Numerical Analysis

Mathematics of Scientific Computing

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Basic Concepts

- V stand for the set of all infinite sequences of complex numbers, such as $x = [x_1, x_2, x_3, ...]$ $y = [y_1, y_2, y_3, ...]$, A sequence is a complex-valued function defined on the set of positive integers $N = \{1, 2, 3, ...\}$. $x(n)$ is the value of the function x at the argument n.
- In the set V, we define two operations:

$$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3, ...]$$

$$\lambda x = [\lambda x_1, \lambda x_2, \lambda x_3, ...]$$

$$(x + y)_n = x_n + y_n$$

$$(\lambda x)_n = \lambda x_n$$

  with the adoption of these definitions, V becomes a vector space.
- 0 element in V, namely,$0 = [0, 0, 0, ...]$

Basic Concepts

- **Shift operator** or **Displacement operator** denoted by E and defined by the equation

$$Ex = [x_2, x_3, x_4, \ldots] \quad where \quad x = [x_1, x_2, x_3, \ldots]$$

$$(Ex)_n = x_{n+1}$$

$$(EEx)_n = x_{n+2}$$

$$(E^k x)_n = x_{n+k}$$

- **Linear differece operator** can be expressed as linear combinations of powers of E. The general form of such an operator is

$$L = \sum_{i=0}^{m} c_i E^i$$

Basic Concepts

- E^0 is defined as the identity operator,

$$(E^0 x)_n = x_n$$

- The linear difference operator make up a linear subspace in the set of all linear operators from V to V. The powers of E provide a basis for this subspace.
- we could write $L = p(E)$.
  Where p is a polynomial called the **characteristic polynomial** of L and defined by

$$p(\lambda) = \sum_{i=0}^{m} c_i \lambda^i$$

- The set $\{x : Lx = 0\}$ is a linear subspace of V; it is called the **null space** of L.

Basic Concepts

- Let us consider a concrete example of L , say by taking $c_0 = 2$, $c_1 = -3$, $c_2 = 1$,and all other $c_i = 0$. The resulting equation is a linear difference equation,can be written in three forms:

$$(E^2 - 3E^1 + 2E^0)x = 0$$

$$x_{n+2} - 3x_{n+1} + 2x_n = 0 \qquad (n \geq 1)$$

$$p(E)x = 0 \qquad p(\lambda) = \lambda^2 - 3\lambda + 2$$

Basic Concepts

- It is easy to generate sequences that solve. we can choose x_1 and x_2 arbitarily and then determine x_3,x_4,... we can obtain the solutions

$$[1, 0, -2, -6, -14, -30, \ldots]$$

$$[1, 1, 1, 1, \ldots]$$

$$[2, 4, 8, 16, \ldots]$$

- Two solutions are obviously of the form $x_n = \lambda^n$, with $\lambda = 1$ or $\lambda = 2$. Putting the $x_n = \lambda^n$ in the yields.

$$\lambda^{n+2} - 3\lambda^{n+1} + 2\lambda^n = 0$$

$$\lambda^n(\lambda - 1)(\lambda - 2) = 0$$

- The other solution of the type sought-namely, $[0,0,0,...]$, this we call the **trivial solution**. The solutions $u_n = 1$ and $v_n = 2^n$ are a basis for the solution space.
- Let x be any solution, $x = \alpha u + \beta v$, the equation means that $x_n = \alpha u_n + \beta v_n$ for all n.
- For $n = 1$ and $n = 2$, we have

$$x_1 = \alpha + 2\beta$$

$$x_2 = \alpha + 4\beta$$

- because the determinant of the matrix is not 0,Equation uniquely determines α and β.

- If this equation is true for indices less than n, then it is true for n because

$$x_n = 3x_{n-1} - 2x_{n-2} = 3(\alpha u_{n-1} + \beta v_{n-1}) - 2(\alpha u_{n-2} + \beta v_{n-2}) = \alpha u_n + \beta v_n$$

- this example illustrates the case of simple roots of the characteristic polynomial.

下一节内容

Theorem on Null Space

- If p is a polynomial and λ is a root of p, then one solution of the difference equation $p(E)x = 0$ is $[\lambda, \lambda^2, \lambda^3, ...]$. If all the root of p are simple and nonzero, then each solution of the difference equation is a linear combination of such special solutions.

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Multiple Roots

- When p has multiple roots, slove a difference equation $p(E)x = 0$. Define $x(\lambda) = [\lambda, \lambda^2, \lambda^3, ...]$. If p is any polynomial, we have seen that

$$p(E)x(\lambda) = p(\lambda)x(\lambda)$$

A differentiation with respect to λ yields

$$p(E)x'(\lambda) = p'(\lambda)x(\lambda) + p(\lambda)x'(\lambda)$$

If λ is a multiple root of p, then $p(\lambda) = p'(\lambda) = 0$, $x(\lambda)$ and $x'(\lambda)$ are solutions of the difference equence.

Multiple Roots

- If λ is a root of p having multiplicity k, then the following sequences are solutions of the difference equation $p(E)x = 0$:

$$x(\lambda) = [\lambda, \lambda^2, \lambda^3, ...]$$

$$x'(\lambda) = [1, 2\lambda, 3\lambda^2, ...]$$

$$x''(\lambda) = [0, 2, 6\lambda, ...]$$

$$...$$

$$x^{(k-1)}(\lambda) = \frac{d^{(k-1)}}{d\lambda^{(k-1)}}[\lambda, \lambda^2, \lambda^3, ...]$$

Theorem on Basic for Null Space

- Let p be a polynomial satisfying $p(0) \neq 0$. Then a basis for the null space of $p(E)$ is obtained as follows: With each root λ of p having multiplicity k, associate the basic solutions $x(\lambda)$, $x'(\lambda)$,...,$x^{(k-1)}(\lambda)$, where $x(\lambda) = [\lambda, \lambda^2, \lambda^3, ...]$

EXAMPLE 1

- Determine the general solution of this difference equation: $4x_n + 7x_{n-1} + 2x_{n-2} - 3x_{n-3} = 0$
- Solution $p(\lambda) = 4\lambda^3 + 7\lambda^2 + 2\lambda - 1 = (\lambda + 1)^2(4\lambda - 1)$
  p has a double root at -1 and a simple root at $\frac{1}{4}$. The basic solutions are

$$x(-1) = [-1, 1, -1, 1, ...]$$
$$x'(-1) = [1, -2, 3, -4, ...]$$
$$x(\frac{1}{4}) = [\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, ...]$$

- The general solution is $x_n = \lambda(-1)^n + \beta n(-1)^{(n-1)} + \gamma(\frac{1}{4})^n$

下一节内容

Stable Difference Equations

- An element $x = [x_1, x_2, ...]$ of V is said to be bounded if there is a constant c such that $|x_n| \leq c$ for all n. A difference equation of the form $p(E)x = 0$ is said to be **stable** if all of its solutions are bounded.

- Theorem on Stable Difference Equations
  For a polynomial p satisfying $p(0) \neq 0$, these properties are equivalent:
  1.The difference equation $p(E)x = 0$ is stable.
  2.All roots of satisfy $|z| \leq 1$, and all multiple roots satisfy $|z| < 1$

EXAMPLE 2

- Determine whether this difference equation is stable:

$$4x_n + 7x_{n-1} + 2x_{n-2} - x_{n-3} = 0$$

- Solution The given equation is of the form $p(E)x = 0$, where $p(\lambda) = 4\lambda^3 + 7\lambda^2 + 2\lambda - 1$.
  p has a double root at -1 and a simple root at $\frac{1}{4}$. The equation is therefore unstable.