Numerical Analysis

Mathematics of Scientific Computing

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目录

内容提要

下一节内容

A general type of iterative process for solving the system $Ax = b$
A certain matrix Q, called the **splitting matrix**,

$$Qx = (Q - A)x + b$$
$$Qx^{(k)} = (Q - A)x^{(k-1)} + b$$
$$x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b$$

Our objective is to choose Q so that these two conditions are met:
1. The sequence $[x^{(k)}]$ is easily computed.
2. The sequence $[x^{(k)}]$ converges rapidly to a solution.

THEOREM 1 (Theorem on Iterative Method Convergence)

If $\| I - Q^{-1}A \| < 1$ *for some subordinate matrix norm, then the sequence produced by the Equation* $Qx^{(k)} = (Q - A)x^{(k-1)} + b$ *converges to the solution of* $Ax = b$ *for any initial vector* $x^{(0)}$.

Proof.

Since $\| I - Q^{-1}A \| < 1$, A and Q are nonsingular.

$x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b$

The actual solution x satisfies the equation $x = (I - Q^{-1}A)x + Q^{-1}b$

$x^{(k)} - x = (I - Q^{-1}A)(x^{(k-1)} - x)$

$\| x^{(k)} - x \| \leq \| (I - Q^{-1}A) \| \| (x^{(k-1)} - x) \|$

$\| x^{(k)} - x \| \leq \| (I - Q^{-1}A) \|^k \| (x^{(0)} - x) \|$

Since $\| I - Q^{-1}A \| < 1$ We can conclude that $\lim_{k\to\infty} \| x^{(k)} - x \| = 0$ □

下一节内容

Q is chosen to be the identity matrix. The equation $Qx^{(k)} = (Q - A)x^{(k-1)} + b$ in this case read as follows:

$$x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} + r^{(k-1)}$$

where $r^{(k-1)}$ is the residual vector, $r^{(k-1)} = b - Ax^{(k-1)}$. If $\| I - A \| < 1$ for some subordinate matrix norm, the Richardson iteration will produce a solution to $Ax = b$.

An algorithm of Richardson iteration is as follows:

input n, (a_{ij}), (b_i), (x_i), M
for $k = 1$ to M do
for $i = 1$ to n do
$r_i \leftarrow b_i - \sum_{j=1}^{n} a_{ij} x_j$
end do
for $i = 1$ to n do
$x_i \leftarrow x_i + r_i$
end do
end do
output k, (x_i), (r_i)

```
function [k,x,r]=richardson(n,a,b,x,M)
for k=1:M
    for i=1:n
        s=b(i)
        for j=1:n
            s=s-a(i,j)*x(j)
        end
        r(i)=s
    end
    for i=1:n
        x(i)=x(i)+r(i)
    end
end
```

下一节内容

Q is the diagonal matrix whose diagonal entries are the same as those in the matrix $A = (a_{ij})$. $Q^{-1}A$ is a_{ij}/a_{ii}, and hence

$$\| I - Q^{-1}A \|_{\infty} = \max_{1 \le i \le n} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}/a_{ii}|$$

THEOREM 2 (Theorem on Convergence of Jacobi Method)

If A is diagonally dominant, then the sequence produced by the Jacobin iteration converges to the solution of $Ax = b$ for any staring vector.

Proof.

Diagonal dominance means that

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad (1 \leq i \leq n)$$

$$\| I - Q^{-1}A \|_\infty = \max_{1\leq i\leq n} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}/a_{ii}|$$

We then conclude that $\| I - Q^{-1}A \|_\infty < 1$.
By Theorem 1, the Jacobi iteration converges. □

input n, (a_{ij}), (b_i), (x_i), M
for $k = 1$ to M do
  for $i = 1$ to n do
  $u_i \leftarrow (b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j)/a(ii)$
  end do
  for $i = 1$ to n do
  $x_i \leftarrow u_i$
  end do
  output k, (x_i)
end do

```
function [k, x]=jacobi(n,a,b,x,M)
for k=1:M
    for i=1:n
        d=1/a(i,i)
        b(i)=d*b(i)
        for j=1:n
            a(i,j)=d*a(i,j)
        end
    end
    for i=1:n
        u(i)=b(i)
        for j=1:n
         u(i)=u(i)-a(i,j)*x(j)
        end
         u(i)=u(i)+a(i,i)*x(i)
    end
    for i=1:n
        x(i)=u(i)
    end
end
```

下一节内容

Basic Concepts

- For arbitrary linear iterative processes. $x^{(k)} = Gx^{(k-1)} + c$, we set $G = I - Q^{-1}A$ and $c = Q^{-1}b$.
- The **characteristic equation** of A: $det(A - \lambda I) = 0$
- The **spectral radius** of A: $\rho(A) = \max\{|\lambda| : det(A - \lambda I) = 0\}$
- If $S^{-1}AS = B$, the similar matrices matrices have the same eigenvalues.

THEOREM 3 (Theorem on Similar Upper Trigngular Matrices)

Every square matrix is similar to an (possibly complex) upper triangular matrix whose off-diagonal elements are arbitrarily small.

Proof.

Let A be an $n \times n$ matrix. Schur's theoerm states that A is similar to an upper triangular matrix $T = (t_{ij})$, which may be complex. Let $D = diag(\varepsilon, \varepsilon^2, ..., \varepsilon^n)$. The generic element of $B = D^{-1}TD$ is $t_{ij}\varepsilon^{j-i}$. The elements above the diagonal can be made as small as we wish by decreasing ε. □

THEOREM 4 (Theorem on Spectral Radius)

The spectral radius function safisfies the equation

$$\rho(A) = inf_{\|\cdot\|} \parallel A \parallel$$

in which the infimum is take over all subordinate matrix norms.

Proof.

First, we prove that $\rho(A) \leq inf_{\|\cdot\|} \parallel A \parallel$
Let λ be any eigenvalue of A. Select a nonzero eigenvector x corresponding to λ.
$|\lambda| \parallel x \parallel = \parallel \lambda x \parallel = \parallel Ax \parallel \leq \parallel A \parallel \parallel x \parallel$
$|\lambda| \leq \parallel A \parallel$. We have $\rho(A) \leq inf_{\|\cdot\|} \parallel A \parallel$

□

Proof.

Second, we prove that $inf_{\|\cdot\|} \parallel A \parallel \leq \rho(A)$
For the reverse inequality, we use Theorem 3. We have
$\parallel S^{-1}AS \parallel_\infty = \parallel D + T \parallel_\infty \leq \parallel D \parallel_\infty + \parallel T \parallel_\infty$
Since D has the eigenvalues of A on its diagonal, it follows that
$\parallel D \parallel_\infty = \max_{1\leq i\leq n} |\lambda| = \rho(A)$
$\parallel S^{-1}AS \parallel_\infty \leq \rho(A) + \varepsilon$
$\parallel A \parallel'_\infty = \parallel S^{-1}AS \parallel_\infty$
$inf_{\|\cdot\|} \parallel A \parallel \leq \rho(A) + \varepsilon$
$inf_{\|\cdot\|} \parallel A \parallel \leq \rho(A)$
So $\rho(A) = inf_{\|\cdot\|} \parallel A \parallel$ $\square$

THEOREM 5 (Theorem on Necessary and Sufficient Conditions for Iterative Method Convergence)

For the iteration formual $x^{(k)} = Gx^{(k-1)} + c$ to produce a sequence converging to $(I - G)^{-1}c$, for any starting vector $x^{(0)}$, it is necessary and sufficient that the spectral radius of G be less than 1.

Proof.

First, Suppose that $\rho(G) < 1$. There is a subordinate matrix norm such that $\| G \| < 1$. We write

$x^{(1)} = Gx^{(0)} + c$

$x^{(2)} = G^2 x^{(0)} + Gc + c$

$x^{(3)} = G^3 x^{(0)} + G^2 c + Gc + c$

The general formula is $x^{(k)} = G^k x^{(0)} + \sum_{j=0}^{k-1} G^j c$

$\| G^k x^{(0)} \| \leq \| G^k \| \| x^{(0)} \| \leq \| G \|^k \| x^{(0)} \| \rightarrow 0$ as $k \rightarrow \infty$

We have $\sum_{k=1}^{\infty} G^j c = (I - G)^{-1} c$

$\lim_{k\rightarrow\infty} x^{(k)} = (I - G)^{-1} c$ $\square$

Proof.

Second, suppose that $\rho(G) \geq 1$.
Select u and λ, so that $Gu = \lambda u \quad |\lambda| \geq 1 \quad u \neq 0$.
Let $c = u$ and $x^{(0)} = 0$.
By the Equation $x^{(k)} = G^k x^{(0)} + \sum_{j=0}^{k-1} G^j c$,
$x^{(k)} = \sum_{j=0}^{k-1} G^j u = \sum_{j=0}^{k-1} \lambda^j u$.
If $\lambda = 1$, $x^{(k)} = ku$, and this diverges as $k \to \infty$.
If $\lambda \neq 1$, then $x^{(k)} = (\lambda^k - 1)(\lambda - 1)^{-1} u$, and this diverges also because $\lim_{k\to\infty} \lambda^k$ does not exist. $\square$

COROLLARY 1 (Iterative Method Convergence Corollary)

The iteration formual $Qx^{(k)} = (Q - A)x^{(k-1)} + b$*, will produce a sequence converging to the solution of* $Ax = b$*, for any* $x^{(0)}$*, if* $\rho(I - Q^{-1}A) < 1$

下一节内容

- Let us examine Gauss-Seidel iteration in more detail. It is defined by letting Q be the lower triangular part of A, including the diagonal.
- An algorithm for the Gauss-Seidel iteration follows:
  input n, (a_{ij}), (b_i), (x_i), M
  for $k = 1$ to M do
  for $i = 1$ to n do
  $x_i \leftarrow (b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}x_j)/a_{ii}$
  end do
  output k, (x_i)
  end do

THEOREM 6 (Theorem on Gauss-Seidel Method Convergence)

If A is diagonally dominant, then the Gauss-Seidel method converges for any starting vector.

Proof.

By Corollary 1, it suffices to prove that $\rho(I - Q^{-1}A) < 1$. To this end, let λ be any eigenvalue of $I - Q^{-1}A$. Let x be a corresponding eigenvector. We assume, that $\| x \|_\infty = 1$. We have now $(I - Q^{-1}A)x = \lambda x$ or $Qx - Ax = \lambda Qx$.

Since Q is the lower triangular part of A, including its diagonal,

$$-\sum_{j=i+1}^{n} a_{ij}x_j = \lambda \sum_{j=1}^{i} a_{ij}x_j \qquad (1 \le i \le n)$$

$$\lambda a_{ii}x_i = -\lambda \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}x_j \qquad (1 \le i \le n)$$

Select an index i such that $|x_i| = 1 \ge |x_j|$ for all j. Then,

$|\lambda||a_{ii}| \le |\lambda| \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|$

$|\lambda| \le \{\sum_{j=i+1}^{n} |a_{ij}|\}\{|a_{ij}| - \sum_{j=1}^{i-1} |a_{ij}|\}^{-1} < 1$ □

下一节内容

Basic Concepts

In the space of complex $n-$vectors.

- The **conjugate transpose** of y, $y^* = (\bar{y_1}, \bar{y_2}, ..., \bar{y_n})$
- The **inner product** $\langle x, y\rangle = y^* x = \sum_{i=1}^{n} x_i \bar{y}_i$
- $\langle x, x\rangle > 0 \qquad (\text{if } x \neq 0)$
  $\langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle$
  $\langle x, y\rangle = \overline{\langle y, x\rangle}$
- The **conjugate transpose** of A, $A^* = (\bar{a_{ji}})$
- **Hermitian:** $A^* = A$
- If A is Hermitian, then $\langle Ax, y\rangle = \langle x, Ay\rangle$

THEOREM 7 (Theorem on SOR Method Convergence)

In the SOR method, suppose that splitting matrix Q *is chosen to be* $\alpha D - C$*, where* α *is a real parameter,* D *is any positive definite Hermitian matrix, and* C *is any matrix safisfying* $C + C^{*} = D - A$*. If* A *is positive definite Hermitian, if* Q *is nonsigular, and if* $\alpha > 1/2$*, then the SOR iteration converges for any starting vector.*

Proof.

We let $G = I - Q^{-1}A$ and attempt to establish that the spectal radius of G satisfies $\rho(G) < 1$. Let $Gx = \lambda x$, and $y = (I - G)x$.

$$y = x - Gx = x - \lambda x = Q^{-1}Ax$$

$$Q - A = (\alpha D - C) - (D - C - C^*) = \alpha D - D + C^*$$

We have

$$(\alpha D - C)y = Qy = Ay \tag{1}$$

$$(D - C - C^*)y = AGx \tag{2}$$

$$\alpha\langle Dy, y\rangle - \langle Cy, y\rangle = \langle Ax, y\rangle \tag{3}$$

$$\alpha\langle y, Dy\rangle - \langle y, Dy\rangle + \langle y, C^*y\rangle = \langle y, AGx\rangle \tag{4}$$

On adding Equations(3) and (4), We obtain

$$2\alpha\langle Dy, y\rangle - \langle y, Dy\rangle = \langle Ax, y\rangle + \langle y, AGx\rangle \tag{5}$$

Proof.

$$(2\alpha - 1)\langle Dy, y\rangle = \langle Ax, y\rangle + \langle y, AGx\rangle$$

Since $y = (1-\lambda)x$ and $Gx = \lambda x$, Equation(5) yields

$$(2\alpha - 1)|1-\lambda|^2\langle Dx, x\rangle = (1-\bar{\lambda})\langle Ax, x\rangle + \bar{\lambda}(1-\lambda)\langle x, Ax\rangle = (1-|\lambda|^2)\langle Ax, x\rangle \tag{6}$$

If $\lambda \neq 1$, the left side of the equation (6) is positive. Hence, the right side must also be positive, and $|\lambda| < 1$.
On the other hand, if $\lambda = 1$, then $y = 0$ from $y = (1-\lambda)x$ and $Ax = 0$.
This contradicts the condition $\langle Ax, x\rangle > 0$ for any $x \neq 0$.
Thence, $\rho(G) < 1$ and the SOR method converges. □

A common choice for D and C in the SOR method is to let D be the diagonal of A and $-C$ be the lower triangular part of A, excluding the diagonal.

下一节内容

Suppose that A is partitioned into $A = D - C_L - C_U$, where $D = diag(A)$, C_L is the negitive of the strictly lower triangular part of A, and C_U is the negitive of the stictly upper triangular part of A.

$Qx^{(k)} = (Q - A)x^{(k-1)} + b$

$x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b$

$G = I - Q^{-1}A$

Richardson:

$$\begin{cases} Q = I \\ G = I - A \end{cases}$$

$$x^{(k)} = (I - A)x^{(k-1)} + b$$

Jacobi

$$\begin{cases} Q = D \\ G = D^{-1}(C_L + C_U) \end{cases}$$

$$Dx^{(k)} = (C_L + C_U)x^{(k-1)} + b$$

Gauss-Seidel

$$\begin{cases} Q = D - C_L \\ G = (D - C_L)^{-1} C_U \end{cases}$$

$$(D - C_L)x^{(k)} = C_U x^{(k-1)} + b$$

SOR

$$\begin{cases} Q = \omega^{-1}(D - \omega C_L) \\ G = (D - \omega C_L)^{-1}(\omega C_U + (1-\omega)D) \end{cases}$$

$$(D - \omega C_L)x^{(k)} = \omega(C_U x^{(k-1)} + b) + (1-\omega)Dx^{(k-1)}$$

下一节内容

Consider the iteration formula $x^{(k)} = Gx^{(k-1)} + c$
We introduce a parameter $\gamma \neq 0$ and embedded in a one-parameter family of iteration methods given by

$$x^{(k)} = \gamma(Gx^{(k-1)} + c) + (1-\gamma)x^{(k-1)} = G_\gamma x^{k-1} + \gamma c$$

where

$$G_\gamma = \gamma G + (1-\gamma)I$$

Then by taking a limit, we get $x = \gamma(Gx + c) + (1-\gamma)x$ or $x = Gx + c$.

THEOREM 8 (Theorem on Eigenvalue of $p(A)$)

If λ is an eigenvalue of a matrix A and if p is a polynomial, then $p(\lambda)$ is an eigenvalue of $p(A)$.

Proof.

Let $Ax = \lambda x$ with $x \neq 0$. Then $A^2x = \lambda Ax = \lambda^2 x$, we have

$$A^k x = \lambda^k x$$

$$p(A)x = \sum_{k=0}^{m} c_k A^k x = \sum_{k=0}^{m} c_k \lambda^k x = p(\lambda)x$$

□

THEOREM 9 (Theorem on Optimal Extrspolation Parameters)

If the only information avaliable about the eigenvalues of G is that they lie in the interval $[a, b]$, and if $1 \notin [a, b]$, then the best choice for γ is $2/(2-a-b)$. With this value of γ, $\rho(G_\gamma) \leq 1 - |\gamma| d$, where d is the distance from 1 to $[a, b]$

Proof.

The eigenvalues of G in an interval $[a, b]$

The eigenvalues of $G_\gamma = \gamma G + (1-\gamma)I$ in the interval $[\gamma a + 1 - \gamma, \gamma b + 1 - \gamma]$

Since $1 \notin [a, b]$, $a > 1$ or $b < 1$

Since $a \leq b < 1$, it follows that $\gamma > 0$ and $d = 1 - b$.

$\gamma a + 1 - \gamma \leq \lambda \leq \gamma b + 1 - \gamma$

$\lambda \leq \gamma b + 1 - \gamma = 1 + \gamma(b-1) = 1 - \gamma d$

$\lambda \geq \gamma a + 1 - \gamma = \gamma(a + b - 2) + 1 + \gamma(1-b) = -1 + \gamma d$

We have proved that $-1 + \gamma d \leq \lambda \leq 1 - \gamma d$ □