Frequency Domain Analysis

Required knowledge

- Fourier-series and Fourier-transform.
- Measurement and interpretation of transfer function of linear systems.
- Calculation of transfer function of simple networks (first-order, high- and low-pass RC network).

Introduction

Signals are often represented in frequency domain by their spectrum, frequency, harmonic components, amplitude and phase. Time- and frequency-domain representations are mutually equivalent, and the Fourier transform can be used to transform signals between the two domains. Fourier transform exists for almost all practical signals which are used in electrical engineering practice. Frequency domain representation often simplifies the solution of several practical problems. It offers a compact and expressive form of signal representation by allowing the separation of spectral components. Frequency-domain representation can be effectively used in measurement of signal parameters, signal transmission, infocommunication, system design, etc.

One of the most important classes of signals is the class of periodic signals. Periodic signals are often used as excitation signals since they produce periodic signal with the same frequency at the output of the system the parameters of which are to be measured. Periodic signals are easy to observe with simple instruments like oscilloscopes, moreover averaging can also be effectively used to increase the signal-to-noise ratio. System parameters can be determined by measuring the amplitude gain (or attenuation) and phase shift between the output and input. Fourier transform allows the characterization of systems in a simple algebraic form instead of differential equations connected to time-domain representation.

Aim of the Measurement

During the measurement, the students study the method of signal analysis in frequency domain. They compare time domain algorithms to frequency domain ones. After finishing the measurement, they will be able to use frequency domain tools to describe properties of signals which are no trivial to be detected in time-domain. This laboratory lecture demonstrates how the students can apply their knowledge of signal and systems in order to solve engineering problems.

Theoretical background

Fourier series

Real-valued periodic signals can be decomposed into linear combination of sine and cosine functions. This trigonometrical series is referred to as Fourier series of signals, and it has the following form (T stands for the period, and $\omega = 2\pi / T$ denotes the angular frequency):

$$u(t) = U_0 + \sum_{k=1}^{\infty} (U_k^A \cos k\omega t + U_k^B \sin k\omega t)\,, \tag{4-1}$$

where the coefficients can be calculated using the following equations:

$$U_0 = \frac{1}{T}\int_0^T u(t)dt\,,\ U_k^A = \frac{2}{T}\int_0^T u(t)\cos(k\omega t)dt\,,\ U_k^B = \frac{2}{T}\int_0^T u(t)\sin(k\omega t)dt\,. \tag{4-2}$$

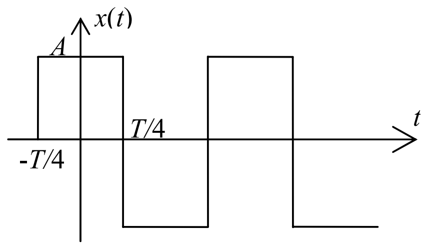
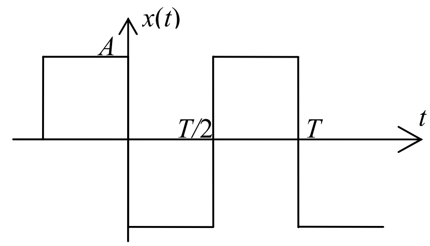
These operations are based on the orthogonality of trigonometric functions on the interval [0…T].

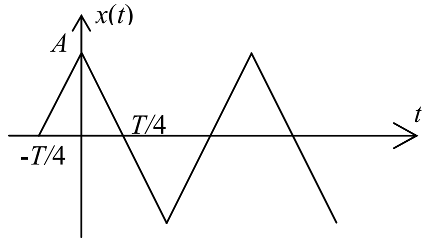
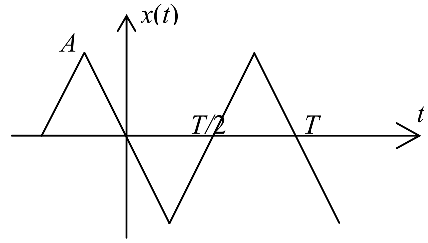
Fourier series have also a simpler form where complex-valued coefficients and complex exponential basis function are used:

$$u(t) = \sum_{k=-\infty}^{\infty} \overline{U}_k^C e^{jk\omega_0 t}\ \text{, where } \overline{U}_k^C = \frac{1}{T}\int_0^T u(t)e^{-jk\omega t}dt,\ k = 0, \pm 1, \pm 2... \tag{4-3}$$

For real-valued signals: $\overline{U}_k^C = \text{conjugate}(\overline{U}_{-k}^C)$, i.e., Fourier components form complex conjugate pairs.

The Fourier series of some practically important signals are summarized in the following table.

	
$x(t) = A\frac{4}{\pi}\left[\cos(\omega t) - \frac{1}{3}\cos(3\omega t) + \frac{1}{5}\cos(5\omega t) - ...\right]$	$x(t) = A\frac{4}{\pi}\left[\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + ...\right]$
$U_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ A\frac{4}{\pi}\left((-1)^{(k-1)/2}\frac{1}{k}\right) & \text{if } k \text{ is odd} \end{cases}$	$U_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ A\frac{4}{\pi}\frac{1}{k} & \text{if } k \text{ is odd} \end{cases}$

	
$x(t) = A\frac{8}{\pi}\left[\cos(\omega t)+\frac{1}{3^2}\cos(3\omega t)+\frac{1}{5^2}\cos(5\omega t)+\dots\right]$	$x(t) = A\frac{8}{\pi}\left[\sin(\omega t)-\frac{1}{3^2}\sin(3\omega t)+\frac{1}{5^2}\sin(5\omega t)-\dots\right]$
$U_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ A\frac{8}{\pi^2}\frac{1}{k^2} & \text{if } k \text{ is odd} \end{cases}$	$U_k = \begin{cases} 0 & \text{if } k \text{ is even} \\ A\frac{8}{\pi^2}\left((-1)^{(k-1)/2}\frac{1}{k^2}\right) & \text{if } k \text{ is odd} \end{cases}$

Fourier series of some periodic signals.

In this Section we show how a Fourier series can be expressed more concisely if we introduce the complex number $\mathrm{i} = \sqrt{-1}$. By utilising the Euler formula:

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

we can replace the trigonometric functions by complex exponential functions. By also combining the Fourier coefficients a_n and b_n into a complex coefficient c_n through

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n)$$

we find that, for a given periodic signal, both sets of constants can be found in one operation. We also obtain Parseval's Theorem which has important applications in electrical engineering. The complex formulation of a Fourier series is an important pre-cursor of the Fourier transform which attempts to Fourier analyse non-periodic functions.

1. Complex Exponential Form of a Fourier Series

So far we have discussed the **trigonometric** form of a Fourier Series i.e. we have represented functions of period T in the terms of sinusoids, and possibly a constant term, using

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right).$$

If we use the angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

We obtain the more concise form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t).$$

We have seen that the Fourier coefficients are calculated using the following integrals:

$$a_n = \tfrac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\omega_0 t \;\mathrm{d}t \qquad n = 0, 1, 2, \ldots \tag{1}$$

$$b_n = \tfrac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin n\omega_0 t \;\mathrm{d}t \qquad n = 1, 2, \ldots \tag{2}$$

An alternative, more concise form, of a Fourier Series is available using complex quantities. This form is quite widely used by Engineers, for example in Circuit Theory and Control Theory, and, in this course, will lead naturally into a later Section on the Fourier Transform.

2. Revision of the exponential form of a complex number

Recall that a complex number in Cartesian form

$$z = a + \mathrm{i}b,$$

where a and b are real numbers and $\mathrm{i}^2 = -1$, can be written in **polar** form

$$z = r(\cos\theta + \mathrm{i}\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and θ, the argument or phase of z, is such that

$$a = r\cos\theta \qquad\qquad b = r\sin\theta.$$

A more concise version of the polar form of z can be obtained by defining a **complex exponential** quantity $e^{\mathrm{i}\theta}$ by

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

(This is sometimes known as Euler's relation.) The polar angle θ is normally expressed in **radians**.
Replacing i by $-$i we obtain

$$e^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta$$

Using these results we can redraft an expression of the form

$$a_n \cos n\theta + b_n \sin n\theta$$

in terms of complex exponentials.
(This expression, with $\theta = \omega_0 t$, is of course the n'th harmonic of a trigonometric Fourier Series.)

Using the results from the last Key Point (with $n\theta$ instead of θ) rewrite

$$a_n \cos n\theta + b_n \sin n\theta$$

in complex exponential form.

Now collect the terms in $e^{in\theta}$ and in $e^{-in\theta}$

We can now write this expression in more concise form by defining

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n)$$

which has complex conjugate

$$c_n^* = \frac{1}{2}(a_n + \mathrm{i}b_n).$$

Clearly, we can now rewrite the trigonometric Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n e^{\mathrm{i}n\omega_0 t} + c_n^* e^{-\mathrm{i}n\omega_0 t}\right) \tag{3}$$

A neater, and particularly concise, form of this expression can be obtained as follows:
Firstly write $\frac{a_0}{2} = c_0$ (which is consistent with the general definition of c_n since $b_0 = 0$)
The second term in the summation

$$\sum_{n=1}^{\infty} c_n^* e^{-\mathrm{i}n\omega_0 t} = c_1^* e^{-\mathrm{i}\omega_0 t} + c_2^* e^{-2\mathrm{i}\omega_0 t} + \ldots$$

can be written, if we define $c_{-n} = c_n^* = \frac{1}{2}(a_n + \mathrm{i}b_n)$, as

$$c_{-1}e^{-\mathrm{i}\omega_0 t} + c_{-2}e^{-2\mathrm{i}\omega_0 t} + c_{-3}e^{-3\mathrm{i}\omega_0 t} + \ldots = \sum_{n=-1}^{-\infty} c_n e^{\mathrm{i}n\omega_0 t}$$

Hence (3) can be written

$$c_0 + \sum_{n=1}^{\infty} c_n e^{\mathrm{i}n\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{\mathrm{i}n\omega_0 t}$$

or, finally, in the extremely concise form

$$\sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}n\omega_0 t}.$$

The **complex Fourier coefficients** c_n can be readily obtained as follows using (1) and (2) for a_n, b_n.
Firstly

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\mathrm{d}t \tag{4}$$

For $n = 1, 2, 3, \ldots$ we have

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)(\cos n\omega_0 t - \mathrm{i}\sin n\omega_0 t)\mathrm{d}t$$

$$\text{i.e.} \qquad c_n = \tfrac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\mathrm{i}n\omega_0 t}\mathrm{d}t \tag{5}$$

Also for $n = 1, 2, 3, \ldots$ we have

$$c_{-n} = c_n^* = \frac{1}{2}(a_n + \mathrm{i}b_n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{\mathrm{i}n\omega_0 t}\mathrm{d}t$$

This last expression is equivalent to stating that for $n = -1, -2, -3, \ldots$

$$c_n = \tfrac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\mathrm{i}n\omega_0 t}\mathrm{d}t \tag{6}$$

The 3 equations (4), (5), (6) can thus all be contained in the one expression

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\mathrm{i}n\omega_0 t}\mathrm{d}t \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The results of this discussion are summarised in the following Key Point.

Key Point

Fourier Series in Complex Form

A function $f(t)$ of period T has a complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}n\omega_0 t} \qquad \text{where} \qquad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\mathrm{i}n\omega_0 t}\mathrm{d}t$$

For the special case $T = 2\pi$ so that $\omega_0 = 1$ these formulae become particularly simple

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\mathrm{d}t.$$

Note the extremely concise form of these results.