Lecture 9.5

Quasi-Linear Setting / Environment

The outcomes are not arbitrary. In particular, an outcome is a tuple $(k, t_1, \ldots, t_n)$. The first component k is an allocation which belongs to a set $\mathcal{K}$ of set of allocations. t_i is the money received by player i.

The set of outcomes:

$$X = \left\{ (k, t_1, \ldots, t_n) \;\middle|\; k \in \mathcal{K},\ \underbrace{\sum_{i=1}^{n} t_i \leq 0}_{\text{since no external supply of money.}} \right\}$$

The utility function has the following structure.

$$\begin{aligned} u_i(x, \theta_1, \ldots, \theta_n) &= u_i(k, t_1, \ldots, t_n, \theta_1, \ldots, \theta_n) \\ &= u_i(k, t_1, \ldots, t_n, \theta_i) \\ &= v_i(k, \theta_i) + t_i \end{aligned}$$

$v_i(k, \theta_i)$ is the valuation of allocation k to player i when its type is θ_i.

utility = valuation + payment.

A social choice function $f(\theta_1, \dots, \theta_n)$ has the following structure:-

$$f(\theta_1, \dots, \theta_n) = \left(k(\theta_1, \dots, \theta_n), \ \left(t_i(\theta_1, \dots, \theta_n) \right)_{i \in [n]} \right)$$

The social choice function consists of two functions.

(i) Allocation function: $k : \bigtimes_{i=1}^{n} \Theta_i \longrightarrow \mathcal{K}$

(ii) payment functions: $t_j : \bigtimes_{i=1}^{n} \Theta_i \longrightarrow \mathbb{R}$

Q: How does any efficient social choice function look like in a quasi-linear environment?

Allocative Efficiency (AE): An allocation function is called allocatively efficient if

$\forall\ (\theta_1, \ldots, \theta_n) \in \bigtimes_{i=1}^{n} \Theta$, we have

$$k(\theta_1, \ldots, \theta_n) \in \underset{k \in \mathcal{K}}{\operatorname{argmax}} \sum v_i(k, \theta_i)$$

equivalently,

$$\sum_{i=1}^{n} v_i\left(k(\theta_1, \ldots, \theta_n), \theta_i\right) = \max_{k \in \mathcal{K}} \sum_{i=1}^{n} v_i(k, \theta_i)$$

Budget Balanced (BB): The payment functions $(t_i(\cdot))_{i \in [n]}$ are called strongly budget balanced (SBB) if

$$\forall (\theta_1, \ldots, \theta_n) \in \bigtimes_{i=1}^{n} \Theta_i ,$$

$$\sum_{i=1}^{n} t_i(\theta_1, \ldots, \theta_n) = 0$$

The payment functions $(t_i(\cdot))_{i \in [n]}$ are called weakly budget balanced if

$$\forall (\theta_1, \ldots, \theta_n) \in \bigtimes_{i=1}^{n} \Theta_i$$

$$\sum_{i=1}^{n} t_i(\theta_1, \ldots, \theta_n) \leq 0$$

Observation: If we have at least two players, then no social choice function is dictatorial in a quasi-linear environment.

Proof: Let $n (\geq 2)$ be the number of players. If possible, let us assume that there exists a social choice function $f(\cdot)$ where a player, say player d, is a dictator.

Let $\varepsilon > 0$ be any positive real number, θ any type profile. Suppose $f(\theta) = (k(\theta), (t_d(\theta), t_{-d}(\theta)))$

Consider the outcome,

$$x = \left(k(\theta), \left(t_j(\theta) - \epsilon\right)_{j \in [n], j \neq d}, \quad t_d(\theta) + (n-1)\epsilon\right)$$

$$u_d(x, \theta_d) = \underbrace{v_d(k(\theta)) + t_d} + (n-1)\epsilon$$

$$= u_d(f(\theta), \theta_d) + (n-1)\epsilon$$

$$> u_d(f(\theta), \theta_d)$$

This contradicts our assumption that d is the dictator of the social choice function f. $\blacksquare$