Polynomial Local Search (PLS)

An abstract problem in PLS complexity class is defined by the following three algorithms.

i) An algorithm to pick an initial solution.

ii) An algorithm to compute the value of a solution.

iii) An algorithm to determine if a solution is a local optimal or executes a "local move" which improves the value of the solution.

PLS = the set of all abstract search problems.

<u>Observation:</u> The problem finding a PSNE in a congestion game belongs to PLS. More generally, the PSNE problem for finite potential games is in PLS.

Reduction in PLS class: A PLS problem P_1 reduces to another PLS problem P_2 in polynomial time if we have the following two algorithms. $\quad P_1 \leq P_2$

(i) An algorithm A to construct from every instance x of P_1 to an instance $A(x)$ of P_2

(ii) Another algorithm B to construct a solution of x from a solution of $A(x)$.

Fact: Local weighted maximum cut is PLS-complete.

Theorem: PSNE problem for congestion games is PLS-complete.

Proof: Need to show:

i) membership in PLS ←

ii) PLS-hardness.

Since congestion games are finite potential games, PSNE for ______ belongs to PLS.

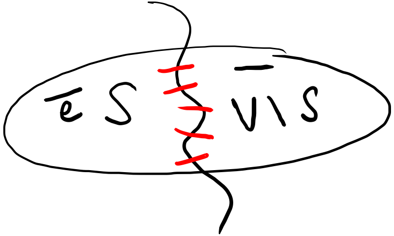
To show PLS-hardness, we reduce from local weighted max-cut.

Let $(G=(V,E), w)$ be any instance of local weighted max-cut.

- Set of players (N) : V.

- Set of resources : $\{ r_e , \bar{r}_e : e \in E \}$
- Strategy set of player v : $\{ \{ r_e : e \in \delta(v) \} , \{ \bar{r}_e : e \in \delta(v) \} \}$
- Cost of r_e or $\bar{r}_e$: 0 if at most one player uses it ; $w(e)$ otherwise.

Recall, $$\Phi(s) = \sum_{e \in E} \sum_{i=1}^{f(e)} c_e(i).$$



$\Rightarrow$ Players in S play $\{ r_e : e \in \delta(v) \}$

——— $V \setminus S$ " $\{ \bar{r}_e : e \in \delta(v) \}$

A cut of weight $w(S, \bar{S})$ corresponds to a strategy profile of potential $\sum_{e \in E} w(e) - w(S, V \setminus S)$.

Locally maximum cut $\longleftrightarrow$ local minimum of potential function.

Given a PSNE, if player v plays $\{\overline{r_e} : e \in E\}$, then put v in $V \setminus S$, otherwise put v in S.

Lecture 5.2

Theorem: PSNE problem for congestion games is PLS-complete. Moreover, the cuts are in one-to-one correspondence with strategy profiles.

Fact: There exist an instance of local weighted max cut where we need exponentially many local moves to find a local max-cut.

Corollary: The best-response dynamic can make exponentially many iterations to converge to a PSNE.

Theorem: PSNE problem is PLS-complete even for symmetric congestion games.

Proof: Membership in PLS follows from the fact that congestion games are potential games.

To prove PLS-hardness, we reduce from PSNE problem for congestion games.

Let $\Gamma = \langle N, E, (S_i)_{i \in N}, (c_e : \mathbb{N} \rightarrow \mathbb{R})_{e \in E} \rangle$ be any congestion game. We now construct a symmetric congestion game

$$\Gamma' = \langle N', E', (S_i')_{i \in N'}, (c_e' : \mathbb{N} \rightarrow \mathbb{R})_{e \in E'} \rangle$$

Reduced instance

- $N' = N$
- $E' = E \cup \{r_i : i \in N\}$
- $S_i' = \bigcup_{i \in N} \{ s_i \cup \{r_i\} : s_i \in S_i \} \quad \forall i \in N'$
- $c_e' = c_e \quad \forall e \in E, \quad c_e'(\ell) = \begin{cases} 0 & \text{if } \ell \leq 1 \\ \infty & \text{o/w.} \end{cases} \quad \forall e \in \{r_i : i \in N\}$

Need an algorithm to construct a PSNE of the congestion game from a PSNE of the symmetric congestion game.

Let $s' = (s'_i)_{i \in N}$ be any PSNE of Γ'. Observe that each resource r_i is used by exactly one player in the strategy profile s'. Otherwise, there would exist another resource r_j which no player is using. Any of the multiple players using r_i can unilaterally deviate to any strategy containing r_j which reduces the cost of the deviating player. However this contradicts our assumption that s' is a PSNE. We define a strategy profile s of T as follows: $s = (s_i)_{i \in N}$.

$$s_i = s'_i \quad \text{such that} \quad \exists j \in N, \; s'_j = s_i \cup \{r_i\}$$

The values of the potential function are the same in both s and s'. $\blacksquare$

Corollary: The best response dynamic can take exponential number of iteration to find a PSNE even for symmetric congestion games.

Q. What is the complexity of finding an MSNE of a finite strategic form game?

Lecture 5.3

Finding an MSNE

Qn: What is the computational complexity of finding an MSNE in a normal form game?

(i) do there exist at least two MSNEs?

(ii) does there exist an MSNE where player 1 receives at least some minimum utility?

(iii) does there exist an MSNE where both the players receive certain minimum utility?

(iv) does there exist any MSNE which involves some strategies of player 1?

(v) does there exist any MSNE which does not involve some strategies of player 1?

⋮

All the above problems are NP-complete.

Functional NP (FNP): This class consists of all problems in NP with the extra requirement that for Yes instances, the algorithm also needs to output a certificate.

Can we show MSNE problem for bimatrix games is FNP-complete? NO!

Theorem: If MSNE for bimatrix games is FNP-complete, then

$$NP \overset{\supseteq}{\underset{\subseteq}{=}} \text{co-}NP.$$

Proof: Enough to prove $NP \subseteq \text{co-}NP$.

Assumption: MSNE for bimatrix games is FNP-complete.

$\Rightarrow$ There exist a reduction from functional SAT to MSNE.

$\Rightarrow$ i) An algorithm A to contruct an intance of MSNE A(f) from any Boolean formula f.

ii) An algorithm B to map every MSNE s^* of A(f) to either

a) yes and a satisfying assignment (certificate) for f

or b) no.

claim: SAT $\in$ co-NP.

Proof: To prove, there exist an efficiently verifiable certificate for the No instances of SAT also. Let f be a No instance.

We claim $(\underline{A(f)}, \underline{s^*})$ a certificate, and the algorithm B is the poly-time verifier. $\blacksquare$

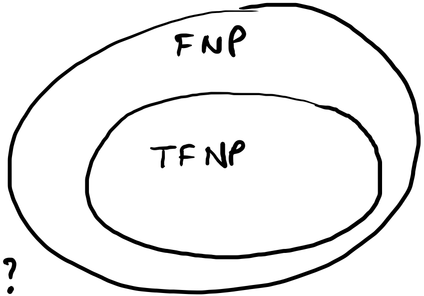
Computational Complexity of Finding an MSNE

- Functional NP.

Theorem: If MSNE is FNP-complete, then NP = co-NP.

Total FNP (TFNP): The set of all problems in FNP which have only yes instances.

Examples: MSNE, factoring.



Theorem: If any problem in TFNP is FNP-complete, then we have NP = co-NP.

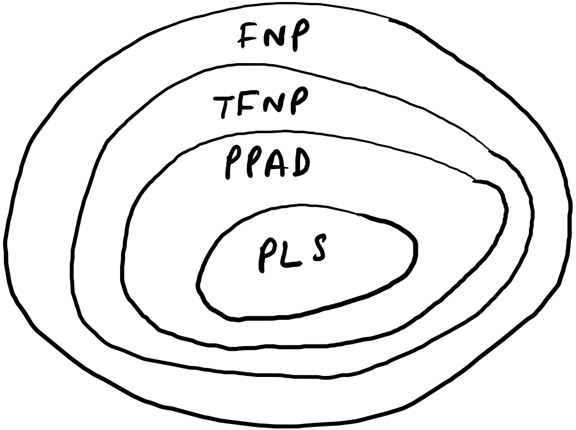
Q: Can we show MSNE is TFNP-complete?

No!

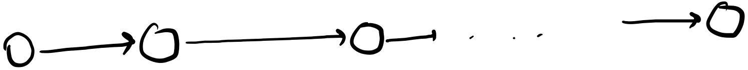
Because TFNP is a "semantic" complexity class whereas most other complexity classes, e.g. NP, XP, PSPACE, FNP are "syntactic" complexity class.

For TFNP, we do not have any complete problem.

Way-out: Define appropriate "syntactic" subclass of TFNP which contains MSNE.



PPAD (Polynomial Parity Argument on Directed graphs)



PLS: Finding a sink in an acyclic graph.

PPAD: search for sink or a source node in a directed graph

Definition: PPAD consists of search problems which has the following two algorithms:

(i) An algorithm to pick an initial solution.

(ii) Given an intermediate solution, an algorithm to find the "next intermediate solution" or output that the current solution is a local optimum.

Although not immediately clear, there exists algorithms for computing an MSNE (e.g. Lemke-Howson's algorithm) that traverse certain directed graph on the strategy profiles. Due to this, MSNE problem for bimatrix games belongs to PPAD.

Can we show that MSNE is PPAD-complete?

YES!

BROWER'S PROBLEM: Finding a stationary (fixed) point of a function. $f : [0,1) \rightarrow [0,1]$, continuous. Such functions has a fixed point. A point $x \in [0,1]$ is called a fixed point if $f(x) = x$.

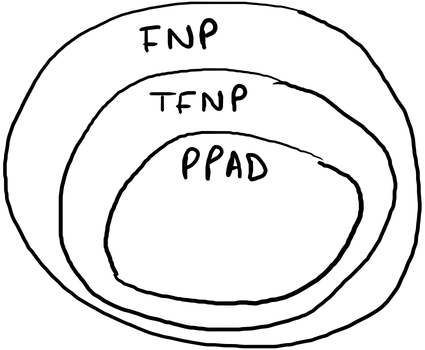
SPERNER'S PROBLEM:

PAPADMITRIOU AND DASKALAKIS. "ACM DESSERTATION AWARD"

Lecture 5.5

Complexity of finding an MSNE in a bimatrix game.

Polynomial Parity Argument on Directed graphs (PPAD)

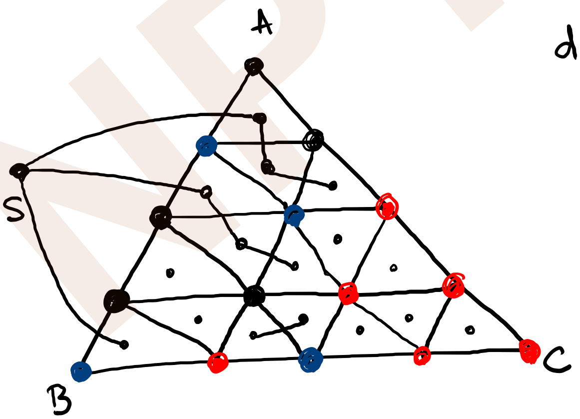


__Theorem:__ MSNE problem for bimatrix games is PPAD-complete.

Sperner's Lemma

Lemma:

There exists at least one trichromatic triangle.



degree of any vertex

0/1/2

Proof: Construct graph with baby triangles as vertices.

We also have a special vertex

The degree of a trichromatic triangle must be 1.

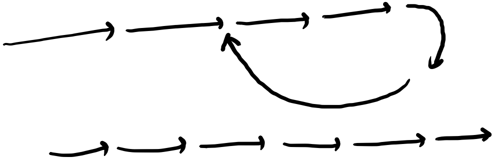
Fact: Any undirected graph has an even number of vertices of odd degree.

- The degree of S must be an odd number.

Hence, there exists at least one vertex other than S whose degree is odd. Equivalently, there exists a trichromatic baby triangle. $\square$

Sperner's Problem: Given an oracle access to a sub-divided simplex $\left(= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0, \sum_{i=1}^{n} x_i = 1\} \right)$ in n-dimension, find a colorful baby simplex.

Sperner's problem $\in$ PPAD.



In any directed graph where the out-degree of every vertex is at most 1, if there exists a source node, then there exists a sink.

It turns out that the edges in the proof of Sperner's lemma can be directed appropriately such that any trichromatic baby triangle corresponds to sink nodes.

<u>Theorem</u>: Sperner's problem is PPAD-complete.