Numerical Analysis

Mathematics of Scientific Computing

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目录

内容提要

下一节内容

bisection method

- Intermediate-Value Theorem:If f is a continuous function on the interval $[a, b]$ and if $f(a)f(b) < 0$, then f must have a zero in (a, b).
- The process of bisection method:
  If $f(a)f(b) < 0$, then $c = (a + b)/2$
  If $f(a)f(c) < 0$, then f has a zero in $[a, c]$, then $b \leftarrow c$
  If $f(b)f(c) < 0$, then f has a zero in $[c, b]$, then $a \leftarrow c$
  If $f(a)f(c) = 0$, then c is the zero of f
  repeat this process
- The bisection method finds one zero but not all the zeros in the interval $[a, b]$.

Bisection Algorithm

- **Bisection pseudocode need some additional explanation:**
  First, $c \leftarrow a + (b-a)/2$ rather than $c \leftarrow (a+b)/2$
  second, $sign(f(c)) \neq sign(f(a))$ rather than $f(a)f(c) < 0$
- **three stopping criteria**
  M give the maximum number of steps that present the computation going into an infinite loop.
  When the value of $f(c)$ is small enough, the calculation can be stopped.
  When $b-a$ is small enough, the calculation can be stopped.

The bisection algorithm can be written as follows

- input a, b, M, δ, ε
  $u \leftarrow f(a)$
  $v \leftarrow f(b)$
  $e \leftarrow (b-a)$
  output a, b, u, v
  If $sign(u) = sign(v)$ then stop
  for $k = 1$ to M do
  $e \leftarrow e/2$
  $c \leftarrow a + e$
  $w \leftarrow f(c)$
  output k, c, w, e
  If $|e| < \delta$ or $|w| < \varepsilon$ then stop

- If $sign(w) \neq sign(u)$ then

  $b \leftarrow c$

  $v \leftarrow w$

  else

  $a \leftarrow c$

  $u \leftarrow w$

  end if

  end do

EXAMPLE 1

- Use the bisection method to find the root of the equation $e^x = sinx$ closest to 0.

```
function f=equation(x)
    f=exp(1)^x-sin(x);
```

```
clc
a=-4;b=-3;n=16;p=10^(-5); q=10^(-5);
[c,w]=Bisection(a,b,n,p,q)
```

EXAMPLE 1

```
function [c,w]=Bisection(a,b,n,p,q)
u=equation(a);
v=equation(b);
if sign(u)==sign(v)
    error('function has same sign at both end points')
end
for i=1:n
    e=b-a;
    c=a+0.5*e
    w=equation(c)
 if abs(w)<p || abs(e)<q
    break
 end
 if sign(w)==sign(u)
     a=c;u=w;
 else
     b=c; v=w;
```

Error Analysis

- Let us denote the successive intervals that arise in the process by $[a_0, b_0]$,$[a_1, b_1]$, and so no.

$$a_0 \le a_1 \le a_2 \le a_3 \le \ldots \le b_0$$

$$b_0 \ge b_1 \ge b_2 \ge b_3 \ge \ldots \ge a_0$$

$$b_{n+1} - a_{n+1} = 1/2(b_n - a_n)(n \ge 0)(1)$$

If we apply Equation(1) repeatedly, we find that

$$b_n - a_n = 2^{-n}(b_0 - a_0)$$

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{-n}(b_0 - a_0) = 0$$

$$r = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

- then,by taking a limit in the inequality $f(a_n)f(b_n) \leq 0$, we obtain $[f(r)]^2 \leq 0$, whence $f(r) = 0$
  The best estimate of the root at this stage is not a_n or b_n but the midpoint of the interval: $c_n = (a_n + b_n)/2$
  The error is the bounded as follows:

$$|r - c_n| \leq 1/2(b_n - a_n) = 2^{-(n+1)}(b_0 - a_0)$$

Theorem on bisection method

THEOREM 1 (Theorem on bisection method)

If $[a_0, b_0]$,$[a_1, b_1]$,...,$[a_n, b_n]$,...*denote the intervals in the bisection method, then the limits* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist, are equal,and repredsent a zero of* f. *If* $r = \lim_{n\to\infty} c_n$ *and* $c_n = 1/2(a_n + b_n)$,*then*

$$|r - c_n| \leq 2^{-(n+1)}(b_0 - a_0)$$

EXAMPLE 2

- Suppose that the bisection method is started with the interval $[50, 63]$.How many steps should be taken to compute a root with relative accuracy of one part in 10^{-12}?
  relative accuracy means that

$$|r - c_n|/|r| \leq 10^{-12}$$

We know that $r \geq 50$,so

$$|r - c_n|/50 \leq 10^{-12}$$

$$|r - c_n| \leq 2^{-(n+1)}(63 - 50)$$

$$2^{-(n+1)} * (13/15) \leq 10^{-12}$$

We conclude that $n \geq 37$

下一节内容

Newton's Method

- We have a fnction f whose zeros are to be determined numerically. Let r be a zero of f and let x be an approximation to r. If f' exists and is continuous, then by Taylor's theorem,

$$0 = f(r) = f(x+h) = f(x) + hf'(x) + O(h^2)$$

where $h = r - x$, ignore the $O(h^2)$, then $h = -f(x)/f'(x)$

$$r = h + x = x - f(x)/f'(x)$$

Newton's method begins with an estimate x_0 of r and then defines inductively

$$x_{n+1} = x_n - f(x_n)/f'(x_n)(n \geq 0)$$

Newton's Algorithm

- input x_0, M, δ, ε
  $v \leftarrow f(x_0)$
  output 0,x_0,v
  If $|v| < \varepsilon$ then stop
  for $k = 1$ to M do
  $x_1 \leftarrow x_0 - v/f'(x_0)$
  $v \leftarrow f(x_1)$
  output k,x_1,v
  If $|x_1 - x_0| < \delta$ or $|v| < \varepsilon$ then stop
  $x_0 \leftarrow x_1$
  end do

EXAMPLE 3

- Use Newton's method,to find the negative zero of the function $f(x) = e^x - 1.5 - tan^{-1}x$

```
function f=newf(x)
f=exp(1)^(x)-1.5-atan(x);
```

```
function df=newdf(x)
df=exp(1)^x-(1+x^2)^(-1);
```

```
clc
x0=-7;M=10;p=10^(-8);q=10^(-8);
[x1,v]=newton(x0,M,p,q)
```

EXAMPLE 3

```
function [x1,v]=newton(x0,M,p,q)
v=newf(x0)
if abs(v)<p
    return
end
for k=1:M
    x1=x0-v/newdf(x0)
    v=newf(x1)
 if abs(x1-x0)<p || abs(v)<q
     return
 end
 x0=x1;
end
```

Graphical Interpretation

- Newton's method involves linearizing the function.Replace f by the first two in his Taylor series.

$$f(x) = f(c) + f'(c)(x - c) + 1/2f''(c)(x - c)^2 + ...$$

then the linearization (at c) produces the linear function:

$$l(x) = f(c) + f'(c)(x - c)$$

l is a good approximation to f in the vicinity of c, and we have $l(c) = f(c)$ and $l'(c) = f'(c)$

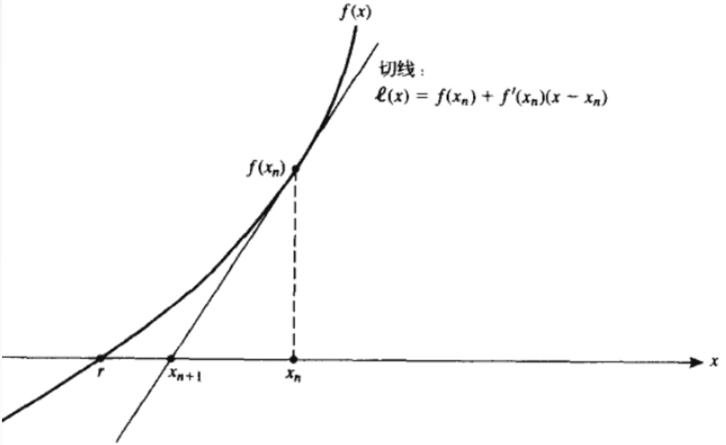

图 3-4 牛顿法的图形解释

Error Analysis

- Now we shall analyze the errors in Newton's method.

$$e_n = x_n - r$$

Let us assume that f'' is continuous and r is a simple zero of f, so that $f(r) = 0 \neq f'(r)$. From the definition of the Newton iteration, We have

$$e_{n+1} = x_{n+1} - r = x_n - \frac{f(x_n)}{f'(x_n)} - r = e_n - \frac{f(x_n)}{f'(x_n)} = \frac{e_n f'(x_n) - f(x_n)}{f'(x_n)}$$

$$0 = f(r) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{1}{2} e_n^2 f''(\xi_n)$$

$$e_n f'(x_n) - f(x_n) = \frac{1}{2} f''(\xi_n) e_n^2$$

$$e_{n+1} = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} e_n^2 \approx \frac{1}{2} \frac{f''(r)}{f'(r)} e_n^2 = C e_n^2$$

Theorem on Newton's Method

THEOREM 2 (Theorem on Newton's Method)

Let f'' be continuous and let r be a simple zero of f. Then there is a neighborhood of r and a constant C such that if Newton's method is started in that neighborhood, the successive points become steadily closer to r and satisfy

$$|x_{n+1} - r| \leq C(x_n - r)^2$$

- In some situations Newton's iteration can be guaranteed to converge from an arbitary starting point.

Theorem on Newton's Method for a Convex Function

THEOREM 3 (Theorem on Newton's Method for a Convex Function)

If f belongs to $C^2(R)$, is increasing, is convex, and has a zero, then the zero is unique, and the Newton iteration will converge to it from any starting point.

- proof: f is convex $\to f''(x) > 0$, f is increasing $\to f' > 0$ on the R
  By Equation, $e_{n+1} = \frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)}e_n^2$, $e_{n+1} > 0$
  $e_{n+1} = x_{n+1} - r \to x_n > r$ for $n \geq 1$
  f is increasing, $f(x_n) > f(r) = 0$. by Equation $e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)}$,
  $e_{n+1} < e_n$
  Thus ,the sequences $[e_n]$ and $[x_n]$ are decreasing and bounded below $(0, r)$. Therefore, the limits $e^* = \lim_{n\to\infty} e_n$ and $x^* = \lim_{n\to\infty} x_n$ exist. $e^* = e^* - \frac{f(x^*)}{f'(x^*)}$, Whence $f(x^*) = 0$ and $x^* = r$.

Implicit Functions

- For the equation $G(x, y) = 0$,If x is prescribed, the equation $G(x, y) = 0$ can be solved for y using Newton's method. Form a suitable staring point y_0, we define $y_1, y_2, \dots$ by, this method can be used to construct a table of the function $y(x)$.

$$y_{k+1} = y_k - \frac{G(x, y_k)}{\frac{\partial G}{\partial y}(x, y_k)}$$

If the table contains an entry $(x_n . y_n)$, we can start the Newton iteration with (x_{n+1}, y_n), the result will the value y_{n+1}.

EXAMPLE 4

- Produce a table of x versus y, where y is defined implicitly as a function of x. Use $G(x,y) = 3x^7 + 2y^5 - x^3 + y^3 - 3$ and start at $x = 0$, proceeding in steps of 0.1 to $x = 10$.

```
function f=F(x,y)
f=3*x^7+2*y^5-x^3+y^3-3;
```

```
function fy=Fy(y)
fy=10*y^4+3*y^2;
```

```
x=0;y=1;h=0.1;M=100;N=4;
[i,x,y,G]=newton4(x,y,h,M,N)
```

EXAMPLE 4

```
function [i,x,y,G]=newton4(x,y,h,M,N)
for i=1:M
    x = x+h
   for j=1:N
        f=F(x,y);
        fy=Fy(y);
        y = y-f/fy;
    end
    y=y
    G=f
end
```

Systems of Nonlinear Equations

- Newton's method for systems of nonlinear equations follows **linearize** and **solve**,let us illustrate with a pair of equations involving two variables:

$$\begin{cases} f_1(x_1,x_2)=0 \\ f_2(x_1,x_2)=0 \end{cases}$$

linear terms in the Tayor expansion

$$\begin{cases} 0=f_1(x_1+h_1,x_2+h_2)\approx f_1(x_1,x_2)+h_1\frac{\partial f_1}{\partial x_1}+h_2\frac{\partial f_1}{\partial x_2} \\ 0=f_2(x_1+h_1,x_2+h_2)\approx f_2(x_1,x_2)+h_1\frac{\partial f_2}{\partial x_1}+h_2\frac{\partial f_2}{\partial x_2} \end{cases}$$

J is the **Jacobian matrix** of f_1 and f_2:

$$\mathbf{J}=\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

- the solution is

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = -J^{-1} \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

Newton's method for two nonlinear equations in two variables is

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} h_1^{(k)} \\ h_2^{(k)} \end{bmatrix}$$

下一节内容

Secant Method

- **The drawback of Newton's method** is need the derivative of the function at zero.
- How to slove ?
  1.Steffensen's iteration $x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)}$
  2.Scant Method

Secant Method

- The Newton iteration is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, replace $f'(x_n)$ by a **difference quotient**,such as

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$x_{n+1} = x_n - f(x_n)[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}]$$

Secant Algorithm

- input a, b, M, δ, ε $fa \leftarrow f(a)$; $fb \leftarrow f(b)$
  output 0,a,fa
  output 1,b,fb
  for $k = 2$ to M do
  If $|fa| > |fb|$ then
  $a \leftrightarrow b$; $fa \leftrightarrow fb$
  end if
  $s \leftarrow (b - a)/(fb - fa)$
  $b \leftarrow a$
  $fb \leftarrow fa$
  $a \leftarrow a - fa * s$
  $fa \leftarrow f(a)$
  output k,a,fa
  If $|fa| < \delta$ or $|b - a| < \varepsilon$ then stop
  end do

EXAMPLE 5

Use the secant method to find a zero of the function
$f(x) = x^3 - sinhx + 4x^2 + 6x + 9$

```
 function f=sca(x)
f=x^3-sinh(x)+4*x^2+6*x+9;
```

```
clc
a=8;b=7;M=10;p=10^(-8);q=10^(-8);
[a,fa]=scant(a,b,M,p,q)
```

EXAMPLE 5

```
function [a,fa]=scant(a,b,M,p,q)
fa=sca(a);fb=sca(b)
for k=1:M
    if abs(fa)>abs(fb)
        c=b;b=a;a=c
        fd=fb;fb=fa; fa=fd
    end
 s=(b-a)/(fb-fa);
 b=a;
 fb=fa;
 a=a-fa*s
 fa=sca(a)
 if abs(fa)<p || abs(b-a)<q
     break
 end
end
```

Error Analysis

- $|e_{n+1}| \approx A|e_n|^{(1+\sqrt{5})/2}$
- since $(1+\sqrt{5})/2 \approx 1.62 < 2$, the rapidity of convergence of the secant method is not as good as Newton's method but is better than the bisection method.