Lecture 10.1

Characterization of Ex-post Efficient Social choice Functions in a Quasi-Linear Environment

Theorem: A social choice function in a quasi-linear environment is ex-post efficient if and only if it is allocatively efficient and strictly budget balanced.

Proof: (If part) Let $f(\cdot)$ be a social choice function which is allocatively efficient and strictly budget balanced.

Let $\theta \in \bigtimes_{i=1}^{n} \Theta_i$,

$$\sum_{i=1}^{n} u_i(\underbrace{f(\theta)}_{(k(\theta),\,(t_i(\theta))_{i\in[n]})}, \theta_i) = \sum_{i=1}^{n} \left(v_i(k(\theta), \theta_i) + t_i(\theta) \right)$$

$$= \sum_{i=1}^{n} v_i(k(\theta), \theta_i) + \underbrace{\boxed{\sum_{i=1}^{n} t_i(\theta)}}_{=0}$$

$$= \sum_{i=1}^{n} v_i(k(\theta), \theta_i)$$

$$\geqslant \sum_{i=1}^{n} v_i(k, \theta_i) \qquad \forall (k, (t_i')) \in \mathcal{X}$$

$$= \sum_{i=1}^{n} v_i(k, \theta_i) + \underbrace{\sum_{i=1}^{n} t_i'}_{= 0 \ \because \text{SBB.}}$$

$$= \sum_{i=1}^{n} \left(v_i(k, \theta_i) + t_i' \right)$$

$$= \sum_{i=1}^{n} u_i(x, \theta_i), \quad \text{where } x = (k, (t_i')_{i \in [n]})$$

$$\sum_{i=1}^{n} u_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} u_i(x, \theta_i) \quad \forall x \in \mathcal{X}.$$

Hence $f(\cdot)$ is ex-post efficient.

(Only if part) Let f be an ex-post efficient social choice function.

$$f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$$

We will first prove that $k(\cdot)$ is allocatively efficient. For the sake of finding a contradiction, suppose $k(\cdot)$ is <u>not</u> allocatively efficient. Then there exists a type profile $\theta \in \Theta$ and an allocation $k' \in K$ such that

$$\sum_{i=1}^{n} v_i(k(\theta), \theta_i) < \sum_{i=1}^{n} v_i(k', \theta_i)$$

There exists a player $j \in [n]$ such that

$v_j(k(\theta), \theta_j) < v_j(k', \theta_j)$. Let us define

$$\epsilon = \sum_{i=1}^{n} v_i(k', \theta_i) - \sum_{i=1}^{n} v_i(k(\theta), \theta_i) > 0$$

Let us consider the following outcome.

$$x = \Big(k', \ \big(t_i = t_i(\theta) + \boxed{v_i(k(\theta), \theta_i) - v_i(k', \theta_i)}\big)_{i \in [n],\, i \neq j},$$
$$t_j = t_j(\theta) + \boxed{v_j(k(\theta), \theta_j) - v_j(k', \theta_j)} + \varepsilon \Big)$$

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} t_i(\theta) + \underbrace{\sum_{i=1}^{n} v_i(k(\theta),\theta_i) - \sum_{i=1}^{n} v_i(k',\theta_i) + \epsilon}_{=0}$$

$$= \sum_{i=1}^{n} t_i(\theta)$$

$$\leq 0$$

Hence $x \in \mathcal{X}$.

$$u_i(x,\theta_i) = u_i(f(\theta),\theta_i) \quad \forall\, i \in [n] \setminus \{j\}$$

$$u_j(x,\theta_j) > u_j(f(\theta),\theta_j).$$

Hence $k(\cdot)$ must be allocatively efficient.

Suppose $t_1(\cdot), \ldots, t_n(\cdot)$ is not strongly budget balanced.

Then, there exists $\theta \in \times_{i=1}^{n} \Theta_i$ such that

$$\sum_{i=1}^{n} t_i(\theta) < 0$$

Consider an outcome x defined as follows.

$$x = \left(k(\theta),\ t_1 = t_1(\theta) - \sum_{i=1}^{n} t_i(\theta),\ \left(t_i = t_i(\theta)\right)_{i \neq 1} \right)$$

$u_1(x, \theta_1) > u_1(f(\theta), \theta_1)$, $u_i(x, \theta_i) = u_i(f(\theta), \theta_i)$

Hence $f(\cdot)$ must be strictly budget balanced. ∎