Lecture 2.1

<u>Theorem</u> (Indifference Principal) Given $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i \in N}\rangle$, a mixed strategy profile $(\sigma_i^*)_{i\in N} \in \mathop{\times}_{i\in N} \Delta(S_i)$ is an MSNE if and only if

$$\Big(\text{For each } i \in N,\ s_i \in S_i,\ \sigma_i^*(s_i) \neq 0 \Rightarrow u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)\ \forall s_i' \in S_i\Big)$$

Proof: (If part) Suppose $(\sigma_i^*)_{i \in N}$ be a mixed strategy profile which satisfies the equivalent condition.

To show: $(\sigma_i^*)_{i \in N}$ is an MSNE.

Fix any player $i \in N$,

$$u_i(\sigma_i, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i) \prod_{\substack{j \in N, \\ j \neq i}} \sigma_j^*(s_j) \, u_i(s_i, s_{-i})$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i) \, u_i(s_i, \sigma_{-i}^*)$$

A convex combination of $\{u_i(s_i, \sigma_{-i}^*) : s_i \in S_i\}$

Convex combination of a set $\{a_1, \ldots, a_n\}$ of numbers is $\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n$, where $\lambda_1, \ldots, \lambda_n \geq 0$

$\lambda_1 + \ldots + \lambda_n = 1$

Obs: A convex combination $\lambda_1 a_1 + \ldots + \lambda_n a_n$ is maximized

if $\left(\lambda_i \neq 0 \Rightarrow a_i = \max\{a_1, \ldots, a_n\} \quad \forall\, i \in [n] =: \{1, \ldots, n\} \right)$

$$u_i(\sigma_i, \sigma_{-i}^*) \leq \sum_{s_i \in S_i} \sigma_i^*(s_i)\, u_i(s_i, \sigma_{-i}^*)$$

$$= u_i(\sigma_i^*, \sigma_{-i}^*)$$

(Only if part) Suppose $(\sigma_i^*)_{i \in N}$ is an MSNE of Γ.

"Proof by contradiction"

Let us assume $\exists\, i \in N$, $s_i \in S_i$ such that

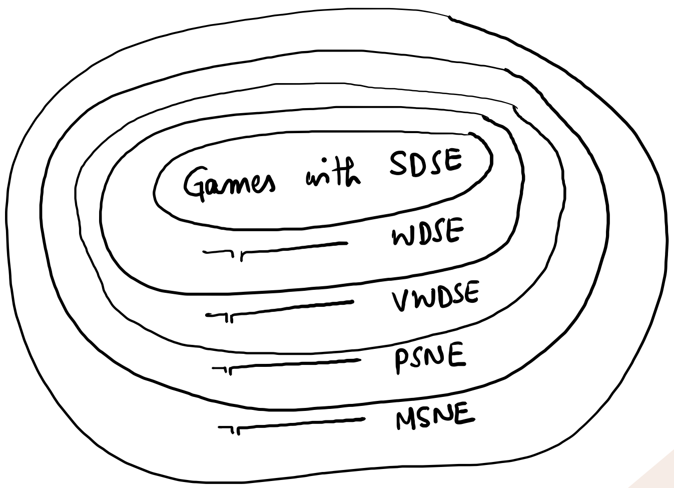
$\sigma_i^*(s_i) \neq 0$ and $u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$ for some $s_i' \in S_i$, $s_i \neq s_i'$, w.l.o.g assume s_i' maximizes $u_i(\cdot, \sigma_{-i}^*)$

Consider another mixed strategy which puts entire probability mass on s_i', which is a pure strategy

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \underbrace{\sum_{s_i \in S_i} \sigma_i^*(s_i'') \, u_i(s_i'', \sigma_{-i}^*)}_{\text{Convex combination of } \{u_i(s_i'', \sigma_{-i}^*) : s_i'' \in S_i\}}$$

$$< u_i(s_i', \sigma_{-i}^*)$$

$\blacksquare$


Games with SDSE
WDSE
VWDSE
PSNE
MSNE