Minmax Theorem: $A \in \mathbb{R}^{m \times n}$. There exist mixed strategies $x^* = (x_1^*, \ldots, x_m^*)$ and $y^* = (y_1^*, \ldots, y_n^*)$ s.t.

$$\max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y.$$

Proof: LP1 and LP2 are duals of each other.

LP1 is clearly feasible.

$$\min_{i,j} A_{ij} \leq \max_{x \in \Delta([m])} x A y^* \leq \max_{i,j} A_{ij}$$

$\mathrm{OPT}(LP1) = \mathrm{OPT}(LP2)$ by strong duality

$$\mathrm{OPT}(LP1) = \min_{j \in [n]} \sum_{i=1}^{m} A_{ij} x_i^* = \max_{j \in [n]} x^* A e_j = \min_{x \in \Delta([m])} \max_{j \in [n]} x A e_j$$

$$\mathrm{OPT}(LP2) = \max_{i \in [m]} \sum_{j=1}^{n} A_{ij} y_j^*$$

$$= \max_{i \in [m]} e_i A y^*$$

$$\max_{x \in \Delta([m])} x A y^* = \max_{i \in [m]} e_i A y^* = \max_{j \in [n]} x^* A e_j = \min_{y \in \Delta([n])} x^* A y$$

$\blacksquare$

<u>Corollary</u>: In every matrix game, there exists an MSNE. The mixed strategies of both the players guarantee their security level.

<u>Proof</u>: Let $x^* = (x_1^*, \dots, x_m^*)$ be a solution to LP1

$y^* = (y_1^*, \dots, y_n^*)$ ——"—— LP2

By minmax Theorem,

$$\underline{\max_{x \in \Delta([m])} x A y^*} = \min_{y \in \Delta([n])} x^* A y$$

<u>claim:</u> (x^*, y^*) is an MSNE.

$$x^* A y^* \leq \max_{x \in \Delta([m])} x A y^* = \min_{y \in \Delta([n])} x^* A y$$

$\Rightarrow$ The column player does not have any incentive to deviate from y^* given the row player plays x^*.

$$x^* A y^* \geq \min_{y \in \Delta([n])} x^* A y = \max_{x \in \Delta([m])} x A y^*$$

$\Rightarrow$ The row player also does not have any incentive to deviate given column player plays y^*.

Hence (x^*, y^*) is an MSNE.

x^* and y^* are solutions of LP1 and LP2 respectively.

An MSNE of a two player zero-sum game can be computed in polynomial time.

$$x^* A y^* \leq \max_{x \in \Delta([m])} x A y^* = \max_{x \in \Delta([m])} \min_{j \in [n]} x A e_j = \underline{v}$$

$\Rightarrow$ utility of row player is at most its value.

$\Rightarrow$ Utility of the row player in (x^*, y^*) is its value.

$$x^* A y^* \geq \min_{y \in \Delta([n])} x^* A y = \min_{y \in \Delta([n])} \max_{i \in [m]} e_i A y = \overline{v}$$

$\Rightarrow$ Utility of the column player in (x^*, y^*) is at most its value.

$\Rightarrow$ —"— exactly its value. $\blacksquare$