# Nash Equilibrium

**Definition:** Game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ a strategy profile $(s_i^*)_{i \in N} \in S$ is called a pure strategy Nash equilibrium if (PSNE)

$\forall i \in N,$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \quad \forall s_i' \in S_i$$

$(A, A), (B, B)$ are PSNEs for battle of sexes and coordination games.

**Observation:** (i) Every SDSE is a WDSE

(ii) Every WDSE is a VWDSE

(iii) Every VWDSE is a PSNE

Matching pennies, rock-paper-scissor games does not have any PSNE.

**Mixed strategy:** a probability distribution over the strategy set of a player. $\sigma_i \in \Delta(S_i),\ i \in N$

$$u_i\left((\sigma_i)_{i \in N}\right) = \sum_{(s_1, \ldots, s_n) \in S} \sigma_1(s_1) \cdots \sigma_n(s_n) \cdot u_i(s_1, \ldots, s_n)$$

# Mixed Strategy Nash Equilibrium (MSNE)

**Definition:** Game $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$ a mixed strategy profile $(\sigma_i^*)_{i\in N}$ is called an MSNE if

$\forall i \in N,$

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Delta(S_i)$$

← infinitely many conditions

**Nash Theorem:** Every finite game has at least one MSNE.

## Characterization of MSNE (Indifference Principle)

**Theorem:** Given $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ a mixed strategy profile $(\sigma_i^*)_{i \in N} \in \times_{i \in N} \Delta(S_i)$ is an MSNE for $\Gamma$ if and only if the following holds for all the players, $i \in N$.

$$\forall s_i \in S_i, \ \sigma_i(s_i) \neq 0 \Rightarrow u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*) \quad \forall s_i' \in S_i$$

That is, $\sigma_i$ puts entire probability mass only on "best-response" strategies against $\sigma_{-i}^*$.

<u>Check</u>:– $\left(\{A:\frac{1}{2}, B:\frac{1}{2}\}, \{A:\frac{1}{2}, B:\frac{1}{2}\}\right)$ is the <u>unique</u> MSNE for the matching pennies game.

– $\left(\{R:\frac{1}{3}, P:\frac{1}{3}, S:\frac{1}{3}\}, \{R:\frac{1}{3}, P:\frac{1}{3}, S:\frac{1}{3}\}\right)$ is the <u>unique</u> MSNE for the rock-paper-scissor game.