned}$$

This shows that s is a polynomial of degree at most n. For a nonzero complex number z_0, the condition $p(z_0) = 0$ is equivalent to the condition $s(1/z_0) = 0$.

THEOREM 5 (Theorem on Localization of Roots)

If all the roots of s are in the disk $\{z : |z| \leqslant \rho\}$, then all the nonzero roots of p are outside the disk $\{z : |z| < \rho^{-1}\}$.

Contents

Horner's Algorithm

- If a polynomial p and a complex number z_0 are given, Horner's algorithm will produce the number $p(z_0)$ and the polynomial

$$q(z) = \frac{p(z) - p(0)}{z - z_0}$$

From this equation we have

$$p(z) = (z - z_0)q(z) + p(z_0) \tag{4}$$

Let the unknown polynomial q be represented by

$$q(z) = b_0 + b_1 z + \ldots + b_{n-1} z^{n-1}$$

Then this form for $q(z)$ and the analogous for $p(z)$ are substituted into Equation(4).

Horner's Algorithm

- The coefficients of like powers of z on the two sides of the equation can be set equal to each other. These equations arise from doing so:

$$
\begin{aligned}
b_{n-1} &= a_n \\
b_{n-2} &= a_{n-1} + z_0 b_{n-1} \\
&\vdots \\
b_0 &= a_1 + z_0 b_1 \\
p(z_0) &= a_0 + z_0 b_0
\end{aligned}
$$

- The calculation can be carried out in the following arrangement

$$
\begin{array}{c|ccccc}
 & a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \\
z_0 & & z_0 b_{n-1} & z_0 b_{n-2} & \ldots & z_0 b_0 \\
\hline
 & b_{n-1} & b_{n-2} & b_{n-3} & \ldots & b_{-1}
\end{array}
$$

Horner's Algorithm

```
function b=Horner1(a,z_0) % P(z) = a_0 + a_1*z +...+ a_n*z^n
n=length(a);              % poly degree plus one: N+1
b(n)=a(n);                % initial condition
for k=n-1:-1:1            % iterative Horner's algorithm
b(k)=a(k)+z_0*b(k+1);
end
b=b(1)
```

Horner's Algorithm

- A application of Horner's algorithm is in finding the Taylor expansion of a polynomial about any point. Let $p(z)$ be as in Equation(3), and suppose that we desire the cofficient c_k in the equation

$$\begin{aligned} p(z) &= a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \\ &= c_n (z - z_0)^n + c_{n-1} (z - z_0)^{n-1} + \ldots + c_0 \end{aligned}$$

Notice that $p(z_0) = c_0$. The algorithm yields the polynomial

$$q(z) = \frac{p(z) - p(z_0)}{z - z_0} = c_n (z - z_0)^{n-1} + c_{n-1} (z - z_0)^{n-2} + \ldots + c_1$$

This shows c_1 can be obtained by applying Horner's algorithm to the polynomial q with point z_0 because $c_1 = q(z_0)$. This process is repeated until all coefficients c_k are found.

Horner's Algorithm

- We call the algorithm described the complete Horner's algorithm.

```
function a=Horner2(a,z_0)
n=length(a);
for k=1:n-1
    for j=n-1:-1:k
        a(j)=a(j)+(z_0)*a(j+1)
    end
end
```

Horner's Algorithm

- We will show how Newton's iteration can be carried out on a polynomial. Here is a code that produces $\alpha = p(z_0)$ and $\beta = p'(z_0)$.

```
function [alpha,beta]=Horner3(a,z_0)
n=length(a);
alpha=a(n);
beta=0;
for k=n-1:-1:1
     beta=alpha+z_0*beta;
     alpha=a(k)+z_0*alpha;
end
```

Horner's Algorithm

- Then a code for taking M steps in Newton's method on the given polynomial, starting at z_0, would look like this:

```
function x=NewtonH(a,z_0,M,epsilon)
for j=1:M
    [alpha,beta]=Horner3(a,z_0);
    alph=vpa(alpha,6);
    bet=vpa(beta,6);
    z_1=z_0-alpha/beta;
    x=z_1;
    x=vpa(x,5)
    if abs(z_1-z_0)<epsilon
        break
    end
     z_0=z_1;
end
```

Theorem on Horner's Method

THEOREM 6 (Theorem on Horner's Method)

Let $p(x) = a_n x^n + \ldots + a_1 x + a_0$. *Define pairs* (α_j, β_j) *for* $j = n, n-1, \ldots, 0$ *by the algorithm*

$$\begin{cases} (\alpha_n, \beta_n) = (a_n, 0) \\ (\alpha_j, \beta_j) = (a_j + x\alpha_{j+1}, \alpha_{j+1} + x\beta_{j+1}) \quad (n-1 \geqslant j \geqslant 0) \end{cases}$$

Then $\alpha_0 = p(x)$ *and* $\beta_0 = p'(x)$.

Theorem on Successive Newton Iterates

THEOREM 7 (Theorem on Successive Newton Iterates)

Let x_k and x_{k+1} be two successive iterates when Newton's method is applied to a polynomial p of degree n. Then there is a root of p within distance $n|x_k - x_{k+1}|$ of x_k in the complex plane.

Proof.

Let $r_1, r_2, \ldots, r_n$ be the roots of p. Then $p(z) = c\prod_{j=1}^{n}(z - r_j)$. The correction term in the Newton iteration is $-p(z)/p'(z)$. The derivative of p is

$$p'(z) = c\sum_{k=1}^{n}\prod_{\substack{i=1\\i\neq k}}^{n}(z - r_i) = \sum_{k=1}^{n} p(z)/(z - r_k) = p(z)\sum_{k=1}^{n}(z - r_k)^{-1}$$

□

Theorem on Successive Newton Iterates

Proof.

For any z (playing the role of x_k) there is an index j for which $|z - r_j| \leqslant n|p(z)/p'(z)|$.
If no index j satisfies the desired inequality, then for all j, $|z - r_j| > n|p(z)/p'(z)|$. From this it would follow that

$$|z - r_j|^{-1} < \frac{1}{n}|p'(z)/p(z)| = \frac{1}{n}|\sum_{k=1}^{n}(z - r_k)^{-1}| \leqslant \frac{1}{n}\sum_{k=1}^{n}|(z - r_k)^{-1}|$$

But this is not possible because the average of n numbers cannot be greater than each of them. □

Contents

Theorem on Real Quadratic Factor

THEOREM 8 (Theorem on Real Quadratic Factor)

If p is a polynomial whose coefficients are all real, and if w is a nonreal root of p, then $\overline{w}$ is also a root, and $(z-w)(z-\overline{w})$ is a real quadratic factor of p.

Proof.

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, with all a_k being real. Since w is a root of p, we have

$$0 = a_n w^n + a_{n-1} w^{n-1} + \ldots + a_1 w + a_0$$

Take the conjugate of both sides, the result is

$$0 = a_n \overline{w}^n + a_{n-1} \overline{w}^{n-1} + \ldots + a_1 \overline{w} + a_0$$

Theorem on Quotient and Remainder

THEOREM 9 (Theorem on Quotient and Remainder)

If the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ *is divided by the quadratic polynomial* $z^2 - uz - v$*, then the quotient and remainder*

$$q(z) = b_n z^{n-2} + b_{n-1} z^{n-3} + \ldots + b_3 z + b_2$$
$$r(z) = b_1 (z - u) + b_0$$

can be computed recursively by setting $b_{n+1} = b_{n+2} = 0$ *and then using*

$$b_k = a_k + u b_{k+1} + v b_{k+2} \qquad (n \geqslant k \geqslant 0)$$

Theorem on Quotient and Remainder

Proof.

The relationship between p, q, and r is expressed by

$$p(z) = q(z)(z^2 - uz - v) + r(z)$$

In detail, this equation reads:

$$\sum_{k=0}^{n} a_k z^k = (\sum_{k=2}^{n} b_k z^{k-2})(z^2 - uz - v) + b_1(z - u) + b_0$$

If we equate the coefficients of z^k on the two sides of this equation,

$$\begin{aligned} a_k &= b_k - ub_{k+1} - vb_{k+2} \qquad (0 \leqslant k \leqslant n-2) \\ a_{n-1} &= b_{n-1} - ub_n \\ a_n &= b_n \end{aligned}$$

Bairstow's Method

Proof.

In the division process above, b_0 and b_1 are functions of u and v, and we wirte $b_0 = b_0(u, v)$ and $b_1 = b_1(u, v)$. In order that q be a factor of p, the remainder r should vanish; this leads to the two equations

$$b_0(u, v) = 0$$
$$b_1(u, v) = 0$$

This pair of simultaneous nonlinear equation is solved by Newton's method. We require the paitial derivatives

$$c_k = \frac{\partial b_k}{\partial u} \qquad d_k = \frac{\partial b_{k-1}}{\partial v} \qquad (0 \leqslant k \leqslant n)$$

□

Bairstow's Method

Proof.

These are obtained by differentiating the recurrence relation already established for b_k in Theorem 9.

$$\begin{aligned} c_k &= b_{k=1} + uc_{k+1} + vc_{k+2} \qquad (c_{n+1} = c_n = 0) \\ d_k &= b_{k=1} + ud_{k+1} + vd_{k+2} \qquad (d_{n+1} = d_n = 0) \end{aligned}$$

Since these recurrence relation generate the same two sequences, we need the first. Starting values are assigned to u and v. We seek corrections, donoted by δu and δv, so that the equations

$$b_0(u + \delta u, v + \delta v) = b_1(u + \delta u, v + \delta v) = 0$$

are sure. □

Bairstow's Method

Proof.

We linearize these equations by writing

$$b_0(u,v) + \frac{\partial b_0}{\partial u}\delta u + \frac{\partial b_0}{\partial u}\delta v = 0$$
$$b_1(u,v) + \frac{\partial b_1}{\partial u}\delta u + \frac{\partial b_1}{\partial u}\delta v = 0$$

In view of the preceding remarks, this system becomes

$$\begin{bmatrix} c_0 & c_1 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix} = - \begin{bmatrix} b_0(u,v) \\ b_1(u,v) \end{bmatrix}$$

□

Bairstow's Method

Proof.

The solution of this system follows:

$$\begin{aligned}\delta u &= (c_1 b_1 - c_2 b_0)/J \\ \delta v &= (c_1 b_0 - c_0 b_1)/J \\ J &= c_0 c_2 - c_1^2\end{aligned}$$

Notice that J is the Jacobian determinant for the pair of nonlinear function $b_0(u, v)$ and $b_1(u, v)$. □

Bairstow's Method

```
function [u,v,b]=Bairstow(a,u,v,M)
n=length(a);b(n)=a(n);c(n)=0;c(n-1)=a(n);
for j=1:M
    b(n-1)=a(n-1)+u*b(n);
    for k=n-2:-1:1
        b(k)=a(k)+u*b(k+1)+v*b(k+2);
        c(k)=b(k+1)+u*c(k+1)+v*c(k+2);
    end
    J=c(1)*c(3)-c(2)^2;
    u=u+(c(2)*b(2)-c(3)*b(1))/J;
    v=v+(c(2)*b(1)-c(1)*b(2))/J;

end
   u=vpa(u,14)
   v=vpa(v,14)
```

Theorem on the Jacobian in Bairstow's Method

THEOREM 10 (Theorem on the Jacobian in Bairstow's Method)

Let (u_0, v_0) be a point such that the roots of $z^2 - u_0 z - v_0$ are simple roots of p. The Jacobian in Bairstow's method is not 0 at (u_0, v_0).