

Lecture 2.3

Theorem: $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ any game, and
 $(\sigma_i^*)_{i \in N}$ be an MSNE. Then

$$\forall i \in N, \quad u_i(\sigma_i^*) \geq v_i$$

The above inequalities are all tight for
two-person zero-sum games.

Let A be the utility matrix of the row player.

Then $-A$ is the utility matrix of the column player.

value of the row player,

$$\underline{v} = \max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^m \sigma(i) A_{ij} \rightarrow \text{maximin value}$$

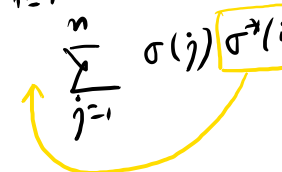
The value of the column player

$$\bar{v} = - \min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^n \sigma(j) A_{ij} \rightarrow \text{minimax value}$$

Lemma:

$$\max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^m \sigma(i) A_{ij} \leq \min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^n \sigma(j) A_{ij}$$

Proof:

$$\begin{aligned} \text{Let } \max_{\sigma \in \Delta([m])} \min_{j \in [n]} \sum_{i=1}^m \sigma(i) A_{ij} &= \min_{j \in [n]} \sum_{i=1}^m \sigma^*(i) A_{ij} \\ &= \min_{\sigma \in \Delta([n])} \sum_{j=1}^n \sum_{i=1}^m \sigma(j) \sigma^*(i) A_{ij} \\ &= \min_{\sigma \in \Delta([n])} \sum_{i=1}^m \sum_{j=1}^n \sigma(j) \sigma^*(i) A_{ij} \end{aligned}$$


$$= \min_{\sigma \in \Delta([n])} \underbrace{\sum_{i=1}^m \sigma^*(i) \left(\sum_{j=1}^n \sigma(j) A_{ij} \right)}_{\text{convex combination}}$$

$$\leq \min_{\sigma \in \Delta([n])} \max_{i \in [m]} \sum_{j=1}^n \sigma(j) A_{ij}$$

Minimax Theorem: $A \in \mathbb{R}^{m \times n}$, there exists $x^* = (x_1^*, \dots, x_m^*) \in \Delta([m])$ and $y^* = (y_1^*, \dots, y_n^*) \in \Delta([n])$ such that

$$\max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y$$

Linear Program of the row player:

LP1 maximize $\min_{j \in [n]} \sum_{i=1}^m A_{ij} x_i$

st: $\sum_{i=1}^m x_i = 1$
 $x_i \geq 0 \quad \forall i \in [m]$

max t
 st. $t \leq \sum_{i=1}^m A_{ij} x_i \quad \forall j \in [n]$
 $\sum_{i=1}^m x_i = 1$
 $x_i \geq 0 \quad \forall i \in [m]$

Linear Program for the Column Player

LP2

$$\text{minimize } \max_{i \in [m]} \sum_{j=1}^n A_{ij} y_j$$

$$\text{s.t.: } \sum_{j=1}^n y_j = 1$$

$$y_j \geq 0 \quad \forall j \in [n]$$

minimize

$$\text{s.t.: } w \geq \sum_{j=1}^n A_{ij} y_j \quad \forall i \in [m]$$

$$\sum_{j=1}^n y_j = 1$$

$$y_j \geq 0 \quad \forall j \in [n].$$

Claim: LP1 and LP2 are duals of each other.

Strong Duality Theorem: Let LP1 and LP2 are two linear programs which are duals of each other. If LP1/LP2 is feasible and bounded, then LP2/LP1 is also feasible and bounded and $\text{opt}(\text{LP1}) = \text{opt}(\text{LP2})$

