Instructions

The following questions should be answered by hand, and you must show all your work. You will be asked to verify some of your solutions using Python, and for other questions you may find it helpful to use Python's linear algebra functionality. For those portions, please use a Jupyter Notebook. Your final submission should be a single PDF containing both your handwritten answers and an exported version of your Jupyter Notebook.

Assignment

1. **Matrix and vector operations (20 pts)**
   Suppose we have the following the matrices A and B and the vectors $\vec{x}$ and $\vec{y}$.

   $$A = \begin{bmatrix} 1 & 4 & 1 \\ -2 & 2 & 1 \\ 3 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

   Compute the following by hand, and verify your answers using `numpy`.

   (a) $3\vec{x} + 2\vec{y}$

   (b) $A\vec{x}$

   (c) AB

   (d) $||\vec{x}||$

   (e) $||\vec{y}||$

   (f) $\vec{x} \cdot \vec{y}$

   (g) The angle θ between $\vec{x}$ and $\vec{y}$

2. **Matrix size (10 pts)**
   Suppose A, B, and C are matrices that satsify $A + BB^\top = C$. Determine which of the following statements are necessarily true.
   (a) A is square
   (b) A and B have the same dimensions
   (c) A, B, and C have the same number of rows
   (d) B has more rows than columns

3. **Currency Exchange (10 pts)**
   Consider n currencies, labeled $1, \ldots, n$. These might correspond to USD, JPY, EUR, etc. At any point in time the exchange rates among the n currencies are given by an $n \times n$ exchange matrix R, where component R_{ij} is the amount of currency i you can buy for one unit of currency j. All entries of R are positive, and you can assume that comission charges are included, so we have $R_{ji}R_{ij} < 1$ for all $i \neq j$, and $R_{ij} = 1$ for $i = j$.

   Suppose $\vec{x} \in \mathbb{R}^n$ is a vector with nonnegative entries that represents the amounts of each of the currencies we hold. Now define $\vec{y} = R\vec{x}$. What does each of the components y_i represent qualitatively, and why? Your answer must be in words, though you can include math to support your answer if you like.

4. **Matrices for signal processing (20 pts)**
   Consider a time series vector $\vec{x} \in \mathbb{R}^n$ with each of its components x_i as the value of a sensor measurement at time i, for $i = 1, \ldots, n$. Each of the transformations described below takes the input $\vec{x}$ and turns it into an output $\vec{y}$ to perform a common signal processing operation. For each transformation, find the matrix A that results in $\vec{y} = A\vec{x}$.
   (a) 2x up-conversion with linear interpolation: we want $\vec{y} \in \mathbb{R}^{2n-1}$. For i odd, $y_i = x_{(i+1)/2}$. For i even, $y_i = (x_{i/2} + x_{i/2+1})/2$. This produces $\vec{y}$ with twice the sampling rate of $\vec{x}$ by inserting new samples in between the original samples via linear interpolation.

(b) 2x down-sampling: Assume that n is even, we want $\vec{y} \in \mathbb{R}^{n/2}$ with $y_i = x_{2i}$.

(c) 2x down-sampling with averaging: Assume that n is even, we want $\vec{y} \in \mathbb{R}^{n/2}$ with $y_i = (x_{2i-1} + x_{2i})/2$

5. **Marketing Budget Allocation (15 pts)**
Potential customers are divided into m market segments, which are groups of customers with similar demographics, e.g., college-educated women from age 25-29. A company markets its products by purchasing advertising in a set of n channels, i.e., specific TV or radio shows, magazines, web sites, blogs, direct mail, etc. The ability of each channel to deliver impressions or views by potential customers is characterized by the reach matrix $R \in \mathbb{R}^{m \times n}$, where component R_{ij} is the number of views of customers in segment i for each dollar spent on channel j. We assume that the total number of views in each market segment is the sum of the views from each channel, and that the views from each channel scale linearly with spending. The n-vector $\vec{c}$ will denote the company's purchases of advertising, in dollars, in the n channels. The m-vector $\vec{v}$ gives the total number of impressions in the m market segments due to the advertising in all channels. Finally, we introduce the m-vector $\vec{a}$, where component a_i gives the profit in dollars per impression in market segment i. The entries of R, $\vec{c}$, $\vec{v}$, and $\vec{a}$ are all nonnegative.

Answer the following questions about the vectors and matrices described above. For each question, you must also explain *how* you arrived at your answer.

Hint: You can express the sum of a vector $\vec{x} \in \mathbb{R}^n$ as $\mathbf{1}^\top \vec{x}$, where $\mathbf{1}^\top$ is a row vector of size $1 \times n$ with every component equal to 1.

(a) Express the total amount of money the company spends on advertising using vector/matrix notation

(b) Express $\vec{v}$ using vector/matrix notation in terms of the other vectors and matrices.

(c) Express the total profit from all market segments using vector/-

matrix notation.

(d) How would you find the single channel most effective at reaching market segment 3, in terms of impressions per dollar spent?

(e) What does it mean if R_{35} is very small compared to the other entries of R?

6. **Running Hot (25 pts)**
We want to monitor the temperature of a CPU at two critical locations. These temperatures are given by the vector $T = (T_1, T_2)$ (degrees C), and are affine functions* of the power dissipated by the three processor cores, given by the vector $P = (P_1, P_2, P_3)$ (Watts). We set up an experiment, and take four measurements: in the first, all cores are idling and dissipate 10W. In the next three measurements, one of the cores is set to 100W while the other two idle. In each experiment we measure the temperatures at locations 1 and 2. The results of this experiment are shown in Table 1.

P_1	P_2	P_3	T_1	T_2
10W	10W	10W	21°	23°
100W	10W	10W	47°	35°
10W	100W	10W	39°	47°
10W	10W	100W	32°	53°

Table 1

Say we operate all cores at the same power p. What is the maximum value of p such that neither T_1 nor T_2 exceeds 65°?

*Note: An affine function is just a linear function plus an offset. For example, $f(x) = mx + b$ is an affine function because it has a linear scaling coefficient m applied to x, and a constant offset b. People often call such a function linear, but they are technically wrong; a linear function would just be $f(x) = mx$.