Lecture 2.3

Theorem: $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$ any game, and $(\sigma_i^*)_{i\in N}$ be an MSNE. Then

$$\forall i \in N, \quad u_i((\sigma_i^*)_{i\in N}) \geq v_i$$

The above inequalities are all tight for two-person zero-sum games.

Let A be the utility matrix of the row player.

Then $-A$ is —— ———— column ″ .

$A \in \mathbb{R}^{m \times n}$

value of the row player,

$$\underline{v} = \boxed{\max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^{m} \sigma(i) A_{ij}} \rightarrow \text{maxmin value}$$

The value of the column player

$$\bar{v} = - \boxed{\min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^{n} \sigma(j) A_{ij}} \rightarrow \text{minmax value}$$

Lemma:

$$\max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^{m} \sigma(i) A_{ij} \leq \min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^{n} \sigma(j) A_{ij}$$

Proof: Let

$$\max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^{m} \sigma(i) A_{ij} = \min_{j \in [n]} \sum_{i=1}^{m} \sigma^*(i) A_{ij}$$

$$= \min_{\sigma \in \Delta([n])} \sum_{j=1}^{n} \sum_{i=1}^{m} \sigma(j)\, \sigma^*(i)\, A_{ij}$$

$$= \min_{\sigma \in \Delta([n])} \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma(j)\, \boxed{\sigma^*(i)}\, A_{ij}$$

$$= \min_{\sigma \in \Delta([n])} \underbrace{\sum_{i=1}^{m} \sigma^*(i) \left(\sum_{j=1}^{n} \sigma(j) A_{ij} \right)}_{\text{convex combination}}$$

$$\leq \min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^{n} \sigma(j) A_{ij}$$

■

Minmax Theorem: $A \in \mathbb{R}^{m \times n}$, there exists $x^* = (x_1^*, \ldots, x_m^*) \in \Delta([m])$ and $y^* = (y_1^*, \ldots, y_n^*) \in \Delta([n])$ such that

$$\max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y$$

Linear Program of the row player:

LP1

maximize $\min_{j \in [n]} \sum_{i=1}^{m} A_{ij} x_i$

s.t : $\sum_{i=1}^{m} x_i = 1$

$x_i \geq 0 \quad \forall i \in [m]$

max t

s.t. $t \leq \sum_{i=1}^{m} A_{ij} x_i \quad \forall j \in [n]$

$\sum_{i=1}^{m} x_i = 1$

$x_i \geq 0 \quad \forall i \in [m]$

Linear Program for the column player

LP2

minimize $\max_{i \in [m]} \sum_{j=1}^{n} A_{ij} y_j$

s.t: $\sum_{j=1}^{n} y_j = 1$

$y_j \geq 0 \quad \forall j \in [n]$

minimize w

s.t: $w \geq \sum_{j=1}^{n} A_{ij} y_j \quad \forall i \in [m]$

$\sum_{j=1}^{n} y_j = 1$

$y_j \geq 0 \quad \forall j \in [n].$

Claim: LP1 and LP2 are duals of each other.

Strong Duality Theorem: Let LP1 and LP2 are two linear programs which are duals of each other. If LP1/LP2 is feasible and bounded, then LP2/LP1 is also and

$OPT(LP1) = OPT(LP2)$