

Numerical Analysis

Mathematics of Scientific Computing

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Limit, Continuity, and Derivative

If f is a real-valued function of a real variable, then the limit of the function f at c (if it exists) is defined as the following:

$$\lim_{x \rightarrow c} f(x) = L$$

means that to each positive ε there corresponds a positive δ such that the distance between $f(x)$ and L is less than ε whenever the distance between x and c is less than δ ; that is $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

If there is no number L with this property, the limit of f at c does not exist.

Limit, Continuity, and Derivative

If f is defined only on a specified subset X of the real line, the definition of limit is modified so that $|f(x) - L| < \varepsilon$ whenever $x \in X$ and $0 < |x - c| < \delta$.

The function f is said to be continuous at c if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Limit, Continuity, and Derivative

THEOREM 1 Intermediate-Value Theorem for Continuous Functions

On an interval $[a, b]$, a continuous function assumes all values between $f(a)$ and $f(b)$.

The derivative of f at c (if it exists) is defined by the equation

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

If f is a function for which $f'(c)$ exists, we say that f is differentiable at c , then f must be continuous at c , $f'(x)$ exists and $\lim_{x \rightarrow c} f(x) = f(c)$.

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Taylor's Theorem

THEOREM 2 Taylor's Theorem with Lagrange Remainder

If $f \in C^n[a, b]$ and if $f^{(n+1)}$ exists on the open interval (a, b) , then for any points c and x in the closed interval $[a, b]$,

$$f(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(c)(x - c)^k + E_n(x)$$

where, for some point ξ between c and x , the error term is

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - c)^{(n+1)}$$

Here “ ξ between c and x ” means that either $c < \xi < x$ or $x < \xi < c$ depending on the particular values of x and c involved.

Taylor's Theorem

An important special case arises when $c = 0$. The equation becomes the *Maclaurin series* for $f(x)$:

$$f(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(0) x^k + E_n(x)$$

where

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) x^{(n+1)}$$

Taylor's Theorem

THEOREM 3 Mean-Value Theorem

If f is in $C[a,b]$ and if f' exists on the open interval (a,b) , then for x and c in the closed interval $[a,b]$,

$$f(x) = f(c) + f'(\xi)(b - a)$$

where ξ is between c and x . Taking $x = b$ and $c = a$ and rearranging, we have the important equation

$$f(b) - f(a) = f'(\xi)(b - a) \quad \text{where } a < \xi < b$$

Taylor's Theorem

THEOREM 4 Rolle's Theorem.

If f is continuous on $[a,b]$, if f' exists on (a,b) , and if $f(a) = f(b)$, then $f'(\xi) = 0$ for some ξ in the open interval (a,b) .

This is an immediate consequence of an equation above. (Actually, in a formal development, Rolle's Theorem is proved first, and from it, Taylor's Theorem is derived.) In both Rolle's Theorem and the Mean-value Theorem, there may be more than one point ξ in the interval $[a,b]$ that satisfies the given equations.

Taylor's Theorem

THEOREM 5 Taylor's Theorem with Integral Remainder

If $f \in C^{(n+1)}[a,b]$, then for any points x and c in the closed interval $[a,b]$,

$$f(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(c)(x-c)^k + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)(x-t)^n dt$$

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Other Forms of Taylor's Formula

THEOREM 6 Alternative Form of Taylor's Theorem

If $f \in C^{(n+1)}[a, b]$, then for any points x and $x + h$ in the closed interval $[a, b]$,

$$f(x + h) = \sum_{k=0}^n \frac{h^k}{k!} f^{(k)}(x) + E_n(h)$$

where

$$E_n(h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

In which the point ξ lies between x and $x + h$. In detail,

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \dots + \frac{h^n}{n!}f^{(n)}(x) + E_n(h)$$

It is an important form for many applications.

Other Forms of Taylor's Formula

THEOREM 7 Taylor's Theorem in Two Variables

Let $f \in C^{(n+1)}([a,b] \times [c,d])$. If (x, y) and $(x+h, y+k)$ are points in the rectangle $[a,b] \times [c,d] \subseteq R^2$, then

$$f(x+h, y+k) = \sum_{i=0}^n \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(x, y) + E_n(h, k)$$

where

$$E_n(h, k) = \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{(n+1)} f(x + \theta h, y + \theta k)$$

in which θ lies between 0 and 1.

Other Forms of Taylor's Formula

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^0 f(x, y) = f(x, y)$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^1 f(x, y) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)(f(x, y))$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) = \left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)(x, y)$$

and so on. Letting $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x\partial y}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$, we can write the first few terms of (5) as

$$f(x+h, y+k) = f + (hf_x + kf_y) + \frac{1}{2}(h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}) + \dots$$

where on the right-hand side the function f and each of the following partial derivatives are evaluated at (x, y) .