Computing Equilibrium

Lecture 3.5

Computational task:

Input: A finite game $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N} \rangle$

Output: An MSNE

Is it possible to have an MSNE where some probability values are irrational numbers even when input consists of only rational numbers?

YES if the number of players in Γ is at least 3.

If the number of players is 2, then the answer is NO.

$\varepsilon > 0$

ε-MSNE: Suppose all the utility values are in between 0 and 1. Then a mixed strategy profile $(\sigma_i^*)_{i \in N} \in \bigtimes_{i \in N} \Delta(S_i)$ is called an MSNE if unilateral deviation by any player can benefit it by at most ε.

ε-NASH:

Input: A normal form game $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$

Output: An ε-MSNE.

Support Enumeration: Suppose we have only 2 players. Their utility matrices are $A, B \in \mathbb{R}^{m\times n}$. We are looking for an MSNE $(\sigma_1^*, \sigma_2^*) \in \Delta([m]) \times \Delta([n])$.

We guess the supports I and J of σ_1^* and σ_2^* respectively.

That is $\mathcal{I} = \{ i \in [m] \mid \sigma_1^*(i) \neq 0 \} \subseteq 2^{[m]} \setminus \{\emptyset\}$

$\mathcal{J} = \{ j \in [n] \mid \sigma_2^*(j) \neq 0 \} \subseteq 2^{[n]} \setminus \{\emptyset\}$

Let $\sigma_1^* = (x_1, \dots, x_m)$, $\sigma_2^* = (y_1, \dots, y_n)$.

$\forall i \in \mathcal{I}, \quad \sum_{j=1}^{n} a_{ij} \cdot y_j = u,$

$\forall i \in [m] \setminus \mathcal{I}, \quad \sum_{j=1}^{n} a_{ij} y_j \leq u,$

$y_1 + y_2 + \dots + y_n = 1,$

$\forall j \in [n] \quad y_j \geq 0, \quad \forall j \in [n] \setminus \mathcal{J}, \; y_j = 0$

$\forall j \in \mathcal{J}, \quad \sum_{i=1}^{m} b_{ij} \cdot x_i = v,$

$\forall j \in [n] \setminus \mathcal{J}, \quad \sum_{i=1}^{m} b_{ij} x_i \leq v,$

$x_1 + \dots + x_m = 1,$

$x_1, \dots, x_m \geq 0,$

$\forall i \in [m] \setminus \mathcal{I}, \; x_i = 0$

Iterate over all possible $2^m - 1$ non-empty subsets I of $[m]$

and ——"—— $2^n - 1$ ——"—— J of $[n]$,

solve the linear program. If the LP is feasible for some I and J then, $((x_1^*, \ldots, x_m^*), (y_1^*, \ldots, y_n^*))$ is an MSNE where $(x_1^*, \ldots, x_m^*, y_1^*, \ldots, y_n^*, u^*, v^*)$ is a feasible solution.

Proof of correctness follows from indifference principal.

Run time: $O\left(2^m \cdot 2^n \cdot \text{poly(input size)} \right).$ $\blacksquare$