Lecture 2.1

Theorem (Indifference Principal) Given $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a mixed strategy profile $(\sigma_i^*)_{i \in N} \in \underset{i \in N}{\times} \Delta(S_i)$ is an MSNE if and only if

$$\left(\text{For each } i \in N,\ s_i \in S_i,\ \sigma_i^*(s_i) \neq 0 \Rightarrow u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)\ \forall s_i' \in S_i \right)$$

Proof: (<u>If part</u>) Suppose $(\sigma_i^*)_{i \in N}$ be a mixed strategy profile which satisfies the equivalent condition.

<u>To show:</u> $(\sigma_i^*)_{i \in N}$ is an MSNE.

Fix any player $i \in N$,

$$u_i(\sigma_i, \underline{\sigma}_{-i}^*) = \sum_{s_i \in S_i} \sum_{\underline{s}_{-i} \in \underline{S}_{-i}} \sigma_i(s_i) \prod_{\substack{j \in N, \\ j \neq i}} \sigma_j^*(s_j)\, u_i(s_i, \underline{s}_{-i})$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, \underline{\sigma}_{-i}^*)$$

A convex combination of $\{u_i(s_i, \underline{\sigma}_{-i}^*) : s_i \in S_i\}$

Convex combination of a set $\{a_1, \ldots, a_n\}$ of numbers is $\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n$, where $\lambda_1, \ldots, \lambda_n \geq 0$

$\lambda_1 + \ldots + \lambda_n = 1$

Obs: A convex combination $\lambda_1 a_1 + \ldots + \lambda_n a_n$ is maximized if ($\lambda_i \neq 0 \Rightarrow a_i = \max\{a_1, \ldots, a_n\}$ $\forall i \in [n] =: \{1, \ldots, n\}$)

$$u_i(\sigma_i, \sigma_{-i}^*) \leq \sum_{s_i \in S_i} \sigma_i^*(s_i)\, u_i(s_i, \sigma_{-i}^*)$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*)$$

(<u>Only if part</u>) Suppose $(\sigma_i^*)_{i \in N}$ is an MSNE of Γ.

"Proof by contradiction"

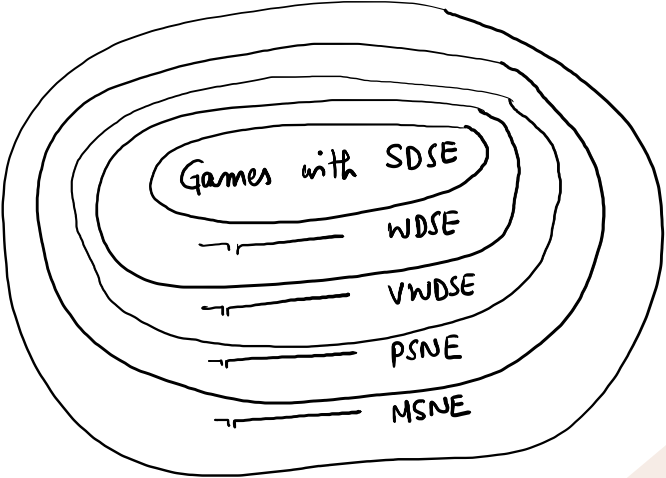
Let us assume $\exists i \in N, s_i \in S_i$ such that $\sigma_i^*(s_i) \neq 0$ and $u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$ for some $s_i' \in S_i$, $s_i \neq s_i'$, w.l.o.g assume s_i' maximizes $u_i(\cdot, \sigma_{-i}^*)$

Consider another mixed strategy which puts entire probability mass on s_i', which is a pure strategy

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \underbrace{\sum_{s_i'' \in S_i} \sigma_i^*(s_i'')\, u_i(s_i'', \sigma_{-i}^*)}_{\text{Convex combination of } \{u_i(s_i'', \sigma_{-i}^*) : s_i'' \in S_i\}}$$

$$< u_i(s_i', \sigma_{-i}^*)$$

$\blacksquare$


Games with SDSE
WDSE
VWDSE
PSNE
MSNE

Lecture 2.2

Two Person Zero Sum Game

Matching pennies:

	A	B
A	1, −1	−1, 1
B	−1, +1	1, −1

Rock-paper-Scissor:

	Rock	Paper	Scissor
Rock	0, 0	−1, +1	1, −1
Paper	1, −1	0, 0	−1, 1
Scissor	−1, 1	1, −1	0, 0

Strictly competitive games / Win-loss game / matrix game

Security of a player:

Unique PSNE : (B, B)

	A	B	max
A	2, 2	2.5, 1	2
B	-100, 2	3, 3	-100

maximum utility that a player can guarantee without assuming anything about other players.

Security level of the row player is 2.

"Reasoning" from security level of the row player, we can predict that players will play (A, A) which is not any equilibrium we have seen so far.

Security of a player in pure strategies.

Def^n: $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

security level of player i in pure strategies or value

$$v_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

maxmin value of player i in pure strategies.

The value of both the players in pure strategies in the matching pennies game is -1.

Security level in mixed strategies

$$\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$$

$$v_i = \sup_{\sigma_i \in \Delta(S_i)} \ \inf_{\sigma_{-i} \in \times_{j \in N, j \neq i} \Delta(S_j)} u_i(\sigma_i, \sigma_{-i})$$

maxmin value in mixed strategies.

Value of both the players in mixed strategies in matching pennies game is 0.

<u>obs:</u> $$v_i = \sup_{\sigma_i \in \Delta(S_i)} \min_{s_{-i} \in S_{-i}} u_i(\sigma_i, s_{-i})$$

<u>Proof:</u> Follows from maximization of a convex combination

Theorem: $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$. Let $(\sigma_i^*)_{i\in N}$ be an MSNE. Then,

$$\forall i \in N, \quad u_i((\sigma_i^*)_{i\in N}) \geqslant v_i$$

Proof: Follows immediately from the definitions. $\blacksquare$

Lecture 2.3

Theorem: $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ any game, and $(\sigma_i^*)_{i \in N}$ be an MSNE. Then

$$\forall i \in N, \quad u_i\left((\sigma_i^*)_{i \in N}\right) \geq v_i$$

The above inequalities are all tight for two-person zero-sum games.

Let A be the utility matrix of the row player.

Then $-A$ is ———— column ″ .

$$A \in \mathbb{R}^{m \times n}$$

value of the row player,

$$\underline{v} = \boxed{\max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^{m} \sigma(i) A_{ij}} \rightarrow \text{maxmin value}$$

The value of the column player

$$\bar{v} = - \boxed{\min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^{n} \sigma(j) A_{ij}} \rightarrow \text{minmax value}$$

Lemma:

$$\max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^{m} \sigma(i) A_{ij} \leq \min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^{n} \sigma(j) A_{ij}$$

Proof: Let

$$\max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^{m} \sigma(i) A_{ij} = \min_{j \in [n]} \sum_{i=1}^{m} \sigma^*(i) A_{ij}$$

$$= \min_{\sigma \in \Delta([n])} \sum_{j=1}^{n} \sum_{i=1}^{m} \sigma(j) \sigma^*(i) A_{ij}$$

$$= \min_{\sigma \in \Delta([n])} \sum_{i=1}^{m} \sum_{j=1}^{n} \sigma(j) \boxed{\sigma^*(i)} A_{ij}$$

$$= \min_{\sigma \in \Delta([n])} \underbrace{\sum_{i=1}^{m} \sigma^*(i) \left(\sum_{j=1}^{n} \sigma(j) A_{ij} \right)}_{\text{convex combination}}$$

$$\leq \min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^{n} \sigma(j) A_{ij} \qquad \blacksquare$$

Minmax Theorem: $A \in \mathbb{R}^{m \times n}$, there exists $x^* = (x_1^*, \ldots, x_m^*) \in \Delta([m])$ and $y^* = (y_1^*, \ldots, y_n^*) \in \Delta([n])$ such that

$$\max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y$$

Linear Program of the row player:

LP1

maximize $\min_{j \in [n]} \sum_{i=1}^{m} A_{ij} x_i$

s.t : $\sum_{i=1}^{m} x_i = 1$

$x_i \geq 0 \quad \forall i \in [m]$

max t

s.t. $t \leq \sum_{i=1}^{m} A_{ij} x_i \quad \forall j \in [n]$

$\sum_{i=1}^{m} x_i = 1$

$x_i \geq 0 \quad \forall i \in [m]$

Linear Program for the Column Player

LP2

minimize $\max_{i \in [m]} \sum_{j=1}^{n} A_{ij} y_j$

s.t: $\sum_{j=1}^{n} y_j = 1$

$y_j \geq 0 \quad \forall j \in [n]$

minimize w

s.t: $w \geq \sum_{j=1}^{n} A_{ij} y_j \quad \forall i \in [m]$

$\sum_{j=1}^{n} y_j = 1$

$y_j \geq 0 \quad \forall j \in [n].$

Claim: LP1 and LP2 are duals of each other.

Strong Duality Theorem: Let LP1 and LP2 are two linear programs which are duals of each other. If LP1/LP2 is feasible and bounded, then

LP2/LP1 is also and

OPT(LP1) = OPT(LP2)

Minmax Theorem: $A \in \mathbb{R}^{m \times n}$. There exist mixed strategies $x^* = (x_1^*, \ldots, x_m^*)$ and $y^* = (y_1^*, \ldots, y_n^*)$ s.t.

$$\max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y.$$

Proof: LP1 and LP2 are duals of each other.

LP1 is clearly feasible.

$$\min_{i,j} A_{ij} \leq \max_{x \in \Delta([m])} x A y^* \leq \max_{i,j} A_{ij}$$

$OPT(LP1) = OPT(LP2)$ by strong duality

$$OPT(LP1) = \min_{j \in [n]} \sum_{i=1}^{m} A_{ij} x_i^* = \max_{j \in [n]} x^* A e_j = \min_{x \in \Delta([m])} \max_{j \in [n]} x A e_j$$

$$OPT(LP2) = \max_{i \in [m]} \sum_{j=1}^{n} A_{ij} y_j^*$$

$$= \max_{i \in [m]} e_i A y^*$$

$$\max_{x \in \Delta([m])} x A y^* = \max_{i \in [m]} e_i A y^* = \max_{j \in [n]} x^* A e_j = \min_{y \in \Delta([n])} x^* A y.$$

$\blacksquare$

Corollary: In every matrix game, there exists an MSNE. The mixed strategies of both the players guarantee their security level.

Proof: Let $x^* = (x_1^*, \dots, x_m^*)$ be a solution to LP1

$y^* = (y_1^*, \dots, y_n^*)$ ——— " ——— LP2

By minmax Theorem,

$$\underline{\max_{x \in \Delta([m])} x A y^*} = \min_{y \in \Delta([n])} x^* A y$$

claim: (x^*, y^*) is an MSNE.

$$x^* A y^* \leqslant \max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y$$

$\Rightarrow$ The column player does not have any incentive to deviate from y^* given the row player plays x^*.

$$x^* A y^* \geqslant \min_{y \in \Delta([n])} x^* A y = \max_{x \in \Delta([m])} x A y^*$$

$\Rightarrow$ The row player also does not have any incentive to deviate given column player plays y^*.

Hence (x^*, y^*) is an MSNE.

x^* and y^* are solutions of LP1 and LP2 respectively.

An MSNE of a two player zero-sum game can be computed in polynomial time.

$$x^* A y^* \leq \max_{x \in \Delta([m])} x A y^* = \max_{x \in \Delta([m])} \min_{j \in [n]} x A e_j = \underline{v}$$

$\Rightarrow$ utility of row player is at most its value.

$\Rightarrow$ Utility of the row player in (x^*, y^*) is its value.

$$x^* A y^* \geq \min_{y \in \Delta([n])} x^* A y = \min_{y \in \Delta([n])} \max_{i \in [m]} e_i A y = \bar{v}$$

$\Rightarrow$ Utility of the column player in (x^*, y^*) is at most its value.

$\Rightarrow$ exactly its value. $\blacksquare$

Lecture 2.5

Maxmin and Minmax Values Coincide in Matrix Games

Theorem: $A \in \mathbb{R}^{m \times n}$.

$$\max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy = \min_{y \in \Delta([n])} \max_{x \in \Delta([m])} xAy$$

Proof: We have already shown before,

$$\max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy \leq \min_{y \in \Delta([n])} \max_{x \in \Delta([m])} xAy$$

<u>To show:</u> $\max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy \geq \min_{y \in \Delta([n])} \underbrace{\max_{x \in \Delta([m])} xAy}_{}$. ✓

Let x^* be an optimal solution of LP1

y^* — — LP2

$\min_{y \in \Delta([n])} \max_{x \in \Delta([m])} xAy \geq \min_{y \in \Delta([n])} x^* Ay = \max_{x \in \Delta([m])} xAy^* \geq \max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy$

$\blacksquare$

Theorem: $A \in \mathbb{R}^{m \times n}$. Then the utility of both the players in any MSNE is exactly their values.

Proof: (x^*, y^*) be any MSNE.

$$v_r = \max_{x \in \Delta([m])} \min_{y \in \Delta([n])} xAy, \quad v_c = -\min_{y \in \Delta([n])} \max_{x \in \Delta([m])} xAy$$

From minmax theorem, we know that $v_r = -v_c$ for the matrix game A.

Let (x^*, y^*) be an MSNE

$$x^* A y^* \leq \max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y \leq \max_{x \in \Delta([m])} \min_{y \in \Delta([n])} x A y = v_r$$

$$x^* A y^* \leq v_r \quad , \quad x^* A y^* \geq v_r \quad \Rightarrow \quad x^* A y^* = v_r$$

$$-x^* A y^* \leq -\min_{y \in \Delta([n])} x^* A y = -\max_{x \in \Delta([m])} x A y^* \leq -\min_{y \in \Delta([n])} \max_{x \in \Delta([m])} x A y = v_c$$

$$-x^* A y^* \leq v_c \quad , \quad -x^* A y^* \geq v_c \quad \Rightarrow \quad -x^* A y^* = -v_c .$$

$\blacksquare$

		A	B
$x_A \to$	A	1, −1	−1, 1
$x_B \to$	B	−1, −1	1, −1

Security level of row player.

$$\max_{\substack{x_A, x_B : \\ x_A + x_B = 1, \\ x_A, x_B \geq 0}} \boxed{\min\{x_A - x_B,\ x_B - x_A\}}$$

Solution $x_A = \frac{1}{2}$, $x_B = \frac{1}{2}$

$\left(A : \frac{1}{2}, B : \frac{1}{2}\right)$ value guaranteeing strategy for the row player.

Similarly, $\left(A : \frac{1}{2}, B : \frac{1}{2}\right)$ column player.

$\left\{ \left(A:\frac{1}{2}, B:\frac{1}{2}\right), \left(A:\frac{1}{2}, B:\frac{1}{2}\right) \right\}$ is an MSNE.

<u>Corollary:</u> $A \in \mathbb{R}^{m \times n}$. $x^* \in \Delta([m])$, $y^* \in \Delta([n])$. Then, (x^*, y^*) is an MSNE if and only if x^* and y^* are value guaranteeing strategies for the row and column player respectively.

Q: Give an example of a matrix game with 10 MSNEs.

		Rock	Paper	Scissor
x_R	→ Rock	0,0	−1, 1	1, −1
x_P	→ Paper	1, −1	0,0	−1, 1
x_S	→ Scissor	−1, 1	1, −1	0,0

$$\max_{x_R, x_P, x_S} \boxed{\min\{x_P - x_S,\ x_S - x_R,\ x_R - x_P\}}$$

$$\boxed{\begin{array}{l} x_R + x_P + x_S = 1 \\ x_R, x_P, x_S \geq 0 \end{array}}$$

maximize if

$x_P = x_S = x_R = \frac{1}{3}$

$\{(R:\frac{1}{3}, P:\frac{1}{3}, S:\frac{1}{3}), (R:\frac{1}{3}, P:\frac{1}{3}, S:\frac{1}{3})\}$ unique MSNE for the rock-paper-scissor game.