Lecture 6.5

Multiplicative Weight Algorithm

1. $w_0(a) = 1 \quad \forall a \in A$
2. for $t = 1, \dots, T$ {
3. the commited probability distribution is $p_t(a) \propto w_{t-1}(a)$
4. Picks $a \sim p_t$ and receives an utility of $\pi_t(a)$

5. After knowing π_t, update $w_t(a) = w_{t-1}(a) \cdot (1+\varepsilon)^{\pi_t(a)}$ $\forall a \in A$

}

<u>Theorem:</u> Let $|A| = n$. Then the MW algorithm has external regret $O\left(\sqrt{\frac{\log n}{T}}\right)$.

<u>Proof:</u> $\Gamma_t := \sum_{a \in A} w_t(a)$ the sum of the weight values in the t-th iteration.

Expected utility: $\displaystyle\sum_{t=1}^{T} \sum_{a \in A} p_t(a) \cdot \pi_t(a) = \sum_{t=1}^{T} \boxed{\sum_{a \in A} \frac{w_{t-1}(a)}{\Gamma_{t-1}(a)} \cdot \pi_t(a).}$

Benchmark (OPT): $\displaystyle\max_{a \in A} \sum_{t=1}^{T} \pi_t(a) = \sum_{t=1}^{T} \pi_t(a^*)$

$$\begin{aligned}
\Gamma_T &= \sum_{a \in A} w_t(a) \\
&\geq w_t(a^*) \\
&= w_{t-1}(a^*) \cdot (1+\varepsilon)^{\pi_t(a^*)} \\
&\vdots \\
&= \underbrace{w_0(a^*)}_{1} \cdot \prod_{t=1}^{T} (1+\varepsilon)^{\pi_t(a^*)}
\end{aligned}$$

$$\Gamma_T \geq \prod_{t=1}^{T} (1+\varepsilon)^{\pi_t(a^*)}$$

$$= (1+\varepsilon)^{\sum_{t=1}^{T} \pi_t(a^*)}$$

$$= (1+\varepsilon)^{OPT} \quad \cdots\cdots \Rightarrow \quad \boxed{\Gamma_t \geq (1+\varepsilon)^{OPT}} \quad \text{——— (1)}$$

$$\Gamma_T = \sum_{a \in A} \omega_T(a)$$

$$= \sum_{a \in A} \omega_{T-1}(a) \cdot \underbrace{(1+\varepsilon)^{\pi_t(a)}}$$

$$\leq \sum_{a \in A} \omega_{T-1}(a) \left(1 + \varepsilon\, \pi_t(a)\right)$$

$\left[\because (1+\varepsilon)^x \leq 1 + \varepsilon x \text{ if } x \in [0,1]\right]$

$$= \sum_{a \in A} w_{T-1}(a) + \varepsilon \underbrace{\sum_{a \in A} w_{T-1}(a)\, \pi_t(a)}$$

$$= \Gamma_{T-1} + \varepsilon \cdot \Gamma_{T-1} \underbrace{\boxed{\sum_{a \in A} \frac{w_{T-1}(a)}{\Gamma_{T-1}} \cdot \pi_t(a)}}_{\nu_T}$$

Expected utility in the t-th iteration.

$$= \Gamma_{T-1} \left(1 + \varepsilon \nu_T \right)$$

$$\vdots$$

$$\leq \Gamma_0 \prod_{t=1}^{T} \left(1 + \varepsilon \nu_t \right)$$

$$= n \prod_{t=1}^{T} \left(1 + \varepsilon \nu_t \right)$$

$$T_T \leq n \cdot \prod_{t=1}^{T} (1 + \varepsilon \nu_t) \quad \text{——— (2)}$$

From (1) and (2)

$$(1+\varepsilon)^{OPT} \leq n \cdot \prod_{t=1}^{T} (1 + \varepsilon \nu_t)$$

$$\Rightarrow \quad OPT \ln(1+\varepsilon) \leq \ln n + \sum_{t=1}^{T} \ln(1 + \varepsilon \nu_t)$$

$$\Rightarrow \quad OPT(\varepsilon - \varepsilon^2) \leq \ln n + \sum_{t=1}^{T} \varepsilon \nu_t$$

$$\Rightarrow \quad OPT(1-\varepsilon) \leq \frac{\ln n}{\varepsilon} + \boxed{\sum_{t=1}^{T} \nu_t}$$

Expected utility.

$\varepsilon \in [0,1]$

$$\ln(1+\varepsilon) \geq \varepsilon - \varepsilon^2$$

$$\ln(1+\varepsilon) \leq \varepsilon$$

$$\Rightarrow \quad OPT - \sum_{t=1}^{T} \nu_t \leq \varepsilon\, OPT + \frac{\ln n}{\varepsilon}$$

$$\leq \underline{\varepsilon T} + \underline{\frac{\ln n}{\varepsilon}} \qquad [\because OPT \leq T]$$

$$\leq 2\sqrt{T \cdot \ln n} \qquad \left[\text{Put } \varepsilon = \sqrt{\frac{\ln n}{T}}\right]$$

$$\Rightarrow \quad \frac{1}{T}\left(OPT - \sum_{t=1}^{T} \nu_t\right) \leq 2\sqrt{\frac{\ln n}{T}}$$