Tilings Day 1

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Inhaltsverzeichnis

Tiles and Tilings

We kick off discussion with a small assumption (at least for today). Whatever we speak of from here on out is based on a finite or infinite subset of $\mathbb{R}^n$.

Definition (Tile)

A *tile* is a closed set $T \subset \mathbb{R}^n$, with a non-empty interior and a piecewise linear boundary, which we think of as a building block.

Definition (Tiling)

A *tiling* of $X \subseteq \mathbb{R}^2$ by a collection of tiles $\mathcal{T}$ is a family $(T_i)_{i \in I}$ of tiles $T_i \in \mathcal{T}$ such that:

1. $X = \bigcup_{i \in I} T_i$,
2. $\text{int}(T_i) \cap \text{int}(T_j) = \varnothing$ for $i \neq j$.

Some More Definitions

Definition (Prototile)

A *prototile* is a tile up to congruence (rotations, translations, sometimes reflections). A tiling is *monohedral* if it uses only one prototile.

Definition (Lattice)

A *lattice* Λ in $\mathbb{R}^2$ is a discrete additive subgroup of the form

$$\Lambda = \{n_1 v_1 + n_2 v_2 : n_1, n_2 \in \mathbb{Z}\},$$

where v_1, v_2 are linearly independent vectors. A tiling is often called *lattice periodic* if it is invariant under translations by all vectors in some lattice Λ.

Examples of Tiles and Tiling Regions

Some types of tiles:

1. Domino Tile
2. Lozenge Tile
3. Trihex Tile

Some examples of tiling regions:

1. $m \times n$ Rectangle
2. Hexagons
3. a, b-Benzels

Periodic and Aperiodic Tilings I

Definition

A tiling $\mathcal{T}$ of $\mathbb{R}^2$ is *periodic* if there exists a nonzero vector v such that translating all tiles by v leaves the tiling invariant:

$$\mathcal{T} + v = \mathcal{T}.$$

If no such nonzero vector exists, the tiling is called *aperiodic*.

Periodic and Aperiodic Tilings II

Example

- Periodic Tilings:
  - **Square tiling**: Repeats every unit translation; simplest lattice tiling.
  - **Hexagonal (honeycomb) tiling**: 6-fold rotational symmetry; appears in crystals.

  Key idea: There exists a nonzero translation vector v such that the whole tiling is invariant under translation by v.
- Aperiodic Tilings:
  - **Penrose tilings (kite & dart / rhombus)**: No translation symmetries; exhibit 5-fold rotational order; discovered in the 1970s.
  - **The "Hat" Monotile (2023)**: Discovered by Smith, Myers, Kaplan, Goodman-Strauss. First aperiodic monotile that forces aperiodicity with no matching rules.

  Key idea: No nonzero translation vector preserves the tiling, even though local patterns repeat in complex, hierarchical ways.

Regular, Semi-Regular, and Irregular Tilings I

Definition (Regular Tilings)

A regular tiling of the plane is a tiling where:

- All tiles are congruent regular polygons (same number of sides, same angles), and
- Every vertex configuration is identical (the same number and type of polygons meet in the same cyclic order at every vertex).

There are exactly three regular tilings of the Euclidean plane:

- Equilateral triangles (6 at each vertex),
- Squares (4 at each vertex),
- Regular hexagons (3 at each vertex).

Regular, Semi-Regular, and Irregular Tilings II

Definition (Vertex-Type)

The vertex-type of a vertex v is a cyclically ordered tuple of integers $k = [k_1, k_2, \ldots, k_d]$ where

- d is the degree of the vertex, and
- the k_i's are the sizes of the d polygons in cyclic order around v.

Definition (Semi-Regular Tilings)

A tiling is semi-regular if all vertices have the same vertex-type. We call it a k-tiling.

Some examples:

- $(3, 12, 12)$ - one triangle and two dodecagons meeting at each vertex,
- $(3, 3, 4, 3, 4)$,
- $(4, 8, 8)$

Regular, Semi-Regular, and Irregular Tilings III

Definition (Irregular Tilings)

An irregular tiling is any tiling that is not regular or semi-regular.
This means one of the following holds:

- Tiles are not all regular polygons;
- More than one type of vertex configuration exists;
- Tiles may not meet edge-to-edge.

Some examples are:

- General polyomino tilings (like domino tilings),
- Penrose tilings (even though they are highly structured, their vertex configurations vary $\rightarrow$ not semi-regular).

Fun-fact: There are 21 possible vertex-types, of which exactly 11 extend to semi-regular tilings.

Fault and Fault-Free Tilings

Definition (Faults (Rectangular Case))

Consider an $m \times n$ rectangular board tiled by dominos. A *fault line* is a vertical or horizontal line between two columns or rows that cuts the board into two smaller rectangles, such that no domino crosses this line.

Definition (Fault-Free Tilings (Rectangular Case))

A tiling with no fault lines is called *fault-free*.

Similarly we can define vertical and horizontal faults.

Substitution Tilings

Definition (Substitution Tilings)

A *substitution* on a set of prototiles $\{T_1, \ldots, T_k\}$ is a rule that replaces each T_i by a patch of tiles similar to $T_1, \ldots, T_k$ (scaled by a fixed factor). Iterating the substitution produces larger and larger patches, and in the limit one obtains a tiling of the plane. Tilings arising in this way are called *substitution tilings*.

Newer tiles arising from this are called reptiles.

Definition (Reptile)

A reptile or k-reptile is a tile that can itself be tiled with k scaled-down replicas of itself, all congruent to one another.

Generating Functions: Basic Idea I

Definition (Generating Function (GF))

A *generating function* is a way to encode a sequence

$$(a_0, a_1, a_2, \dots)$$

as a single power series.

Definition (Ordinary Generating Function (OGF))

The *ordinary generating function (OGF)* of (a_n) is

$$A(x) = \sum_{n \geq 0} a_n x^n.$$

Think of $A(x)$ as a "container" storing the entire sequence in a compact algebraic form.

Generating Functions: Basic Idea II

Example

Consider the Fibonacci sequence:

$$f_n = f_{n-1} + f_{n-2}, \qquad f_0 = 1,\ f_1 = 1.$$

The generating function is:

$$F(x) = \sum_{n \geq 0} f_n x^n = \frac{1}{1 - x - x^2}.$$

- **Shifts**:

$$\sum_{n\geq 0} a_{n+1}x^n = \frac{A(x) - a_0}{x}$$

Algebraic Operations with GF

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$$\sum_{n \geq 0} a_{n+1} x^n = \frac{A(x) - a_0}{x}$$

- **Sum**: If $c_n = a_n + b_n$ then

$$C(x) = A(x) + B(x).$$

Algebraic Operations with GF

- **Shifts**:

$$\sum_{n\geq 0} a_{n+1}x^n = \frac{A(x) - a_0}{x}$$

- **Sum**: If $c_n = a_n + b_n$ then

$$C(x) = A(x) + B(x).$$

- **Convolution**: If $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ then

$$C(x) = A(x)\, B(x).$$

More Algebraic Operations with GF

- **Linear recurrences**: If $a_n = \sum_{i=1}^{d} c_i a_{n-i}$, then the OGF is a rational function:

$$A(x) = \frac{P(x)}{1 - c_1 x - c_2 x^2 - \cdots - c_d x^d}.$$

More Algebraic Operations with GF

- **Linear recurrences**: If $a_n = \sum_{i=1}^{d} c_i a_{n-i}$, then the OGF is a rational function:
$$A(x) = \frac{P(x)}{1 - c_1 x - c_2 x^2 - \cdots - c_d x^d}.$$

Play around and you can find many more such relations. It's just a google search away!!

Generating Functions in Tilings I

- In tilings, a_n often counts the number of ways to tile an $m \times n$ region.
- Decomposing tilings column-by-column leads to recurrence relations.
- Therefore, their generating functions $T(x)$ are *rational functions*.
- $T(x)$ encodes all tilings.
- The recurrence encodes the *local* structure of tiling.
- The GF reveals the *global* combinatorial structure.
- The denominator of the rational function contains structural information about the tilings.

Generating Functions in Tilings II

Example

For domino tilings of a $2 \times n$ rectangle:

$$T_n = T_{n-1} + T_{n-2}, \qquad T_0 = 1,\ T_1 = 1.$$

Thus the generating function is:

$$T(x) = \sum_{n \geq 0} T_n x^n = \frac{1}{1 - x - x^2}.$$

And this rational function is exactly the same as the one we get from the recurrence relation for Fibonacci sequence.

Definitions I

Definition (Graph)

A graph G is a pair $G = (V, E)$, where V is a set of vertices (or nodes), and E is a set of edges. An edge is typically an unordered pair $\{u, v\}$ with $u, v \in V$, $u \neq v$.
If edges are ordered pairs (u, v), the graph is called a directed graph (or digraph).

Definitions II

Definition (Walk)

A walk in a graph G is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that every consecutive pair (v_{i-1}, v_i) is an edge in the graph. The length of the walk is the number of edges used.

Definition (Cycle)

A cycle is a walk $v_0, v_1, \ldots, v_k$ such that $v_0 = v_k$ and all vertices $v_0, v_1, \ldots, v_{k-1}$ are distinct.

Adjacency Matrix I

Definition (Adjacency Matrix)

Let $G = (V, E)$ be a finite graph with vertices $V = \{v_1, \ldots, v_n\}$. The *adjacency matrix* of G is the $n \times n$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

For directed graphs, $a_{ij} = 1$ if there is an edge $v_i \to v_j$.

Adjacency Matrix II

Example

Consider the graph with edges v_1–v_2, v_2–v_3, v_1–v_3. Its adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Adjacency Matrix III

Theorem (Significance of Powers of A)

For any graph with adjacency matrix A, the entry $(A^k)_{ij}$ equals the number of walks of length k from vertex v_i to vertex v_j.

$$(A^k)_{ij} = \#\{\text{walks of length } k \text{ from } v_i \text{ to } v_j\}.$$

Thus A^k encodes the k-step connectivity structure of the graph.

Proof. **DIY**

The Transfer Matrix Method

- To count tilings of an $m \times n$ rectangle by dominos, we build the tiling **column-by-column**.
- At any vertical cut between columns k and $k+1$, some cells may be *occupied by horizontal dominos* that started in previous columns.
- A **state** is a binary vector of length m:

$$s = (b_1, b_2, \ldots, b_m), \qquad b_i \in \{0, 1\}.$$

  - $b_i = 0$: the cell in row i is free.
  - $b_i = 1$: the cell is already occupied (overhang from the left).
- Legal placements of dominos in the next column determine which states can follow which.

The Transition Graph of States I

- Let S be the set of all *valid* states for width m.
- Construct a directed graph G:

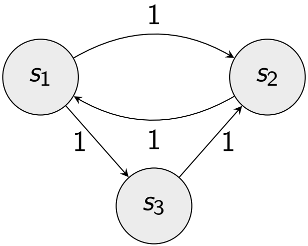
$$s_i \longrightarrow s_j$$

  if we can legally place dominos in column $k+1$ to go from state s_i to state s_j.
- The corresponding **transfer (transition) matrix** A has entries

$$A_{ij} = \begin{cases} 1, & \text{if } s_i \to s_j \text{ is legal,} \\ 0, & \text{otherwise.} \end{cases}$$

- Rows/columns are indexed by the states $s_1, \ldots, s_{|S|}$.

The Transition Graph of States II



Example of a small transition graph for width $m = 2$ or 3.

Encoding Tilings Using Matrix Powers

- Let A be the transfer matrix on the state set S.
- Let e_{start} be the standard basis vector for the all-zero, no-overhang state.
- Let e_{end} be the same vector (we must end with no overhang).
- After placing n columns, the number of valid tilings is:

$$T_n = e_{\text{start}}^{\top} A^n \, e_{\text{end}}.$$

- **Interpretation:**
  - A encodes all legal transitions from one column to the next.
  - A^n counts all legal sequences of n transitions.
  - Selecting the $(\text{start}, \text{end})$ entry isolates the full tilings.

Example: Domino Tilings of a $2 \times n$ Rectangle

- For width $m = 2$, the only valid states are:

$$s_0 = (0,0), \qquad s_1 = (1,1).$$

- The transitions are:

$$s_0 \to s_0, \qquad s_0 \to s_1, \qquad s_1 \to s_0.$$

- Thus the transfer matrix is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

- The number of domino tilings of $2 \times n$ is:

$$T_n = (A^n)_{00} = F_{n+1},$$

the $(n+1)$-st Fibonacci number.

The entire tiling problem reduces to powers of a 2×2 matrix.

Generating Function via the Transfer Matrix I

Let A be the transfer (transition) matrix for domino tilings of an $m \times n$ board.

- Let e_{start} be the basis vector for the all-zero state (no overhang).
- Let e_{end} be the same vector (tiling must end with no overhang).
- The number of tilings of length n is

$$T_n = e_{\text{start}}^{\top} A^n \, e_{\text{end}}.$$

Generating Function via the Transfer Matrix II

Generating function:

$$T(x) = \sum_{n\geq 0} T_n x^n = e_{\text{start}}^{\top} \left(\sum_{n\geq 0} (xA)^n \right) e_{\text{end}} = e_{\text{start}}^{\top} (I - xA)^{-1} e_{\text{end}}.$$

- $(I - xA)^{-1}$ expands formally as

$$(I - xA)^{-1} = I + xA + x^2A^2 + x^3A^3 + \cdots ,$$

  so the $(\text{start}, \text{end})$ entry collects all valid n-column sequences.
- $T(x)$ is always a **rational function** because $(I - xA)^{-1}$ is rational.

Fault-Free Tilings and the Functional Equation

Definition

Let

$$T_n = \text{total number of domino tilings of an } m \times n \text{ rectangle,}$$

$$V_n = \text{number of } \textit{vertically fault-free} \text{ tilings of the } m \times n \text{ rectangle.}$$

Remark

Let the generating functions be

$$T(x) = \sum_{n \geq 0} T_n x^n,$$

$$V(x) = \sum_{n \geq 1} V_n x^n \ \textit{(since, } V_0 = 0\textit{)}$$

Decomposition by the First Vertical Fault

Every tiling of an $m \times n$ rectangle falls into exactly one of two cases:

1. It has **no** vertical faults: contributes V_n.
2. It has at least one fault. Let k be the position of the *first* vertical fault.

If the first fault occurs after column k:

- The left region is an $m \times k$ rectangle and must be **vertically fault-free**, contributing V_k.
- The right region is an $m \times (n-k)$ rectangle with **arbitrary** tilings, contributing T_{n-k}.

Thus by idea of convolution, for all $n \geq 1$:

$$T_n = V_n + \sum_{k=1}^{n-1} V_k \, T_{n-k}.$$

Proof of the Functional Equation I

We start from

$$T_n = V_n + \sum_{k=1}^{n-1} V_k T_{n-k} = \sum_{k=1}^{n} V_k T_{n-k}.$$

Multiply by x^n and sum $n \geq 1$:

$$\sum_{n \geq 1} T_n x^n = \sum_{n \geq 1} \sum_{k=1}^{n} V_k T_{n-k} x^n.$$

Left side:

$$\sum_{n \geq 1} T_n x^n = T(x) - T_0 = T(x) - 1$$

Right side:

$$\sum_{n \geq 1} \sum_{k=1}^{n} V_k T_{n-k} x^n = (V(x) - V_0) T(x) = V(x) T(x).$$

Proof of the Functional Equation II

Thus:

$$T(x) - 1 = V(x)T(x).$$

Rearrange:

$$T(x) - V(x)T(x) = 1.$$

Solve for $T(x)$:

$$\begin{aligned} & T(x)(1 - V(x)) = 1 \\ \implies \quad & T(x) = \frac{1}{1 - V(x)}. \end{aligned}$$

Problems I

- **State graph for $3 \times n$ domino tilings.** For $m = 3$, describe all valid states (binary strings of length 3 compatible with domino overhangs). Construct the directed transition graph and its adjacency matrix A. Express the generating function as

$$T(x) = e_{\text{start}}^{\top}(I - xA)^{-1}e_{\text{end}}.$$

  Compute first 10 terms of $T(x)$ explicitly.

Problems II

- **State graph for $4 \times n$ and substitution.** For $m = 4$, describe all valid states (trinary strings of length 4 compatible with tromino overhangs). Construct the directed transition graph and its adjacency matrix A. Now consider that we define a substitution rule that every tromino tile decomposes into a domino tile and an unit square. How does the state space, directed transition graph, and its adjacency matrix change?