DATA 5100 Assignment 5

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Ques. 1: $A = \begin{bmatrix} 1 & 4 & 1 \\ -2 & 2 & 1 \\ 3 & 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Solution:

(a) $3\vec{x} + 2\vec{y}$

$$3 \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$$

(b) $A\vec{x}$

$$\underset{3\times3}{\begin{bmatrix} 1 & 4 & 1 \\ -2 & 2 & 1 \\ 3 & 1 & 4 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$$\cancel{\begin{bmatrix} 1+(-2)+3 \\ 8+4+2 \\ 3+3+12 \end{bmatrix}} \quad \begin{bmatrix} 1+8+3 \\ -2+4+3 \\ 3+2+12 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 2 \\ 14 \\ 18 \end{bmatrix}} \quad \begin{bmatrix} 12 \\ 5 \\ 17 \end{bmatrix}$$

(c) AB

$$\begin{bmatrix} 1 & 4 & 1 \\ -2 & 2 & 1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3+4+2 & 1+20+1 & 2+24+2 \\ -6+2+2 & -2+10+1 & -4+12+2 \\ 9+1+8 & 3+5+4 & 6+6+8 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 22 & 28 \\ -2 & 9 & 10 \\ 18 & 12 & 20 \end{bmatrix}$$

(d) $\|\vec{x}\|$

Solving using Euclidean norm formula

$$\|\vec{x}\| = \sqrt{1^2+2^2+3^2}$$

$$= \sqrt{1+4+9}$$

$$= \sqrt{14}$$

$$\|\vec{x}\| \approx 3.741$$

(e) $\|\vec{y}\|$ Again solving using Euclidean norm formula

$$\|\vec{y}\| = \sqrt{0^2+1^2+0^2}$$

$$= \sqrt{1}$$

$$\|\vec{y}\| = 1$$

(f) $\vec{x} \cdot \vec{y}$

Dot product

$\Rightarrow (1)(0) + (2)(1) + (3)(0)$

$\Rightarrow 0 + 2 + 0$

$\Rightarrow 2$

(g) Angle θ between $\vec{x}$ and $\vec{y}$.

Will use the angle between two vectors:

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}$$

$$\cos\theta = \frac{2}{\sqrt{14} \cdot 1} \leftarrow \text{From (f) ans.}$$

$$= \frac{2}{\sqrt{14}}$$

$\cos\theta \approx 0.5345$

To get θ value, inverse $\cos\theta$..

$\theta = \cos^{-1}(0.5345)$

$\theta \approx$ ~~57.69~~ 57.69°

Ques. 2:

Soln: Considering dimensions involved for each matrix.

i) $A + BB^T = C$

$A \to m \times n$, $p \times p$; $B \to p \times q$; $C \to m \times n$, $p \times p$

ii) BB^T

$B^T \to q \times p$

So, $BB^T \to p \times p \to$ As B has p rows and B^T has p columns.

So, BB^T is a square matrix of $p \times p$.

→ A must also be $p \times p$, makes it a square matrix.
→ and C will also be $p \times p$ as it is similar.

(a) A is Square ⟹ True
- A must be a square matrix with dimension $p \times p$.

(b) A and B have same dimension ⟹ False
- A has dimension $p \times p$,
- B has dimension $p \times q$

Considering that there is no statement saying $p = q$.

So, A and B do not necessarily have same dimensions.

(c) A, B and C have the same number of rows ⟹ True
- Since A and C are both $p \times p$, having p rows.
- B has dimension $p \times q$, having p rows.

- So, all three matrices A, B and C have common number of rows

(d) B has more rows than columns ⇒ It is not always True;
as it always on dimensions of B.
· B has dimensions $p \times q$, but there no statement that ~~pxq~~ $p > q$.
· So, B doesn't have more rows than columns
· p could be less than, equal to or greater than q.

An Quick Example :
Matrix Dimensions
- A → 3×3
- B → 3×2
- C → 3×3

(a) For $A + BB^T = C$ to be valid,
$A = BB^T =$ Same dimension $= 3 \times 3$.
∴ A must be a square, So, it is True.

(b) A and B have the same dimensions
Here $A = 3 \times 3$ and $B = 3 \times 2$
So, this statement is False.

(c) A, B and C have the same number of rows.
$A = \underline{3} \times 3$ $B = \underline{3} \times 2$ $C = \underline{3} \times 3$
We can see same no. of rows.
So, this statement is True.

(d) B has more rows than columns.
$B \rightarrow 3 \times 2$ → So, more rows than columns.
If $B \rightarrow 2 \times 3$, this couldn't be true.
So, we can say Statement is not always True.
As it always depends ~~of~~ on dimensions of B.

Soln. 3

Ques. 3 • n currencies labelled as $1, 2, \ldots, n$, eg. USD, JPY, EU

• $n \times n$ exchange matrix R, $R_{ij} \rightarrow$ amt. of currency i that can be bought with 1 unit of currency j.

• Each diagonal element $R_{ii} = 1 \rightarrow$ exchanging a currency for itself doesn't change itself.

~~Diagonal element $R_{ii} = 1$~~

• For $i \neq j$, $R_{ji} R_{ij} < 1$, $\rightarrow$ bidirectional exchanges always lose a small amount due to commission charges.

Vector $\vec{x} \in \mathbb{R}^n$ with contents as $y_1, y_2, \ldots y_n$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$ then $x_j \rightarrow$ amt. of currency j we hold.

$\vec{y} = R\vec{x}$, $\vec{y} \rightarrow$ vector in $\mathbb{R}^n$ with components $y_1, y_2, \ldots, y$

Each component y_i in $\vec{y}$ represents an amount in currency i,

Calculating of y_i : $y_i = \sum_{j=1}^{n} R_{ij} x_j$

This sum means that each y_i is obtained by taking x_j and converting each amount x_j into currency i using the rate R_{ij}.

Qualitative Meaning of y_i :

$y_i \rightarrow$ Total equivalent amount of currency i if you could obtain if to exchange each currency j that can be hold into currency i.

Example : if i = USD, $y_{usd} \rightarrow$ Total amt. of USD

$\vec{y} = R\vec{x}$ → allows to get our total holdings in any currency.
here considers each row i of R.

y_i → expresses portfolio's total value measured in currency i.
- it is valuable as for example: if we want to access portfolio value in USD, we can consider y_{USD}. etc.

Example:

Assume/consider $R = \begin{bmatrix} 1 & 0.009 & 1.2 \\ 110 & 1 & 130 \\ 0.83 & 0.0077 & 1 \end{bmatrix}$

where i → currency you get for 1 unit of ← currency j.

$\vec{x} = \begin{bmatrix} 100 \\ 10000 \\ 50 \end{bmatrix} \Rightarrow$ 100 USD
10,000 JPY
50 EUR

Calculating $\vec{y} = Rx$:

$y_{USD} = 1 \times 100 + 0.009 \cdot 10000 + 1.2 \cdot 50 = 250$
$y_{JPY} = 110 \times 100 + 1 \cdot 10000 + 130 \cdot 50 = 27500$
$y_{EUR} = 0.83 \times 100 + 0.0077 \times 10000 + 1 \times 50 = 210$

Conclusion:

$y_1 = 250$: Total equivalent in USD if convert all holding in USD
$y_2 = 27500$: ———— JPY ———— JPY
$y_3 = 210$: ———— EUR ———— EUR.

Each y_i shows portfolio's value in one individual currency.

Solution 4:

(a) 2x Up-Conversion with Linear Interpolation:

Input vector : $\vec{x} \in R^n$
Output vector : $\vec{y} \in R^{(2n-1)}$

Transformation Rule:
For Odd i : $y_i = x_{\frac{i+1}{2}}$; Original sample value
For Even i : $y_i = \dfrac{x_{\frac{i}{2}} + x_{\frac{i}{2}+1}}{2}$; Interpolate by averagin

Matrix Representation (A)
A is a $(2n-1) \times n$ matrix.
Example for $n=3$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) 2x Down-Sampling

Input Vector : $\vec{x} \in R^n$ (n is even)
Output Vector : $\vec{y} \in R^{(n/2)}$

Transformation Rule:
$y_i = x_{2i}$ (Considering every second sample from $\vec{x}$).

Matrix Representation (A)
A is an $(n/2) \times n$ matrix.

Example for $n=4$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(C) 2x Down-Sampling with Averaging :

Input Vector : $\vec{x} \in R^n$ (n is even).
Output Vector : $\vec{y} \in R^{(n/2)}$

Transformation Rule :

$$y_i = \frac{x_{2i} + x_{2i}}{2}$$ (Average Pairs of adjacent samples)

Matrix Representation (A) :

A is an $(n/2) \times n$ matrix

Example for $n=4$:

$$A = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Solution 5:

(a) Total Money Spent on Advertising

$\vec{c}$: n-vector representing the company's spending in each of the n channels.

Formula: $1^T c \Rightarrow$ Total Spend.

Total money spent is the sum of all entries in $\vec{c}$.

$1^T \rightarrow$ row vector

Example: (for n=3).

If $\vec{c} = [100, 200, 300]$

$1^T c = 100 + 200 + 300$

$= 600$ dollars.

(b) Total Number of Impression ($\vec{v}$)

Formula: $V = Rc$

vi for each market segment i is dot product of i-th row of R and spend vector c.

$\vec{c}$: n-vector of channel spending.

Example: (for m=2, n=3):

If $R = \begin{bmatrix} 10 & 20 & 30 \\ 5 & 15 & 25 \end{bmatrix}$ and $\vec{c} = [100, 200, 300]$

$$\vec{v} = \begin{bmatrix} 10 & 20 & 30 \\ 5 & 15 & 25 \end{bmatrix} \times \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix} = \begin{bmatrix} 14000 \\ 11000 \end{bmatrix}.$$

(C) Total Profit from All Market Segments

$\vec{v}$: m-vector of total impressions per market segment.

$\vec{a}$: m-vector, where a_i is profit per impression in segment i.

Formula :

$$\text{Total Profit} = a^T v = a^T Rc$$

Total profit in terms per impression a_i the reach matrix R, and the advertising spend c.

Example :

$$\text{Total Profit} = [2\ 3] \times \begin{bmatrix} 14000 \\ 11000 \end{bmatrix}$$

$$= 28000 + 33000$$

$$= 61000 \text{ dollars.}$$

(d) Finding the Most Effective channel for Market Segment 3

R_{ij} → no. of impressions per dollar spent in channel j for segment i.

Formula : Best channel = $\arg\max_j R_{3j}$

means index j where entry R_{3j} is the largest.

Example :

If row 3 of R = [2, 5, 10], channel 3 is with 10 impressions per dollar is most effective.

(e) Meaning of a Small R_{35} Value :

If R_{35} is very small compared to other value in row 3,
it means channel 5 is not effective at reaching market segment 3.

So, spending on channel 5 will yield very few impressions in segment 3 compared to other channels.

Considered → $R_{35} = 0.1$