Lecture 9.1

Bayesian Game Induced by a Mechanism

$$M = \langle N = [n], \mathcal{X}, (\Theta_i)_{i \in N}, (S_i)_{i \in N}, P \in \Delta(\Theta), \underline{g : \times_{i=1}^{n} S_i \to \mathcal{X}}, u_i : \underline{\mathcal{X} \times \Theta} \to \mathbb{R} \rangle$$

induces a Bayesian game

$$\Gamma_M = \langle N = [n], (\Theta_i)_{i \in N}, (S_i)_{i \in N}, P, U_i : \times_{i=1}^{n} S_i \times \Theta \to \mathbb{R} \rangle$$

$$U_i(\underline{(s_i)_{i \in N}}, \underline{(\theta_i)_{i \in N}}) = u_i(g((s_i)_{i \in N}), (\theta_i)_{i \in N})$$

Implementation of a Social Choice Function

$$f : \Theta \longrightarrow \mathcal{X}$$

Definition: We say that an indirect mechanism $M = ((S_i)_{i \in N}, g(\cdot))$ implements a social choice function $f : \Theta \to \mathcal{X}$ if there exists an "equilibrium" $(s_i^*(\cdot))_{i \in N}$ in the induced Bayesian game Γ_M such that

$$\forall (\theta_1, \ldots, \theta_n) \in \Theta, \quad g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = f(\theta_1, \ldots, \theta_n)$$

If the equilibrium in the above definition is a very weakly dominant strategy equilibrium, then we say that the mechanism M implements the social choice function f in dominant strategy equilibrium.

In this case, the mechanism M is called dominant strategy incentive compatible (DSIC) with f.

If the equilibrium is a pure strategy Bayesian Nash equilibrium, then we say that the mechanism M implements the social choice function f in Bayesian Nash equilibrium. In this case, we call M to be Bayesian Incentive Compatible (BIC) with f.

<u>Examples:</u>

Buying auction, i.e. one buyer and n potential sellers.

(i) $f_{fp} : \Theta_0 \times \dots \times \Theta_n \longrightarrow \mathcal{X}$ $\quad f_{fp}(\theta_0, \dots, \theta_n) = (a_0, \dots, a_n, p_0, \dots, p_n)$

defined as follows:

if $\theta_0 < \min_{i \in [n]} \theta_i$: $a_i = 0$, $p_i = 0$ $\forall$ $i \in \{0, 1, \dots, n\}$

otherwise: $a_0 = 1$, $a_j = -1$ where $j \in \underset{i \in [n]}{\operatorname{argmin}} \{\theta_i\}$

$a_i = 0$ $\forall$ $i \in [n] \setminus \{j\}$.

$p_0 = \theta_j$, $p_j = -\theta_j$, $p_i = 0$ $\forall$ $i \in [n] \setminus \{j\}$.

(2) $f_{sp} : \Theta_0 \times \cdots \times \Theta_n \longrightarrow \mathcal{X}$ $\qquad f_{sp}(\theta_0, \ldots, \theta_n) = (a_0, a_1, \ldots, a_n, p_0, p_1, \ldots, p_n)$.

defined as :

if $\theta_0 < \min_{i \in [n]} \theta_i$: $a_0 = 0, a_i = 0, p_0 = 0, p_i = 0 \quad \forall i \in \{1, \ldots, n\}$

otherwise

$a_0 = 1, \quad a_j = -1 \quad$ where $j \in \underset{i \in [n]}{\operatorname{argmin}} \{\theta_i\}$

$a_i = 0 \quad \forall \ i \in [n] \setminus \{j\}$

$p_0 = \min_{i \in [n] \setminus \{j\}} \theta_i \quad , \quad p_j = -p_0 \, , \quad p_i = 0 \quad \forall i \in [n] \setminus \{j\}$

We have seen that the direct mechanism implements f_{sp} in dominant strategies. However the direct mechanism does not implement the social choice function f_{fp} even in Bayesian Nash equilibrium!