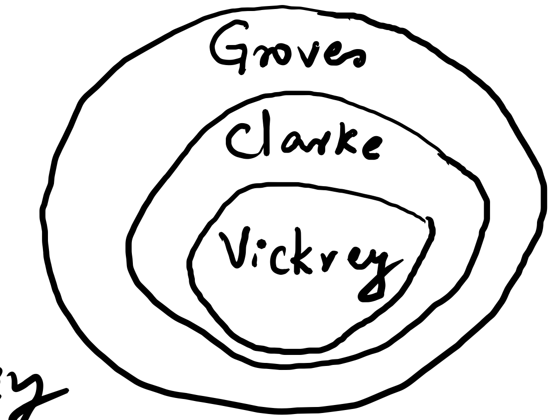
Lecture 10.2

Grove's Mechanism

Vickrey-Clarke-Groves (VCG) mechanism.



Theorem: Let $f(\cdot) = (k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ be a allocatively efficient social choice function. Then $f(\cdot)$ is dominant strategy incentive compatible if the payment functions satisfy the following:

$$t_i(\theta_i, \underline{\theta}_{-i}) = \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*(\theta_i, \theta_{-i})) + h_i(\underline{\theta}_{-i}) \quad , \forall i \in [n]$$

where $h_i : \Theta_{-i} \longrightarrow \mathbb{R}$

<u>Proof:</u> For the sake of finding a contradiction, let us assume that f is <u>not</u> DSIC. Then there exist $(\theta_i, \underline{\theta}_{-i}) \in \Theta$ and $\theta_i' \in \Theta_i$ such that

$$u_i(f(\theta_i, \underline{\theta}_{-i}), \theta_i) < u_i(f(\theta_i', \underline{\theta}_{-i}), \theta_i)$$

$$v_i\left(k^*(\theta_i,\underline{\theta}_{-i}),\theta_i\right) + \underbrace{t_i(\theta_i,\underline{\theta}_{-i})} < v_i\left(k^*(\theta_i',\underline{\theta}_{-i}),\theta_i\right) + \underbrace{t_i(\theta_i',\underline{\theta}_{-i})}$$

$$\Rightarrow \sum_{j=1}^{n} v_j\left(k^*(\theta_i,\underline{\theta}_{-i}),\theta_j\right) + \cancel{h_i(\underline{\theta}_{-i})} < \sum_{j=1}^{n} v_j\left(k^*(\theta_i',\underline{\theta}_{-i}),\theta_j\right) + \cancel{h_i(\underline{\theta}_{-i})}$$

$$\Rightarrow \sum_{j=1}^{n} v_j\left(\underline{\underline{k^*(\theta_i,\underline{\theta}_{-i})}},\theta_j\right) < \sum_{j=1}^{n} v_j\left(\underline{\underline{k^*(\theta_i',\underline{\theta}_{-i})}},\theta_j\right)$$

This contradicts our assumption that $k^*(\cdot)$ is allocatively efficient. $\blacksquare$

Remarks:

- Groves Theorem only provides sufficient condition for a payment rule to make an allocatively efficient allocation rule DSIC.

- For any player i, and any type profile θ_{-i} of other players, the money received by player i depends on its type θ_i only through the allocation function, $k^*(\theta_i, \theta_{-i})$.

If player i changes its type from θ_i to θ_i', then its change in payment

$$t_i(\theta_i, \underline{\theta}_{-i}) - t_i(\theta_i', \underline{\theta}_{-i}) = \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*(\theta_i, \underline{\theta}_{-i}), \theta_j) - \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*(\theta_i', \underline{\theta}_{-i}), \theta_j)$$

The change of payment is the amount of externality imposed by player i on other players.

Clarke (Pivotal) Mechanism

Clarke mechanism is a Grove's mechanism with

$$h_i(\theta_{-i}) = -\sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*_{-i}(\theta_{-i}), \theta_j)$$

For each player $i \in [n]$, let $k^*_{-i}(\cdot)$ be an allocatively efficient rule.

$$t_i(\theta) = \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*(\theta), \theta_j) - \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*_{-i}(\theta_{-i}), \theta_j)$$

Hence, the money received by player i is the total value of an allocatively efficient allocation in the presence of i minus the total value of an allocatively efficient allocation in the absence of i.

$$t_i(\theta) = \underbrace{\sum_{j \in [n]} v_j(k^*(\theta), \theta_j) - \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*_{-i}(\theta_{-i}), \theta_j)}_{\gamma_i \geqslant 0} - v_i(k^*(\theta), \theta_i)$$

γ_i is a "discount" given to player i