Lecture 10.1

Characterization of Ex-post Efficient Social choice Functions in a Quasi-Linear Environment

Theorem: A social choice function in a quasi-linear environment is ex-post efficient if and only if it is allocatively efficient and strictly budget balanced.

<u>**Proof:**</u> (If part) Let $f(\cdot)$ be a social choice function which is allocatively efficient and strictly budget balanced.

Let $\theta \in \bigtimes_{i=1}^{n} \Theta_i$,

$$\sum_{i=1}^{n} u_i(\underbrace{f(\theta)}_{\left(k(\theta),\, (t_i(\theta))_{i \in [n]}\right)}, \theta_i) = \sum_{i=1}^{n} \left(v_i(k(\theta), \theta_i) + t_i(\theta) \right)$$

$$= \sum_{i=1}^{n} v_i(k(\theta), \theta_i) + \underbrace{\boxed{\sum_{i=1}^{n} t_i(\theta)}}_{=0}$$

$$= \sum_{i=1}^{n} v_i(k(\theta), \theta_i)$$

$$\geq \sum_{i=1}^{n} v_i(k, \theta_i) \qquad \forall (k, (t_i')) \in \mathcal{X}$$

$$= \sum_{i=1}^{n} v_i(k, \theta_i) + \underbrace{\sum_{i=1}^{n} t_i'}_{= 0 \ \because \text{SBB.}}$$

$$= \sum_{i=1}^{n} \left(v_i(k, \theta_i) + t_i' \right)$$

$$= \sum_{i=1}^{n} u_i(x, \theta_i), \quad \text{where } x = (k, (t_i')_{i \in [n]})$$

$$\sum_{i=1}^{n} u_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} u_i(x, \theta_i) \quad \forall\ x \in \mathcal{X}.$$

Hence $f(\cdot)$ is ex-post efficient.

(Only if part) Let f be an ex-post efficient social choice function.

$$f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$$

We will first prove that $k(\cdot)$ is allocatively efficient. For the sake of finding a contradiction, suppose $k(\cdot)$ is <u>not</u> allocatively efficient. Then there exists a type profile $\theta \in \Theta$ and an allocation $k' \in K$ such that

$$\sum_{i=1}^{n} v_i(k(\theta), \theta_i) < \sum_{i=1}^{n} v_i(k', \theta_i)$$

There exists a player $j \in [n]$ such that

$$v_j(k(\theta), \theta_j) < v_j(k', \theta_j).$$ Let us define

$$\epsilon = \sum_{i=1}^{n} v_i(k', \theta_i) - \sum_{i=1}^{n} v_i(k(\theta), \theta_i) > 0$$

Let us consider the following outcome,

$$x = \Big(k', \Big(t_i = t_i(\theta) + \boxed{v_i(k(\theta), \theta_i) - v_i(k', \theta_i)}\Big)_{i \in [n], i \neq j},$$
$$t_j = t_j(\theta) + \boxed{v_j(k(\theta), \theta_j) - v_j(k', \theta_j)} + \epsilon\Big)$$

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} t_i(\theta) + \underbrace{\sum_{i=1}^{n} v_i(k(\theta), \theta_i) - \sum_{i=1}^{n} v_i(k', \theta_i) + \epsilon}_{=0}$$

$$= \sum_{i=1}^{n} t_i(\theta)$$

$$\leq 0$$

Hence $x \in \mathcal{X}$.

$$u_i(x, \theta_i) = u_i(f(\theta), \theta_i] \quad \forall\, i \in [n] \setminus \{j\}$$

$$u_j(x, \theta_j) > u_j(f(\theta), \theta_j).$$

Hence $k(\cdot)$ must be allocatively efficient.

Suppose $t_1(\cdot), \ldots, t_n(\cdot)$ is <u>not</u> strongly budget balanced.

Then, there exists $\theta \in \bigtimes_{i=1}^{n} \Theta_i$ such that

$$\sum_{i=1}^{n} t_i(\theta) < 0$$

Consider an outcome x defined as follows.

$$x = \left(k(\theta),\ t_1 = t_1(\theta) - \sum_{i=1}^{n} t_i(\theta),\ \left(t_i = t_i(\theta)\right)_{i \neq 1} \right)$$

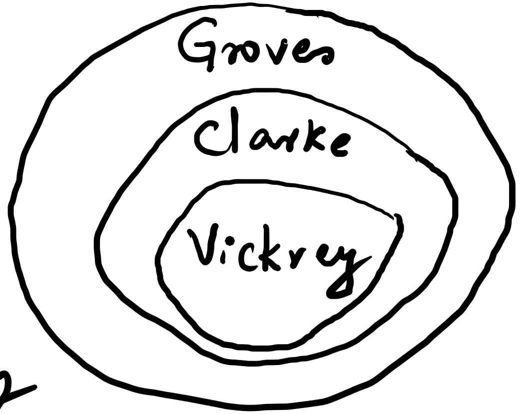
$$u_1(x, \theta_1) > u_1(f(\theta), \theta_1), \quad u_i(x, \theta_i) = u_i(f(\theta), \theta_i)$$

Hence $f(\cdot)$ must be strictly budget balanced.

Lecture 10.2

Grove's Mechanism

Vickrey-Clarke-Groves (VCG) mechanism.



Theorem: Let $f(\cdot) = (k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ be a allocatively efficient social choice function. Then $f(\cdot)$ is dominant strategy incentive compatible if the payment functions satisfy the following:

$$t_i(\theta_i, \underline{\theta}_{-i}) = \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*(\theta_i, \theta_{-i})) + h_i(\underline{\theta}_{-i}), \quad \forall i \in [n]$$

where $h_i : \Theta_{-i} \longrightarrow \mathbb{R}$

<u>Proof:</u> For the sake of finding a contradiction, let us assume that f is <u>not</u> DSIC. Then there exist $(\theta_i, \underline{\theta}_{-i}) \in \Theta$ and $\theta_i' \in \Theta_i$ such that

$$u_i(f(\theta_i, \underline{\theta}_{-i}), \theta_i) < u_i(f(\theta_i', \underline{\theta}_{-i}), \theta_i)$$

$$v_i\left(k^*(\theta_i,\theta_{-i}),\theta_i\right) + \underbrace{t_i(\theta_i,\theta_{-i})} < v_i\left(k^*(\theta_i',\theta_{-i}),\theta_i\right) + \underbrace{t_i(\theta_i',\theta_{-i})}$$

$$\Rightarrow \sum_{j=1}^{n} v_j\left(k^*(\theta_i,\theta_{-i}),\theta_j\right) + \cancel{h_i(\theta_{-i})} < \sum_{j=1}^{n} v_j\left(k^*(\theta_i',\theta_{-i}),\theta_j\right) + \cancel{h_i(\theta_{-i})}$$

$$\Rightarrow \sum_{j=1}^{n} v_j\left(\underline{\underline{k^*(\theta_i,\theta_{-i})}},\theta_j\right) < \sum_{j=1}^{n} v_j\left(\underline{\underline{k^*(\theta_i',\theta_{-i})}},\theta_j\right)$$

This contradicts our assumption that $k^*(\cdot)$ is allocatively efficient. $\blacksquare$

Remarks:

- Groves Theorem only provides sufficient condition for a payment rule to make an allocatively efficient allocation rule DSIC.

- For any player i, and any type profile θ_{-i} of other players, the money received by player i depends on its type θ_i only through the allocation function, $k^*(\theta_i, \theta_{-i})$.

If player i changes its type from θ_i to θ_i', then its change in payment

$$t_i(\theta_i, \underline{\theta}_{-i}) - t_i(\theta_i', \underline{\theta}_{-i}) = \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*(\theta_i, \underline{\theta}_{-i}), \theta_j) - \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*(\theta_i', \underline{\theta}_{-i}), \theta_j)$$

The change of payment is the amount of externality imposed by player i on other players.

Clarke (Pivotal) Mechanism

Clarke mechanism is a Groves mechanism with

$$h_i(\theta_{-i}) = -\sum_{\substack{j \in [n] \\ j \neq i}} v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)$$

For each player $i \in [n]$, let $k^*_{-i}(\cdot)$ be an allocatively efficient rule.

$$t_i(\theta) = \sum_{\substack{j \in [n] \\ j \neq i}} v_j\left(k^*(\theta), \theta_j\right) - \sum_{\substack{j \in [n] \\ j \neq i}} v_j\left(k^*_{-i}(\theta_{-i}), \theta_j\right)$$

Hence, the money received by player i is the total value of an allocatively efficient allocation in the presence of i minus the total value of an allocatively efficient allocation in the absence of i.

$$t_i(\theta) = \underbrace{\sum_{j \in [n]} v_j(k^*(\theta), \theta_j) - \sum_{\substack{j \in [n] \\ j \neq i}} v_j(k^*_{-i}(\theta_{-i}), \theta_j)}_{\gamma_i \geq 0} - v_i(k^*(\theta), \theta_i)$$

γ_i is a "discount" given to player i

Lecture 10.3

Examples of VCG Mechanism

Example 1: (Vickrey auction for selling Multiple Identical Objects)

- 3 identical items with one seller
- 5 buyers each wanting to buy one item.
- We use any allocatively efficient allocation rule.

Set of allocations $(\mathcal{X}) = \{(1,1,1,0,0), (1,0,1,1,0), \ldots\}$

allocation rule: choose three buyers having highest valuation

We will use Clarke's payment rule. Since the allocation rule is allocatively efficient, we know due to <u>Grove's Theorem</u> that the resulting mechanism is dominant strategy incentive compatible. This allows us to assume wlog that the buyers report their true valuation of the item to the mechanism designer.

- The valuations of the item to the players are 20, 15, 12, 10, 8.

The allocation chosen : $(1,1,1,0,0)$

The valuation of $(1,1,1,0,0)$ to player 1 is 20

〃 2 is 15

〃 3 is 12

〃 4 is 0

〃 5 is 0

Sum of valuations of $(1,1,1,0,0)$ is $20+15+12 = 47$.

Payments:

Payment received by player 1 $= (15+12) - (15+12+10) = -10$

——— " ——— 2 $= (20+12) - (20+12+10) = -10$

——— " ——— 3 $= (20+15) - (20+15+10) = -10$

——— " ——— 4 $= (20+15+10) - (20+15+10) = 0$

——— " ——— 5 $= (20+15+10) - (20+15+10) = 0$

Vickrey discount to player 1 $= 20 - 10 = 10$

—— " —— $2 = 15 - 10 = 5$

—— " —— $3 = 12 - 10 = 2$

—— " —— $4 = 0 - 0 = 0$

—— " —— $5 = 0 - 0 = 0$.

Example 2: (Combinatorial Auction)

- One seller having two items, say A and B.
- Three buyers having the following valuations:

	{A}	{B}	{A, B}
Player 1	*	*	12
Player2	5	*	*
player3	*	4	*

Allocatively efficient allocation : {A, B} to player 1.

Payment received by player 1 $= (0+0) - (5+4) = -9$

2 $= (12+0) - (12+0) = 0$

3 $= (12+0) - (12+0) = 0.$

Vickrey discount to player 1 $= 12 - 9 = 3$.

———— 〃 ———— $2 = 0 - 0 = 0$

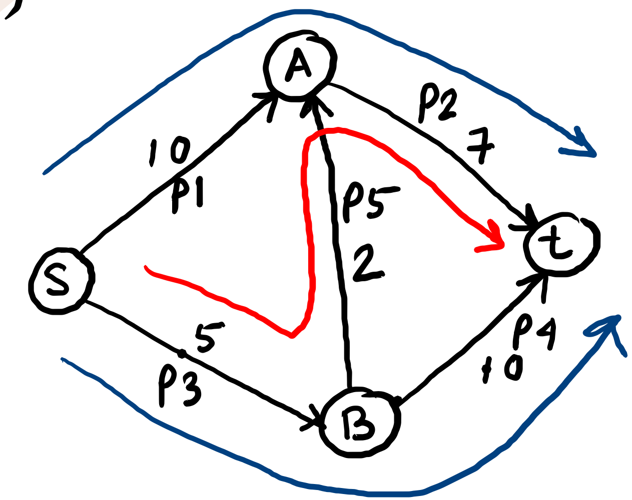
———— 〃 ———— $3 = 0 - 0 = 0$.

Examples of VCG Mechanism

Lecture 10.4

Example 3: (Strategic Network Formation)

Each link is owned by a strategic player. The delay/cost of each link is the private type of the player.



Set of allocation : $\{(1,1,0,0,0), (0,0,1,1,0), (0,1,1,0,1)\}$

$(1,1,0,0,0) \to -17$, $(0,0,1,1,0) \to -15$, $(0,1,1,0,1) \to -14$

The allocatively efficient allocation : $S \to B \to A \to t$ i.e.

$(0, 1, 1, 0, 1)$.

Payment received by agent 1 $= (-14) - (-14) = 0$

—— " —— 2 $= (-5-2) - (-5-10) = 8$

—— " —— 3 $= (-2-7) - (-10-7) = 8$

—— " —— 4 $= (-14) - (-14) = 0$

—— " —— 5 $= (-5-7) - (-5-10) = 3$.

Vickrey discount to player 1 is $0 - 0 = 0$

2 is $(-7) + 8 = 1$

3 is $(-5) + 8 = 3$

4 is $0 - 0 = 0$

5 is $(-2) + 3 = 1$

Weighted VCG

Affine Maximizer: An allocation rule $k : \times_{i=1}^{n} \Theta_i \to \mathcal{K}$ is called an affine maximizer if there exists $\mathcal{K}' \subseteq \mathcal{K}$ and $w_1, \ldots, w_n \in \mathbb{R}$ and $c_{k'} \in \mathbb{R}$ $\forall\, k' \in \mathcal{K}'$ such that, for every $\theta \in \Theta$, we have the following.

$$k(\theta) \in \underset{k' \in \mathcal{K}'}{\operatorname{argmax}} \left[c_{k'} + \sum_{i=1}^{n} w_i v_i(k', \theta) \right]$$

An affine maximizer with $c_{k'} = 0 \ \forall k' \in \mathcal{K}'$ and $w_1, \ldots, w_n = 1$ is an allocatively efficient rule.

Groves Payment for Affine Maximizer Let $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ be a social choice function where $k(\cdot)$ is an affine maximizer with parameters $w_1, \ldots, w_n, \mathcal{K}', \{c_{k'}\}_{k' \in \mathcal{K}'}$.

Then $f(\cdot)$ is dominant strategy incentive compatible if the payment rules satisfy the following.

$\forall\, i \in [n]$,

$$t_i(\theta_i, \underline{\theta}_{-i}) = \sum_{\substack{j \in [n] \\ j \neq i}} \left[\frac{w_j}{w_i} v_j(k^*(\theta), \theta_j) + \frac{c_{k^*(\theta)}}{w_i} \right] + h_i(\underline{\theta}_{-i}) .$$

for any $h_i : \Theta_{-i} \longrightarrow \mathbb{R}$.

Lecture 10.5

Theorem: $f(\cdot) = \left(k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot) \right)$.

$k^*(\cdot)$ is an affine maximizer.

$$\forall i \in [n], \quad t_i(\theta) = \sum_{\substack{j \in [n] \\ j \neq i}} \left[\frac{w_j v_j(k^*(\theta), \theta_j)}{w_i} + \frac{c_{k^*(\theta)}}{w_i} \right] + h_i(\theta_{-i}),$$

$h_i : \Theta_{-i} \longrightarrow \mathbb{R}$.

Proof: Suppose $f(\cdot)$ is not DSIC.

Then there exist $i \in [n]$, $\theta_i \in \Theta_i$, $\underline{\theta}_{-i} \in \underline{\Theta}_{-i}$, $\theta_i' \in \Theta_i'$ such that

$$u_i\left(f(\theta_i', \underline{\theta}_{-i}), \theta_i\right) > u_i\left(f(\theta_i, \underline{\theta}_{-i}), \theta_i\right)$$

$$\Rightarrow v_i\left(k^*(\theta_i', \underline{\theta}_{-i}), \theta_i\right) + p_i(\theta_i', \underline{\theta}_{-i}) > v_i\left(k^*(\theta_i, \underline{\theta}_{-i}), \theta_i\right) + p_i(\theta_i, \underline{\theta}_{-i})$$

$$\Rightarrow v_i\left(k^*(\theta_i', \underline{\theta}_{-i}), \theta_i\right) + \sum_{\substack{j \in [n] \\ j \neq i}} \left[\frac{w_j}{w_i} v_j\left(k^*(\theta_i', \underline{\theta}_{-i}), \theta_j\right) + \frac{c_{k^*(\theta_i', \underline{\theta}_{-i})}}{w_i}\right] >$$

$$v_i\left(k^*(\theta_i, \underline{\theta}_{-i}), \theta_i\right) + \sum_{\substack{j \in [n] \\ j \neq i}} \left[\frac{w_j}{w_i} v_j\left(k^*(\theta_i, \underline{\theta}_{-i}), \theta_j\right) + \frac{c_{k^*(\theta_i, \underline{\theta}_{-i})}}{w_i}\right]$$

$$\Rightarrow \sum_{j\in[n]} w_j v_j\big(\underline{k^*(\theta_i', \theta_{-i})}, \theta_j\big) + C_{k^*}(\theta_i', \theta_{-i}) > \sum_{j\in[n]} w_j v_j\big(\underline{k^*(\theta_i, \theta_{-i})}, \theta_j\big) + C_{k^*}(\theta_i, \theta_{-i})$$

$\Rightarrow$ In the type profile (θ_i, θ_{-i}), the allocation rule $k^*(\cdot)$ does <u>not</u> satisfy the condition of affine maximizer

$\Rightarrow$ This contradicts our assumption that $k^*(\cdot)$ is an affine maximizer. $\square$

<u>Roberts' Theorem:</u> If $|\mathcal{K}| \geq 3$, the allocation rule $k^*(\cdot)$ is an onto function, the valuations are arbitrary, and the allocation rule $k^*(\cdot)$ is implementable in VWDSE using some payment rule, then $k^*(\cdot)$ is an affine maximizer.

Characterization of DSIC Mechanisms

A social choice function $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ is DSIC if and only if the following conditions hold for every $i \in [n]$ and $\underline{\theta}_i \in \Theta_i$

(i) The payment $t_i(\theta)$ depends on θ_i only via $k(\theta_i, \underline{\theta}_{-i})$. That is, for $\theta_i, \theta_i' \in \Theta_i$ and $\underline{\theta}_{-i} \in \Theta_{-i}$ such that $k(\theta_i, \underline{\theta}_{-i}) = k(\theta_i', \underline{\theta}_{-i})$, then $t_i(\theta_i, \underline{\theta}_{-i}) = t_i(\theta_i', \underline{\theta}_{-i})$

(ii) The allocation rule simultaneously optimizes for all the players. That is, for every $\theta_i \in \Theta_i$, $\underline{\theta}_{-i} \in \Theta_{-i}$,

$$k(\theta_i, \underline{\theta}_{-i}) \in \underset{k \in K(\cdot, \underline{\theta}_{-i})}{\operatorname{argmax}} \left[v_i(k, \theta_i) + t_i(\theta_i, \underline{\theta}_{-i}) \right]$$

for every $i \in [n]$

(If part)

<u>Proof:</u> Let $\theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$

$$
\begin{aligned}
u_i(\theta_i, \theta_{-i}) &= v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \\
&= v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(k(\theta_i, \theta_{-i}), \theta_{-i}) \\
&\geq v_i(k(\theta_i', \theta_{-i}), \theta_i) + t_i(k(\theta_i', \theta_{-i}), \theta_{-i})
\end{aligned}
$$

(Only if part)

Let $f(\cdot)$ be DSIC.

$i \in [n]$, $\theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ s.t. $k(\theta_i, \theta_{-i}) = k(\theta_i', \theta_{-i})$

Suppose, $t_i(\theta_i, \theta_{-i}) > t_i(\theta_i', \theta_{-i})$

$$\Rightarrow \quad \underbrace{v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i})} > v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i})$$

$$= \underbrace{v_i(k(\theta_i', \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i})}$$

Contradicts DSIC of $f(\cdot)$ at (θ_i', θ_{-i}).

$\exists\ \theta_i, \theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ such that

$$v_i(k(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) < v_i(k(\theta_i', \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i})$$

Contradicts DSIC property of $f(\cdot)$ at (θ_i, θ_{-i}).