Lecture 1.1 : Introduction

Game theory : "study" of "games".

<u>Game</u> : any "system" involving multiple "self-interested/selfish", "intelligent" players/ agents.

<u>Ex</u>: (Grading game)

players: students of algorithmic game theory course

Action set of each player: $\{0, 1, \ldots, 100\}$

Outcome : grade of each students. grades will be ralative

Ex: (Prisoner's dilemma)

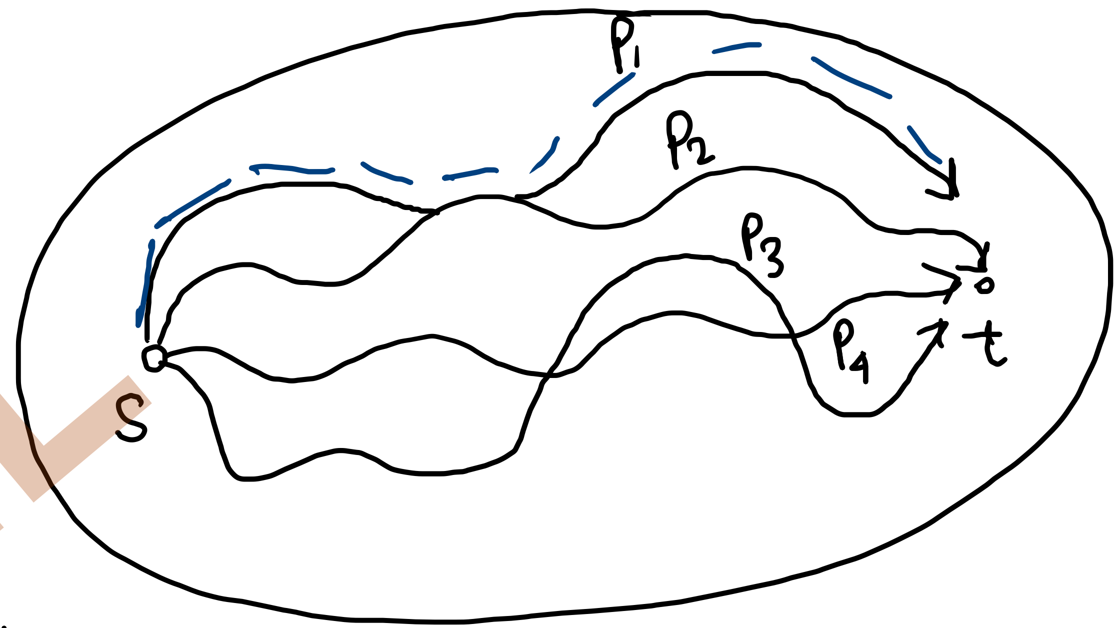
players: 2 people

Action/strategy set: { confess(c), not confess(nc) }

player 2 \ player 1	c	nc
c	-5, -5	-10, -1
nc	-1, -10	-2, -2

Ex: (Congestion game)

players: n commuters wanting to go from s to t.



Strategy set: set of all s-t paths.

Outcome: time required to traverse an edge. This is proportional to the number of players using that edge.

Strategic form game / Normal form game.

Def^n: A normal form game Γ is defined as a tuple $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$

- N is the set of players
- S_i is the strategy set of player $i \in N$
- $u_i : \times_{i\in N} S_i \longrightarrow \mathbb{R}$

For prisoner's dilemma:

u_1, u_2:

2 \ 1	c	nc
c	$-5, -5$	$-10, -1$
nc	$-1, -10$	$-2, -2$

Two important questions

(1) as a player, how should one play?

(2) can we predict the outcome of the game?

Equivalently, can we predict the $\underbrace{\text{strategy profile}}_{(s_i)_{i \in N} \in \times_{i \in N} S_i}$ played by the players?

Lecture 1.2

Big Assumptions of Game Theory

Utility: Each player has a utility function and the only objective of the players is to maximize his/her utility.

Rationality / selfishness: players only wants to maximize his/her utility.

Intelligence: players have infinite computational power.

<u>Common Knowledge</u>: The game, i.e. the set of players, the strategy set of each player, and the utility function of each player is "<u>common knowledge</u>"

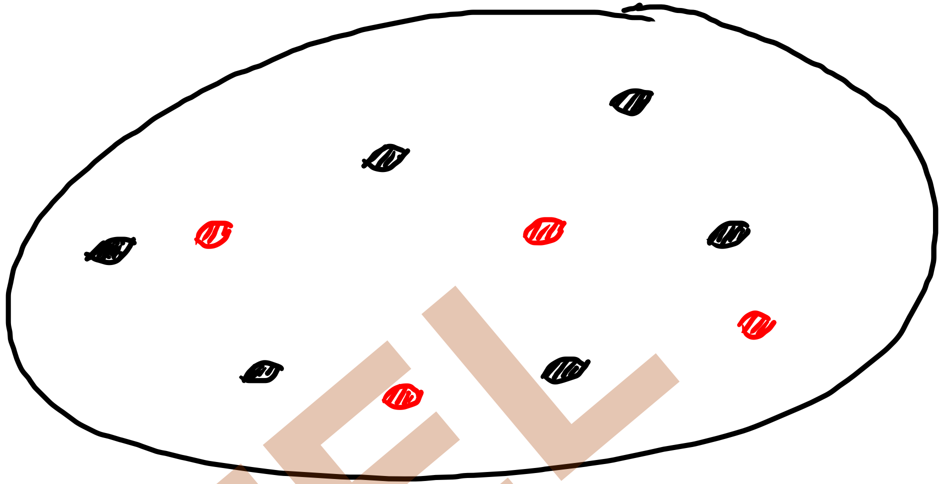
Some information I is a common knowledge of the players means each player knows I, each player knows that each player knows I, ... ("each player knows that")^{i} each player knows I.

Puzzle:

50 black-eyed people

50 red-eyed people.

no reflecting surface.



"there is at least one red-eyed people in the island".

Solution: On the 50-th day, all 50 red-eyed people will suicide.

Ex: (Battle of Sexes)

player 2 (wife)

(Coordination games)

player 1 (husband)

	A	B
A	2, 1	0, 0
B	0, 0	1, 2

Ex:

	A	B
A	10, 10	0, 0
B	0, 0	1, 1

"Anti-cordination game" / Zero sum game / Strictly competitive game

Ex: Matching Pennies

	A	B
A	1, -1	-1, +1
B	-1, +1	1, -1

Sum of utilities of the players in each strategy profile is zero.

Ex: Rock-paper-Scissors

	Rock	Paper	Scissor
Rock	0, 0	-1, 1	1, -1
paper	1, -1	0, 0	-1, 1
Scissor	-1, 1	1, -1	0, 0

Lecture 1.3

Ex: Tragedy of commons:

- $N = \{1, 2, \ldots, n\}$
- $S_i = \{0, 1\}, \; \forall i \in N$
- Utility: $u_i(s_1, \ldots, s_i, \ldots, s_n) = u_i(s_i, s_{-i})$

$$= s_i - \left[\frac{5(s_1 + s_2 + \ldots + s_n)}{n}\right]$$

$$(s_1, \ldots, s_i, \ldots, s_n) \equiv (s_i, s_{-i})$$

Ex: (Auction)

- Players : $N = \{1, 2, \dots, n\}$ (n sellers)
- Strategy set : $S_i = \mathbb{R}_{\geq 0}$, $\forall i \in N$
- Valuation of the item to player i : $v_i \in \mathbb{R}_{\geq 0}$, $\forall i \in N$
- Allocation function. $a : \times_{i \in N} S_i \longrightarrow \mathbb{R}^N$ $\qquad a_i = \begin{cases} 1 & \text{if } i \text{ "wins"} \\ 0 & \text{o/w.} \end{cases}$

  $(s_1, \dots, s_n) \longmapsto (a_1, \dots, a_n)$
- Payment $p : \times_{i \in N} S_i \longrightarrow \mathbb{R}^N$ $\qquad p_i$ = money received by player i.

  $(s_1, \dots, s_n) \longmapsto (p_1, \dots, p_n)$

First price payment rule:

$$p_i = \begin{cases} s_i & \text{if } a_i = 1 \\ 0 & \text{o/w} \end{cases}$$

Second price payment rule:

$$p_i = \begin{cases} \min\limits_{\substack{j \neq i \\ j \in N}} s_j & \text{if } a_i = 1 \\ 0 & \text{o/w.} \end{cases}$$

Utility function:

$$u_i(s_i, s_{-i}) = a_i(p_i - v_i)$$

Dominant Strategy Equilibrium

	c	nc
c	-5, -5	-1, -10
nc	-10, -1	-2, -2

irrespective of what other player players, playing "c" is strictly always better. "c" is called a strongly dominant strategy.

Def^n: In a normal form game $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$, a strategy $s_i^* \in S_i$ is called a strongly dominant strategy for player i if

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \left(= \underset{\substack{j\in N \\ j\neq i}}{\times} S_j\right), \forall s_i \in S_i, s_i \neq s_i^*$$

Strongly dominant strategy equilibrium (SDSE)

Defn.: Given $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ a strategy profile $(s_i^*)_{i \in N} \in S \left(:= \underset{i \in N}{\times} S_i\right)$ is called a strongly dominant strategy equilibrium if each s_i^* is a strongly dominant strategy for player i.

Ex: (C, C) is a SDSE for the prisoner's dilemma game.

Weakly Dominant Strategy

Definition: Given $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$ a strategy $s_i^* \in S_i$ is called a weekly dominant strategy if

(i) $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \left(:= \bigtimes_{j\in N, j\neq i} S_j\right)$
$\forall s_i \in S_i$

(ii) $\forall s_i \in S_i \setminus \{s_i^*\}, \exists s_{-i} \in S_{-i}$ such that

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$$

A strategy profile $(s_i^*)_{i\in N}$ is called a weakly dominant strategy equilibrium (WDSE) if s_i^* is a weakly dominant strategy for player i, for all $i\in N$

Very Weakly Dominant Strategy

A strategy $s_i^* \in S_i$ is called a very weakly dominant strategy if

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}, \ \forall s_i \in S_i$$

A strategy profile $(s_i^*)_{i \in N}$ is called a very weakly dominant strategy equilibrium (VWDSE) if each s_i^* is a very weakly dominant strategy.

Theorem: Bidding valuations is a weakly dominant strategy equilibrium.

Proof: To prove the result, we need to show that bidding valuation is a weakly dominant strategy for player i, $i \in N$. Let v_i be the valuation of player i. Let $s_{-i} \in S_{-i}$ be any strategy profile of other players. $\theta_i = \min_{j \in N, j \neq i} s_j$

Case I: Player i wins the auction. $\Rightarrow s_i^* \leq \theta_i$, $s_i^* = v_i \leq \theta_i$

$$u_i(s_i^*, s_{-i}) = \theta_i - v_i \geq 0 ,$$

$$u_i(s_i', s_{-i}) \leq \theta_i - v_i \leq u_i(s_i^*, s_{-i})$$

Case II : (Player i does not win)

$u_i(s_i^*, s_{-i}) = 0$, $\Rightarrow$ $\boxed{v_i \geq \theta_i}$

$s_i' \in S_i$ (any) if player i loses in (s_i', s_{-i}), then,

$u_i(s_i', s_{-i}) = 0 = u_i(s_i^*, s_{-i})$

if player i wins in (s_i', s_{-i}), then

$u_i(s_i', s_{-i}) = \theta_i - v_i \leq 0 = u_i(s_i^*, s_{-i})$

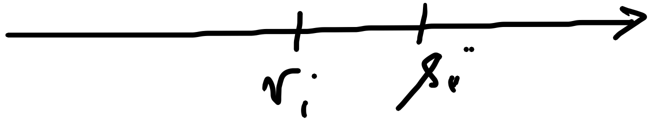
Hence bidding valuation maximizes utility of player i.

Let $s_i \in S_i$, $s_i \neq s_i^*$, need to show, $\exists\, s_{-i}$ s.t.

$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$

If $\underline{s_i > v_i}$

Let $s_j = \frac{v+s_i}{2} \quad \forall j \in N, j \neq i$

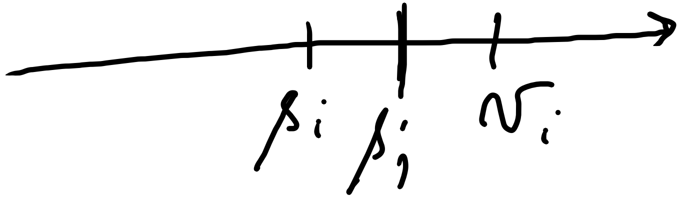


$$u_i(s_i, s_{-i}) = 0 ,$$

$$u_i(s_i^*, s_{-i}) = \frac{v+s_i}{2} - v_i = \frac{s_i - v_i}{2} > 0 = u_i(s_i, s_{-i})$$

If $\boxed{s_i < v_i}$

Let $s_j = \frac{v+s_i}{2} \quad \forall j \in N, j \neq i$



$$u_i(s_i, s_{-i}) = \frac{v+s_i}{2} - v_i = \frac{s_i - v_i}{2} < 0 = u_i(s_i^*, s_{-i}).$$

Nash Equilibrium

Definition: Game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ a strategy profile $(s_i^*)_{i \in N} \in S$ is called a pure strategy Nash equilibrium (PSNE) if

$\forall i \in N,$

$$u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i', s_{-i}^*) \quad \forall s_i' \in S_i$$

$(A, A), (B, B)$ are PSNEs for battle of sexes and coordination games.

Observation: (i) Every SDSE is a WDSE
(ii) Every WDSE is a VWDSE
(iii) Every VWDSE is a PSNE

Matching pennies, rock-paper-scissor games does not have any PSNE.

Mixed strategy: a probability distribution over the strategy set of a player. $\sigma_i \in \Delta(S_i),\ i \in N$

$$u_i\left((\sigma_i)_{i\in N}\right) = \sum_{(s_1,\ldots,s_n)\in S} \sigma_1(s_1)\cdots\sigma_n(s_n)\cdot u_i(s_1,\ldots,s_n)$$

Mixed Strategy Nash Equilibrium (MSNE)

Definition: Game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ a mixed strategy profile $(\sigma_i^*)_{i \in N}$ is called an MSNE if

$\forall i \in N,$

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Delta(S_i)$$

infinitely many condition

Nash Theorem: Every finite game has at least one MSNE.

Characterization of MSNE (Indifference Principle)

Theorem: Given $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$ a mixed strategy profile $(\sigma_i^*)_{i\in N} \in \bigtimes_{i\in N} \Delta(S_i)$ is an MSNE for Γ if and only if the following holds for all the players, $i \in N$.

$\forall s_i \in S_i,\ \sigma_i(s_i) \neq 0 \Rightarrow u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*) \quad \forall s_i' \in S_i$

That is, σ_i puts entire probability mass only on "best-response" strategies against σ_{-i}^*.

<u>Check</u>:- $\left(\{A:\frac{1}{2}, B:\frac{1}{2}\}, \{A:\frac{1}{2}, B:\frac{1}{2}\}\right)$ is the <u>unique</u> MSNE for the matching pennies game.

- $\left(\{R:\frac{1}{3}, P:\frac{1}{3}, S:\frac{1}{3}\}, \{R:\frac{1}{3}, P:\frac{1}{3}, S:\frac{1}{3}\}\right)$ is the <u>unique</u> MSNE for the rock-paper-scissor game.