Price of Anarchy

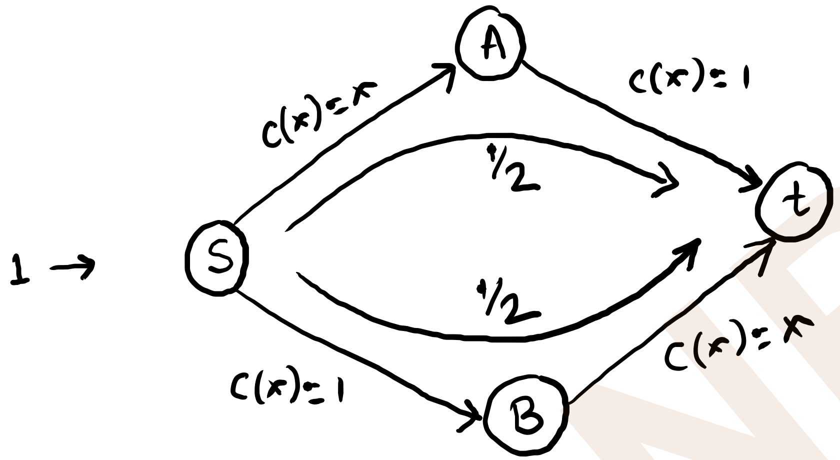
Lecture 7.4

$\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

Social welfare function: $w : \bigtimes_{i \in N} S_i \longrightarrow \mathbb{R}_{>0}$

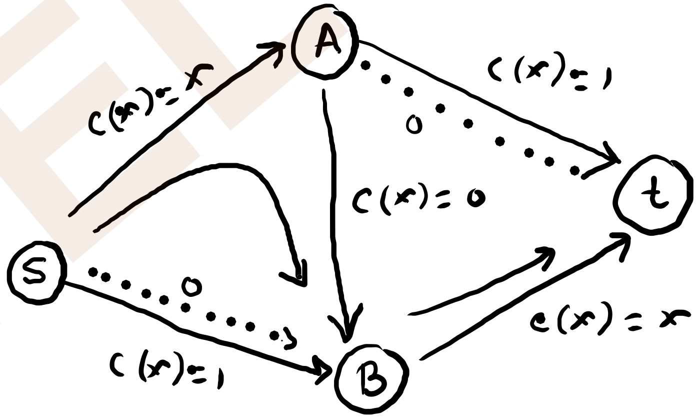
Definition (PoA): $$\mathrm{PoA} = \max_{s^{eq} \in \mathrm{PSNE}(\Gamma)} \frac{\max_{s \in \bigtimes_{i \in N} S_i} w(s)}{w(s^{eq})}$$

Braess's Paradox



$\left(S \to A \to t : \frac{1}{2} \, , \; S \to B \to t : \frac{1}{2}\right)$ is a PSNE

Total delay $= \frac{3}{2}$



$S \to A \to B$ always take a cost of x (≤ 1) which is at most the cost of $S \to B$

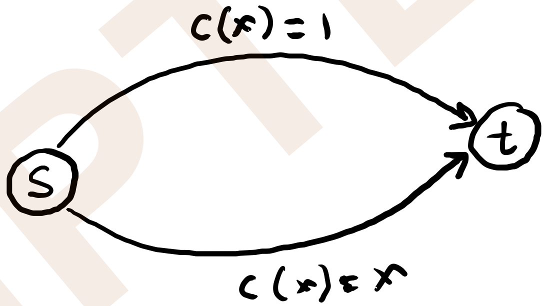
$S \to A \to B \to t$ is a weakly dominant strategy. Thus $(S \to A \to B \to t : 1)$ is a WDSE.

Total delay $= 2$

If the social welfare function is $\frac{1}{\text{total delay}}$, then

PoA is $\geq \frac{2}{3/2} = \frac{4}{3}$

Pigou's Network:

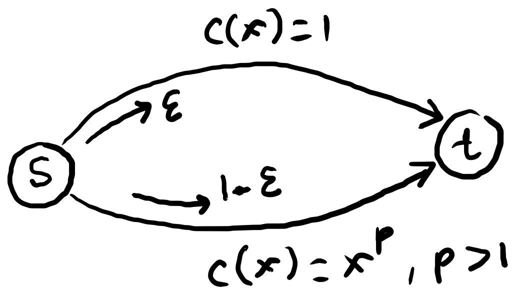


The bottom edge weakly dominates the top edge. So all traffic using the bottom edge is a WDSE.

Total cost = 1

$\frac{1}{2}$ traffic using the top edge and the other $\frac{1}{2}$ traffic using the bottom edge incurs a cost of $\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}$.

$\text{PoA} \geq \frac{4}{3}$



All traffic following the bottom path is a WDSE. Total cost is 1.

Total cost is $\varepsilon\cdot 1 + (1-\varepsilon)\cdot(1-\varepsilon)^p$, $\varepsilon > 0$

$$= \varepsilon + (1-\varepsilon)^{p+1}$$

$$\text{PoA} \geq \frac{1}{\varepsilon + \underline{\underline{(1-\varepsilon)}}^{p+1}} \longrightarrow \infty \text{ as } p \to \infty$$

As non-linearity in the cost function increases, the PoA also increases.

PoA for Selfish Networks

Pigou's Network:

Define:

- It has two vertices: source s and sink t
- ________ edges from s to t.
- Traffic rate is r (>0)
- The cost of one edge is $c(x)$; $c(\cdot)$ is a non-negative, non-decreasing, continuous function

- The cost of the other edge is $c(r)$.