Numerical Analysis

Mathematics of Scientific Computing

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2013 年 9 月 6 日

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Limit, Continuity, and Derivative

If f is a real-valued function of a real variable, then the limit of the function f at c (if it exists) is defined as the following:

$$\lim_{x \to c} f(x) = L$$

means that to each positive ε there corresponds a positive δ such that the distance between $f(x)$ and L is less than ε whenever the distance between x and c is less than δ; that is $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.
If there is no number L with this property, the limit of f at c does not exist.

Limit, Continuity, and Derivative

If f is defined only on a specified subset X of the real line, the definition of limit is modified so that $|f(x)-L| < \varepsilon$ whenever $x \in X$ and $0 < |x-c| < \delta$.

The function f is said to be continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

Limit, Continuity, and Derivative

THEOREM 1 Intermediate-Value Theorem for Continuous Functions

On an interval [a,b], a continuous function assumes all values between $f(a)$ and $f(b)$.

The derivative of f at c (if it exists) is defined by the equation

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

If f is a function for which $f'(c)$ exists, we say that f is differentiable at c, then f must be continuous at c, $f'(x)$ exists and $\lim_{x \to c} f(x) = f(c)$.

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Taylor's Theorem

THEOREM 2 Taylor's Theorem with Lagrange Remainder

If $f \in C^n[a, b]$ and if $f^{(n+1)}$ exists on the open interval (a,b), then for any points c and x in the closed interval [a,b],

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x - c)^k + E_n(x)$$

where, for some point ξ between c and x, the error term is

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - c)^{(n+1)}$$

Here"ξ between c and x"means that either $c < \xi < x$ or $x < \xi < c$ depending on the particular values of x and c involved.

Taylor's Theorem

An important special case arises when $c = 0$. The equation becomes the ***Maclaurin series*** for $f(x)$:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0) x^k + E_n(x)$$

where

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) x^{(n+1)}$$

Taylor's Theorem

THEOREM 3 Mean-Value Theorem

If f is in C[a,b] and if f' exists on the open interval (a,b), then for x and c in the closed interval [a,b],

$$f(x) = f(c) + f'(\xi)(b - a)$$

where ξ is between c and x. Taking $x = b$ and $c = a$ and rearranging, we have the important equation

$$f(b) - f(a) = f'(\xi)(b - a) \qquad \textit{where } a < \xi < b$$

Taylor's Theorem

THEOREM 4 Rolle's Theorem.

If f is continuous on [a,b], if f' exists on (a,b), and if $f(a) = f(b)$, then $f'(\xi) = 0$ for some ξ in the open interval (a,b).

This is an immediate consequence of an equation above. (Actually, in a formal development, Rolle's Theorem is proved first, and from it, Talor's Theorem is derived.) In both Rolle's Theorem and the Mean-value Theorem, there may be more than one point ξ in the interval [a,b] that satisfies the given equations.

Taylor's Theorem

THEOREM 5 Taylor's Theorem with Integral Remainder

If $f \in C^{(n+1)}$[a,b], then for any points x and c in the closed interval [a,b],

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)(x-t)^n \mathrm{d}t$$

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Other Forms of Taylor's Formula

THEOREM 6 Alternative Form of Taylor's Theorem

If $f \in C^{(n+1)}$[a,b], then for any points x and $x+h$ in the closed interval [a,b],

$$f(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x) + E_n(h)$$

where

$$E_n(h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

In which the point ξ lies between x and $x+h$. In detail,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots + \frac{h^n}{n!} f^{(n)}(x) + E_n(h)$$

It is an important form for many applications.

Other Forms of Taylor's Formula

THEOREM 7 Taylor's Theorem in Two Variables

Let $f \in C^{(n+1)}([\mathrm{a,b}]\times[\mathrm{c,d}])$. If (x, y) and $(x+h, y+k)$ are points in the rectangle $[\mathrm{a,b}]\times[\mathrm{c,d}] \subseteq R^2$, then

$$f(x+h, y+k) = \sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{i} f(x+y) + E_n(h, k)$$

where

$$E_n(h, k) = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{(n+1)} f(x+\theta h, y+\theta k)$$

in which θ lies between 0 and 1.

Other Forms of Taylor's Formula

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^0 f(x,y)=f(x,y)$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^1 f(x,y)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)(x,y)$$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x,y)=\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

and so on. Letting $f_x=\frac{\partial f}{\partial x}$, $f_y=\frac{\partial f}{\partial y}$, $f_{xx}=\frac{\partial^2 f}{\partial x^2}$, $f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$, $f_{yy}=\frac{\partial^2 f}{\partial y^2}$, we can write the first few terms of (5) as

$$f(x+h,y+k)=f+(hf_x+kf_y)+\frac{1}{2}(h^2 f_{xx}+2hkf_{xy}+k^2 f_{yy})+\dots$$

where on the right-hand side the function f and each of the following partial derivatives are evaluated at (x,y).