Chapter 7

$$
\begin{array}{c}
\binom{0}{0}\\
\binom{1}{0}\quad\binom{1}{1}\\
\binom{2}{0}\quad\binom{2}{1}\quad\binom{2}{2}\\
\binom{3}{0}\quad\binom{3}{1}\quad\binom{3}{2}\quad\binom{3}{3}\\
\binom{4}{0}\quad\binom{4}{1}\quad\binom{4}{2}\quad\binom{4}{3}\quad\binom{4}{4}\\
\binom{5}{0}\quad\binom{5}{1}\quad\binom{5}{2}\quad\binom{5}{3}\quad\binom{5}{4}\quad\binom{5}{5}\\
\binom{6}{0}\quad\binom{6}{1}\quad\binom{6}{2}\quad\binom{6}{3}\quad\binom{6}{4}\quad\binom{6}{5}\quad\binom{6}{6}\\
\binom{7}{0}\quad\binom{7}{1}\quad\binom{7}{2}\quad\binom{7}{3}\quad\binom{7}{4}\quad\binom{7}{5}\quad\binom{7}{6}\quad\binom{7}{7}\\
\binom{8}{0}\quad\binom{8}{1}\quad\binom{8}{2}\quad\binom{8}{3}\quad\binom{8}{4}\quad\binom{8}{5}\quad\binom{8}{6}\quad\binom{8}{7}\quad\binom{8}{8}\\
\binom{9}{0}\quad\binom{9}{1}\quad\binom{9}{2}\quad\binom{9}{3}\quad\binom{9}{4}\quad\binom{9}{5}\quad\binom{9}{6}\quad\binom{9}{7}\quad\binom{9}{8}\quad\binom{9}{9}\\
\binom{10}{0}\quad\binom{10}{1}\quad\binom{10}{2}\quad\binom{10}{3}\quad\binom{10}{4}\quad\binom{10}{5}\quad\binom{10}{6}\quad\binom{10}{7}\quad\binom{10}{8}\quad\binom{10}{9}\quad\binom{10}{10}
\end{array}
$$

Combinatorics

7.1 Definition of Factorial

The falling product of first n natural numbers is called the "n factorial" and is denoted by n! or $\underline{|n}$.

That is, $n! = n(n-1)\,(n-2)\,\ldots\,3 \times 2 \times 1$

For example, $4! = 4 \times 3 \times 2 \times 1 = 24$; $\quad 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$;

$$\frac{(2n)!}{n!} = \frac{1\cdot 2\cdot 3\cdots(2n-1)(2n)}{n!} \quad \text{(by using the definition of factorials)}$$

$$= \frac{\{1\cdot 3\cdot 5\cdots(2n-1)\}\{2\cdot 4\cdot 6\cdots 2n\}}{n!} = \frac{\{1\cdot 3\cdot 5\cdots(2n-1)\}\,2^n\,n!}{n!}$$

(By taking 2 out from all terms of the second factor in Numerator)

$$= \{1\cdot 3\cdot 5\cdots(2n-1)\}\;2^n$$

Factorials of proper fractions and of negative integers are not defined. Factorial n is defined only for whole numbers.

7.1.1 Properties of Factorial

(a) $0! = 1$ (by definition)

(b) $n! = 1 \times 2 \times \cdots \times (n-1) \times n = [1 \times 2 \times \cdots \times (n-1)]\,n = (n-1)!\,n$

Thus, $n! = n\,((n-1)!)$

(c) If two factorials, *i.e.*, $x!$ and $y!$ are equal, then

$$(x, y) = (0, 1) \text{ or } (1, 0) \text{ or } (k, k)\ \forall\ k \in \mathbb{N}_0$$

(d) $n!$ ends in 0, for all $n > 4$. (Number of 5's in $n!$, $n > 4$, is always less than the number of 2's. Therefore for every 5, there is a 2. Hence $n!$, $n > 4$, ends in 0).

Example 1 *If* $\dfrac{n!}{2!(n-2)!}$ *and* $\dfrac{n!}{4!(n-4)!}$ *are in the ratio* 2:1*, then find the value of n.*

Solution: $$\frac{\dfrac{n!}{2!(n-2)!}}{\dfrac{n!}{4!(n-4)!}} = \frac{2}{1} \Rightarrow \frac{n! \times 4!(n-4)!}{2!(n-2)! \times n!} = \frac{2}{1} \Rightarrow \frac{4 \times 3}{(n-2) \times (n-1)} = \frac{2}{1}$$

$$\Rightarrow (n-2)(n-3) = 6 \Rightarrow n^2 - 5n = 0 \Rightarrow n = 0, 5$$

But, for $n = 0$, $(n - 2)!$ and $(n - 4)!$ are not meaningful

So, $n = 5$.

7.2 Basic Counting Principles

7.2.1 Addition Principle

Let $A = \{ a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ be sets.

Let A and B be disjoint (or mutually exclusive) set, *i.e.*, $A \cap B = \phi$ (the empty set).

Then an element of A or an element of B can be chosen in $n + m$ ways.

It can be extended as

Let a set A_i have k_i elements and any two sets A_i's be disjoint, $i = 1, 2, \ldots, n$. Then any element of A_1 or A_2 or … or A_n can be chosen in $k_1 + k_2 + \cdots + k_n$ ways.

In set theoretic notation, the extended form is stated as:

If A_i, $i = 1, 2, \ldots, n$, are n finite pair-wise disjoint (or mutually exclusive) sets, *i.e.*, $A_i \cap A_j = \phi$ for $i \neq j$; $i, j = 1, 2, \ldots n$; then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|$$

That is, the cardinality of the union of finite number of pair-wise disjoint finite sets is the sum of the cardinalities of the individual sets.

Here $|A_i|$ is the number of elements of the set A_i. Other notations for number of elements of the set A_i are $n(A_i)$ *or* $\#(A_i)$, etc.

In other words:

If there are

n_1 ways for the event E_1 to occur

n_2 ways for the event E_2, to occur

…

…

…

n_k ways for the event E_k, to occur

where $k \geq 1$, and if these are pair-wise disjoint (or mutually exclusive), then the number of ways for at least one of the events $E_1, E_2, \ldots, E_k$ to occur is $n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i$.

Example 2 *There are 15 gates to enter a city from north and 10 gates to enter the city from east. In how many ways a person can enter the city?*

Solution: Number of ways to enter the city from north = 15.

Number of ways to enter the city from east = 10.

A person can enter the city from north or from east.

So, number of ways to enter the city = 15 + 10 = 25.

Example 3 *There are 15 students is a class in which 10 are boys and 5 are girls. The class teacher selects either a boy or a girl for monitor of the class. In how many ways the class teacher can make this selection?*

Solution: A boy can be selected for the post of monitor in 10 ways.

A girl can be selected for the post of monitor in 5 ways.

Number of ways in which either a boy or a girl can be selected = 10 + 5 = 15.

Example 4 *Find the number of two digit numbers (having different digits) which are divisible by 5.*

Solution: Any number of required type either ends in 5 or in 0. Number of two digit numbers (with different digits) ends with 5 is 8 and that of ends with 0 is 9.
Hence, by addition principle the required number of numbers is 8 + 9 = 17.

7.2.2 Multiplication Principle

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ be sets.

An ordered pair (a, b), where $a \in A$, $b \in B$, can be formed in $n \times m$ ways.

It can further be extended as

Let a set A_i have k_i elements, $i = 1, 2, \ldots, n$.

An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for each i, can be formed in $k_1 \times k_2 \times k_3 \times \cdots \times k_n$ ways.

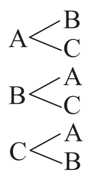
In set theoretic notation, the above principle is stated as:

$\prod_{i=1}^{r} A_i = A_1 \times A_2 \times \cdots \times A_r = \{(a_1, a_2, a_3, \ldots, a_r) : a_i \in A_i, i = 1, 2, 3, \ldots, r\}$ denotes the cartesian product of the finite sets $A_1, A_2, \ldots, A_r$ then $\left|\prod_{i=1}^{r} A_i\right| = \prod_{i=1}^{r} |A_i|$.

In other words:

If an event E can be decomposed into r *ordered* sub events $E_1, E_2, \ldots, E_r$ and if there are n_1 ways (independent to other sub events) for E_1 to occur, n_2 ways (independent to other sub events) for the event E_2 to occur, …, n_r ways (independent to other sub events) for E_r to occur, then the total number of ways for the event E to occur is given by $n_1 \times n_2 \times \cdots \times n_r$.

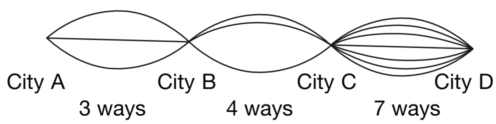
Example 5 *A Hall has 3 gates. In how many ways can a man enter the hall through one gate and come out through a different gate?*



Solution: Suppose the gates are A, B and C. Now there are 3 ways (A, B or C) of entering into the hall. After entering into the hall, the man come out through a different gate in 2 ways. Hence, by the multiplication principle, total number of ways is $3 \times 2 = 6$ ways.

Example 6 *There are 3 routes to travel from City A to City B and 4 routes to travel from City B to City C and 7 routes from C to D. In how many different ways (routes) a man can travel from City A to City D via City B and City C.*

Solution:



The man can perform the task of travelling from City A to City B in ways = 3.

The man can perform task of travelling from City B to City C in ways = 4.

Similarly from City C to City D in ways = 7.

Using fundamental principle of counting, total routes to travel from A to D via B and via C = $m \times n \times p = 3 \times 4 \times 7 = 84$ routes.

Example 7 *If* $S = \{a, b, c, \ldots, x, y, z\}$, *find the number of five-letter words that can be formed from the elements of the set S, such that the first and the last letters are distinct vowels and the remaining three are distinct consonants.*

Solution:

Place:					
Number of choices:	5	21	20	19	4

As there are 5 vowels and 21 consonants, position 1 and 5 can be filled in 5 and 4 ways respectively and 2, 3, 4 can be filled in 21, 20 and 19 ways respectively. Therefore, the total number of ways

$$= 5 \times 4 \times 21 \times 20 \times 19$$
$$= 400 \times 399 = 159600.$$

Example 8 *A city has* 12 *gates. In how many ways can a person enter the city through one gate and come out through a different gate?*

Solution: Since, there are 12 ways to enter into the city. After entering into the city, the man can come out through a different gate in 11 ways.
Hence, by the fundamental principle of counting.
Total number of ways is $12 \times 11 = 132$ ways.

Example 9 *A basket contains* 12 *apples and* 10 *oranges. Ram takes an apple or an orange. Then Shyam takes an apple and an orange. In which case does shyam have more choice: When Ram takes an apple or when he takes an orange? (Consider apples and similarly oranges are distinguishable.) In how many ways both of them can take the fruits?*

Solution:

Case 1: Ram takes an apple
Shyam has to take one apple and one orange from 11 apples and 10 oranges.
Number of ways in which Shyam can take his fruits = $11 \times 10 = 110$.

Case 2: Ram takes an orange
Shyam has to take one apple and one orange from 12 apples and 9 oranges.
Number of ways in which Shyam can take his fruits = $12 \times 9 = 108$.
Shyam has more choice when Ram takes an apple.
Using addition principle, number of ways in which both can take a fruit
$= 12 \times 110 + 10 \times 108$
$= 1320 + 1080 = 1400$

Example 10 *A number lock has* 3 *concentric rings on which the digits* 0, 1, 2, …, 9 *are engraved. Only one particular arrangement on the rings, say ABC, against an arrow opens the lock. What is the number of unsuccessful attempts to open the lock?*

Solution: Total number of numbers formed by the digits 0, 1, 2, …, 9 on the three rings $= 10 \times 10 \times 10$ (by multiplication principle) and number of successful attempts = 1
$\Rightarrow$ Number of unsuccessful attempts = $10^3 - 1$
$= 999$

Note: Here the method for counting used is called **i**ndirect method of counting**.**)

Example 11 *A binary sequence consists of 0's or 1's only. Find the number of binary sequences having n terms.*

Solution: Since every term of the binary sequence has two options (0 or 1), therefore the number of binary sequences of n terms = $\underbrace{2\times 2\times 2\times\cdots\times\ 2}_{\text{n times}} = 2^n$ (using multiplication principle).

Example 12 *How many* (i) 5 *digit* (ii) 3-*digit numbers can be formed using* 1, 2, 3, 7, 9 *without any repetition of digits.*

Solution:

(i) **5-digit numbers:**
Making a 5 digit number is equivalent to filling 5 places.
The last place (unit's place) can be filled in 5 ways using any of the five given digits.
The ten's place can be filled in four ways using any of the remaining 4 digits.
The number of choices for other places can be calculated in the same way.
Number of ways to fill all five places
$= 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$
$\Rightarrow$ 120 five-digit numbers can be formed.

Place:					
Number of choices:	1	2	3	4	5

(ii) **3-digit numbers:**
Making a three-digit number is equivalent to filling three places (unit's, ten's, hundred's).
Number of ways to fill all the three places $= 5 \times 4 \times 3 = 60$
$\Rightarrow$ 60 three-digit numbers can be formed.

Place:			
Number of choices:	1	2	3

Example 13 *How many* 3-*letter words can be formed using a, b, c, d, e if:*
(i) *Repetition is not allowed*
(ii) *Repetition is allowed?*

Solution:

(i) **Repetition is not allowed:**
The number of words that can be formed is equal to the number of ways to fill the three places.
First place can be filled in five ways using any of the five letters (a, b, c, d, e).
Similarly second and third places can be filled using 4 and 3 letters respectively.
$\Rightarrow$ Total number of ways to fill $= 5 \times 4 \times 3 = 60$.
Hence 60 words can be formed.

Place:			
Number of choices:	5	4	3

(ii) **Repetition is allowed:**
The number of words that can be formed is equal to the number of ways to fill the three places.
First place can be filled in five ways (a, b, c, d, e).
If repetition is allowed, each of the remaining places can be filled in five ways using a, b, c, d, e.
Total number of ways to fill $= 5 \times 5 \times 5 = 125$.
Hence 125 words can be formed.

Place:			
Number of choices:	5	5	5

Example 14 *How many four-digit numbers can be formed using the digits* 0, 1, 2, 3, 4, 5 *without repetition?*

Solution: For a four-digit number, we have to fill four places and 0 cannot appear in the first place (thousand's place).

Place: | | | | |

Number of choices: 5 5 4 3

For the first place, there are five choices (1, 2, 3, 4, 5); Second place can then be filled in five ways (0 and remaining four-digits); Third place can be filled in four ways (remaining four-digits); Fourth place can be filled in three ways (remaining three-digits).

Total number of ways $= 5 \times 5 \times 4 \times 3 = 300$

$\Rightarrow$ 300 four-digits numbers can be formed.

Example 15: *In how many ways can six persons be arranged in a row?*

Solution: Arranging a given set of n different objects is equivalent to fill n places. So arranging six persons along a row is equivalent to fill 6 places.

Place: | | | | | | |

Number of choices: 6 5 4 3 2 1

Number of ways to fill all places $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$.

Example 16 *How many 5-digit odd numbers can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition?*

Place: | | | | | |

Number of choices: 4 4 3 2 3

Solution: Making a 5-digit number is equivalent to fill 5 places

To make odd numbers, fifth place can be filled by either of 1, 3, 5, *i.e.*, 3 ways.

Number of ways first place can be filled in = 4 (excluding 0 and the odd number used for the fifth place).

Similarly second, third and fourth places can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting, total number of ways to fill 5 places.

= Total 5-digit odd numbers that can be formed $= 4 \times 4 \times 3 \times 2 \times 3 = 288$ ways.

Example 17 *How many 5-digit numbers divisible by* 2 *can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition.*

Solution: To find 5-digit numbers divisible by 2,

We will make 2 cases. In first case, we will find number of numbers divisible by 2 ending with either 2 or 4. In second case, we will find even numbers ending with 0.

Place: | | | | | |

Number of choices: 4 4 3 2 2

Case 1: Even numbers ending with 2 or 4:

Making a 5 digit number is equivalent to filling 5 places

Fifth place can be filled by 2 or 4, *i.e.*, 2 ways.

First place can be filled in 4 ways (excluding 0 and the digit used to fill fifth place)

Similarly places second, third and fourth can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting, total number of ways to fill all 5 places together $= 4 \times 4 \times 3 \times 2 \times 2 = 192$. (1)

Place: | | | | | |

Number of choices: 5 4 3 2 1

Case 2: Even numbers ending with 0:

Making a 5-digit number is equivalent to fill 5 place.

Fifth place is filled by 0, hence can be filled in 1 way.

First place can be filled in 5 ways (Using either of 1, 2, 3, 4, 5).

Similarly places second, third and fourth can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting, total number of ways to fill 5 places $= 5 \times 4 \times 3 \times 2 \times 1 = 120$ (2)

Combining (1) and (2),

Total number of 5 digit numbers divisible by 2 $= 192 + 120 = 312$.

Example 18 *How many 5-digit numbers divisible by 4 can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition?*

Solution: Making a 5-digit number is equivalent to fill 5 places.
A number would be divisible by 4 if the last 2 places are filled by either of 04, 12, 20, 24, 32, 40, 52.

Case 1:

Place:	☐	☐	☐	☐☐
Number of choices:	4	3	2	3

Last 2 places are filled by either of 04, 20, 40.
Fourth and fifth places can be filled in 3 ways. (either of 04, 20, 40).
First place can be filled in 4 ways (excluding the digits used to fill fourth and fifth place.
Similarly second and third place can be filled in 3 and 2 ways respectively.
Using fundamental principle of counting, total number of ways to fill 5 places
$= 4 \times 3 \times 2 \times 3 = 72$ ways (1)

Case 2:

Place:	☐	☐	☐	☐☐
Number of choices:	3	3	2	4

Last 2 places are filled by either of 12, 24, 32, 52
Fourth and fifth place can be filled in 4 ways (either 12, 24, 32, 52).
First place can be filled in 3 ways (excluding 0 and the digits used to fill fourth and fifth place)
Similarly, second and third place can be filled in 3 and 2 ways respectively.
Using fundamental principle of counting, total number of ways to fill 5 place
$= 3 \times 3 \times 2 \times 4 = 72$ ways. (2)
Combining (1) and (2),
Total number of ways to fill 5 places = Total 5-digit numbers divisible by 4
$= 72 + 72 = 144$.

Example 19 *How many six-digit numbers divisible by* 25 *can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition?*

Solution: Numbers divisible by 25 must end with 25 or 50.

Case 1: Number ending with 25

Place:	☐	☐	☐	☐	☐	☐
Number of choices:	3	3	2	1	1	1

Using fundamental principle of counting, total 6 digit numbers divisible by 25 ending with 25
$= 3 \times 3\,! = 18$ numbers are possible.

Case 2: Number ending with 50

Place:	☐	☐	☐	☐	☐	☐
Number of choices:	4	3	2	1	1	1

Using fundamental principle of counting, total 6 digited numbers divisible by 25 ending with 50
$= 4! = 24$ numbers are possible.
Hence, total numbers of multiples of 25
$= 18 + 24 = 42$.

Example 20 *Find the number of* 4*-digit numbers greater than* 3400*, when digits are chosen from* 1, 2, 3, 4, 5, 6 *with repetition allowed.*

Place:				
Number of choices:	3	6	6	6

Place:	3			
Number of choices:	1	3	6	6

Solution: To count the number of numbers greater than 3400, we consider the following two cases:

Case 1: Thousand's place filled by 4 or 5 or 6
(That is, thousand's place can be filled in 3 ways) each digit (of last three digits) has 6 options (*i.e.*, they can be filled by any of 1, 2, 3, 4, 5, 6). Using multiplication principle, the number of such numbers $= 3 \times 6 \times 6 \times 6 = 648$

Case 2: Thousand's place filled by 3 and hundred's place filled by 4 or 5 or 6.
(That is, thousand's place can be filled in 1 way and hundred's place can be filled in 3 ways)

Using multiplication principle, the number of such numbers $= 1 \times 3 \times 6 \times 6 = 108$

Cases I and II are *mutually exclusive* (*i.e., cannot occur together*) *and exhaustive* (*i.e., all possibilities are covered*)

$\therefore$ Using addition principle, the number of 4-digit numbers greater than 3400, (formed by 1, 2, 3, 4, 5, 6) $= 648 + 108 = 756$.

Example 21 *Find the number of odd integers between* 30,000 *and* 80,000 *in which no digit is repeated.*

Solution:
Let *abcde* be the required odd integers.

a can be chosen from 3, 4, 5, 6 and 7 and *e* can be chosen from 1, 3, 5, 7, 9. Note that 3, 5 and 7 can occupy both the positions *a* and *e*.

So, let us consider the case where one of 3, 5, 7 occupies the position *a*.

Case 1: If *a* gets one of the values 3, 5, 7, then there are 3 choices for *a*, but then, *e* has just four choices as repetition is not allowed. Thus, *a* and *e* can be chosen in this case in $3 \times 4 = 12$ ways.

The 3 positions *b*, *c*, *d* can be filled from among the remaining 8 digits in $8 \times 7 \times 6$ ways. Total number of ways in this case $= 12 \times 8 \times 7 \times 6 = 4{,}032$.

Case 2: If *a* takes the values 4 or 6, then there are two choices for *a* and there are five choices for *e*.

There are again eight choices altogether for the digits *b*, *c*, *d* which could be done in $8 \times 7 \times 6$ ways.

Therefore in this case, the total numbers are $2 \times 5 \times 8 \times 7 \times 6 = 3{,}360$.

Hence, total number of odd numbers between 30,000 to 80,000, without repetition of digits is $4{,}032 + 3{,}360 = 7{,}392$.

Example 22 *A number of four digits is to be formed from* 1, 2, 3, 4, 5 *and* 6. *Find the number of* 4-*digit numbers*

(i) *if repetition of a digit is allowed.*
(ii) *if no repetition of a digit is allowed.*
(iii) *How many of the numbers are divisible by 4, if*
  (a) *repetition is allowed?*
  (b) *repetition is not allowed?*

Solution:

(i) Since each digit of a 4 - digit number can be one from 1, 2, 3, 4, 5, 6, therefore using multiplication principle, the number of 4 digit numbers (repetition is allowed) $= 6 \times 6 \times 6 \times 6 = 6^4$

(ii) Using multiplication principle, the number of 4-digit numbers (repetition is not allowed) $= 6 \times 5 \times 4 \times 3 = 360$

(iii) If a number is divisible by 4, then the last two digits must form one of the following numbers :

12, 16, 24, 32, 36, 44, 52, 56, 64 (9 in all)

(a) Number of numbers, divisible by 4 (repetition is allowed) $= 6 \times 6 \times 9 = 324$

(b) Number of numbers, divisible by 4 (repetition is not allowed) $= 4 \times 3 \times 8 = 96$

Note that in (a), to fill the places of last two digits (considered it as one 2-digit number) 9 options are available as stated above.

Note that in (b), since repetition is not allowed, so the number formed by the last two digits cannot be 44. So it can be one from the remaining 8 options.

Example 23 *Find the sum of all five-digit numbers that can be formed using digits* 1, 2, 3, 4, 5 *if repetition is not allowed?*

Solution: There are $5! = 120$ five digit numbers and there are 5 digits. Hence by symmetry or otherwise we can see that each digit will appear in any place (unit's or ten's or …) $\frac{5!}{5}$ times.

Let $X =$ Sum of digits in any place

$$\Rightarrow X = \frac{5!}{5}\times 5 + \frac{5!}{5}\times 4 + \frac{5!}{5}\times 3 + \frac{5!}{5}\times 2 + \frac{5!}{5}\times 1$$

$$\Rightarrow X = \frac{5!}{5}\times (5+4+3+2+1) = \frac{5!}{5}(15) = 5!\times 3$$

$\Rightarrow$ The sum of the all numbers $= X + 10X + 100X + 1000X + 10000X$

$= X(1 + 10 + 100 + 1000 + 10000)$

$= 5! \times 3\ (1 + 10 + 100 + 1000 + 10000)$

$= 120 \times 3\ (11111) = 3999960.$

Example 24 *Find the sum of the four digit numbers obtained in all possible permutations of the digits* 1, 2, 3, 4.

Solution: There are 4! (= 24) 4-digit numbers made up of 1, 2, 3, 4. In these 24 numbers, in unit place all 1, 2, 3, 4 appear 3! (=6) times. Similarly, in the ten's, hundred's, thousand's places too, they appear 6 times.

Sum $= 6(4+3+2+1) + 10 \times 6\ (4+3+2+1) + 100 \times 6(4+3+2+1) + 1000 \times 6(4+3+2+1)$

$= 60 + 600 + 6000 + 60000 = 66,660$

Example 25 *Find the sum of 5-digit numbers obtained by permuting* 0, 1, 2, 3, 4.

Solution: There are 5! (= 120) 5-digit numbers made up of 0, 1, 2, 3, 4. In all these 120 numbers in unit's place all 0, 1, 2, 3, 4 appear 4! (= 24) times. Similarly in ten's, hundred's, thousand's and ten thousand's places too they appear 24 times.

Sum of 5-digit numbers made up of 0, 1, 2, 3, 4

$= 24(1 + 2 + 3 + 4) + 10 \times 24(1 + 2 + 3 + 4) + 100 \times 24(1 + 2 + 3 + 4)$
$+ 1000 \times 24\ (1 + 2 + 3 + 4) + 10000 \times 24(1 + 2 + 3 + 4)$

$= 240 + 2400 + 24000 + 240000 + 2400000 = 26,66,640.$

Required sum = Sum of 5-digit numbers made up of 0, 1, 2, 3, 4 – sum of 4 digit numbers made up of 1, 2, 3, 4

= 26,66,640 – 66660 {Obtained from previous example}

= 25,99,980.

Example 26 *Find the sum of all four digit numbers that can be formed using the digits* 0, 1, 2, 3, 4, *no digits being repeated in any number.*

Solution: Required sum of numbers = [Sum of four digit numbers using 0, 1, 2, 3, 4, allowing 0 in first place] – [Sum of three digit numbers using 1, 2, 3, 4].

$$= \frac{5!}{5}[0+1+2+3+4][1+10+10^2+10^3] - \frac{4!}{4}(1+2+3+4)(1+10+10^2)$$

$$= 24 \times 10 \times 1111 - 6 \times 10 \times 111 = 259980.$$

Example 27 *Let S be the set of natural numbers whose digits are chosen from* {1, 2, 3, 4} *such that*

(i) *When no digits are repeated, find $n(S)$ and the sum of all numbers in S.*
(ii) *When S is the set of up to 4-digit numbers where digits are repeated. Find $|S|$ and also find the sum of all the numbers in S.*

Solution:

(i) S consists of single-digit numbers, two-digit numbers, three-digit numbers and four-digit numbers.
Total number of single-digit numbers = 4
Total number of two-digit numbers = $4 \times 3 = 12$
(Since repetition is not allowed, there are four choices for tens place and three choices for units place.)
Total number of three-digit numbers = $4 \times 3 \times 2 = 24$
Total number of four-digit numbers = $4 \times 3 \times 2 \times 1 = 24$
$\therefore \qquad n(S) = 4 + 12 + 24 + 24 = 64.$
Now, for the sum of these 64 numbers, sum of all the single-digit numbers = $1 + 2 + 3 + 4 = 10$.
(Since there are exactly 4 digits 1, 2, 3, 4 and their numbers are 1, 2, 3 and 4.)
Now,
The total number of two-digit numbers is 12.
The digits used in units place are 1, 2, 3 and 4.
In the 12 numbers, each of 1, 2, 3 and 4 occurs thrice in units digit $\left(\frac{12}{4} = 3\right)$.
Again in tens place, each of these digits occurs thrice.
So, sum of these 12 numbers
$= 30 \times (1 + 2 + 3 + 4) + 3 \times (1 + 2 + 3 + 4)$
$= 300 + 30 = 330.$
The number of three-digit numbers is 24.
So, the number of times each of 1, 2, 3, 4 occurs in each of units, tens and hundreds place is $\frac{24}{4} = 6$.
So, the sum of all these three-digit numbers is
$100 \times 6(1 + 2 + 3 + 4) + 10 \times 6(1 + 2 + 3 + 4) + 1 \times 6(1 + 2 + 3 + 4)$
$= 6{,}000 + 600 + 60 = 6{,}660.$
Similarly, for the four-digit numbers, the sum is computed as
$1000 \times 6(1 + 2 + 3 + 4) + 100 \times 6(1 + 2 + 3 + 4) + 10 \times 6(1 + 2 + 3 + 4)$
$+ 1 \times 6(1 + 2 + 3 + 4) = 60{,}000 + 6{,}000 + 600 + 60 = 66{,}660$
[Since there are 24 four-digit numbers, each of 1, 2, 3, 4 occurs in each of the four digits in $\frac{24}{4} = 6$ times.]

So, the sum of all the single-digit, two-digit, three-digit and four-digit numbers = 10 + 330 + 6,660 + 66,660 = 73,660.

(ii) (a) There are just four single-digit numbers 1, 2, 3 and 4.
(b) There are $4 \times 4 = 16$ two-digit numbers, as digits can be repeated.
(c) There are $4 \times 4 \times 4 = 64$ three-digit numbers.
(d) There are $4 \times 4 \times 4 \times 4 = 256$ four-digit numbers.

So, total number of numbers up to four-digit numbers that could be formed using the digits 1, 2, 3 and 4 is 4 + 16 + 64 + 256 = 340. Sum of the 4 single-digit numbers = 1 + 2 + 3 + 4 = 10. To find the sum of 16 two-digit numbers each of 1, 2, 3, 4 occur in each of units and tens place $= \frac{16}{4} = 4$ times.

So, the sum of all these 16 numbers is

$$= 10 \times 4(1 + 2 + 3 + 4) + 4(1 + 2 + 3 + 4)$$
$$= 400 + 40 = 440.$$

Similarly, the sum of all the 64 three-digit numbers

$$= 100 \times \frac{64}{4} \times (1+2+3+4) + 10 \times \frac{64}{4} \times (1+2+3+4) + 1 \times \frac{64}{4} \times (1+2+3+4)$$
$$= 16,000 + 1,600 + 160 = 17,760.$$

Again the sum of all the 256 four-digit numbers

$$= 1000 \times \frac{256}{4} \times (1+2+3+4) + 100 \times \frac{256}{4} \times (1+2+3+4)$$
$$+ 10 \times \frac{256}{4} \times (1+2+3+4) + 1 \times \frac{256}{4} \times (1+2+3+4)$$
$$= 6,40,000 + 64,000 + 6,400 + 640 = 7,11,040$$

Therefore, the sum of all the numbers is
=10 + 440 + 17,760 + 7,11,040 = 7,29,250.

Build-up Your Understanding 1

1. How many four digit numbers can be made by using the digits 1, 2, 3, 7, 8, 9 when
   (i) repetition of a digit is allowed?
   (ii) repetition of a digit is not allowed?
2. Find the total number of 9-digit numbers of different digits.
3. Find the total number of 4 digit number that are greater than 3000, that can be formed by using the digits 1, 2, 3, 4, 5, 6 (no digit is being repeated in any number).
4. How many numbers greater than 1000 or equal to, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4, repetition of digits being allowed?
5. How many numbers between 400 and 1000 (both exclusive) which can be made with the digits 2,3,4,5,6,0 if
   (i) repetition of digits not allowed?
   (ii) repetition of digits is allowed?
6. A variable name in a certain computer language must be either an alphabet or a alphabet followed by a decimal digit. Find the total number of different variable names that can exist in that language.

7. Tanya typed a six-digit number, but the two 1's she typed did not show. What appeared was 2006. Find the number of different 6-digit numbers she would have typed.
8. A letter lock consists of three rings each marked with fifteen different letters. It is found that a man could open the lock only after he makes half the number of possible unsuccessful attempts to open the lock . If each attempt takes 10 seconds. Then find the minimum time he must have spent.
9. Find the number of 6-digit numbers that can be formed using 1, 2, 3, 4, 5, 6, 7 so that digits do not repeat and terminal digits are even.
10. Find the total number of numbers that can be formed by using all the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places.
11. Find the number of 6-digit numbers which have 3 digits even and 3 digits odd, if each digit is to be used atmost once.
12. Find the number of 4-digits numbers that can be made with the digits 1, 2, 3, 4 and 5 in which at least two digits are identical.
13. Find the number of 5-digit telephone numbers having atleast one of their digits is repeated.
14. Find the number of 3-digit numbers having only two consecutive digits identical.
15. Find the number of different matrices that can be formed with elements 0, 1, 2 or 3, each matrix having 4 elements.
16. Find the number of 6-digit numbers in which sum of the digits is even.
17. Find the number of 5-digit numbers divisible by 3 which can be formed using 0, 1, 2, 3, 4, 5 if repetition of digits is not allowed.
18. Find the number of 4-digit numbers divisible by 3 that can be formed by four different even digits.
19. Find the number of 5-digit numbers divisible by 6 which can be formed using 0, 1, 2, 3, 4, 5 if repetition of digits is not allowed.
20. Find the number of 5-digit numbers divisible by 4 which can be formed using 0, 1, 2, 3, 4, 5, when the repetition of digits is allowed
21. Natural numbers less than 10^4 and divisible by 4 and consisting of only the digits 0, 1, 2, 3, 4 and 5 (no repetition) are formed . Find the number of ways of formation of such number.
22. Find the number of natural numbers less than 1000 and divisible by 5 which can be formed with the ten digits, each digit not occurring more than once in each number.
23. Two numbers are chosen from 1, 3, 5, 7,…, 147, 149 and 151 and multiplied together. Find the number of ways which will give us the product a multiple of 5.
24. A 7-digit number divisible by 9 is to be formed by using 7 digits out of digits 1, 2, 3, 4, 5, 6, 7, 8, 9. Find the number of ways in which this can be done.
25. Find the number of 9-digits numbers divisible by nine using the digits from 0 to 9 if each digit is used atmost once.
26. Among 9! permutations of the digits 1, 2, 3,…, 9. Consider those arrangements which have the property that if we take any five consecutive positions, the product of the digits in those positions is divisible by 7. Find the number of such arrangements.
27. Find the number of distinct results which can be obtained when n distinct coins are tossed together.
28. Three distinct dice are rolled. Find the number of possible outcomes in which at least one die shows 5.
29. A telegraph has 'm' arms and each arm is capable of 'n' distinct positions including the position of rest. Find the total number of signals that can be made.
30. Find the number of possible outcomes in a throw of n distinct dice in which at least one of the dice shows an odd number.

31. Find the number of times the digit 5 will be written when listing integers from 1 to 1000.

32. Find the number of times of the digits 3 will be written when listing the integer from 1 to 1000.

33. If 33! is divisible by 2^n, then find the maximum value of n.

34. Let $E = \left[\frac{1}{3}+\frac{1}{50}\right]+\left[\frac{1}{3}+\frac{2}{50}\right]+\left[\frac{1}{3}+\frac{3}{50}\right]+\cdots$ upto 50 terms, then find the exponent of 2 in $(E)!$.

35. 3-digit numbers in which the middle one is a perfect square are formed using the digits 1 to 9 . Find their sum.

36. Find the sum of all the 4-digit even numbers which can be formed by using the digits 0, 1, 2, 3, 4 and 5 if repetition of digits is allowed.

37. Find sum of 5-digit numbers that can be formed using 0, 0, 1, 2, 3, 4.

38. Find sum of 5-digit numbers that can be formed using 0, 0, 1, 1, 2, 3.

39. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (that is 1, 16, 31, etc.) This process is continued until a number is reached which has already been marked, then find the all unmarked numbers.

40. Let S be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Find the number of subsets A of S such that. $x \in A$ and $2x \in S \Rightarrow 2x \in A$.

7.3 Combinations

7.3.1 Definition of Combination

Let A, B, C, D be four distinct objects. The number of ways in which we can select two objects out of A, B, C and D is six and these are AB, AC, AD, BC, BD and CD.

These ways of selection of two objects from four different objects are also known as combinations of A, B, C and D taken two at a time or we can say grouping of A, B, C and D taken two at a time.

Similarly $\{a, b, c\}$, $\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$ are *all* the selections of 3 objects from a, b, c, d. So we say that the number of ways of selecting 3 objects out of given 4 objects is 4 or the number of combinations of 3 objects out of given 4 objects is 4.

Note:
By changing the relative positions of objects, we do not get any new combinations. Combination (selection or group) of objects A, B is same as combination of objects B, A. Thus we treat AB and BA as same combination (selection or group). Formally

A combination of objects is merely a selection (suppress order) from a given lot of objects, i.e., a combination is just a set, elements of which are not arranged in a particular way.

7.3.2 Theorem

The number of selections of r objects at a time out of n distinct, is $\frac{n!}{r!(n-r)!}$.

This number is denoted as ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$.

Proof:

$\binom{n}{r-1}$ represents the number of selections of $r-1$ objects out of n distinct objects.

Number of ways to select rth object from remaining $n-(r-1)$ objects is $n-(r-1)$.

By multiplication principle, the number of ways to select r objects out of n distinct objects is apparently $\binom{n}{r}\cdot(n-r+1)$.

However, each selection is counted r times. Note that we are aiming at counting the unordered selections.

For example, $\{a, b, c\}$ or $\{b, a, c\}$ or $\{c, a, b\}$ are to be considered as one selection (not 3 selections)

Therefore ${}^nC_r = {}^nC_{r-1}\,\frac{n-r+1}{r}$. (recurrence relation)

$$ {}^nC_{r-1} = {}^nC_{r-2}\cdot\frac{n-(r-1)+1}{r-1} = {}^nC_{r-2}\frac{n-r+2}{r-1} $$

$$ {}^nC_{r-2} = {}^nC_{r-3}\cdot\frac{n-r+3}{r-2}, \text{ etc.} $$

$$ \therefore\ {}^nC_r = {}^nC_1\cdot\frac{(n-1)(n-2)\cdots(n-r+1)}{r(r-1)\cdots 2\cdot 1} $$

$$ = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)\cdots 2\cdot 1} \quad (\text{Note that } {}^nC_1 = n) $$

$$ = \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 2\cdot 1}{(r(r-1)\cdots 2\cdot 1)((n-r)(n-r-1)\cdots 2\cdot 1)} $$

$$ {}^nC_r = \frac{n}{r!(n-r)!} $$

In general
$$ \binom{n}{r} = \begin{cases} \dfrac{n!}{r!(n-r)!}, & 0 \le r \le n;\ r, n \in \mathbb{N}_0 \\ 0 \quad , & \text{for } r < 0 \text{ or } r > n;\ n \in \mathbb{N}_0, r \in \mathbb{Z} \end{cases} $$

Note: $\binom{0}{0}$ is defined as 1.

7.3.3 Properties of $\binom{n}{r}$; $0 \le r \le n$; $r, n \in \mathbb{N}_0$

(i) $\binom{n}{0} = \binom{n}{n} = 1$

(ii) $\binom{n}{r} = \binom{n}{n-r}$

(iii) If $\binom{n}{r} = \binom{n}{k}$ then $r = k$ or $n - r = k$

(iv) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

(v) $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$ or $r\binom{n}{r} = n\binom{n-1}{r-1}$

(vi) $\frac{1}{r+1}\binom{n}{r} = \frac{1}{n+1}\binom{n+1}{r+1}$

(vii) $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$ or $\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{n-r+1}{r}$

(viii) $\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}$

(ix) (a) If n is even, $\binom{n}{r}$ is greatest for $r = \frac{n}{2}$.

(b) If n is odd, $\binom{n}{r}$ is greatest for $r = \frac{n-1}{2}, \frac{n+1}{2}$.

In general $\binom{n}{r}$ is maximum at $r = \left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil$

Combinatorial proof of (v):

Consider a group of n people. A committee of r people is to be selected, out of these selected r people one chairperson is nominated.

This can be done in following two ways:

(i) Select r people from n people and select one person for chairperson from selected r people.

This can be done in $\binom{n}{r} \times \binom{r}{1}$ ways.

(ii) Another alternative is to select one person as the chairperson from n people and select remaining $(r - 1)$ people from remaining $(n - 1)$ people.

This can be done in $\binom{n}{1} \times \binom{n-1}{r-1}$ ways.

$$\Rightarrow r\binom{n}{r} = n\binom{n-1}{r-1}$$

Students are advised to develop the combinatorials proofs of the remaining properties.

Example 28 *If* ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ *and* ${}^nC_{r+1} = 126$, *then find r.*

Solution:

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{84}{36}$$

$$\Rightarrow \frac{n-r+1}{r} = \frac{7}{3} \qquad \left(\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right)$$

$$\Rightarrow 3n - 3r + 3 = 7r$$

$$\Rightarrow 10r - 3n = 3 \qquad (1)$$

and

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-(r+1)+1}{(r+1)} = \frac{126}{84} \qquad \left(\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right)$$

$$\Rightarrow \frac{n-r}{r+1} = \frac{3}{2} \Rightarrow 2n - 2r = 3r + 3$$

$$\Rightarrow 5r - 2n = -3 \Rightarrow 10r - 4n = -6 \qquad (2)$$

Subtracting Eq. (2) from Eq. (1), we get $n = 9$

$$10r - 27 = 3\ 10 \Rightarrow r = 30 \Rightarrow r = 3$$

Example 29 *There were some men and two women participating in a chess tournament. Every participant played two games with every other participant. The number of games that the men played among themselves exceed by 66 that of the games which the men played with the two women. What was the total number of participants? How many games were played in all?*

Solution: Let the number of men participants be m.

The number of games which men have played among themselves is $2\binom{m}{2} = m(m-1)$.

The number of games which the men played with each of the two women is $2m \times 2$.

[m men played $2 \times m$ game with the first woman and another $2 \times m$ game with the second woman.]

According to the data given

$$\begin{aligned} m(m-1) - 2 \times 2m &= 66 \\ \Rightarrow m^2 - 5m - 66 &= 0 \\ \Rightarrow (m-11)(m+6) &= 0 \\ \Rightarrow m &= 11 (m = -6 \text{ is not acceptable}) \end{aligned}$$

So, there are totally $11 + 2 = 13$ players.

The number of games played is $2 \times {}^{13}C_2 = 2 \times \frac{13 \times 12}{1.2} = 156.$

7.3.4 Some Applications of Combinations

7.3.4.1 Always Including *p* Particular Objects in the Selection

The number of ways to select r objects from n distinct objects where p particular objects should always be included in the selection $= {}^{n-p}C_{r-p}$.

Logic:

We can select p particular objects in 1 way. Now from remaining $(n - p)$ objects we select remaining $(r - p)$ objects in ${}^{n-p}C_{r-p}$ ways.

Using fundamental principle of counting, number of ways to select r objects where p particular objects are always included

$$= 1 \times {}^{n-p}C_{r-p} = {}^{n-p}C_{r-p}.$$

Example 30 *In how many ways a team of* 11 *players be selected from a list of* 16 *players where two particular players should always be included in the team.*

Solution: Number of ways to make a team of 11 players from 16 players always including 2 particular players $= {}^{16-2}C_{11-2} = {}^{14}C_9$.

7.3.4.2 Always Excluding *p* Particular Objects in the Selection

The number of ways to select r objects from n different objects where p particular objects should never be included in the selection $= {}^{n-p}C_r$.

Logic:

As p particular objects are never to be selected, selection should be made from remaining $n - p$ objects. Therefore r objects can be selected from $(n - p)$ different objects in ${}^{n-p}C_r$ ways.

Example 31 *In how many ways a team of* 11 *players can be selected from a list of* 16 *players such that* 2 *particular players should never be included in the selection.*

Solution: The number of ways to select a team of 11 players from a list of 16 players, always excluding 2 particular players = ${}^{(16-2)}C_{11} = {}^{14}C_{11}$.

Example 32 *A mixed doubles tennis game is to be arranged from 5 married couples. In how many ways the game can be arranged if no husband and wife pair is included in the same game?*

Solution: To arrange the game we have to do the following operations.

(i) Select two men from from 5 men in ${}^5C_2$ ways.

(ii) Select two women from 3 women excluding the wives of the men already selected. This can be done in ${}^3C_2$ ways.

(iii) Arrange the 4 selected persons in two teams. If the selected men are M_1 and M_2 and the selected women are W_3 and W_4, this can be done in 2 ways :

M_1W_3 play against M_2W_4

M_2W_3 play against M_1W_4

Hence the number of ways to arrange the game

$$= {}^5C_2\,{}^3C_2\,(2) = 10 \times 3 \times 2 = 60.$$

7.3.4.3 Exactly or Atleast or Atmost Constraint in the Selection

There are problems in which constraints are to select exactly or minimum (atleast) or maximum (atmost) number of objects in the selection. In these problems, we should always make cases to select objects. If we do not make cases, we will get wrong answer. Following illustrations will show you how to make cases to solve problems of this type.

Example 33 *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicketkeepers? Assume that the team of 11 players requires 5 batsmen, 3 all-rounders, 2-bowlers and 1 wicketkeeper.*

Solution: Divide the selection of team into four operations.

(i) Selection of batsman can be done (5 from 10) in ${}^{10}C_5$ ways.

(ii) Selection of bowlers can be done (2 from 8) in ${}^8C_2$ ways.

(iii) Selection of all-rounders can be done (3 from 5) in ${}^5C_3$ ways.

(iv) Selection of wicketkeeper can be done (1 from 2) in ${}^2C_1$ ways.

$\Rightarrow$ The team can be selected in ${}^{10}C_5 \times {}^8C_2 \times {}^5C_3 \times {}^2C_1$ ways $= \dfrac{10! \times 8 \times 7 \times 10 \times 2}{5!5!2!} = 141120.$

Example 34 *In a group of 80 persons of an association, a chairman, a secretary and three members are to be elected for the executive committee. Find in how many ways this could be done.*

Solution: This would be done in:

Chairman can be elected in ${}^{80}C_1$ ways,

Secretary can be elected in ${}^{79}C_1$ ways and the three members can be elected in ${}^{78}C_3$ ways.

So, the total number of ways in which this executive committee can be selected is

$$\begin{aligned} {}^{80}C_1 \times {}^{79}C_1 \times {}^{78}C_3 &= 80 \times 79 \times \frac{78 \times 77 \times 76}{1 \times 2 \times 3} \\ &= 80 \times 79 \times 13 \times 77 \times 76 \\ &= 800{,}320 \text{ ways.} \end{aligned}$$

Example 35 *A box contains 5 distinct red and 6 distinct white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?*

Solution: The selection of balls from 5 red and 6 white balls will consist of any of the following possibilities.

Red Balls (out of 5)	2	3	4
White Balls (out of 6)	4	3	2

If the selection contains 2 red and 4 white balls, then it can be done in ${}^5C_2\,{}^6C_4$ ways.
If the selection contains 3 red and 3 white balls then it can be done in ${}^5C_3\,{}^6C_3$ ways.
If the selection contains 4 red and 2 white balls then it can be done in ${}^5C_4\,{}^6C_2$ ways.

Any one of the above three cases can occur. Hence the total number of ways to select the balls.

$$\begin{aligned}&= {}^5C_2\,{}^6C_4 + {}^5C_3\,{}^6C_3 + {}^5C_4\,{}^6C_2\\ &= 10(15) + 10(20) + 5(15)\\ &= 425.\end{aligned}$$

Example 36 *In how many ways a team of 5 members can be selected from 4 ladies and 8 gentlemen such that selection includes at least 2 ladies?*

Solution: As the selection includes 'atleast' constraint, we make cases to find total number of teams.

Ladies in the team	Gentlemen in the team	Number of ways to select team
2	3	${}^4C_2 \times {}^8C_3$
3	2	${}^4C_3 \times {}^8C_2$
4	1	${}^4C_4 \times {}^8C_1$

Combining all cases shown in the table, total number of ways to select a team of 5 members

$$= {}^4C_2 \times {}^8C_3 + {}^4C_3 \times {}^8C_2 + {}^4C_2 \times {}^8C_1 = 456.$$

Example 37 *In a company there are 12 job vacancies. Out of 12, 3 are reserved for 'reserved category' candidates and rest 9 are open for all. In how many ways these 12 vacancies can be filled by 5 from 'reserved category' and 10 from general category candidates?*

Solution: There are 12 vacancies. As 3 are reserved for 'reserved category' candidates, it means we have to select 12 candidates (to fill 12 vacancies) such that selection should include at least 3 candidates from 'reserved category'. As rest 9 vacancies are open for all, it means 'reserved category' candidates can also take these vacancies.

As selection includes atleast constraint, we need to make following cases:

Reserved category	General category candidates	Number of ways to select
3	9	${}^5C_3 \times {}^{10}C_9$
4	8	${}^5C_4 \times {}^{10}C_8$
5	7	${}^5C_5 \times {}^{10}C_7$

Combining all cases shown above, we get, number of ways to fill 2 vacancies

$$\begin{aligned}&= {}^5C_3 \times {}^{10}C_9 + {}^5C_4 \times {}^{10}C_8 + {}^5C_5 \times {}^{10}C_7\\ &= 100 + 225 + 120 = 445 \text{ ways.}\end{aligned}$$

Example 38 *A man has 7 relatives, 4 of them ladies and 3 gentlemen; his wife has 7 relatives, 3 of them are ladies and 4 gentlemen. In how many ways they can invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?*

Solution: The possible ways of selecting 3 ladies and 3 gentleman for the party can be analysed with the help of the following table.

Man's relative		Wife's relative		
Ladies (4)	Gentleman (3)	Ladies (3)	Gentleman (4)	Number of ways
3	0	0	3	${}^4C_3\,{}^3C_0\,{}^3C_0\,{}^4C_3 = 16$
2	1	1	2	${}^4C_2\,{}^3C_1\,{}^3C_1\,{}^4C_2 = 324$
1	2	2	1	${}^4C_1\,{}^3C_2\,{}^3C_2\,{}^4C_1 = 144$
0	3	3	0	${}^4C_0\,{}^3C_3\,{}^3C_3\,{}^4C_0 = 1$

Total number of ways to invite = 16 + 324 + 144 + 1 = 485.

7.3.4.4 Selection of One or More Objects

7.3.4.4.1 From n Distinct Objects

The number of ways to select one or more objects from n different objects or we can say, selection of at least one object from n different objects = $2^n - 1$.

Logic:

The number of ways to select 1 object from n different objects = ${}^nC_1$

The number of ways to select 2 objects from n different objects = ${}^nC_2$

… … …

… … …

… … …

The number of ways to select n objects from n different objects = ${}^nC_n$

Combining all above cases, we get the number of ways to select at least one (one or more) object from n different objects

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_4 + \cdots + {}^nC_n$

$= 2^n - 1$ [Using sum of binomial coefficients in the expansion of $(1 + x)^n = 2^n$]

Alternate logic:

Let us assume $a_1, a_2, a_3, \ldots, a_n$ be n distinct objects.

We have to make our selection from these n objects.

We can make out selection from a_1 object in 2 ways.

This is because either we will choose a_1 or we would not choose a_1. Similarly selection of $a_2, a_3, \ldots, a_n$ can be done in 2 ways each.

Objects	a_1	a_2	a_3	a_4	…	a_n
Ways	2	2	2	2	…	2

Using fundamental principle of counting, the total number of ways to make selection from $a_1, a_2, ..., a_n$

$$= 2 \times 2 \times 2 \times 2 \ldots n \text{ times}$$
$$= 2^n$$

But the above selection includes a case where we have not selected any object. On subtracting this case from 2^n we get, the number of ways to select atleast one (one or more) object from n different objects = $2^n - 1$

Notes:

1. The number of ways to select 0 or more objects from n distinct objects $= 2^n$
2. The number of ways to select at least 2 objects from n distinct objects $= 2^n - 1 - {}^nC_1$
3. The number of ways to select at least r objects from n distinct objects $= 2^n - 1 - {}^nC_1 - {}^nC_2 - {}^nC_3 - \cdots - {}^nC_{r-1}$ or ${}^nC_r + {}^nC_{r+1} + {}^nC_{r+2} + \cdots + {}^nC_n$.

7.3.4.4.2 From n Identical Objects

The number of ways to select one or more objects (or at least one object) from n identical object $= n$.

Logic:

To select r objects from n identical objects, we cannot use $\binom{n}{r}$ formula here, as all objects are not distinct. In fact, all objects are identical. It means we cannot choose objects. It does not matter which r objects we take as all objects are identical.

The number of ways to select 1 object from n identical objects = 1

The number of ways to select 2 object from n identical objects = 1

… … …

… … …

… … …

The number of ways to select n objects from n identical objects = 1.

Combining all above cases, we get

Total number of ways to select 1 or more objects fron n identical objects

$$= 1 + 1 + \cdots n \text{ times} = n$$

Notes:

1. The number of ways to select r objects from n identical objects is 1.
2. The number of ways to select 0 or more objects from n identical objects $= n + 1$.
3. The number of ways to select at least 2 objects from n identical objects $= n - 1$.
4. The number of ways to select atleast r objects from n identical objects is $n - (r - 1) = n - r + 1$
5. The total number of selections of some or all out of $(p + q + r)$ objects where p are alike of one kind, q are alike of second kind and rest r are alike of third kind is $(p + 1)(q + 1)(r + 1) - 1$. [Using fundamental principle of counting]

7.3.4.4.3 From Objects Which are not All Distinct from Each Other

The number of ways to select one or more objects from $(p + q + r \cdots + n)$ objects where p objects are alike of one kind, q are alike of second kind, r are alike of third kind, … and remaining n are distinct from each other

$$= [(p + 1)(q + 1)(r + 1) \cdots 2^n] - 1.$$

Logic:

The numbers of ways to select 0 or more objects from p alike objects of one kind $= p + 1$

The number of ways to select 0 or more objects from q alike objects of second kind $= q + 1$

The number of ways to select 0 or more objects from r alike objects of third kind $= r + 1$

… … …

… … …

… … …

The number of ways to select 0 or more objects from n distinct objects $= 2^n$

Combining all cases and using fundamental principle of counting, we get:

Total number of ways to select 0 or more objects

$$= [(p + 1)\,(q + 1)\,(r + 1) \ldots\,]2^n \qquad (1)$$

But above selection includes a case where we have not selected any object. So we need to subtract 1 from the above result if we want to select at least one object.

Therefore, the total number of ways to select one or more objects (at least one) from p alike of one kind, q alike of another kind, r alike to third kind … and n distinct objects

$$= [(p + 1)\,(q + 1)\,(r + 1) \ldots\,]2^n - 1$$

Notes:

1. The number of ways to select 0 or more objects from p alike of one kind, q alike of second kind, r alike of third kind and n distinct objects $= (p + 1)\,(q + 1)\,(r + 1)\,2^n$.
2. The number of ways to select objects from p alike of one kind, q alike of second kind and r alike of third kind and n distinct objects such that selection includes at least one object each of first, second, and third kind and atleast one object from n different kind $= pqr(2^n - 1)$.
3. The number of ways to select objects from p alike of one kind, q alike of second kind and r alike of third kind and n distinct objects such that selection includes at least one object of each kind $= pqr$.

Example 39 *A man has 5 friends. In how many ways can he invite one or more of them to a party?*

Solution: If he invites one person to the party, number of ways $= {}^5C_1$

If he invites two persons to the party, number of ways $= {}^5C_2$

Proceeding on the similar pattern, total number of ways to invite

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$
$$= 5 + 10 + 10 + 5 + 1 = 31$$

Alternate Method:

To invite one or more friends to the party, he has to take 5 decisions, one for every friend.

Each decision can be taken in two ways, invited or not invited.

Hence, the number of ways to invite one or more

= (number of ways to make 5 decisions - 1)

$$= 2 \times 2 \times 2 \times 2 \times 2 - 1 = 2^5 - 1 = 31$$

Note that we have to subtract 1 to exclude the case, when all are not invited.

Example 40 *Prove that there are* $2(2^{n-1} - 1)$ *ways of dealing n distinct cards to two persons.* (*The players may receive unequal number of cards and each one receiving at least one card*).

Solution: Let us number the cards for the moment. Let us accept the case where all the cards go to one of the two players, also with just two cards, we have the possibilities,

AA AB BA BB (1)

Here, AA means A gets card 1 and also card 2,

AB means A gets card 1 and B gets card 2,

BA means B gets card 1 and A gets card 2, BB means B gets card 1 and also card 2.

Thus, for two cards we have four possibilities.

For three cards

$$\text{AAA, ABA, BAA, BBA, AAB, ABB, BAB, BBB} \tag{2}$$

That is, for three cards there are $2^3 = 8$ possibilities. Here, if the third card goes to A, then, in Eq. (1) annex A at the end, thus getting

$$\text{AAA, ABA, BAA, BBA.}$$

Thus, the possibilities doubled, when a new card (third card) is included.

In fact just with one card it may either go to A or B.

By annexing the second card, it may give

AA	BA	giving (1)
AB	BB	

Thus, every new card doubles the existing number of possibilities of distributing the cards.

Hence, the number of possibilities with n cards is 2^n. But this includes the 2 distributions where one of them gets all the cards, and the other none.

So, total number of possibilities is $2^n - 2 = 2(2^{n-1} - 1)$.

Note: We can look at the same problem in the following way. The above distribution of cards is the same as number of possible n-digit numbers where only two digits 1 and 2 are used, and each digit must be used at least once. This is $2^n - 2 = 2(2^{n-1} - 1)$.

Aliter: Since n cards are dealt with and each player must get at least one card, player 1 can get r cards and player 2 get $(n - r)$ cards where $1 \le r \le n - 1$. Now, player 1 can get r cards in $C(n, r)$ ways. Total number of ways of dealing cards to players 1 and 2

$$= \sum_{r=1}^{n-1} C(n,\ r) = \sum_{r=0}^{n} C(n,\ r) - C(n,\ 0) - C(n,\ n) = 2^n - 2.$$

Example 41 *Find the number of ways in which one or more letters can be selected from the letters:*

A A A A B B B C D E

Solution: The given letters can be divided into five following categories: (AAAA), (BBB), C, D, E

To select at least one letter, we have to take five decisions—one for every category. Selections from (AAAA) can be made in 5 ways: include no A, include one A, include AA, include AAA, include AAAA.

Similarly, selections from (BBB) can be made in 4 ways, and selections from C, D, E can be made in $2 \times 2 \times 2$ ways.

$\Rightarrow$ Total number of selections $= 5 \times 4 \times (2 \times 2 \times 2) - 1 = 159$

(excluding the case when no letter is selected).

Example 42 *The question paper in the examination contains three sections: A, B, C. There are 6, 4, 3 questions in sections A, B, C respectively. A student has the freedom to answer any number of questions attempting at least one from each section. In how many ways can the paper be attempted by a student?*

Solution: There are three possible cases:

Case 1: Section A contains 6 questions. The student can select at least one from these in $2^6 - 1$ ways.

Case 2: Section B contains 4 questions. The student can select at least one from these in $2^4 - 1$ ways.

Case 3: Section C can similarly be attempted in $2^3 - 1$ ways.

Hence, total number of ways to attempt the paper

$$= (2^6 - 1)(2^4 - 1)(2^3 - 1)$$
$$= 63 \times 15 \times 7 = 6615.$$

Example 43 *Find the number of factors (excluding 1 and the expression itself) of the product of $a^7 b^4 c^3 d e f$ where a, b, c, d, e, f are all prime numbers.*

Solution: A factor of expression $a^7 b^4 c^3 d e f$ is simply the result of selecting none or one or more letters from 7 a's, 4 b's, 3 c's, d, e, f

The collection of letters can be observed as a collection of 17 objects out of which 7 are alike of one kind (a's), 4 are of second kind (b's), 3 are of third kind (c's) and 3 are distinct (d, e, f).

The number of selections $= (1 + 7)(1 + 4)(1 + 3)\, 2^3 = 8 \times 5 \times 4 \times 8 = 1280$.

But we have to exclude two cases :

(i) When no letter is selected, (ii) When all letters are selected.

Hence the number of factors $= 1280 - 2 = 1278$.

Example 44 *Find the number of positive divisors of $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r}$, where $p_1, p_2, \ldots, p_r$ are distinct prime numbers and $k_1, k_2, \ldots, k_r$ are positive integers.*

Solution: A divisor d of n is of the form

$$d = p_1^{l_1} \cdot p_2^{l_2} \cdots p_r^{l_r} \text{ where } 0 \le l_i \le k_i, i = 1, 2, \ldots, r.$$

Associate each divisor d of n with an r tuple $(l_1, l_2, \ldots, l_r)$ such that $0 \le l_i \le k_i$. Therefore, the number of divisors is the same as the number of r tuples $(l_1, l_2, \ldots, l_r)$, $0 \le l_i \le k_i$, $i = 1, 2, \ldots, r$.

Since l_1, can have $k_1 + 1$ possible values $0, 1, 2, \ldots, k_1$ similarly l_2, can have $k_2 + 1$ values and so on. The number of r-triples $(l_1, l_2, \ldots, l_r)$ is

$$(k_1 + 1) \times (k_2 + 1) \times (k_3 + 1) \times \cdots \times (k_r + 1) = \prod_{i=1}^{r} (k_i + 1)$$

That is the total number of divisors of

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r} \text{ is } (k_1 + 1)(k_2 + 1) \cdots (k_r + 1) = \prod_{i=1}^{r} (k_i + 1).$$

Note: Also refer article 6.6 on page 6.13 of number theory chapter.

7.3.4.5 Selection of r Objects from n Objects when All n Objects are not Distinct

In this problem type we will discuss how to select r objects from n objects when all n objects are not distinct.

For example, selection of 3 letters from letters AABBBC.

To find number of ways to select, it is possible to derive a formula that can be applied in all such cases.

Instead of formula, we will discuss a method (procedure) that should be applied to find selections.

The method involves making cases based on alike items in the selection. You should go through the following examples to learn how to apply this 'method of cases' to find selections of r objects from n objects when all n objects are not distinct.

Example 45 *In how many ways* 3 *letters can be selected from letters AABBBC.*

Solution: The given letters include AA, BBB, C, *i.e.*, 2A letters, 3B letters and 1C letter.
To find number of selections, we will make the following cases based on alike letters we choose in the selection.

Case 1: All 3 letters are alike
3 alike letters can be selected from given letters in only 1 way, *i.e.*, BBB.
$\Rightarrow$ The number of selections with all 3 letters alike = 1 (1)

Case 2: 2 alike and 1 distinct letter
2 alike letters can be selected from 2 sets of alike letters (AA, BB) in ${}^2C_1$ ways.
1 distinct letter (distinct from selected alike letters) can be selected from remaining letters in ${}^2C_1$ ways. (either A or B).
Using fundamental principle of counting, total number of selections with 2 alike and 1 distinct letter

$$= {}^2C_1 \times {}^2C_1 = 4 \text{ ways} \tag{2}$$

Case 3: All letters distinct
All 3 letters distinct can be selected from 3 distinct letters (A, B, C) in 1 way.
$\Rightarrow$ Total number of ways to select 3 distinct letters is 1 way (3)
Combining (1), (2) and (3).
Total number of ways to select 3 letters from given letters = 1 + 4 + 1 = 6.

Example 46 *In how many ways* 4 *letters can be selected from the letters of the word INEFFECTIVE?*

Solution: INEFFECTIVE contains 11 letters: EEE, FF, II, C, T, N, V
We will make following cases to select 4 letters.

Case 1: 3 alike and 1 distinct
3 alike letters can be selected from 1 set of 3 alike letters (EEE) in 1 way.
$\Rightarrow$ The number of ways to select 3 alike letters = 1
$\Rightarrow$ The number of ways to select 1 distinct letters = 6
$\Rightarrow$ Total ways = 6 × 1 = 6 (1)

Case 2: 2 alike and 2 alike
'2 alike and 2 alike' means we have to select 2 groups of 2 alike letters (EE, FF, II) in ${}^3C_2$ ways.
$\Rightarrow$ The number of ways to select '2 alike and 2 alike' letters = ${}^3C_2 = 3$. (2)

Case 3: 2 alike and 2 distinct
1 group of 2 alike letters can be selected from 3 sets of 2 alike letters (EE, FF, II) in ${}^3C_1$ ways.
2 distinct letters can be selected from 6 distinct letters (C, T, N, V, remaining 2 sets of two letters alike) in ${}^6\mathrm{C}_2$ ways.
The number of ways to select '2 alike and 2 distinct letters'

$${}^3C_1 \times {}^6C_2 = 3 \times 15 = 45 \tag{3}$$

Case 4: All distinct letters
All distinct letters can be selected from 7 distinct letters (I, E, F, N, C, T, V) in ${}^7C_4$ ways.
$\Rightarrow$ The number of ways to select all distinct letters = ${}^7C_4 = 35$ (4)

Combining (1), (2), (3), and (4), we get,

Total number of ways to select 4 letters from the letter of the word 'INEFFECTIVE' $= 6 + 3 + 45 + 35 = 89$.

Example 47 *In how many ways a child can select 5 balls from 5 red, 4 black, 3 white, 2 green, 1 yellow balls? (Assume balls of the same colour are identical)*

Solution: It is given that child can select 5 balls from RRRRR BBBB WWW GG Y balls. We will make following cases:

(i) **All alike:**
There is one group of all alike balls (5 red balls)
$\Rightarrow$ Number of ways to choose 1 group $= {}^1C_1 = 1$

(ii) **4 alike and 1 distinct:**
There are 2 groups of 4 alike balls (4 red balls, 4 black balls) and after selecting one group, there are 4 distinct balls left from where we require to choose one ball.
$\Rightarrow$ Number of ways to select **'4 alike and 1 distinct'** $= {}^2C_1 \times {}^4C_1 = 8$

(iii) **3 alike and 2 alike:**
Select 3 alike balls from 3 groups of 3 alike balls (RRR, BBB, WWW) in ${}^3C_1$ ways. Then select 2 alike balls from remaining 3 groups of 2 alike balls in ${}^3C_1$ ways.
$\Rightarrow$ Number of ways to select **'3 alike and 2 alike'**
$= {}^3C_1 \times {}^3C_1 = 9$

(iv) **3 alike and 2 distinct:**
Select one group of 3-alike balls from 3 groups of 3-alike balls in ${}^3C_1$ ways. Select 2 balls from remaining 4 distinct balls in ${}^4C_2$ ways.
$\Rightarrow$ Number of ways to select **'3 alike and 2 distinct'**
$= {}^3C_1 \times {}^4C_2 = 18$

(v) **2 alike, 2 alike and 1 distinct:**
Select 2 groups of 2-alike balls from 4 groups of 2-alike balls in ${}^4C_2$ ways. Further select 1 ball from remaining 3 distinct balls in ${}^3C_1$ ways.
$\Rightarrow$ Number of way to select **'2 alike, 2 alike and 1 distinct'**
$= {}^4C_2 \times {}^3C_1 = 18$

(vi) **2 alike and 3 distinct:**
Select one group of 2-alike balls from 4 groups of 2-alike balls in ${}^4C_1$ ways. Then select 3 balls from remaining 4 distinct balls in ${}^4C_3$ ways.
$\Rightarrow$ Number of ways to select **'2 alike and 3 distinct'**
$= {}^4C_1 \times {}^4C_3 = 16$

(vii) **All distinct:**
Select 5 distinct balls from 5 distinct balls (R, B, W, G, Y) in ${}^5C_5$ ways.
$\Rightarrow$ Number of ways to select **'All distinct'**
$= {}^5C_5 = 1$.
Combining all above cases, total number of ways in which child can select 5 balls $= 1 + 8 + 9 + 18 + 18 + 16 + 1 = 71$ ways.

7.3.4.6 Occurrence of Order in Selection

If n objects are chosen as 'first $(n - 1)$ objects are chosen and then nth object' or 'n objects are chosen one by one' then always ordered selections are made and hence the repetitions. So in the final count, these repetitions are to be deleted.

Example 48 *In how many ways we can select two unit square on an ordinary chess board such that both square neither in same row nor in same column.*

Solution: First square is selected in 64 ways.
After selection of first, we can't select any of the remaining 7 squares which are in the same row with first square and similarly we cannot select any of remaining 7 squares which are in the same column with first square. So number of choices for second square is $64 - 1 - 7 - 7 = 49$.

Hence, apparently, by multiplication principle, number of ways $= 64 \times 49$.

But in this count, repetitions occurred. In fact, each selection is counted twice.

So final answer $= \dfrac{64 \times 49}{2} = 1568$ ways.

Example 49 *Find the number of pairings of a set of* $2n$ *elements* [*e.g.,* $\{(1, 2), (3, 4), (5, 6)\}$ $\{(1, 3), (2, 4), (5, 6)\}$ *are two pairings of the set* $\{1, 2, 3, 4, 5, 6\}$].

Solution: Let $A = \{1, 2, 3, 4, \ldots, 2n - 1, 2n\}$.
A pair having 1 as one element (out of the two elements) can be obtained in $(2n - 1)$ ways. Say, selected element is k (Assuming $k \neq 2$). Similarly a pair having 2 as one element (out of two elements), can be obtained in $2n - 3$ ways, etc.

Number of pairings $= (2n - 1)(2n - 3)\,(2n - 5) \cdots 3 \cdot 1$

Aliter: First pair can be obtained in ${}^{2n}C_2$ ways. Second pair can be obtained in ${}^{2n-2}C_2$ ways.

Third pair can be obtained in ${}^{2n-4}C_2$ ways.

$\vdots$

nth pair can be obtained in ${}^{2}C_2$ ways.

Apparently, by multiplication principle,
number of pairings $= {}^{2n}C_2 \cdot {}^{2n-2}C_2 \cdots {}^{2}C_2$.

But in this count, too many repetitions have been counted. In fact, each pairing is counted $n!$ times.

Required number $= \dfrac{{}^{2n}C_2 \cdot {}^{2n-2}C_2 \cdots {}^{4}C_2 \cdot {}^{2}C_2}{n!}$

(Verify this number is same as $(2n - 1)\,(2n - 3)\,(2n - 5) \cdots 3 \cdot 1$)

7.3.4.7 Points of Intersection between Geometrical Figures

We can use ${}^{n}\mathrm{C}_r$ (number of ways to select r objects from n different objects) to find points of intersection between geometrical figures.

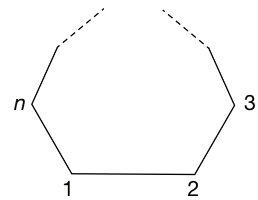
For example:

1. Number of points of intersection of 'n' non-concurrent and non parallel lines is ${}^{n}C_2$.
   Logic: When two lines intersect, we get a point of intersection. Two lines from n distinct lines can be selected in ${}^{n}C_2$ ways. Therefore, number of points of intersection is ${}^{n}C_2$.
2. Number of lines that can be drawn, passing though any 2 points out of n given points in which no three of them are collinear, is ${}^{n}C_2$.
   Logic: A line can be drawn through two points. Two points can be selected from n distinct points in ${}^{n}C_2$ ways. Therefore, number of lines that can be drawn is ${}^{n}C_2$.
3. Number of triangles that can formed, by joining any three points out of n given points in which no three of them are collinear is ${}^{n}C_3$.
   Logic: A triangle is formed using 3 different points. Three points can be selected from n distinct points in ${}^{n}C_3$ ways. Therefore, we can form ${}^{n}C_3$ triangles using n distinct points.

4. Number of diagonals that can be drawn in a 'n' sided polygon is $\dfrac{n(n-3)}{2}$.

   Logic: There are n vertices in a n sided polygon. When two vertices are joined (excluding the adjacent vertices), we get a diagonal. The number of ways to select 2 vertices from n vertices is ${}^nC_2$. But this also includes n sides (when adjacent vertices are selected). Therefore number of diagonals

$$= {}^nC_2 - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$



Aliter: $n-3$ diagonals emerge from each vertex. For example, from vertex named 1, $n-3$ diagonals emerge whose other ends are vertices 3, 4, …, $n-1$.

Number of diagonals apparently, by multiplication principle, is $n(n-3)$ but each diagonal is counted twice.

Required number $= \dfrac{n(n-3)}{2}$.

$\left(\text{Verify that, } {}^nC_2 - n \text{ is same as } \dfrac{n(n-3)}{2}.\right)$

Example 50 *How many triangles can be formed by joining the vertices of a hexagon?*

Solution: Let $A_1, A_2, A_3, \ldots, A_6$ be the vertices of the hexagon. One triangle is formed by selecting a group of 3 points from 6 given vertices.
Number of triangles = Number of groups of 3 each from 6 points.

$$= {}^6C_3 = \frac{6!}{3!\,3!} = 20.$$

Example 51 *There are* 10 *points in a plane, no three of which are in the same straight line, except* 4 *points, which are collinear. Find the*
(i) *number of straight lines obtained from the pairs of these points;*
(ii) *number of triangles that can be formed with the vertices as these points.*

Solution:
(i) Number of straight lines formed joining the 10 points, taking 2 at a time

$$= {}^{10}C_2 = \frac{10!}{2!8!} = 45$$

Number of straight lines formed by joining the four points (which are collinear), taking 2 at a time $= {}^4C_2 = \dfrac{4!}{2!2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.
So, required number of straight lines = 45 – 6 + 1 = 40.

(ii) Number of triangles formed by joining the points, taking 3 at a time

$$= {}^{10}C_3 = \frac{10!}{3!7!} = 120$$

Number of triangles formed by joining the 4 points (which are collinear), taken 3 at a time $= {}^4C_3 = 4$.
But, 4 collinear points cannot form a triangle when taken 3 at a time.
So, required number of triangles = 120 – 4 = 116.

Example 52 *There are* 12 *points in a plane,* 5 *of which are concyclic and out of remaining* 7 *points, no three are collinear and none concylic with previous* 5 *points. Find the number of circles passing through at least* 3 *points out of* 12 *given points.*

Solution: Consider Set A consists of 5 concyclic points. Set B consists of remaining 7 points.

Case 1: Circle passes through 3 points of set B
Number of circles = ${}^7C_3$

Case 2: Circle passes through 2 points of set B and one point of set A
Number of circles = ${}^7C_2 \cdot {}^5C_1$

Case 3: Circle passes through 1 point of set B and two points of set A
Number of circles = ${}^7C_1 \cdot {}^5C_2$

Case 4: Circle passes through no point from set B.
Number of circles = 1

All 4 cases are ***exhaustive and mutually exclusive***.

So, total number of circles

$$= {}^7C_3 + {}^7C_2 \cdot {}^5C_1 + {}^7C_1 \cdot {}^5C_2 + 1$$

$$= \frac{7!}{3!4!} + \frac{7!}{2!5!}.5 + 7.\frac{5!}{2!3!} + 1$$

$$= \frac{7.6.5}{1.2.3} + \frac{7.6}{1.2}.5 + 7.\frac{5.4}{1.2} + 1$$

$$= 35 + 105 + 70 + 1$$

$$= 211.$$

Aliter: Select three points out of 12 in ${}^{12}C_3$ ways. This number includes the number of circles obtained from 3 points out of 5 concyclic points. Note that we get the same circle by selecting any three points out of 5 concyclic points but we count it ${}^5C_3$ times.

Required number $= {}^{12}C_3 - {}^5C_3 + 1$
$= 211.$

Example 53 *In a plane there are* 37 *straight lines, of which* 13 *pass through the point A and* 11 *pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

Solution: The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore instead of getting ${}^{11}C_2$ points, we get only one point B. Hence, the number of intersection points of the lines is ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$.

Example 54 *l_1 and l_2 are two parallel lines; m and n are the points on l_1 and l_2, respectively. Find the number of triangles that could be constructed using these points as vertices.*

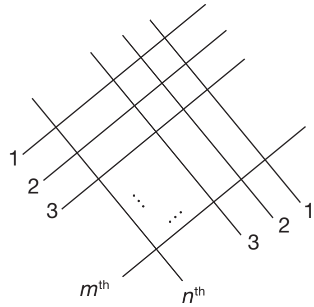
Solution: Any two points on l_1 and a point on l_2 form a triangle; again any two points on l_2 and a point on l_1 also form a triangle.
2 points can be chosen in ${}^mC_2$ ways from m points of l_1 and we have n choices for a point on l_2 and similarly, 2 points can be chosen in ${}^nC_2$ ways from n points of l_2 and in m ways we can choose a point on l_1,

Therefore, the number of triangles formed is given by

$$ {}^{m}C_2 \times n + {}^{n}C_2 \times m = n \times \frac{m(m-1)}{2} + m \times \frac{n(n-1)}{2} = \frac{mn}{2}(m+n-2). $$

Example 55 *If m parallel lines in plane are intersected by a family of n parallel lines. Find the number of parallelogram formed.*

Solution: A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.
Two straight lines from the set of m parallel lines can be chosen in ${}^{m}C_2$ ways and two straight lines from the set of n parallel lines can be chosen in ${}^{n}C_2$ ways. Hence, the number of parallelograms formed.



$$ = {}^{m}C_2 \times {}^{n}C_2 = \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4} $$

Example 56 *In a plane, a set of* 8 *parallel lines intersects a set of n other parallel lines, giving rise to* 420 *parallelograms* (*many of them overlap with one another*). *Find the value of n.*

Solution: If two lines which are parallel to one another (in one direction) intersect another two lines which are parallel, we get one parallelogram. Thus, we can choose $C(8, 2)$ pairs of parallel lines in one direction and the number of parallel lines intersecting there will be $C(n, 2)$ pairs.

So, the number of parallelograms thus obtained is

$$ C(n, 2) \times C(8, 2) = 420 $$

$$ \Rightarrow \quad \frac{n(n-1)}{1.2} \times \frac{8 \times 7}{1.2} = 420 $$

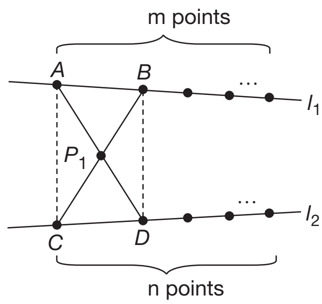
$$ \Rightarrow \quad n(n-1) = 30 $$

$$ \Rightarrow \quad n = 6 \text{ (or } n = -5\text{, which is not admissible)} $$

Thus $n = 6$ is the solution.

Example 57 *Prove that, if each of the m points in one straight line be joined to each of the n points by straight lines terminated by the points then excluding the given points, these lines will intersect in* $\frac{1}{4}mn\,(m-1)(n-1)$ *points.*

Solution: Two straight lines intersect in one point.



So to get one point of intersection, we require two points on the first line (l_1) and two points on the second line (l_2).
For joining A of l_1 to C and D of l_2, they intersect in A, which is not counted as the required point. However, AD and CB intersect at the point P_1, AC and BD intersect only when extended which is also not counted as the required point. Thus to get an intersection, other than the points in l_1 and l_2, we should take two points from each of l_1 and l_2 and joined them in cross pattern.

The number of ways we can choose two points from l_1 in which m points are plotted, is ${}^{m}C_2$. Similarly, we can choose two points from l_2 in ${}^{n}C_2$ ways. For each pair of points from l_1 and l_2, we get one point of intersection.

So, the total number of points when there are ${}^{m}C_2$ pairs from l_1 and ${}^{n}C_2$ pairs from l_2 is

$$^mC_2 \times {}^nC_2 = \frac{m(m-1)}{1.2} \times \frac{n(n-1)}{1.2}$$

$$= \frac{1}{4} mn(m-1)(n-1).$$

Example 58 *Let there be n concurrent lines and another line parallel to one of them. Find the number of different triangles that will be formed by the* $(n+1)$ *lines.*

Solution: The number of triangles = Number of selections of 2 lines from the $(n-1)$ lines which are cut by the last line

$$= {}^{n-1}C_2 = \frac{(n-1)!}{2!(n-3)!} = \frac{(n-1)(n-2)}{2}.$$

Example 59 *Out of 18 points in a plane no three are in the same straight line except five points which are collinear. Find the number of straight lines that can be formed by joining any two of them.*

Solution: The number of straight lines = $^{18}C_2 - (^5C_2 - 1) = 144$.

Example 60 *There are p points in a plane, no three of which are in the same straight line with the exception of q, which are all in the same straight line. Find the number of*

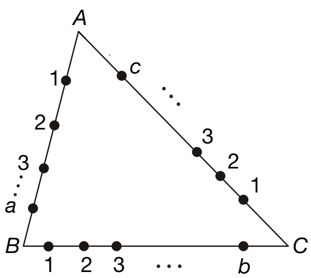
(i) *straight lines*
(ii) *triangles which can be formed by joining them.*

Solution:

(i) If no three of the p points were collinear, the number of straight lines = Number of groups of two that can be formed from p points = $^pC_2$.
Due to the q points being collinear, there is a loss of $^qC_2$ lines that could be formed from these points.
But these points are giving exactly one straight line passing through all of them.
Hence, the number of straight lines = $^pC_2 - {}^qC_2 + 1$.

(ii) If no three points were collinear, the number of triangles = $^pC_3$
But there is a loss of $^qC_3$ triangles that could be formed from the group of collinear points.
Hence the number of triangles formed = $^pC_3 - {}^qC_3$.

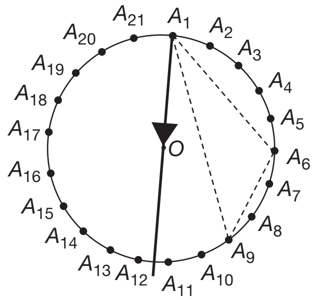
Example 61 *The sides AB, BC and CA of a triangle ABC have a, b and c interior points on them respectively then find the number of triangles that can be constructed using these interior points as vertices.*



Solution: Required number of triangles
= Total number of ways choosing 3 points
− Number of ways of choosing all the 3 points either from AB or BC or CA
$= {}^{a+b+c}C_3 - ({}^aC_3 + {}^bC_3 + {}^cC_3)$

Example 62 *Let* A_i, $i = 1, 2, \ldots, 21$ *be the vertices of a 21-sided regular polygon inscribed in a circle with centre O. Triangles are formed by joining the vertices of the 21-sided polygon. How many of them are acute-angled triangles? How many of them are right-angled triangles? How many of them are obtuse-angled triangles? How many of them are equilateral? How many of them are isosceles?*

Solution: Since this is a regular polygon with odd number of vertices, no two of the vertices are placed diagonally opposite, so there is no right-angled triangle. Hence number of right-angled triangle is zero. Let A be the number of acute-angled triangles. To form a triangle we need to choose 3 vertices out of the 21 vertices which can be done in $C(21,3)=\dfrac{21\times20\times19}{6}=1330$ ways. Since the triangles are either acute or obtuse, we get $A+O=1330$.



Let us find O, the number of obtuse angled triangles first.
Draw one diameter say passing through A_1. Now let us count all obtuse angle triangle on right side of the diameter and having one verfex at A_1. For these triangles we need two more vertex out of A_2 to A_{11}. Which can be seleted in $\binom{10}{2}$ ways.

Hence total number of obtuse angle triangles is $21\cdot\binom{10}{2}=945$

Now acute angle triangles
$$A=1330-945$$
$$=385$$

A triangle $A_iA_jA_k$ is equilateral if A_i, A_j, A_k are equally spaced.

Out of $A_1,\ldots,A_{21}$, we have only 7 such triplets
$A_1A_8A_{15}$, $A_2A_9A_{16},\ldots,A_7A_{14}A_{21}$. Therefore, there are only 7 equilateral triangles.

Consider the diameter A_1OB where B is the point where A_1O meets the circle. If we have an isosceles triangle A_1 as its vertex then A_1B is the altitude and the base is bisected by A_1B. This means that the other two vertices, A_j and A_k, are equally spaced from B.

We have 10 such pairs, so we have 10 isosceles triangles with A_1 as vertex of which one is equilateral.

Because proper isosceles triangles with A_1, as vertex (non-equilateral) are 9, with each vertex A_i, $i=1,2,\ldots,21$ we have 9 such isosceles triangles.

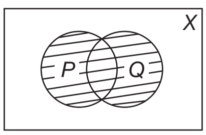
So, total number of isosceles but non-equilateral triangles are $9\times21=189$. But the 7 equilateral triangles are also to be considered as isosceles.

$\therefore$ The total number of isosceles triangles is $189+7=196$.

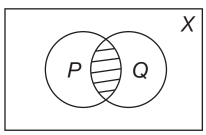
Note: This problem can be generalized to a regular polygon having n vertices. Find the number of acute, obtuse, right, isosceles, equilateral and scalene triangles.

7.3.4.8 Formation of Subsets

In these type of problems, we select elements from a given set to form subsets. We are supposed to form subsets under constraints. For example, two subsets P and Q are to be formed such that $P\cup Q$ has all elements, $P\cap Q$ has no elements, etc. To understand the problems based on this type, read the following examples carefully.

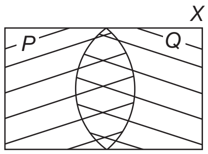


$P\cup Q$



$P\cap Q$

Example 63 *Let X be a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random then find the number of ways to form sets such that $P\cup Q=X$.*



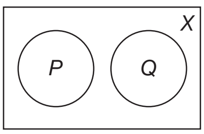
Solution: As $P \cup Q = X$, it means every element would be either included in P or in Q or both so for every element, there are 3 choices.
$\Rightarrow$ Number of ways to select P and Q such that $(P \cup Q = X) = 3^n$.

Example 64 *Let X be a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to choose P and Q such that $P \cup Q$ contains exactly r elements.*

Solution: $P \cup Q$ has r elements. It means r elements out of n elements should be present in either P or in Q or in both. r elements out of n elements can be selected in ${}^nC_r$ ways.
Each of these r elements has 3 choices
$\Rightarrow$ Number of ways to select elements of P and $Q = 3^r$
Each of remaining $(n - \text{r})$ elements has 1 choice, *i.e.*, neither belongs to P nor belongs to $Q \Rightarrow$ Number of ways $= 1^{n-r}$.
$\Rightarrow$ Number of ways to select P and Q such that $P \cup Q$ has exactly r elements

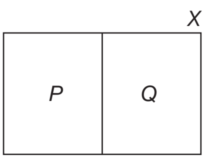
$$= {}^nC_r 3^r (1)^{n-r} = {}^nC_r 3^r.$$

Example 65 *Let X be a set containing n elements. A subset P of set X is chosen at random . The set X then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P \cap Q$ is empty, i.e., $P \cap Q = \phi$.*



Solution: $P \cap Q = \phi$. It means P and Q should be disjoint sets. That is there is no element common in P and Q.
$\Rightarrow$ For every elements in set X there are 3 choices. Either it is selected in P but not in Q or selected in Q but not in P or not selected in both P and Q.
$\Rightarrow$ Number of ways to select P and Q such that $P \cap Q$ is $\phi = 3^n$.

Example 66 *Let X be a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P = \bar{Q}$.*



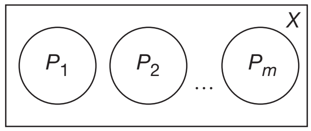
Solution: $P = \bar{Q}$ or Q^C. It means P and Q are complementary sets, *i.e.*, every element present in X is either present in P or Q.
$\Rightarrow$ For every element there are 2 choices to select. Either it will be selected for P or it will be selected for Q.
$\Rightarrow$ Number of ways to select $= 2^n$

Example 67 *Let X be a set containing n elements. A subset P_1 is chosen at random and then set X is reconstructed by replacing the elements of set P_1. A subset P_2 of X is now chosen at random and again set X is reconstructed by replacing the elements of P_2. This process is continued to choose subsets $P_3, P_4, P_5, \ldots, P_m$ where $m \geq 2$. Find numbers of ways to select sets such that:*

(i) $P_i \cap P_j = \phi$ *for* $i \neq j$ *and* $i, j = 1, 2, \ldots, m$.

(ii) $P_1 \cap P_2 \cap P_3 \cap \ldots \cap P_m = \phi$.

Solution:



(i) $P_i \cap P_j = \phi \; \forall \; i \neq j$
Every element in X has $(m + 1)$ choices because either it can be selected for P_1 or P_2 or P_3 or … or P_m or not get selected in any of the sets.
$\Rightarrow$ Number of favourable ways $= (m + 1)\,(m + 1) \ldots n$ times $= (m + 1)^n$

(ii) $P_1 \cap P_2 \cap P_3 \ldots \cap P_m = \phi$.
This means there is no element to be common to all sets $P_1, P_2, P_3 \ldots P_m$.
For each element out of $a_1, a_2, \ldots a_n$ there are $(2^m - 1)$ choices to get selected. It can be selected in any sets but not for all sets together so we subtract 1 from 2^m.
Total ways to select $P_1, P_2, P_3, \ldots, P_m$ such that $P_1 \cap P_2 \ldots \cap P_m = \phi$ is $(2^m - 1)^n$.

7.4 The Bijection Principle

Let A $= \{ a_1, a_2, \ldots, a_n \}$ and $B = \{b_1, b_2, \ldots, b_m \}$.

If $f : A \to B$ is an injective function then $n \leq m$.

If $f : A \to B$ is a surjective function then $n \geq m$.

If $f : A \to B$ is injective and surjective then f is known to be a bijective function. For a bijective function, $n = m$.

Example 68 *What is the total number of subsets of a set containing exactly n elements?*

Solution: It is a well known result, number of subsets $= 2^n$.
Let $S = \{a_1, a_2, a_3, \ldots, a_n \}$ be a set of exactly n elements.
Let P be the set of all subsets of S and Q be the set of all binary sequences of n elements.

Let $A \in P$. Let $f : P \to Q$ be a function that associates a binary sequence with A as follows:

$a_i \in A$, iff ith term of the sequence is 1.

For example, subset $\{a_2, a_4, a_{n-1} \}$ corresponds to binary sequence

$$\begin{array}{ccccccccc} 0 & \underline{1} & 0 & \underline{1} & 0 & 0 \ldots 0 & \underline{1} & 0 \\ & \text{2nd} & & \text{4th} & & & (n-1)\text{th} & \\ & \text{place} & & \text{place} & & & \text{place.} & \end{array}$$

Observe that, for every subset A, there is a binary sequence of n terms and for every binary sequence of n terms as stated above, there is a subset A of S.

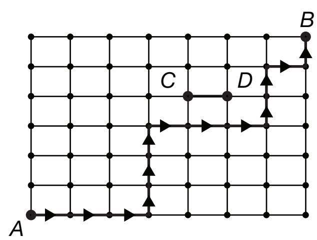
Therefore f is a bijection between P and Q.

Hence, the number of subsets = number of binary sequences $= 2^n$.

Example 69 *Consider a network as shown in the figure. Paths from* A to B consists of the horizontal or vertical line segments.

No diagonal movement is allowed. We can only move left to right or down to up. One sample path from A to B is shown.

(i) *How many paths are there from A to B?*
(ii) *How many paths go via C?*
(iii) *How many paths go via CD?*



Solution: Assign 0 for horizontal line segment of one unit. Assign 1 for vertical line segment of one unit. For example, corresponding to the path shown in the figure, we can write one binary sequence as 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 1.

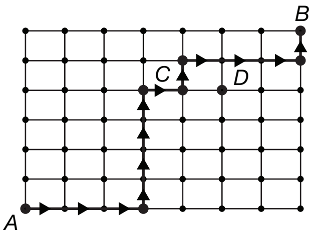
Note that there are 7 horizontal and 6 vertical line segments, of one unit each, in every path from A to B.

(i) Since, for every path between A and B, there is a binary sequence of 7, 0's and 6, 1's and for every sequence we can have corresponding one path made up of horizontal and vertical lines. Therefore there is bijection between the set of all paths from A to B and the set of all binary sequences of 7, 0's and 6, 1's.

$\Rightarrow$Number of paths between A and B = Number of binary sequences

= Number of ways to select 7 places to put 0 out of 13 different places $= \binom{13}{7}$

$= \dfrac{13!}{7!6!}$



(ii) Number of paths through C

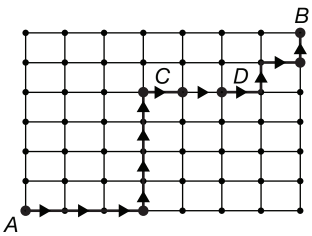
= (Number of paths from A to C) $\times$ (Number of paths from C to B)

= Number of ways to select 4 places to put 0 out of first 8 different places $\times$ Number of ways to select 3 places to put 0 out of next 5 different places

$= \binom{8}{4} \times \binom{5}{3}$

$= \dfrac{8!}{4!4!} \times \dfrac{5!}{3!\,2!}$

(Note that there are 4 horizontal and 4 vertical line segments of one unit each, in every path from A to C. There are 3 horizontal and 2 vertical line segments of one unit each in every path between C and B.)



(iii) Similarly number of paths from D to B $= \dfrac{4!}{2!\times 2!}$

(as there are 2 horizontal and 2 vertical line segments of one unit each in every path between B and D.)

Number of paths containing CD $= \dfrac{8!}{4!\times 4!} \times \dfrac{4!}{2!\times 2!}$.

Note: If a problem, similar to street network, but in three dimensions, is to be solved, we define *ternary sequences* consisting of 0's, 1's and 2's.

For example, number of paths between (0, 0, 0) and (3, 4, 6), consisting of line segments of one unit each in positive directions of the co-ordinate axes $= \dfrac{13!}{3!4!6!}$.

7.5 Combinations with Repetitions Allowed

Here we will discuss combinations of n different objects taken r at a time when each object can be repeated any number of times in a combination.

Suppose three different objects A, B, C are given. We have to select two objects from A, B, C and in our selection we can include A, B, C repeatedly any number of times. This selection can be done in following ways.

AA, BB, CC, AB, AC, BC, *i.e.*, 6 ways.

This number 6 cannot be obtained using formula ${}^nC_r$ as here repetition of objects is allowed. To find answer to this type of problem, where repetition of objects is allowed, we use the following formula:

Number of ways to select r objects from n different objects where each object can be selected any number of times is ${}^nH_r$.

$$^nH_r = \binom{n+r-1}{r}$$

Logic:

Let n different objects be numbered as 1, 2, 3, …, n.

And selected numbers be $a_1, a_2, a_3, \ldots, a_r$, such that

$$1 \le a_1 \le a_2 \le a_3 \le \cdots \le a_r \le n \quad (1)$$

Here we allowed weak inequalities between a_i's, as numbers may be repeated which will correspond to repetition of objects.

Now consider another sequence,

$$a_1, a_2+1, a_3+2, \cdots, a_r+r-1 \quad (2)$$

We can observe following properties in sequence (2):

1. Sequence is strictly increasing
2. Minimum and Maximum element in the sequence can be 1 and $n + r$ -1 respectively.
3. There are $\binom{n+r-1}{r}$ such sequence

(As any r numbers can be selected from 1 to $n + r - 1$)

Now there is a Bijection between sequence (1) and sequence (2)

Hence total number of sequence (1) is also $\binom{n+r-1}{r}$.

Example 70 *In how many ways a person can buy 5 icecreams from a shop in which four different flavours of icecreams are available.*

Solution: Here person can buy all five icecreams of same flavour or in any other combination, *i.e.*, any flavour can be taken 0 or 1 or 2 … or 5 times.

Hence our current problem is selection of 5 icecreams from 4 flavours with repetition allowed, so answer is

$$^4\mathrm{H}_5 = \binom{4+5-1}{5} = \binom{8}{5} = 56.$$

Build-up Your Understanding 2

1. **(a)** Find 'n' if (i) $^{2n}C_3 : {}^nC_2 = 12 : 1$ (ii) $^{25}C_{n+5} = {}^{25}C_{2n-1}$
   (b) Prove that $^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$ if $n > 7$.
2. Find the number of positive integers satisfying the inequality $^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \le 100$.
3. There are 20 questions in a questions paper. If no two students solve the same combination of questions but solve equal number of questions then find the maximum number of students who appeared in the examination.
4. In how many ways can 5 colours be selected out of 8 different colours including red, blue, and green
   (i) if blue and green are always to be included,
   (ii) if red is always excluded,
   (iii) if red and blue are always included but green excluded?

5. The kindergarten teacher has 25 kids in her class. She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once. Find the number of visits, the teacher makes to the garden and also the number of of visits every kid makes.

6. A teacher takes 3 children from her class to the zoo at a time as often as she can, but does not take the same three children to the zoo more than once. She finds that she goes to the zoo 84 more than a particular child goes to the zoo. Find the number of children in her class.

7. A team of four students is to be selected from a total of 12 students. Find the total number of ways in which team can be selected such that two particular students refuse to be together and other two particular students wish to be together only.

8. A women has 11 close friends. Find the number of ways in which she can invite 5 of them to dinner, if two particular of them are not on speaking terms and will not attend together.

9. Four couples (husband and wife) decide to form a committee of four members. Find the number of different committees that can be formed in which no couple finds a place.

10. Find the number of ways in which a mixed double tennis game can be arranged from amongst 9 married couple if no husband and wife plays in the same game.

11. Find the number of ways of choosing a committee of 2 women and 3 men from 5 women and 6 men, if Mr. A refuses to serve on the committee if Mr. B is a member and Mr. B can only serve, if Miss C is the member of the committee.

12. Find the number of ways in which we can choose 3 squares on a chess board such that one of the squares has its two sides common to other two squares.

13. Find the number of ways of selecting three squares on a chessboard so that all the three be on a diagonal line of the board or parallel to it.

14. 5 Indian and 5 American couples meet at a party and shake hands. If no wife shakes hands with her husband and no Indian wife shakes hands with a male, then find the number of hand shakes that takes place in the party.

15. A person predicts the outcome of 20 cricket matches of his home team. Each match can result either in a win, loss or tie for the home team. Find the total number of ways in which he can make the predictions so that exactly 10 predictions are correct.

16. A forecast is to be made of the results of five cricket matches, each of which can be a win, a draw or a loss for Indian team. Find
(i) the number of different possible forecasts.
(ii) the number of forecasts containing 0, 1, 2, 3, 4 and 5 errors respectively.

17. A forecast is to be made of the results of five cricket matches, each of which can be a win or a draw or a loss for Indian team.
Let p = Number of forecasts with exactly 1 error
q = Number of forecasts with exactly 3 errors and
r = Number of forecasts with all five errors
then prove that $2q = 5r$, $8p = q$, and $2(p + r) > q$.

18. In a club election the number of contestants is one more than the number of maximum candidates for which a voter can vote. If the total number of ways is which a voter can vote be 62, then find the number of candidates.

19. Every one of the 10 available lamps can be switched on to illuminate certain Hall. Find the total number of ways in which the hall can be illuminated.

20. In a unique hockey series between India and Pakistan, they decide to play on till a team wins 5 matches . Find the number of ways in which the series can be won by India, if no match ends in a draw.

21. There are n different books and p copies of each in a library. Find the number of ways in which one or more books can be selected.

22. A class has n students. We have to form a team of the students by including atleast two students and also by excluding atleast two students. Find the number of ways of forming the team.

23. If the $(n + 1)$ numbers $a_1, a_2, a_3, \ldots, a_{n+1}$, be all different and each of them is a prime number, then find the number of different factors (other than 1) of $a_1^m \cdot a_2 \cdot a_3 \cdots a_{n+1}$.

24. In a certain algebraical exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation-combination and 6 examples on binomial theorem. Find the number of ways a teacher can select for his pupils atleast one but not more than 2 examples from each of these sets.

25. Find the number of straight lines that can be drawn through any two points out of 10 points, of which 7 are collinear.

26. n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. Find the number of different points at which these lines will cut each other.

27. Eight straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Find The number of parts into which these lines divides the plane.

28. In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70 then find the number of diagonals of the polygon.

29. In a plane there are two families of lines $y = x + r$, $y = -x + r$, where $r \in \{0, 1, 2, 3, 4\}$. Find the number of squares of diagonals of the length 2 formed by the lines.

30. Find the number of triangles whose vertices are at the vertices of an octagon but none of whose side happen to come from the sides of the octagon.

31. Let there be 9 fixed points on the circumference of a circle . Each of these points is joined to every one of the remaining 8 points by a straight line and the points are so positioned on the circumference that atmost 2 straight lines meet in any interior point of the circle. Find the number of such interior intersection points.

32. A bag contains 2 Apples, 3 Oranges and 4 Bananas. Find the number of ways in which 3 fruits can be selected if atleast one banana is always in the combination (Assume fruit of same species to be alike).

33. Find the number of selections of four letters from the letters of the word ASSASSINATION.

34. Find the number of ways to select 2 numbers from $\{0, 1, 2, 3, 4\}$ such that the sum of the squares of the selected numbers is divisible by 5 (repetition of numbers is allowed).

35. Find the number of ways in which we can choose 2 distinct integers from 1 to 100 such that difference between them is at most 10.

36. If a set A has m elements and another set B has n elements then find the number of functions from A to B.

37. Let $A = \{x : x \text{ is a prime number and } x < 30\}$. Find the number of different rational numbers whose numerator and denominator belongs to A.

38. Find the number of all three elements subsets of the set $\{a_1, a_2, a_3, \ldots, a_n\}$ which contain a_3.

39. If the total number of m-element subsets of the set $A = \{a_1, a_2, a_3, \ldots, a_n\}$ is k times the number of m-elements subsets containing a_4, then find n.

40. A set contains $(2n + 1)$ elements. Find the number of subsets of the set which contains at most n elements.

41. Find the number of subsets of the set $A = \{a_1, a_2, \ldots, a_n\}$ which contain even number of elements.

42. 'A' is a set containing 'n' distinct elements. A subset P of 'A' is chosen. The set 'A' is reconstructed by replacing the elements of P. A subset 'Q' of 'A' is again chosen. Find the number of ways of choosing P and Q so that $P \cap Q$ contains exactly two elements.

43. Find the number of ways of choosing triplets (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, \ldots, n, n+1\}$.

44. Find the number of ways in which the number 94864 can be resolved as a product of two factors.

45. Find the sum of the divisors of $2^5 \cdot 3^4 \cdot 5^2$.

46. In the decimal system of numeration, find the number of 6-digits numbers in which the digit in any place is greater than the digit to the left to it.

47. Find the number of 3-digit numbers of the form xyz such that $x < y$ and $z \leq y$.

48. Find the total number of 6-digit numbers $x_1\, x_2\, x_3\, x_4\, x_5\, x_6$ having the property $x_1 < x_2 \leq x_3 < x_4 < x_5 \leq x_6$.

49. The streets of a city are arranged like the lines of a chess board. There are m streets running North to South and 'n' streets running East to West. Find the number of ways in which a man can travel from NW to SE corner going the shortest possible distance.

50. Let there be $n \geq 3$ circles in a plane. Find the value of n for which the number of radical centres, is equal to the number of radical axes. (Assume that all radical axes and radical centre exist and are different)

51. Rajdhani express going from Bombay to Delhi stops at 4 intermediate stations. 10 passengers enter the train during the journey (including Bombay and 4 intermediate stations) with ten distinct tickets of two classes. Find the number of different sets of tickets they may have.

52. Find the number of functions f from the set $A = \{0, 1, 2\}$ into the set $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \leq f(j)$ for $i < j$ and, $i, j \in A$.

53. Show that the number of ways of selecting n-objects out of $3n$-objects, n of which are alike and rest different is $2^{2n-1} + \binom{2n-1}{n-1}$.

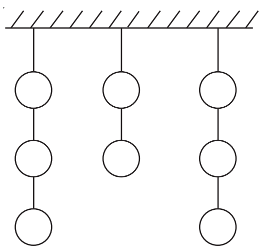
54. Use a combinatorial argument to prove that:
(i) ${}^{2n}C_2 = 2 \cdot {}^{n}C_2 + n^2$ (ii) $r \cdot {}^{n}C_r = n\, {}^{n-1}C_{r-1}$

55. Prove (combinatorially) that
${}^{n}C_1 + 2\,{}^{n}C_2 + 3\,{}^{n}C_3 + \cdots + n\,{}^{n}C_n = n\,2^{n-1}$.

56. Prove (combinatorially) that
${}^{r}C_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^{n}C_r = {}^{n+1}C_{r+1}$, $r \leq n$.

57. In a chess tournament, each participant was supposed to play exactly one game with each of the others. However, two participants withdraw after having played exactly 3 games each, but not with each other. The total number of games played in the tournament was 84. How many participants were there in all?

58. A positive integer n is called strictly ascending if its digits are in the increasing order. For example, 2368 and 147 are strictly ascending but 43679 is not. Find the number of strictly ascending numbers $< 10^9$.



59. The given figure shows 8 clay targets, arranged in 3 columns, to be shot by 8 bullets. Find the number of ways in which they can be shot, such that no target is shot before all the targets below it, if any, are first shot.

60. How many hexagons can be constructed by joining the vertices of a quindecagon (15 sides) if none of the sides of the hexagon is also the side of the 15-gon.

7.6 Definition of Permutation (Arrangements)

A permutation of given objects is an arrangement of the objects in a line or row, unless specified otherwise. These arrangements can be generated by changing the relative positions of objects in the row. Every possible relative order between the objects is taken into account.

For example, if 3 objects are represented as A, B, C, then permutations (arrangements or orders) of A, B, C in a row can be done in the following ways:

ABC, BAC, CAB, ACB, BCA, CBA

It can be observed that these permutations of A, B, C in a row are made by changing relative positions of A, B, C among themselves.

The permutations of A, B, C can also be made by taking not all A, B, C at a time but by just taking 2 objects at a time. This can be done in the following ways;

AB, BA, BC, CB, CA, AC

It can be observed that first, 2 objects are selected and then they are permutated (ordered or arranged) in the row by changing their relative positions among themselves.

Similarly (2, 1, 3, 4, 5), (5, 2, 1, 4, 3), (1, 2, 5, 4, 3), etc. are permutations of 1, 2, 3, 4, 5.

7.6.1 Theorem I

(Number of Permutations (arrangements, order) of n distinct objects taken all at a time)

The total number of permutations of n **distinct** objects $= n!$

Proof:

Let us consider that we have n distinct objects say $a_1, a_2, a_3, \dots, a_n$. We have to find total number of different permutations (arrangements or orders) of these objects along a row.

Every permutation of n objects is equivalent to fill n boxes (which are in a line) with these objects.

Let us consider n boxes

	1	2	3	4	5		$n-1$	n
Boxes:	□	□	□	□	□	…	□	□
Ways:	n	$n-1$	$n-2$	$n-3$	$n-4$		2	1

Box-1 can be filled in n ways by any of the n objects $a_1, a_2, a_3 \dots, a_n$.

Box-2 can be filled in $(n-1)$ ways by any of the remaining $(n-1)$ objects (excluding the object that has been used to fill Box-1).

Similarly, Box-3, Box-4, …, Box-n can be filled in $(n-2)$, $(n-3)$, …, 1 ways respectively.

Using fundamental principle of counting, total number of different ways to fill n boxes

$$= n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$
$$= n!$$

Hence, total number of permutation of n distinct objects is $n!$

Example 71 *Find number of different words which can be formed using all the letters of the word 'HISTORY'.*

Solution: Every way of arranging letters of the HISTORY will give us a word.
Therefore total number of ways to permutate letters H, I, S, T, O, R, Y, in a row

= Total number of words that can be formed using all letters together = 7!

$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

= 5040.

Example 72 *In how many way 5 distinct red balls, 3 distinct black balls and 2 distinct white balls can be arranged along a row?*

Solution: Total number of ways to arrange 10 balls along a row

= Number of permutations of 10 distinct objects in a row

$= \underline{|10}$.

Example 73 *In how many ways can the letters of the word 'DELHI' be arranged so that the vowels occupy only even places?*

Solution: All the letters in the word 'DELHI' are distinct with 2 vowels (E, I) and 3 consonants (D, L, H).
In five letter words, two even places can occupy 'E' and 'I' in 2! ways and remaining 3 places can occupy consonants D, L, H in 3! ways. So, number of words $= (3!) \times (2!) = 12$.

Example 74

(i) *How many words can be made by using letters of the word COMBINE all at a time?*
(ii) *How many of these words begin and end with a vowel?*
(iii) *In how many of these words do the vowels and the consonants occupy the same relative positions as in COMBINE?*

Solution:

(i) The total number of words = arrangements of seven letters taken all at a time = 7! = 5040.

(ii) The corresponding choices for all the places are as follows:

Place	vowel						vowel
Number of choices	3	5	4	3	2	1	2

As there are three vowels (O, I, E), first place can be filled in three ways and the last place can be filled in two ways. The rest of the places can be filled in 5! ways by five remaining letters.
Number of words $= 3 \times 5! \times 2 = 720$.

(iii) Vowels should be at second, fifth and seventh positions.
They can be arranged in 3! ways.
Consonants should be at first, third, fourth and sixth positions.
They can be arranged here in 4! ways.
Total number of words $= 3! \times 4! = 144$.

Example 75

(i) *How many words can be formed using letters of the word EQUATION taken all at a time?*
(ii) *How many of these begin with E and end with N?*
(iii) *How many of these end and begin with a consonants?*
(iv) *In how many of these, vowels occupy the first, third, fourth, sixth and seventh positions?*

Solution:

(i) Number of arrangements taken all at a time = 8! = 40320

$\Rightarrow$ 40320 words can be formed.

(ii) **Places:** **E _ _ _ _ _ _ N**

Choices: **1 6 5 4 3 2 1 1**

Number of words $= 1 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times 1$

$= 6! = 720$ words can be formed.

(iii) There are three consonants and five vowels.

Places: **_ _ _ _ _ _ _ _**

Choices: **3 6 5 4 3 2 1 2**

- First place can be filled in three ways, using any of the three consonants (T, Q, N).
- Last place can be filled in two ways, using any of the remaining two consonants.
- Remaining places can be filled by using remaining six letters

Number of words $= 3 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times 2$

$= 3 \times (6!) \times 2 = 4320$ words.

(iv) Let v: vowels and c: constants

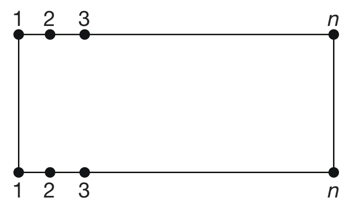
Places: **v c v v c v v c**

Choices: **5 3 4 3 2 2 1 1**

- First, put the vowels in the corresponding places in $5 \times 4 \times 3 \times 2 \times 1 = 5!$ ways
- Put the consonants in remaining three places in $3 \times 2 \times 1 = 3!$ ways

$\Rightarrow$ Number of words $= 5!\ 3! = 120 \times 6 = 720$.

Example 76 *$2n$ people (including A and B) are to be seated across a table, n people on each side (as shown in the figure). Find the number of arrangements so that A, B are neither next to each other nor directly opposite each other.*



Solution:

Case 1: 'A' at a corner seat

Options available for A = 4

Options available for B $= 2n - 3$

Number of arrangements $= 4 \times (2n - 3) \times (2n - 2)\ !$

(Note that remaining $2n - 2$ people in the remaining seats can be seated in $(2n - 2)!$ ways)

Case 2: 'A' not in a corner seat

Options available for A $= 2n - 4$

Options available for B $= 2n - 4$

Number of arrangements $= (2n - 4) \times (2n - 4) \times (2n - 2)!$

Using addition principle, total number of arrangements

$= 4 \times (2n - 3) \times (2n - 2)! + (2n - 4)^2\ (2n - 2)!$

$= (4n^2 - 8n + 4)\ (2n - 2)!$

$= 4(n - 1)^2\ (2n - 2)!$

7.6.2 Theorem 2

(Number of Permutations (arrangements, order) of n distinct objects taken r at a time)

The total numbers of permutations of r objects, out of n distinct objects, is $\dfrac{n!}{(n-r)!}$, $1 \le r \le n$.

This number is denoted as ${}^nP_r$ or $P(n, r)$ or ${}^nA_r$ or $A(n, r)$

Proof:

Let us consider that we have n different objects say $a_1, a_2, a_3, \ldots, a_n$. We have to find number of different permutations (arrangements or orders) of these objects taken only r at a time. (*i.e.*, we have to select r objects and arrange them).

Every arrangement of n objects taken r at a time is equivalent to fill r boxes.

Let us consider r boxes as shown in the figure:

	1	2	3	4	5	…	r
Boxes:						…	
Ways:	n	$n-1$	$n-2$	$n-3$	$n-4$		$n-(r-1)$

Box-1 can be filled in n ways by any of the n objects $a_1, a_2, a_3, \ldots, a_n$.

Box-2 can be filled in $(n-1)$ ways by any of the remaining $(n-1)$ objects (excluding the one that is used to fill Box-1).

Similarly, boxes 3, 4, 5, …, rth can be filled in $(n-2)$, $(n-3)$, …, $n-(r-1)$ ways respectively.

Using fundamental principle of counting, total number of ways to fill r boxes

$= n(n-1)(n-2)(n-3)\ldots(n-r+1)$

Multiply and divide by $\underline{|n-r}$ to get,

Number of ways to permutate n things taken r at a time

$$= \frac{n(n-1)(n-2)(n-3)\cdots(n-r+1))\underline{|n-r}}{\underline{|n-r}}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{\underline{|n-r}}$$

$$= \frac{\underline{|n}}{\underline{|n-r}} \qquad \{\text{Using}: \underline{|n-r} = (n-r)\ (n-r-1)\cdots 3\cdot 2\cdot 1\}$$

$$= {}^nP_r \qquad [\text{read it as 'n P r'}]$$

Alternatively, number of permutation of r objects, out of n distinct objects is equivalent to selecting r objects first out of n distinct which can be selected in $\binom{n}{r}$ ways and then arranging them in a line in $r!$ ways so total ways is $\binom{n}{r}\times r!$

$$\Rightarrow {}^np_r = \binom{n}{r}\times r!$$

$$= \frac{n!}{r!(n-r)!}\times r!$$

$$= \frac{n!}{(n-r)!}$$

Example 77 *If* ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, *find* ${}^rP_2$.

Solution: We have

$$\frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = \frac{30800}{1}$$

$$\Rightarrow \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow 56.55(51-r) = 30800 \Rightarrow r = 41$$

$$\Rightarrow \quad {}^{r}P_2 = {}^{41}P_2 = 41 \cdot 40 = 1640.$$

Example 78 *Prove that* ${}^{n}P_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$.

Solution: RHS $= {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-1-r+1)!} = \frac{(n-1)!}{(n-r-1)!} + \frac{r(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r)!}[n-r+r] = \frac{n!}{(n-r)!}$$

$$= {}^{n}P_r = \text{LHS}.$$

Aliter (Combinatorial): ${}^{n}P_r$ denotes the number of ways of arranging r-objects out of n-objects, in a line. This work can be done in the following way also. Suppose the objects are $a_1, a_2, \ldots, a_n$. First we find the number of permutations, in which a_1 does not appear. Number of such permutations is ${}^{n-1}P_r$. Further we consider those arrangements, in which a_1 necessarily appears. Number of such permutation is $r \cdot {}^{n-1}P_{r-1}$, (as we can arrange $(r-1)$ objects out of $(n-1)$ objects in ${}^{n-1}P_{r-1}$ ways, and then in any such permutation we can fix the position of a_1 in r ways). Now using the principle of addition, the required number is ${}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$.

Example 79 *Find number of different 4 letter words which can be formed using the letters of the word 'HISTORY'.*

Solution: Making a 4-letter word is equivalent to permutation of letters of the word 'HISTORY' taken 4 at a time.

$\Rightarrow$ Number of 4-letter words using letters of the word 'HISTORY'

= Number of permutation of letters H, I, S, T, O, R, Y taken only 4 at a time

$$= {}^{7}P_4 = \frac{\lfloor 7}{\lfloor 7-4} = \frac{\lfloor 7}{\lfloor 3}$$

$$= \frac{7\times 6\times 5\times 4\times \lfloor 3}{\lfloor 3} = 7\times 6\times 5\times 4 = 840.$$

Example 80 *In how many ways* 5 *distinct red balls,* 3 *distinct black balls and* 2 *distinct white balls can be placed in* 3 *distinct boxes such that each box contains only* 1 *ball.*

Solution: Total number of balls = 10. All balls are distinct.

The placement of 10 balls in 3 distinct boxes is equivalent to permutations of 10 distinct balls taken 3 at a time. This is because every arrangement of 3 balls will give a different way of placing 3 balls in 3 distinct boxes.

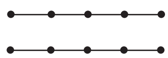
Therefore, total number of ways to place 10 distinct balls in 3 distinct boxes

= Number of permutations of 10 distinct balls taken 3 at a time

$$= {}^{10}P_3 = \frac{\lfloor 10}{\lfloor 10-3} = \frac{\lfloor 10}{\lfloor 7} = \frac{10\times 9\times 8\times 7}{\lfloor 7}$$

$= 10 \times 9 \times 8 = 720$ ways.

Example 81 *In a railway train compartment there are two rows of facing seats, five in each. Out of* 10 *passengers,* 4 *wish to sit looking forward and* 3 *looking towards rear of the train. The other three are indifferent. In how many ways can the passengers take seats?*

Solution:  (Forward) (Row A, say)
(Rear) (Row B, say)

4 people, in row A, can sit in ${}^5P_4$ ways $= \dfrac{5!}{(5-4)!} = 5 \times 4 \times 3 \times 2$ ways

3 people, in row B, can sit in ${}^5P_3$ ways $= \dfrac{5!}{(5-3)!} = 5 \times 4 \times 3$ ways

3 (indifferent) people in remaining 3 seats can sit in ${}^3P_3$ ways
$= 3! = 3 \times 2 \times 1$

By multiplication principle, the total number of ways in which 10 people can sit in rows A and B
$= (5 \times 4 \times 3 \times 2) \times (5 \times 4 \times 3) \times (3 \times 2 \times 1)$
$= (5!)^2 \times 3$
$= 43,200$ ways

Example 82 *A tea party is arranged for* 16 *people along two sides of a long table with* 8 *chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated?*

Solution: Let $A_1, A_2, A_3, \ldots, A_{16}$ be the sixteen persons. Assume that A_1, A_2, A_3, A_4 want to sit on side 1 and A_5, A_6 want to sit on side 2.
The persons can be made to sit if we complete the following operations:

(i) Select 4 chairs from the side 1 in ${}^8C_4$ ways and allot these chairs to A_1, A_2, A_3, A_4 in 4! ways.
(ii) Select two chairs from side 2 in ${}^8C_2$ ways and allot these two chairs to A_5, A_6 in 2! ways.
(iii) Arrange the remaining 10 persons in remaining 10 chairs in 10! ways.
$\Rightarrow$ Hence the total number of ways in which the persons can be arranged

$$= ({}^8C_4 4!)({}^8C_2 2!)(10!)$$

$$= \frac{8!}{4!\,4!} 4! \times \frac{8!\,2!}{2!\,6!} 10! = \frac{8!\,8!\,10!}{4!\,6!}.$$

Note: It is advised to use $\binom{n}{r} \times r!$ instead of ${}^nP_r$ directly as after selecting r objects you can always decide that whether you have to arrange them or not!

7.6.3 Theorem 3

(Permutation of Objects when not all objects are distinct)
Let there be n_1 A_1s, n_2 A_2s, …, n_k A_k's. Then the number of permutations $= \dfrac{(n_1 + n_2 + \cdots + n_k)!}{n_1!\,n_2! \ldots n_k!}$ (This number is known as a ***multinomial coefficient.***)

Numerator of the above formula is factorial of total number of items. Each terms in denominator is factorial of number of objects which are of same type and identical to each other. In earlier sections, we discussed how to permutate n different objects either taking all at a time or just r at a time. In this section, we will discuss how to arrange objects taken all at a time when all object are not distinct from each other.

For example, if we have to permutate A, A, B (Two A letters are identical) then number of permutations would not be same as permutations of 3 distinct objects say A, B, C. This is because two A letters cannot be permutated among themselves. Following are the ways to permutate A, A, B.

AAB, ABA, BAA, *i.e.*, 3 ways. This is not equal to $\lfloor 3$.

So we need to re define the formula we use to arrange n distinct objects.

For a case when all objects are not distinct. The redefined formula is given in theorem 3.

Proof: Total places we need to arrange all A_i's is $n_1 + n_2 + n_3 + \cdots + n_k = n$ (say)

Let us first select n_1 place out of n places to arrange n_1 A_1's this can be done in $\binom{n}{n_1}$ ways and there is only one way to arrange A_1 on these places. Now select n_2 places for A_2's out of remaining $n - n_1$ places.

This can be done in $\binom{n-n_1}{n_2}$ ways and arrange A_2's at these places in 1 way only and so on

$$\Rightarrow \quad \text{Total ways} = \binom{n}{n_1}\cdot 1\binom{n-n_1}{n_2}\cdot 1\cdots\binom{n_k}{n_k}\cdot 1$$

$$= \frac{n!}{n_1!\,n_2!\cdots n_k!}$$

Example 83 *How many different words can be formed by permuting all the letters of the word MATHEMATICS.*

Solution: In the word MATHEMATICS, total letters are 11

Number of 'M' letters = 2

Number of 'A' letters = 2

Number of 'T' letters = 2

Number of different letters = 5 (H, E, I, C, S)

Number of ways to arrange letters of the word 'MATHEMATICS'

$= \dfrac{\lfloor 11}{\lfloor 2 \lfloor 2 \lfloor 2}$ [using the formula given in Theorem 3]

Example 84 *How many different words can be formed by permuting all the letters of the word MISSISSIPPI?*

Solution: The word MISSISSIPPI is formed by 4S's, 4I's' 2P's and 1 M. Required number of different words $= \dfrac{11!}{4!\,4!\,2!\,1!}$ (using theorem 3).

Example 85 *How many n-term binary sequences can be formed of r 0's and (n – r) 1's?*

Solution: Number of binary sequences having n terms (r 0's, $(n-r)$ 1's) $= \dfrac{n!}{r!(n-r)!}$

This number known as a **binomial coefficient**.

Example 86 *How many 9-letter words can be formed by using the letters of the words*
(i) *EQUATIONS* (ii) *ALLAHABAD?*

Solution:
(i) All 9-letters in the word EQUATIONS are different.
Hence number of words = 9! = 362880.
(ii) ALLAHABAD contains LL, AAAA, H, B, D

$$\text{Number of words } = \frac{9!}{2!\,4!} = \frac{9\times 8\times 7\times 6\times 5}{2} = 7560.$$

Example 87 *How many anagrams (rearrangements) can be formed of the word 'PRIYANKA'?*

Solution: Here total letters are 8, in which 2 A's, but the rest are different. Hence the number of words formed = $\frac{8!}{2!}$ = 20160.

As we have to count rearrangements, so remove one word that is 'PRIYANKA'
Hence number of anagrams =20160 – 1= 20159.

Example 88 *Find the number of permutations of* 1, 2, …, 6, *in which*
(i) *1 occurs before 2,*
(ii) *3 occurs before 4,*
(iii) *5 occurs before 6.*

For example, 3 5 1 4 2 6

Solution: Let us use the following terms.
A permutation has property P_1 if 1 occurs before 2. A permutation has property P_2 if 3 occurs before 4. A permutation has property P_3 if 5 occurs before 6.

$$P_1^C \Leftrightarrow \text{not } P_1$$
$$P_2^C \Leftrightarrow \text{not } P_2$$
$$P_3^C \Leftrightarrow \text{not } P_3.$$

So there are 8 possibilities, *e.g.*, $P_1\, P_2^C\, P_3$, $P_1^C\, P_2\, P_3^C$, *etc.*
Number of $P_1\, P_2\, P_3$ = Number of $P_1^C\, P_2\, P_3$ = . . . = Number of $P_1\, P_2\, P_3^C$

$$\Rightarrow \quad \text{Number of permutations having } P_1\, P_2\, P_3 = \frac{6!}{8} = 90.$$

Aliter 1: Assume 1 and 2 as *a*, *a*, 3, 4 as *b*, *b*, 5, 6 as *c*,*c* now arrange *a*, *a*, *b*, *b*, *c*, *c* in a line. This can be done in $\frac{6!}{2!2!2!}$ ways = 90.

Now starting from left first *a* replaced by 1 and second *a* replaced by 2, similarly *b* and *c*, we will get the desired permutation.

Aliter 2: Arrange 1 and 2 in 6 places in ${}^6C_2$ ways.
Now, to arrange 3 and 4 we have ${}^4C_2$ ways and to arrange 5, 6 we have only one way.

Finally by Multiplication Principle total number of ways ${}^6C_2\,{}^4C_2 = \frac{6!}{8}$. = 90.

7.6.3.1 Permutations of n Objects Taken r at a Time when All n Objects are not Distinct

In this section we will discuss how to arrange (permutate) n objects taken r at a time where all n objects are not distinct.

For example, arrangements of letters AABBBC taken 3 at a time.

To find such arrangements, it is not possible to derive a formula that can be applied in all such cases.

So, we will discuss a method (or procedure) that should be applied to find arrangements. The method involves making cases based on alike items that we choose in the arrangement. You should read the following examples to learn how to apply this 'method of cases' to find arrangements of n objects taken r at a time when all objects are not different.

Example 89 *Find the number of 4-letter words, that can be formed from the letters of the word 'ALLAHABAD'.*

Solution: We have four A, two L, and one each of H, B and D.
Four letters from the letters of the word ALLAHABAD would be one of the following types; (i) all same (ii) three same, one distinct (iii) two same, two same (iv) two same, two distinct and (v) all four distinct

Now number of words of type (i) is 1

Number of words of type (ii) is ${}^4C_1 \times \frac{4!}{3!} = 16$

Number of words of type (iii) is $\frac{4!}{2!2!} = 6$

Number of words of type (iv) is ${}^2C_1{}^4C_2 \times \frac{4!}{2!} = 144$

Number of words of type (v) is ${}^5C_4\, 4! = 120$

Thus the required number = 1 + 16 + 6 + 144 + 120 = 287.

Example 90 *Find in how many ways we can arrange letters AABBBC taken 3 at a time.*

Solution: The given letters include AA, BBB, C, *i.e.*, 2 A letters, 3 B letters and 1 C letters.
To find arrangements of 3 letters, we will make following cases based on alike letters we choose in the arrangement.

Case 1: All 3 letters are alike

3 alike letters can be selected from given letters in only 1 way, *i.e.*, BBB.

Further 3 selected letters can be arranged among themselves in $\frac{\lfloor 3}{\lfloor 3} = 1$ way.

$\Rightarrow$ Total number of arrangement with all letters alike = 1 (1)

Case 2: 2 alike and 1 distinct

2 alike letters can be selected from 2 sets of alike letters (AA, BB) in ${}^2C_1$ ways.

1 distinct letter (distinct from selected alike letters) can be selected from remaining letters in ${}^2C_1$ ways. (C, A or B either).

Further 2 alike and 1 distinct selected letters can be arranged among themselves in $\frac{\lfloor 3}{\lfloor 2}$ ways.

$\Rightarrow$ Total number of arrangements with '2 alike and 1 distinct letter'

$$= {}^2C_1 \times {}^2C_1 \times \frac{\lfloor 3}{\lfloor 2} = 2\times 2\times 3 = 12 \quad (2)$$

Case 3: All distinct letters

All 3 letters distinct can be selected from 3 distinct letters (A, B, C) in 1 way.

Further 3 distinct letters can be arranged among themselves in $\lfloor 3$ ways.

$\Rightarrow$ Total number of arrangements with all 3 letters distinct $= 1\times\lfloor 3 = \lfloor 3 = 6$ (3)

Combining (1), (2) and (3)

Total number of permutations of AABBBC taken 3 at a time = 1 + 12 + 6 = 19.

Example 91 *How many 4-letters words can be formed using the letters of the word INEFFECTIVE?*

Solution: INEFFECTIVE contains 11 letters: EEE, FF, II, C, T, N, V.

As all letters are not distinct, we cannot use ${}^nP_r$. The 4-letter words will be among any one of the following cases:

1. 3 alike letters, 1 distinct letter.
2. 2 alike letters, 2 alike letters.
3. 2 alike letters, 2 distinct letters.
4. All distinct letters.

Case 1: 3 alike, 1 distinct

3 alike can be selected in one way, *i.e.*, EEE.

Distinct letters can be selected from F, I, T, N, V, C in ${}^6C_1$ ways.

$\Rightarrow$ Number of groups $= 1 \times {}^6C_1 = 6 \Rightarrow$ Number of words $= = 6\times\frac{4!}{3!\times 1!} = 24.$

Case 2: 2 alike, 2 alike

Two sets of 2 alike can be selected from 3 sets (EE, II, FF) in ${}^3C_2$ ways.

$\Rightarrow$ Number of words $= {}^3C_2 \times \frac{4!}{2!\times 2!} = 18$

Case 3: 2 alike, 2 distinct

$\Rightarrow$ Number of groups $= ({}^3C_1)\times({}^6C_2) = 45 \Rightarrow$ Number of words $= 45\times\frac{4!}{2!} = 540$

Case 4: All distinct

$\Rightarrow$ Number of groups $= {}^7C_4$ (out of E, F, I, T, N, V, C)

$\Rightarrow$ Number of words $= {}^7C_4 \times 4! = 840$

Hence total 4-letter words = 24 + 18 + 540 + 840 = 1422.

7.6.4 Theorem 4

(Arrangement of n distinct objects with repetition of objects)

Total number of ways to permutate n distinct things taken r at a time when objects can be repeated any number of times is n^r.

Proof:

Here we have to arrange n distinct objects in a row taken only r at a time when objects can be repeated any number of times, *i.e.*, repetition of objects is allowed.

Permutation of n objects in a row taken r at a time is equivalent to filling r boxes. Let us consider r boxes as shown in the figure:

	1	2	3	4	…	r
Boxes:	☐	☐	☐	☐	…	☐
Ways:	n	n	n	n		n

Box-1 can be filled in n ways by any of the n objects.

Box-2 can also be filled in n ways as any of the n objects can be used to fill Box-2. This is because, we can reuse the object used to fill Box-1 to fill Box-2 as repetition of objects is allowed.

Similarly Box-3, Box-4, …, Box-r each one can be fill in n ways each.

Using fundamental principle of counting, total number of way to fill n boxes $= n \times n \times n \ldots r$ times $= n^r$.

Example 92 *A child has four pockets and three different marbles. In how many ways can the child put the marbles in his pockets?*

Solution: The first marble can be put into the pocket in 4 ways, so the second can also be put in the pocket in 4 ways so can the third . Thus, the number of ways in which the child can put the marbles $= 4 \times 4 \times 4 = 64$ ways.

Example 93 *In how many ways can 5 letters be posted in 4 letter boxes?*

Solution: Since each letter can be posted in any one of the four letter boxes. So, a letter can be posted in 4 ways. Since there are 5 letters and each letter can be posted in 4 ways. So, total number of ways in which all the five letters can be posted $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

Example 94 *Five person entered the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin*

(i) *at any one of the 7 floors* (ii) *at different floors.*

Solution: Suppose A_1, A_2, A_3, A_4, A_5 are five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$.

(ii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4 and A_5 can leave the cabin in 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7 \times 6 \times 5 \times 4 \times 3 = 2520$.

Example 95 *There are 6 single choice questions in an examination. How many sequence of answers are possible, if the first three questions have 4 choices each and the next three have 5 each?*

Solution: Here we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences $= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$.

Example 96 *Three tourist want to stay in five different hotels. In how many ways can they do so if:*

(i) *each hotel can not accommodate more than one tourist?*
(ii) *each hotel can accommodate any number of tourists?*

Solution:

(i) Three tourists are to be placed in 3 different hotels out of 5. This can be done as:
Place first tourist in 5 ways
Place second in 4 ways
Place third in 3 ways
$\Rightarrow$ Required number of placements $= 5 \times 4 \times 3 = 60$

(ii) To place the tourists we have to do following three operations.
(a) Place first tourist in any of the hotels in 5 ways.
(b) Place second tourist in any of the hotels in 5 ways.
(c) Place third tourist in any of the hotels in 5 ways.
$\Rightarrow$ the required number of placements $= 5 \times 5 \times 5 = 125$.

7.6.5 Some Miscellaneous Applications of Permutations

7.6.5.1 Always Including *p* Particular Objects in the Arrangement

The number of ways to select and arrange (permutate) r objects from n distinct objects such that arrangement should always include p particular objects $= {}^{n-p}C_{r-p} \times r!$.

Logic: First select p particular objects which should always be included in 1 way (1)

Then select remaining $(r - p)$ objects from remaining $(n - p)$ objects in ${}^{n-p}C_{r-p}$ ways. (2)

Finally arrange r selected objects in $r!$ ways (3)

Using fundamental principle of counting, operations (1), (2) and (3) can be performed together in ways

$= 1 \times {}^{n-p}C_{r-p} \times r!$ ways.

7.6.5.2 Always Excluding *p* Particular Objects in the Arrangement

The number of ways to select and arrange r objects from n distinct objects such that p particular objects are always excluded in the selection $= {}^{n-p}C_r \times r!$.

Logic: First exclude p particular objects from n different objects.

Then select r objects from $(n - p)$ different objects in ${}^{n-p}C_r$ ways. (1)

Then permutate r selected objects in r! ways. (2)

Using fundamental principle of counting, operations (1) and (2)can be performed together in ${}^{n-p}C_r \times r!$ ways.

Example 97 *How many three letter words can be made using the letters of the words SOCIETY, so that*

(i) *S is included in each word?* (ii) *S is not included in any word?*

Solution:

(i) To include S in every word, we will use following steps.
Step 1: Select the remaining two letters from remaining 6 letters, *i.e.*, O, C, I, E, T, Y in ${}^6C_2$ ways.

Step 2: Include S in each group and then arrange each group of three in 3! ways.
$\Rightarrow$ Number of words = ${}^6C_2\ 3! = 90$.

(ii) If S is not to be included, then we have to make all the three words from the remaining 6.
$\Rightarrow$ Number of words = ${}^6C_3\ 3! = 120$.

7.6.5.3 'p' Particular Objects Always Together in the Arrangement

The number of ways to arrange n distinct objects such that p particular objects remain together in the arrangement $(n-p+1)!\ p!$

Logic: Make a group of p particular objects that should remain together. Arrange this group of p particular objects and remaining $(n-p)$ objects in $(n-p+1)!$ ways. (1)

Finally arrange p particular objects among themselves in $p!$ ways. (2)

Using fundamental principle of counting operations (1) and (2) can be performed together in $(n-p+1)! \times p!$ ways

Example 98 *How many words can be formed using the letters of the word TRIANGLE so that*

(i) *A and N are always together?* (ii) *T, R, I are always together?*

Solution:

(i) Assume (AN) as a single letter. Now there are seven letters in all:
(AN), T, R, I, G, L, E
Seven letters can be arranged in 7! ways.
All these 7! words will contain A and N together. A and N can now be arranged among themselves in 2! ways (AN and NA).
Hence total number of words = 7! 2! = 10080.

(ii) Assume (TRI) as a single letter.
The letters: (TRI), A, N, G, L, E can be rearranged in 6! ways.
TRI can be arranged among themselves in 3! ways.
Total number of words = 6! 3! = 4320.

Example 99 *How many 5-letter words containing* 3 *vowels and* 2 *consonants can be formed using the letters of the word EQUATIONS so that the two consonants occur together in every word?*

Solution: There are 5 vowels and 3 consonants in EQUATION. To form the words we will use following steps:

Step 1: Select vowels (3 from 5) in ${}^5C_3$ ways.

Step 2: Select consonants (2 from 3) in ${}^3C_2$ ways.

Step 3: Arrange the selected letters (3 vowels and 2 consonants (always together)) in $4! \times 2!$ ways.
Hence the number of words = ${}^5C_3\ {}^3C_2\ 4!\ 2! = 10 \times 3 \times 24 \times 2 = 1440$.

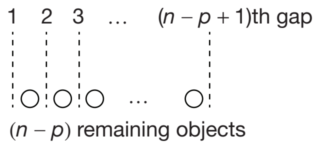
7.6.5.4 'p' Particular Objects Always Separated in the Arrangement

The number of ways to arrange n different objects such that p particular objects are always separated

$$= {}^{n-p+1}C_p \times (n-p)! \times p!$$

Logic: First arrange $n - p$ objects in $(n - p)!$ ways. Now we have to place p particular objects between $(n - p)$ remaining objects so that all p particular objects must be separated from each other.

From figure we can see there are $(n - p + 1)$ gaps (including before and after) between $(n - p)$ objects where we can place p particular objects such that p objects are separated from each other.



Select p gaps from $(n - p + 1)$ gaps for p particular objects in ${}^{n-p+1}C_p$ ways. Now place and arrange p objects in p selected gaps in $p!$ ways. Using fundamental principle of counting, all operations can be performed together in ${}^{n-p+1}C_p \times (n - p)! \times p!$ ways.

Example 100 *There are 9 candidates for an examination out of which 3 are appearing in Mathematics and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two Mathematics candidates are together?*

Solution: Divide the work in two steps.

Step 1: First, arrange the remaining candidates in 6! ways.

Step 2: Place the three Mathematics candidates in the row of six other candidates so that no two of them are together.

***x*:** Places available for Mathematics candidates.

***o*:** Others.

x	o	x	o	x	o	x	o	x	o	x	o	x

In any arrangement of 6 other candidates (o), there are seven places available for Mathematics candidate so that they are not together. Now 3 Mathematics candidates can be placed in these 7 places in $\binom{7}{3} 3!$ ways.

Hence total number of arrangements

$$= 6!\binom{7}{3}3! = 720 \times \frac{7!}{4!} = 151200.$$

Example 101 *In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus (–) signs are together?*

Solution:

Step 1: The plus signs can be arranged in one way (because all are identical).

	+		+		+		+		+		+		+	

A blank box shows available spaces for the minus signs.

Step 2: The 5 minus (–) signs are now to be placed in the 8 available spaces so that no two of them are together.

(i) Select 5 places for minus signs in ${}^8C_5$ ways.

(ii) Arrange the minus signs in the selected places in 1 way (all signs being identical).

Hence number of possible arrangements $= 1 \times {}^8C_5 \times 1 = 56$.

Example 102 *There are 20 stations between stations A and B. In how many ways a train moving from station A to station B can stop at 3 stations between A and B such that no two stopping stations are together?*

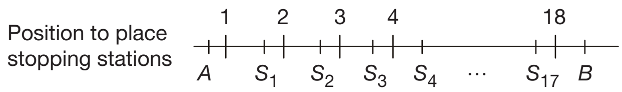
Solution: We have to select 3 stations from 20 stations between A and B so that train can stop at these stations.

According to the question:

There are 3 stopping stations that should be separated from each other, *i.e.*, even no two of them are together.

First separate out 3 stations to the selected from 20 stations, *i.e.*, 17 station left.

Now, we select 3 positions between 17 stations so that we can place 3 stopping stations. There are 18 positions between 17 stations where we can place 3 stopping stations.



Therefore, number of ways to select 3 stations where train can stop

= number of ways to place 3 stopping stations between remaining 17 stations

$= {}^{18}C_3$.

7.6.5.5 Rank of a Word in the Dictionary

In these type of problems, dictionary of words is formed by using all the arrangement of all letters at a time of the given word. The dictionary format means words are arranged in the alphabetical order. You will be supposed to find the rank (position) of the given word or some other word in the dictionary.

Following examples will help you learn how to find the rank in the dictionary.

Example 103 *Find the rank of the word MOTHER in the dictionary order of the words formed by M, T, H, O, E, R.*

Solution: Number of words starting with E, having other letters M, T, H, O, R = 5 ! = 120

Number of words starting with H, having other letters M, T, E, O, R = 5 ! = 120

Number of words having first two letters M,E and other letters O, T, H, R = 4 ! = 24

Number of words having first two letters M,H and other letters T, E, O, R = 4 ! = 24

Number of words having first three letters M,O,E and other letters H, T, R = 3 ! = 6

Number of words having first three letters M,O,H and other letters T, E, R = 3 ! = 6

Number of words having first three letters M,O,R and other letters T, H, E = 3 ! = 6

Number of words having first four letters M,O,T,E and other letters H, R = 2 ! = 2

Total number of words, before MOTHER, in the dictionary order made up of M, O, E, T, H, R = 120 + 120 + 24 + 24 + 6 + 6 + 6 + 2 = 308

$\therefore$ Rank of the word MOTHER = 309.

Example 104 *If all the letters of the word RANDOM are written in all possible orders and these words are written out as in a dictionary, then find the rank of the word RANDOM in the dictionary.*

Solution: In a dictionary the words at each stage are arranged in alphabetical order. In the given problem, we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time, *i.e.*, A will occur 5! times. D, M, N, O will occur in the first place the same number of times.

Number of words starting with A = 5! = 120

Number of words starting with D = 5! = 120

Number of word starting with M = 5! = 120

Number of words starting with N = 5! = 120

Numbers of words starting with O = 5! = 120
After this, words beginning with RA must follow.
Number of words beginning with RAD or RAM = 3!
Now the words beginning with RAN must follow.
First one is RANDMO and the next one is RANDOM.
∴ Rank of RANDOM = 5(5!) + 2(3!) + 2 = 614.

Example 105 *Find the rank of the word 'TTEERL' in the dictionary of words formed by using the letters of the word 'LETTER'.*

Solution: In the dictionary of words formed, we need to count words before the word 'TTEERL' in the dictionary. To count such words, we need to first count words starting with E, L, R, TE, TL, TR and then add 2 to the count for words 'TTEELR' and 'TTEERL'.

Number of words starting with E = Arrangement of letter E, T, T, R, L $= \dfrac{\underline{|5}}{\underline{|2}}$

Number of words starting with L = Arrangement of letters E, T, T, E, R $= \dfrac{\underline{|5}}{\underline{|2}\,\underline{|2}}$

Number of words starting with R = Arrangement of letters E, T, T, E, L $= \dfrac{\underline{|5}}{\underline{|2}\,\underline{|2}}$

Number of words starting with TE = Arrangement of letters T, E, R, L $= \underline{|4}$

Number of words starting with TL = Arrangement of letters E, T, E, R $= \dfrac{\underline{|4}}{\underline{|2}}$

Number of words starting with TR = Arrangement of letters T, E, E, L $= \dfrac{\underline{|4}}{\underline{|2}}$

Rank of TTEERL $= \dfrac{\underline{|5}}{\underline{|2}} + \dfrac{\underline{|5}}{\underline{|2}\,\underline{|2}} + \dfrac{\underline{|5}}{\underline{|2}\,\underline{|2}} + \underline{|4} + \dfrac{\underline{|4}}{\underline{|2}} + \dfrac{\underline{|4}}{\underline{|2}} + 2 = 170$

(Now, try to find the rank of the word COCHIN, in the list, in the dictionary order, of the words made up of C, C, H, I, O, N. Your answer should be 97).

Build-up Your Understanding 3

1. Find the value of r in following equations:
   (i) ${}^{5}P_r = {}^{6}P_{r-1}$ **(ii)** ${}^{10}P_r = 720$ **(iii)** ${}^{20}P_r = 13 \times {}^{20}P_{r-1}$
2. In a railway compartment 6 seats are vacant on a berth. Find the number of ways in which 3 passengers sit on them.
3. Three men have 6 different trousers, 5 different shirts and 4 different caps. Find the number of different ways in which they can wear them.
4. Find the number of words of four letters containing equal number of vowels and consonants (repetition not allowed).
5. Find the number of words that can be formed using 6 consonants and 3 vowels out of 10 consonants and 4 vowels.
6. Find the number of ways in which the letters of the word ARRANGE can be made such that both R's do not come together.
7. Find the number of arrangements of the letters of the word BANANA is which the two 'N's do not appear adjacently.
8. We are required to form different words with the help of the letters of the word INTEGER. Let m_1 be the number of words in which I and N are never together and m_2 be the number of words which begin with I and end with R, then find m_1/m_2.

9. Find the number of arrangements that can be made with the letters of the word MATHEMATICS and also find the number of them, in which the vowels occur together.

10. Find the number of ways in which letters of the word VALEDICTORY be arranged so that the vowels may never be separated.

11. Find the number of different words which can be formed from the letters of the word LUCKNOW when

(i) all the letters are taken.

(ii) all the letters are taken and words begin with L.

(iii) all the letters are taken and the letters L and W respectively occupy the first and last places.

(iv) all the letters are taken and the vowels are always together.

12. Find the number of permutations of the word AUROBIND in which vowels appear in an alphabetical order.

13. If as many more words as possible be formed out of the letters of the word DOGMATIC then find the number of words in which the relative position of vowels and consonants remain unchanged.

14. Find the number of words which can be formed using all letters of the word 'Pataliputra' without changing the relative order of the vowels and consonants.

15. Find the total numbers of words that can be made by writing all letters of the word PARAMETER so that no vowel is between two consonants.

16. Find the total number of permutation of $n(n > 1)$ distinct things taken not more than r at a time and atleast 1, when each thing may be repeated any number of times.

17. Find the number of permutations of n distinct objects taken

(i) atleast r objects at a time

(ii) atmost r objects at a time

(Where repetition of the objects is allowed)

18. If the number of arrangements of $n - 1$ things from n distinct things is k times the number of arrangements of $n - 1$ things taken from n things in which two things are identical then find the value of k.

19. Find the number of different 7-digit numbers that can be written using only the three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number.

20. Six identical coins are arranged in a row. Find the total number of ways in which the number of heads is equal to the number of tails.

21. There are n distinct white and n distinct black balls. Find the number of ways in which we can arrange these balls in a row so that neighboring balls are of different colours.

22. Find number of ways in which 6 girls and 6 boys can be arranged in a line if no two boys or no two girls are together.

23. Find the number of ways in which 3 boys and 3 girls (all are of different heights) can be arranged in a line so that boys as well as girls among themselves are in decreasing order of height (from left to right).

24. Find the number of ways in which 10 candidates $A_1, A_2, \ldots, A_{10}$ can be ranked so that A_1 is always above A_2.

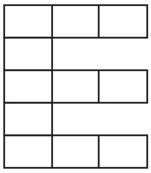
25. Let A be a set of n (≥ 3) distinct elements. Find the number of triples (x, y, z) of the elements of A in which atleast two coordinates are equal.

26. Find the number of ways of arranging m numbers out of 1, 2, 3, …, n so that maximum is $(n - 2)$ and minimum is 2 (repetitions of numbers is allowed) such that maximum and minimum both occur exactly once, $(n > 5, m > 3)$.

27. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4, and then the men select the chairs from amongst the remaining. Find the number of possible arrangements.

28. There are 10 numbered seats in a double decker bus, 6 in the lower deck and 4 on the upper deck. Ten passengers board the bus, of them 3 refuse to go to the upper deck and 2 insist on going up. Find the number of ways in which the passengers can be accommodated.

29. In how many different ways a grandfather along with two of his grandsons and four grand daughters can be seated in a line for a photograph so that he is always in the middle and the two grandsons are never adjacent to each other.



30. Find the number of ways in which A A A B B B can be placed in the squares of the figure as shown, so that no row remains empty.

31. The tamer of wild animals has to bring one by one 5 lions and 4 tigers to the circus arena. Find the number of ways this can be done if no two tigers immediately follow each other.

32. In a conference 10 speakers are present. If S_1 wants to speak before S_2 and S_2 wants to speak after S_3, then find the number of ways all the 10 speakers can give their speeches with the above restriction if the remaining seven speakers have no objection to speak at any number.

33. Find the total number of flags with three horizontal strips, in order, that can be formed using 2 identical red, 2 identical green and 2 identical white strips.

34. Messages are conveyed by arranging 4 white, 1 blue and 3 red flags on a pole. Flags of the same colour are alike. If a message is transmitted by the order in which the colours are arranged then the find the total number of messages that can be transmitted if exactly 6 flags are used.

35. Find number of arrangements of 4-letters taken from the word EXAMINATION.

36. Find number of ways in which an arrangement of 4-letters can be made from the letters of the word PROPORTION.

37. Find the number of permutations of the word ASSASSINATION taken 4 at a time.

38. The letters of the word TOUGH are written in all possible orders and these words are written out as in a dictionary, then find the rank of the word TOUGH.

39. The letters of the word SURITI are written in all possible orders and these words are written out as in a dictionary. What is the rank of the word SURITI?

40. There are 720 permutations of the digits 1, 2, 3, 4, 5, 6. Suppose these permutations are arranged from smallest to largest numerical values, beginning from 1 2 3 4 5 6 and ending with 6 5 4 3 2 1.
(a) What number falls on the 124th position?
(b) What is the position of the number 321546?

41. All the five digits number in which each successive digit exceeds its predecessor are arranged in the increasing order of their magnitude. Find the 97th number in the list.

42. All the 5 digit numbers, formed by permuting the digits 1, 2, 3, 4 and 5 are arranged in the increasing order. Find:
(i) the rank of 35421 (ii) the 100th number.

43. There are 11 seats in a row. Five people are to be seated. Find the number of seating arrangements, if
(i) the central seat is to be kept vacant;
(ii) for every pair of seats symmetric with respect to the central seat, one seat is vacant.

44. Find the number of ways in which six children of different heights can line up in a single row so that none of them is standing between the two children taller than him.

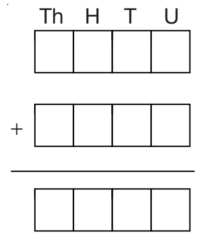
45. Define a 'good word' as a sequence of letters that consists only of the letters A, B and C and in which A never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. If the number of n-letter good words are 384, then find the value of n.

46. There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. Find the number of ways in which they can be arranged in a row so that atleast one ball is separated from the balls of the same colour.

47. Eight identical rooks are to be placed on an 8×8 chess-board. Find the number of ways of doing this, so that no two rooks are in attacking positions.

48. How many arrangements of the 9 letters $a, b, c, p, q, r, x, y, z$ are there such that y is between x and z? (Any two, or all three, of the letters x, y, z, may not be consecutive.)

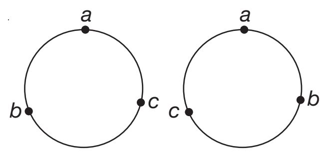
49. In the figure, two 4-digit numbers are to be formed by filling the place with digits. Find the number of different ways in which these places can be filled by digits so that the sum of the numbers formed is also a 4-digit number and in no place the addition is with carrying.



50. Two n-digit integers (leading 0 allowed) are said to be equivalent if one is a permutation of the other. Thus 10075 and 01057 are equivalent. Find the number of 5-digit integers such that no two are equivalent.

7.7 Introduction to Circular Permutation

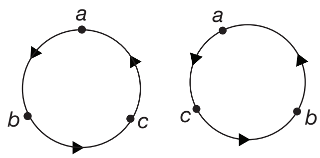
When objects are to be arranged (ordered) in a circle instead of a row, it is known as Circular Permutation. For example, three objects a, b, c can be permutated in a circle as shown in figure:



Number of ways to arrange a, b, c in circle is not same as number of ways to arrange a, b, c in a row.

This is because arrangements abc, bca, cab in a row are same in circle as shown in the figure.

Similarly arrangements acb, cba, bac in a row are same in circle as shown in the figure.

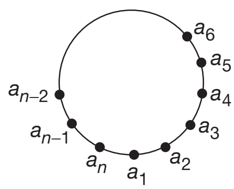


7.7.1 Theorem

The number of *circular permutations* of n distinct objects is $(n - 1)!$

Proof: Let $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ be n distinct objects. Let the total number of circular permutations be x. Consider one of these x permutations as shown in Figure.

Clearly, this circular permutation provides n linear permutations as given below:



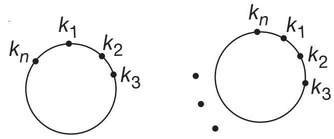
$$a_1, a_2, a_3 \ldots a_{n-1}, a_n$$
$$a_2, a_3, a_4, \ldots a_n, a_1$$
$$a_3, a_4, a_5, \ldots a_n, a_1, a_2$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$a_n, a_1, a_2, a_3, \ldots, a_{n-1}$$

Thus, each circular permutation gives n linear permutations. As there are x circular permutations, the number of linear permutations is xn. But the number of linear permutations of n distinct objects is $n!$.

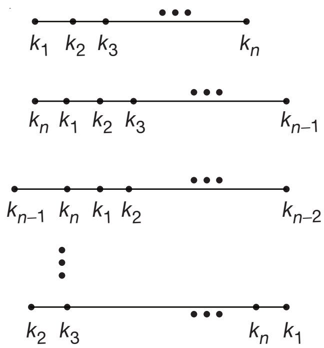
$$\therefore xn = n! \Rightarrow x = \frac{n!}{n} = (n-1)!$$

Aliter 1: Number of linear permutations of n distinct objects $= n!$. Consider two linear permutation of n distinct objects $k_1, k_2, k_3, \ldots, k_n$ and $k_n, k_1, k_2, \ldots, k_{n-1}$.

Consider a corresponding circular permutation as shown in the following figure.



(For example, think of two thread each having n beads)



In fact, both the circular arrangements are same. Not only that, there are more similar looking circular permutations. There are n linear permutations as shown, which give the same circular permutation.

So while counting the number of circular permutations from the number of linear permutations, one circular permutation is counted n times.

$\therefore$ Number of circular permutations $= \dfrac{n!}{n} = (n-1)!$

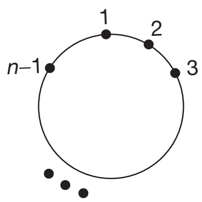
Aliter 2: Let P_n denote the number of circular permutations of n distinct objects.

Note that $P_1 = 1$.

Let $(n-1)$ objects (out of these n objects) be placed on a circle.

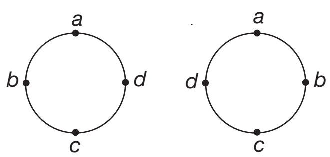
This can be done in P_{n-1} ways.

These $n-1$ objects break the circle into $n-1$ arcs. Now the nth object is to be kept some where on these $(n-1)$ arcs. This can be done in $(n-1)$ ways.



$$
\begin{aligned}
\therefore P_n &= (n-1)\,P_{n-1} \quad \text{(recurrence relation)} \\
&= (n-1)(n-2)\,P_{n-2} \\
&= (n-1)(n-2)(n-3)\,P_{n-3} \text{ and so on} \\
&= (n-1)(n-2)(n-3)\ldots 3.\,2.\,1.\,P_1 \\
&= (n-1)!
\end{aligned}
$$

7.7.2 Difference between Clockwise and Anti-clockwise



In some of the problems we need to consider clockwise and anti-clockwise arrangements of objects as same arrangements. See the adjacent circular permutations. There is a difference of just the cyclic order. In first arrangement a, b, c, d are arranged in anti-clockwise order where as in second they are arranged clockwise order.

If we have to consider these arrangements same (for example, arrangement of flowers in garland or arrangement of beads in a necklace), then we need to divide total circular permutation by 2.

Therefore,

Number of circular permutations of n distinct objects such that clockwise and anti-clockwise arrangements of objects are same $= \dfrac{(n-1)!}{2}$, $n \geq 3$.

Notes:

1. Number of circular permutations of 'n' distinct things taken 'r' at a time $= \dbinom{n}{r}(r-1)!$ (when clockwise and anticlockwise orders are taken as different)
2. If clockwise and anticlockwise orders are taken as same, then the required number of circular permutations $= \dfrac{\dbinom{n}{r}(r-1)!}{2}$, $r \geq 3$.

Example 106 *In how many ways can* 13 *persons out of* 24 *persons be seated around a table.*

Solution: In case of circular table the clockwise and anti-clockwise orders are different, thus the required number of circular permutations $= \dfrac{\binom{24}{13}13!}{13} = \dfrac{24!}{13 \times 11!}$.

Example 107 *Out of ten people,* 5 *are to be seated around a round table and* 5 *are to be seated across a rectangular table. Find the number of ways to do so.*

Solution: First select 5 people out of 10, those who sit around the table. This can be done in ${}^{10}C_5$ ways.

Number of ways in which these 5 people sit around the round table = 4!

Remaining 5 people sit across a rectangular table in 5! ways.

Total number of arrangements

$$= {}^{10}C_5 \times 4! \times 5!$$

$$= \frac{10!}{5!5!} \times 4! \times 5!$$

$$= 10! \times \frac{1}{5} = 9! \times 2.$$

Example 108 *There are* 20 *persons among whom are two brothers. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers.*

Solution: Let B_1 and B_2 be two brothers among 20 persons and let M be a person that will sit between B_1 and B_2. The person M can be chosen from 18 person (excluding B_1 and B_2) in 18 ways. Considering the two brothers B_1 and B_2 and person M as one person and remaining 17 persons separately, we have 18 persons in all. These 18 persons can be arranged around a circle in (18 – 1)! = 17! ways. But B_1 and B_2 can be arranged among themselves in 2! ways.

Hence, the total number of ways = 18 × 17! × 2! = 2 × 18!

Example 109 *In how many ways can a party of* 4 *men and* 4 *women be seated at a circular table so that no two women are adjacent?*

Solution: The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men in (4 – 1)! = 3! ways. Now, 4 vacant seats can be occupied by 4 women in 4! ways.

Hence, the required number of seating arrangements = 3! × 4! = 144.

Example 110 *A round table conference is to be held between* 20 *delegates of* 2 *countries. In how many ways can they be seated if two particular delegates are* (i) *always together?* (ii) *never together?*

Solution:

(i) Let D_1 and D_2 be two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all. These 19 delegates can be seated round a circular table in (19 – 1)! = 18! ways. But two particular delegates can arrange among themselves in 2! ways (D_1 D_2 and D_2 D_1).

Hence, the total number of ways = 18! × 2! = 2 (18!).

(ii) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of seating arrangements of 20 persons around the round table. Clearly 20 persons can be seated around a circular table in $(20 - 1)! = 19!$ ways.

Hence, the required number of seating arrangements $= 19! - 2 \times 18! = 17\,(18!)$.

Alternate Solution:

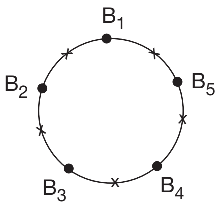
First arrange remaining 18 persons in $(18 - 1)! = 17!$ ways.

Then select two gaps out of 18 gaps between 18 persons on the circle in ${}^{18}C_2$ ways and arrange the two in 2! ways.

Number of ways $= 17! \times {}^{18}C_2 \times 2!$

$= 17\,(18!)$.

Example 111 *In how many different ways can five boys and five girls form a circle such that the boys and girls are alternate?*



Solution: After fixing up one boy on the table the remaining can be arranged in 4! ways. There will be 5 places, one place each between two boys which can be filled by 5 girls in 5! ways.

Hence by the principle of multiplication, the required number of ways

$$= 4! \times 5! = 2880.$$

Example 112 *Find the number of ways to seat 5 boys and 5 girls around a table so that boy B_1 and girl G_1 are not adjacent.*

Solution: Number of ways of arranging 5 boys and 5 girls around a table is

$$(10 - 1)! = 9!.$$

Among these, we have to discard the arrangements where B_1 and G_1 sit together. Consider B_1G_1 as a single entity. There all 9 people can be arranged around a circle in $(9 - 1)! = 8!$ ways.

But the boy B_1 and girl G_1 can either be arranged in B_1G_1 or in G_1B_1 position. So, the number of ways in which boy B_1 and girl G_1 are together is $2 \times 8!$.

Therefore, the number of ways in which boy B_1 and girl G_1 are not together is $9! - 2 \times 8! = 8!(9 - 2) = 7 \times 8! = 2{,}82{,}240$.

Aliter: Exclude G_1 initially. The remaining 9 can be arranged in $(9 - 1)! = 8!$ ways around a circle. Now, there are 9 in-between positions among the 9 people seated around a circle. Of these 9, the two sides of B_1, *i.e.*, his left and right are not suited for G_1 (as B_1 and G_1 must not come together). Hence, there are 7 choices in each of the circular permutations for G_1.

$\therefore$ The total number of ways of arranging the person is 7(8!) ways.

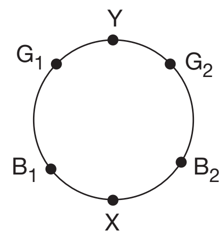
Example 113 *There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together?*

Solution: Five gentlemen can be seated at a round table in $(5 - 1)! = 4!$ ways. Now, 5 places are created in which 4 ladies are to be seated. Select 4 seats for 4 ladies from 5 seats in ${}^5C_4$ ways. Now 4 ladies can be arranged on the 4 selected seats in 4! ways.

Hence, the total number of ways in which no two ladies sit together

$$= 4! \times {}^5C_4 \times \lfloor 4 = (4!)\,5(4!) = 2880.$$

Example 114 *Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible?*



Solution: Let B_1, B_2 and X be three boys and G_1, G_2 and Y be three girls. Since the boy X does not want any girl neighbour. Therefore boy X will have his neighbours as boys B_1 and B_2 as shown in in the figure. Similarly, girl Y has her neighbour as girls G_1 and G_2 as shown figure. But the boys B_1 and B_2 can be arranged among themselves in 2! ways and the girls G_1 and G_2 can be arranged among themselves in 2! ways.
Hence, the required number of arrangements = $2! \times 2! = 4$.

Example 115 *Find the number of ways in which* 8 *distinct flowers can be strung to form a garland so that* 4 *particular flowers are never separated.*

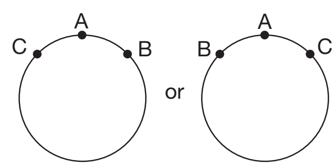
Solution: Considering 4 particular flowers as one group of flower, we have five flowers (one group of flowers and remaining four flowers) which can be strung to form a garland in $\frac{4!}{2}$ ways. But 4 particular flowers can be arranged themselves in 4! ways.
Thus, the required number of ways $= \frac{4! \times 4!}{2} = 288$.

Example 116 *Find the number of arrangements in which g girls and b boys are to be seated around a table, $b \le g$, so that no two boys are together.*

Solution: g girls can be seated around a table in $(g-1)!$
This positioning of g girls create g gaps for b boys to be seated. b boys in those g gaps can be seated in $\binom{g}{b} b!$ ways.

Total number of arrangements $= (g-1)! \times \binom{g}{b} b!$.

Example 117 *Find the number of arrangements of* 10 *people including A, B, C such that B and C occupy the chairs next to A on a circular arrangement.*



Solution: 'A' occupies his chair in 1 way. B and C occupy their chairs in 2 ways.
Remaining 7 people occupy their chairs in 7 ! ways.

Total number of arrangements = $1 \times 2 \times 7\,!$

Aliter: Consider A, B, C as one person so there are 8 person and we can arrange them in 7! ways. Now B and C can interchange their position in 2! ways. So total ways = $2 \times 7!$.

Example 118 *Find the number of ways in which* 12 *distinct beads can be arranged to form a necklace.*

Solution: 12 distinct beads can be arranged among themselves in a circular order in $(12 - 1)! = 11!$ ways. Now in the case of necklace there is no distinction between clockwise and anti-clockwise arrangements. So the required number of arrangements $= \frac{1}{2}(11!)$.

Example 119 *How many necklace of* 12 *beads each can be made from* 18 *beads of various colours?*

Solution: In the case of necklace there is no distinction between the clockwise and anticlockwise arrangements, thus the required number of circular permutations

$$=\frac{\binom{18}{12}12!}{2\times 12}=\frac{18!}{6!\times 24}=\frac{18\times 17\times 16\times 15\times 14\times 13!}{6\times 5\times 4\times 3\times 2\times 1\times 24}=\frac{119\times 13!}{2}.$$

Example 120 *In how many ways can seven persons sit around a table so that all shall not have the same neighbours in any two arrangements?*

Solution: Clearly, 7 persons can sit at a round table in $(7-1)!=6!$ ways. But, in clockwise and anti-clockwise arrangements, each person will have the same neighbours.

So, the required number of ways $=\frac{1}{2}(6!)=360$

Example 121 *If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these things so that no two of them are next to each other is* $\frac{n}{6}(n-4)(n-5)$.

Solution: Let $a_1, a_2, a_3, \ldots, a_n$ be the n distinct objects.
Number of ways to select three objects so that no two of them are consecutive = Total number of ways to select three objects − Number of ways to select three consecutive objects − Number of ways to select three objects in which two are consecutive and one is separated. (1)

Total number of ways to select 3 objects from n distinct objects = ${}^nC_3$ (2)

Select three consecutive objects:
The three consecutive objects can be selected in the following manner.

Select from: $a_1\, a_2\, a_3,\ a_2\, a_3\, a_4,\ a_3\, a_4\, a_5, \ldots, a_{n-1}\, a_n\, a_1,\ a_n\, a_1\, a_2$

So, number of ways in which 3 consecutive objects can be selected from n objects arranged in a circle is n. (3)

Select two consecutive (together) and 1 separated:
The three objects so that 2 are consecutive and 1 is separated can be selected in the following manner:

Take $a_1\, a_2$ and select third object from $a_4, a_5, \ldots, a_{n-1}$, *i.e.*, take $a_1\, a_2$ and select third object in $(n-4)$ ways or in general we can say that select one pair from n available pairs, *i.e.,* $a_1\, a_2\; a_2\, a_3 \ldots a_n\, a_1$ and third object in $(n-4)$ ways.

So, number of ways to select 3 objects so that 2 are consecutive and 1 is separated $= n(n-4)$ (4)

Using (1), (2), (3) and (4), we get:

Number of ways to select 3 objects so that all are separated $={}^nC_3-n-n(n-4)$

$$=\frac{n(n-1)(n-2)}{6}-n-n(n-4)=n\left[\frac{n^2-3n+2-6(n-3)}{6}\right]$$

$$=\frac{n}{6}(n^2-9n+20)=\frac{n}{6}(n-4)(n-5).$$

Build-up Your Understanding 4

1. A cabinet of ministers consists of 11 ministers, one minister being the chief minister. A meeting is to be held in a room having a round table and 11 chairs round it, one of them being meant for the chairman. Find the number of ways in which the ministers can take their chairs such that the chief minister occupying the chairman's place.
2. 20 persons were invited for a party. In how many ways can they and the host be seated at a circular table? In how many of these ways will two particular persons be seated on either side of the host?
3. In how many ways can 7 boys be seated at a round table so that two particular boys are
   (i) next to each other (ii) separated.
4. A round table conference is to be held between 20 delegates of 2 countries. In how many ways can they be seated if two particular delegates
   (i) always sit together (ii) never sit together.
5. There are 20 persons including two brothers. In how many ways can they be arranged on a round table if:
   (i) There is exactly one person between the two brothers.
   (ii) The two brothers are always separated.
   (iii) What will be the corresponding answers if the two brothers were twins (alike in all respects)?
6. $2n$ chairs are arranged symmetrically around a table. There are $2n$ people, including A and B, who wish to occupy the chairs. Find the number of seating arrangements, if:
   (i) A and B are next to each other;
   (ii) A and B are diametrically opposite.
7. The 10 students of Batch B feel they have some conceptual doubt on circular permutation. Mr. Tiwari called them in discussion room and asked them to sit down around a circular table which is surrounded by 13 chairs. Mr. Tiwari told that his adjacent seat should not remain empty. Then find the number of ways, in which the students can sit around a round table if Mr. Tiwari also sit on a chair.
8. Find the number of ways in which 5 boys and 4 girls can be arranged on a circular table such that no two girls sit together and two particular boys are always together.
9. A person invites a party of 10 friends at dinner and place them
   (i) 5 at one round table, 5 at the other round table.
   (ii) 4 at one round table and 6 at other round table.
   Find the ratio of number of circular permutation of case (i) to case (ii).
10. Six persons A, B, C, D, E and F are to be seated at a circular table. Find the number of ways this can be done if A must have either B or C on his right B must have either C or D on his right.
11. Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.
12. Find the number of different garlands, that can be formed using 3 flowers of one kind and 3 flowers of other kind.
13. Find the number of seating arrangements of 6 persons at three identical round tables if every table must be occupied.
14. Let $1 \le n \le r$. The Stirling number of the first kind, $S(m, n)$, is defined as the number of arrangements of m distinct objects around n identical circular tables so that each table contains atleast one object. Show that:
    (i) $S(m, 1) = (m - 1)!$;
    (ii) $S(m, m - 1) = {}^mC_2$, $m \ge 2$.

15. Find the number of different ways of painting a cube by using a different colour for each face from six available colours.

    (Any two colour schemes are called different if one cannot coincide with the other by a rotation of the cube.)

16. Find number of ways in which n things of which r alike and the rest distinct can be arranged in a circle distinguishing between clockwise and anti-clockwise arrangement.

7.8 Division and Distribution of Non-identical Items in Fixed Size

7.8.1 Unequal Division and Distribution of Non-identical Objects

In this section we will discuss ways to divide non-identical objects into groups. For example, if we have to divide three different balls (b_1, b_2, b_3) among 2 boys (B_1 and B_2) such that B_1 gets 2 balls and B_2 gets 1 ball, then

Number of ways to divide balls among boys is 3 ways as shown in the following table.

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2

Instead of writing all ways and counting them, we can make a formula to find number of ways.

First select 2 balls for B_1 in ${}^3C_2$ and then remaining 1 ball for B_2 in ${}^1C_1$ ways.

Total number of ways, using fundamental principle of counting, is

$= {}^3C_2 \times {}^1C_1 = 3 \times 1 = 3$ ways.

If we have to divide 3 non-identical balls among 2 boys such that one boy should get 2 and other boy should get 1, then following are the ways:

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2
b_3	b_1, b_2
$b_1,$	b_2, b_3
b_2	b_3, b_1

Distribution of above 3 ways among 2 boys you can observe that entries are interchanged,between B_1 and B_2

$\Rightarrow$ Total ways to distribute = 6.

Instead of writing all ways and counting them, we can just find number of ways using fundamental principle of counting.

First select 2 balls for B_1 in ${}^3C_2$ ways, then select 1 remaining ball for B_2 in ${}^1C_1$ ways, finally distribute among 2 boys in $\underline{|2}$ ways (ball given to B_1 and B_2 are interchanged) because any boy can get 2 balls and the other 1 ball.

Using fundamental principle of counting, total number of ways

$= {}^3C_2 \times {}^1C_1 \times \underline{|2} = 3 \times 1 \times 2 = 6$ ways.

Now generalising the above cases, we can write the following formula:

1. Number of ways in which $(m + n + p)$ distinct objects can be divided into 3 unequal (groups contain unequal number of objects) **unnumbered groups** containing m, n, p objects $= {}^{m+n+p}C_m \, {}^{n+p}C_n \, {}^{p}C_p = \dfrac{(m+n+p)!}{m!n!p!}$ (Here among m, n, p no two are equal)
2. Number of ways in which $(m + n + p)$ distinct objects can be divided and distribute into 3 unequal **numbered groups** (Here among m, n, p no two are equal) containing m, n, p objects
   = Number of ways to divide $(m + n + p)$ objects in 3 groups × Number of ways to distribute 'division-ways' among groups
   = Number of ways to divide $(m + n + p)$ objects in 3 groups × (Number of groups)!
   $= \dfrac{(m+n+p)!}{m!n!p!} \times 3!$

Above formulae are written for dividing objects into 3 groups but in case groups are more, then also we follow the same approach. For example,

Number of ways to divide 10 non-identical objects in 4 groups (G_1, G_2, G_3, G_4) such that groups G_1, G_2, G_3, G_4 gets 1, 2, 3, 4 objects respectively $= \dfrac{\lfloor 10}{\lfloor 1 \lfloor 2 \lfloor 3 \lfloor 4}$

Number of ways to divide 10 non-identical objects in 4 groups (G_1, G_2, G_3, G_4) such that groups get objects in number 1, 2, 3, 4 (*i.e.*, any group can get 1 object or 2 objects or 3 objects or 4 objects).

= Number of ways to divide and distribute 10 objects in 4 groups containing 1, 2, 3, 4 objects

$$= \frac{\lfloor 10}{\lfloor 1 \lfloor 2 \lfloor 3 \lfloor 4} \times \lfloor 4.$$

7.8.2 Equal Division and Distribution of Non-identical objects

Here we will see formulae to divide and distribute non-identical objects equally in groups, *i.e.*, each group get equal numbers of objects.

1. Number of ways to divide (mn) distinct objects equally in m unnumbered group (each group get n objects)
   $$\binom{mn}{n} \cdot \binom{mn-n}{n} \binom{mn-2n}{n} \cdots \binom{n}{n} \cdot \frac{1}{m!} = \frac{(mn)!}{(n!)^m \, m!}$$
2. Number of ways to divide (mn) objects equally in m numbered group (each group gets n objects)
   $$= \frac{(mn)!}{(n!)^m \, m!} \times m! = \frac{mn!}{(n!)^m}$$

Example 122 *In how many ways,* 12 *distinct objects can be distributed equally in* 3 *groups?*

Solution: Let the groups be labelled as A, B, C. (For our convenience)
Select 4 objects out of 12 to be given to group A in ${}^{12}C_4$ ways. Select 4 objects out of remaining 8 to be given to group B in ${}^{8}C_4$ ways. Rest 4 objects are to be given to group C in one way. (*i.e.*, ${}^{4}C_4$ ways)

Apparently, by multiplication principle, the total number of ways is ${}^{12}C_4 \cdot {}^{8}C_4 \cdot {}^{4}C_4$ but each grouping is counted 3! times ! ${}^{12}C_4 \cdot {}^{8}C_4 \cdot {}^{4}C_4$ is the number of **ordered** grouping.

Understand that, if objects are named as $a_1, a_2, a_3, \ldots, a_{12}$ then the grouping 12 elements as

$(a_1\ a_2\ a_3\ a_4)\ (a_5\ a_6\ a_7\ a_8)\ (a_9\ a_{10}\ a_{11}\ a_{12})$ is same as $(a_1\ a_2\ a_3\ a_4)\ (a_9\ a_{10}\ a_{11}\ a_{12})\ (a_5\ a_6\ a_7\ a_8)$ or same as $(a_9\ a_{10}\ a_{11}\ a_{12})\ (a_1\ a_2\ a_3\ a_4)\ (a_5\ a_6\ a_7\ a_8)$, etc.

$$\therefore \text{ Required number } = \frac{{}^{12}C_4 \cdot {}^{8}C_4 \cdot {}^{4}C_4}{3!} = \frac{12!}{4!8!} \cdot \frac{8!}{4!4!} \cdot \frac{1}{3!} = \frac{12!}{3!(4!)^3}.$$

Example 123 *In how many ways can* 12 *books be equally distributed among* 3 *students?*

Solution: In this question we have to divide books equally among 3 students. So we will use formulae (2) given in section 7.8.2. Where we divided non-identical objects equally among numbered groups as all students are distinct.

Therefore, number of ways to divide and distribute 12 non-identical objects among 3 students equally $= \dfrac{\lfloor 12}{(\lfloor 4)^3}$.

Example 124 *In how many ways we can divide 52 playing cards*
(i) *among 4 players equally?* (ii) *in 4 equal parts?*

Solution:

(i) 52 cards is to be divided equally among 4 players. Each player will get 13 cards. It means we should apply distribution formula. Using formula (2) given in section 7.8.2, we get:

Number of ways to divide playing cards $= \dfrac{\lfloor 52}{(\lfloor 13)^4}$

(ii) As we have to make 4 equal parts, each part consist of 13 cards. We will apply division formula (not distribution). Using formula (1) used in section 7.8.2 we get:

Number of ways to divide 52 cards in 4 parts $= \dfrac{\lfloor 52}{(\lfloor 13)^4} \dfrac{1}{\lfloor 4}$.

7.8.3 Equal as well as Unequal Division and Distribution of Non-identical Objects

Here we will see formulae to divide and distribute non-identical objects into groups such that not all groups contain equal or unequal number of objects, *i.e.*, some groups get equal and some get unequal number of objects.

1. Number of ways to divide $(ma + nb + nc)$ distinct (Out of a, b, c no two numbers are equal) objects in $(m + n + p)$ unnumbered groups such that m groups contains a objects each, n groups contains b objects each, p group contains c objects each

$$\frac{(ma + nb + nc)!}{(a!)^m (b!)^n (c!)^p\ m!\ n!\ p!}$$

Note: We divided by $m!$ because there are m groups containing a objects each (equal number of objects).
We divided by $n!$ also because there are n groups containing b objects each (equal number of objects). We also divided by $p!$ as p groups of are equal size.

2. Number of ways to divide and distribute $(ma + nb + pc)$ distinct objects (out of a, b, c no two numbers are equal) in $(m + n + p)$ numbered groups such that m groups contains a objects each, n groups contains b objects each, p groups contains c object each

$$= \frac{(ma+nb+pc)!}{(a!)^m (b!)^n (c!)^p \ m! \ n! \ p!} \times (m+n+p)!$$

We can make similar formulae for other cases.

Illustration Number of ways to divide 10 objects in 4 groups containing 3, 3, 2, 2 objects

$$\frac{\binom{10}{3}\binom{7}{3}}{2!}\frac{\binom{4}{2}\binom{2}{2}}{2!} = \frac{\lfloor 10}{(\lfloor 2)^2}\frac{1}{(\lfloor 3)^2}\frac{1}{\lfloor 2}\frac{1}{\lfloor 2}$$

Number of ways to divide and distribute completely 10 objects in 4 groups containing 3, 3, 2, 2 objects

$$\frac{\binom{10}{3}\binom{7}{3}}{2!}\frac{\binom{4}{2}\binom{2}{2}}{2!} \times 4! = \left[\frac{\lfloor 10}{(\lfloor 2)^2}\frac{1}{(\lfloor 3)^2}\frac{1}{\lfloor 2}\frac{1}{\lfloor 2}\right] \times \lfloor 4$$

Number of ways to divide and distribute $(m + 2n + 3p)$ distinct in 6 numbered groups such that 3 particular groups get p objects each, 2 particular gets n objects each, one one get m objects

$$= \frac{\lfloor m+2n+3p}{\lfloor m (\lfloor n)^2 (\lfloor p)^3}$$

Example 125 10 *different toys are to be distributed among* 10 *children. Find the total number of ways of distributing these toys so that exactly* 2 *children do not get any toy.*

Solution: It is possible in two mutually exclusive cases;

Case 1: 2 children get none, one child gets three and all remaining 7 children get one each.

Case 2: 2 children get none, 2 children get 2 each and all remaining 6 children get one each.
Using formula (2) given in section 7.8.3, we get:

Case 1: Number of ways $= \left(\dfrac{10!}{(0!)^2 \ 2!\,3!(1!)^7\,7!}\right)10!$

Case 2: Number of ways $= \left(\dfrac{10!}{(0!)^2 2!\,(2!)^2 2!\,(1!)^6\,6!}\right)10!$

Thus total ways $= (10!)^2\left(\dfrac{1}{3!7!2!} + \dfrac{1}{(2!)^4 6!}\right).$

Example 126 *In how many ways can 7 departments be divided among 3 ministers such that every minister gets at least one and atmost 4 departments to control?*

Solution: Let 3 minister be M_1, M_2, M_3.
Following are the ways in which we can divide 7 departments among 3 ministers such that each minister gets at least one and atmost 4.

S.No.	M_1	M_2	M_3
1	4	2	1
2	2	2	3
3	3	3	1

Note: If we have a case (2, 2, 3), then there is no need to make cases (3, 2, 2) or (2, 3, 2) because we will include them when we apply distribution formula to distribute ways of division among ministers.

Case 1: We divide 7 departments among 3 ministers in number 4, 2, 1, *i.e.*, unequal division. As any minister can get 4 departments, any one can get 2 any one can get 1 department, we should apply distribution formula. Using formula (2) given in section 7.8.1, we get:
Number of ways to divide and distribute departments in number 4, 2, 1

$$= \left[\frac{\underline{|7}}{\underline{|4}\,\underline{|2}\,\underline{|1}}\right] \times 3! = 630 \qquad (1)$$

Case 2: It is 'equal as well as unequal' division. As any minister can get any number of departments, we use complete distribution formula. Using formula (2) given in section 7.8.3 we get:
Number of ways to divide and distribute departments in number 2, 2, 3.

$$= \left[\frac{\underline{|7}}{\underline{|2}\,\underline{|2}\,\underline{|3}}\,\frac{1}{\underline{|2}}\right] \times 3! = 630 \qquad (2)$$

Case 3: It is also 'equal as well as unequal' division. As any minister can get any number of departments, we use complete distribution formula. Using formula (2) given in section 7.8.3 we get:
Number of ways to divide and distribute departments in number 3, 3, 1

$$= \left[\frac{\underline{|7}}{(\underline{|3})^2(\underline{|1})}\,\frac{1}{\underline{|2}}\right] \times \underline{|3} = 420 \qquad (3)$$

Combining (1), (2) and (3), we get number of ways to divide 7 departments among 3 minister = 630 + 630 + 420 = 1680 ways.

Build-up Your Understanding 5

1. Find the total number of ways of dividing 15 different things into groups of 8, 4 and 3 respectively.
2. Find the number of ways of distributing 50 identical things among 8 persons in such a way that three of them get 8 things each, two of them get 7 things each and remaining 3 get 4 things each.
3. Find the number of ways in which 14 men be partitioned into 6 committees where two of the committees contain 3 men each, and the others contain 2 men each.
4. If $3n$ different things can be equally distributed among 3 persons in k ways then find the number of ways to divide the $3n$ things in 3 equal groups.

5. Find the number of ways to give 16 different things to three persons A, B, C so that B gets 1 more than A and C gets 2 more than B.
6. Find the number of ways of distributing 10 different books among 4 students S_1, S_2, S_3 and S_4 such that S_1 and S_2 get 2 books each and S_3 and S_4 get 3 books each.
7. Find the number of different ways in which 8 different books can be distributed among 3 students, if each student receives at least 2 books.
8. Find the number of ways in which n different prizes can be distributed amongst m ($<n$) persons if each is entitled to receive at most $n - 1$ prizes.
9. In a school there are two prizes for excellence in physics (Ist and IInd) two in Chemistry (Ist and IInd) and only 1 in Mathematics (Ist). In how many ways can these prizes be awarded to 20 students.
10. In an election three districts are to be canvassed by 2, 3 and 5 men respectively . If 10 men volunteer, then find the number of ways they can be allotted to the different districts.
11. A train time-table must be compiled for various days of the week so that two trains a day depart for three days, one train a day for two days and three trains a day for two days. Assuming all trains are identical how many different time-tables can be compiled?
12. In how many ways can 3 persons stay in 5 hotels? In how many of these each person stays in a different hotel.
13. 'n' different toys have to be distributed among 'n' children. Find the total number of ways in which these toys can be distributed so that exactly one child gets no toy.
14. Find the number of ways in which 7 different books can be given to 5 students if each can receive none, one or more books.
15. There are $(p + q)$ different books on different topics in Mathematics, where $p \neq q$. If L = The number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.
    M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.
    N = The number of ways in which these books are divided into two groups of p books and q books.
    Then prove that $2L = M = 2N$.

7.9 Number of Integral Solutions

7.9.1 Number of Non-negative Integral Solutions of a Linear Equation

Let the given equation be

$$x_1 + x_2 + x_3 + \cdots + x_r = n$$

Let A be the set of all non-negative integral solutions of the given equation and B be the set of all $(n + r - 1)$ term binary sequences containing n, 1's and $(r - 1)$, 0's. Here number of 1's before the first zero is value of x_1, number of 1's between first zero and second zero is value of x_2 and so on, number of 1's after the $r - 1$th zero is the value of x_r.

So for every non-negative integral solution of the equation there is a binary sequence of n, 1's and $(r - 1)$, 0's. And for every binary sequence of n 1's and $(r - 1)$ 0's, we can write a non-negative integral solution. Therefore there is bijection between the sets A and B.

$\Rightarrow$ Number of non negative integral solutions of the equation is same as the number of binary sequences.

Number of non-negative integral solutions $= \dfrac{(n+r-1)!}{n!(r-1)!} = \dbinom{n+r-1}{r-1}$

Example 127 *Find the number of non-negative integral solutions of*

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8$$

Solution: Take a sample solution,

$$\underset{x_1}{\boxed{2}} + \underset{x_2}{\boxed{0}} + \underset{x_3}{\boxed{3}} + \underset{x_4}{\boxed{2}} + \underset{x_5}{\boxed{1}} = 8 \qquad (1)$$

Take a binary sequence of 8, 1's and 4, 0's as

$$110011101101 \qquad (2)$$

which corresponds to the sample solution.

(2) is an arrangement of 12 objects, 8 of which are of one type and 4 of which are of another type.

Total number of such arrangements $= \dfrac{12!}{8!4!}$

= Total number of binary sequences of 8, 1's and 4, 0's.

$$\text{Number of non-negative integral solutions } = \frac{12!}{8!4!} = \frac{12\times 11\times 10\times 9}{4.3.2.1} = 495.$$

Observe that:

1. 0's we have used as demakers or separators. Since there are 4 gaps between the x_is, therefore we need 4 0's.
2. Pocket of x_2 is filled in the sample solution by 0 (that is the value of the variable; students are advised not to get confused between the value zero of a variable and a 0 used in the binary sequence) and the corresponding binary sequence shows a 0 followed by another 0.

Example 128 *Find the number of positive integral solutions of* $x_1 + x_2 + x_3 + x_4 + x_5 = 8$

Solution: Since we are interested in finding the number of positive solutions, therefore each x_i must have minimum value 1. So we take 8 identical coins (*i.e.*, similar to taking 8, 1's basically 8 identical objects to be taken) and 5 pockets of x_i' s.

$$\underset{x_1}{\boxed{}} + \underset{x_2}{\boxed{}} + \underset{x_3}{\boxed{}} + \underset{x_4}{\boxed{}} + \underset{x_5}{\boxed{}} = 8$$

Fill each pocket by one coin. So 3 coins are left, which are now to be filled in the pockets of x_i's.

Now this problem is similar to finding number of binary sequences of 3, 1's and 4, 0's. This number is

$$\frac{7!}{3!4!} = \frac{7\times 6\times 5}{3!} = 35 = \text{Number of positive integral solutions.}$$

7.9.2 Number of Non-negative Integral Solutions of a Linear Inequation

Consider the given inequation as

$$x_1 + x_2 + x_3 + \cdots + x_r \leq n \tag{1}$$

Add a non-negative integer x_{r+1} to get

$$x_1 + x_2 + x_3 + \cdots + x_r + x_{r+1} = n. \tag{2}$$

Number of solutions of Eq. (2)

$$= \binom{n+r}{r} = \frac{(n+r)!}{r!n!}$$

7.9.3 Number of Integral Solutions of a Linear Equation in $x_1, x_2, \ldots, x_r$ when x_i's are Constrained

Consider

$$x_1 + x_2 + x_3 + \cdots + x_r = n \tag{1}$$

where $x_1 \geq a_1, x_2 \geq a_2, \ldots, x_r \geq a_r,$ all a_i's are integers.

Take $x_1 = a_1 + x_1'$

$x_2 = a_2 + x_2'$, etc.,

where $x_1' \geq 0, x_2' \geq 0, \ldots, x_r' \geq 0$

Eq. (1) reduces to

$$(a_1 + a_2 + \ldots + a_r) + x_1' + x_2' + \cdots + x_r' = n$$

$$\Leftrightarrow x_1' + x_2' + \cdots + x_r' = n - (a_1 + a_2 + \cdots + a_r) \tag{2}$$

For every solution of Eq. (1), we can write a corresponding solution of Eq. (2) and for every solution of Eq. (2), we can write a corresponding solution of Eq. (1). Therefore there is a bijection between the sets of solutions of Eqs. (1) and (2).

Number of solutions of Eq. (1) = Number of non-negative integral solutions of Eq. (2)

$$= \frac{(n + r - 1 - (a_1 + a_2 + \cdots + a_r))!}{(r-1)!(n - (a_1 + a_2 + \cdots + a_r))!}$$

Example 129 *Find the number of integral solutions of* $x_1 + x_2 + x_3 + x_4 = 14$, *where* $x_1 \geq -2$, $x_2 \geq 1$, $x_3 \geq 2$ *and* $x_4 \geq 0$.

Solution: Let $x_1 = -2 + x_1'$, $x_2 = 1 + x_2'$, $x_3 = 2 + x_3'$,

Then given equation can be written as

$$x_1' + x_2' + x_3' + x_4 = 13, \quad x_1', x_2', x_3', x_4 \geq 0 \tag{1}$$

Number of non-negative integral solutions of Eq. (1)

$$= \frac{16!}{3!13!}$$

$$= \frac{16 \times 15 \times 14}{1 \times 2 \times 3}$$

$= 560$

= Number of integral solutions of the given equation.

Example 130 *How many integral solutions are there to $x + y + z + t = 29$, when $x \ge 1$, $y \ge 2$, $z \ge 3$ and $t \ge 0$?*

Solution: We have,

$$x \ge 1, y \ge 2, z \ge 3 \text{ and } t \ge 0, \text{ where } x, y, z, t \text{ are integers}$$

Let $u = x - 1$, $v = y - 2$, $w = z - 3$.

Then, $x \ge 1 \Rightarrow u \ge 0$; $y \ge 2 \Rightarrow v \ge 0$; $z \ge 3 \Rightarrow w \ge 0$

Thus, we have

$u + 1 + v + 2 + w + 3 + t = 29 \Rightarrow u + v + w + t = 23$ [where $u \ge 0$; $v \ge 0$; $w \ge 0$]

$\Rightarrow$ The total number of solutions of this equation is

$$^{23+4-1}C_{4-1} = {}^{26}C_3 = 2600.$$

Example 131 *How many integral solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$ when $x_k \ge 0$?*

Solution: We have: $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$

These two equations reduce to

$$x_4 + x_5 = 15 \tag{1}$$

$$\text{and } x_1 + x_2 + x_3 = 5 \tag{2}$$

Since corresponding to each solution of Eq. (1) there are solutions of Eq. (2). So, total number of solutions of the given system of equations.

= Number of solutions of Eq. (1) × Number of solutions of Eq. (2)

$= ({}^{15+2-1}C_1)\,({}^{5+3-1}C_2) = {}^{16}C_1 \times {}^{7}C_2 = 336.$

7.10 Binomial, Multinomial and Generating Function

7.10.1 Binomial Theorem

Blaise Pascal

19 Jun 1623–19 Aug 1662

Nationality: French

Given $n, r \in N$, $0 \le r \le n$, the number $\binom{n}{r}$ or ^nC_r is defined to be the number of r elements subsets of an n elements set. These are also called the binomial coefficients as these occur as the coefficients in the expansion of

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n$$

Some important results related to summation of binomial coefficients:

1. $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r} = \binom{n}{m-r}\binom{n-m+r}{r}$

2. $\sum_{r=0}^{n}\binom{n}{r} = 2^n; \quad \sum_{r\ge 0}\binom{n}{2r} = \sum_{r\ge 0}\binom{n}{2r+1} = 2^{n-1}$

3. $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0$

4. $\sum_{r=0}^{n} r\binom{n}{r} = n \cdot 2^{n-1}$

5. $\sum_{i=0}^{r}\binom{m}{i}\binom{n}{r-i}=\binom{m}{0}\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\cdots+\binom{m}{r}\binom{n}{0}=\binom{m+n}{r}$

(Vandermonde Identity)

6. $\sum_{i+0}^{n}\binom{n}{i}^2=\binom{2n}{n}$

7. $\binom{r}{r}+\binom{r+1}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1}; n,\ r\in N, n\geq r$ (Hockey stick Identity)

8. $\binom{r}{0}+\binom{r+1}{r}+\cdots+\binom{r+k}{k}=\binom{r+k+1}{k}; r, k\in N$

Alexandre-Théophile van der Monde

28 Feb 1735–1 Jan 1796
Nationality: French

7.10.2 Binomial Theorem for Negative Integer Index

Given $n\in\mathbb{N}, x\in(-1, 1)$

then $(1+x)^{-n}=\sum_{r\geq 0}^{\infty}\binom{n+r-1}{r}x^r$

7.10.3 Multinomial Coefficients

Like binomial coefficients, if we consider the expansion of $(x_1+\cdots+x_m)^n$, then we get the following expansion:

$(x_1+x_2+\cdots+x_m)^n=\sum\binom{n}{n_1, n_2, \ldots, n_m}x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$, where the sum is taken over all sequences $(n_1, n_2, \ldots, n_m)$ of non-negative integers with $\sum_{i=1}^{m}n_i=n$.

Here $\binom{n}{n_1, n_2, \ldots, n_m}=\dfrac{n!}{n_1!\cdot n_2!\cdots n_m!}$ is called multinomial coefficient.

7.10.3 Application of Generating Function

For large number of selection of objects we use 'Integral Equation Method followed by generating function'. In this method we group alike objects together and with each group we define a variable representing number of objects selected from the group. Then we add all variables and equate the sum to the total objects to be selected.

For example, if we have to select 3 objects from AAAAABBBBCCC objects, then we make groups of identical objects, group of all A objects, group of all B objects and group of all C objects. Let x_1, x_2, x_3 be the number of A, B, C objects selected respectively.

As total number of objects to be selected is 3, we can make following integral equation:

$$x_1+x_2+x_3=3 \qquad [\text{where } 0\leq x_i\leq 3,\ i=1, 2, 3]$$

Number of solutions of the above integral equation is same as the number of ways to select 3 objects from the given objects. This is because every solution of the equation is a way to select 3 objects.

Number of solutions of the equation

$$=\text{Coefficient of } x^{\text{Sum}} \text{ in } \left[x^{\min(x_1)}+x^{\min(x_1)+1}+\cdots+x^{\max(x_1)}\right]$$
$$\times\left[x^{\min(x_2)}+x^{\min(x_2)+1}+\cdots+x^{\max(x_2)}\right]\times\left[x^{\min(x_3)}+x^{\min(x_3)+1}+\cdots+x^{\max(x_3)}\right]$$

Note: Sum represents right hand side of the equation. For each variable x_1, x_2, x_3 a bracket is formed using the values the variable can take.

$\Rightarrow$ Number of solutions

$$= \text{Coefficient of } x^3 \text{ in } (x^0 + x^1 + x^2 + x^3)^3$$

$$= \text{Coefficient of } x^3 \text{ in } \left[\frac{1-x^4}{1-x}\right]^3$$

$$= \text{Coefficient of } x^3 \text{ in } (1 - x^4)^3 (1 - x)^{-3}$$

$$= \text{Coefficient of } x^3 \text{ in } ({}^3C_0 - {}^3C_1 x^4 + {}^3C_2 x^8 - {}^3C_3 x^{12}) (1 - x)^{-3}$$

$$= \text{Coefficient of } x^3 \text{ in } (1 - x)^{-3} \qquad \text{[as other terms cannot generate } x^3 \text{ term]}$$

$$= {}^{3+3-1}C_3 = {}^5C_3 = 10 \qquad \text{[using: coefficient of } x^r \text{ in } (1 - x)^{-n} = {}^{n+r-1}C_r]$$

Example 132 *In a box there are* 10 *balls,* 4 *red,* 3 *black,* 2 *white and* 1 *yellow. In how many ways can a child select* 4 *balls out of these* 10 *balls? (Assume that the balls of the same colour are identical)*

Solution: Let x_1, x_2, x_3 and x_4 be the number of red, black, white, yellow balls selected respectively.

Number of ways to select 4 balls

= Number of integral solutions of the equation $(x_1 + x_2 + x_3 + x_4) = 4$

Conditions on x_1, x_2, x_3 and x_4:

The total number of red, black, white and yellow balls in the box are 4, 3, 2 and 1 respectively.

So we can take: Max $(x_1) = 4$, Max $(x_2) = 3$, Max $(x_3) = 2$, Max $(x_1) = 1$

There is no condition on minimum number of red, black, white and yellow balls selected, so take:

Min $(x_i) = 0$ for $i = 1, 2, 3, 4$

Number of ways to select 4 balls

$$= \text{Coefficient of } x^4 \text{ in } (1 + x + x^2 + x^3 + x^4) \times (1 + x + x^2 + x^3) \times (1 + x + x^2) \times (1 + x)$$

$$= \text{Coefficient of } x^4 \text{ in } (1 - x^5) (1 - x^4) (1 - x^3) (1 - x^2) (1 - x)^{-4}$$

$$= \text{Coefficient of } x^4 \text{ in } (1 - x^2 - x^3 - x^4) (1 - x)^{-4}$$

$$= \text{Coefficient of } x^4 \text{ in } (1 - x)^{-4} - \text{Coefficient of } x^2 \text{ in } (1 - x)^{-4} - \text{coeff of } x^1 \text{ in } (1 - x)^{-4} - \text{Coefficient of } x^0 \text{ in } (1 - x)^{-4}$$

$$= {}^7C_4 - {}^5C_2 - {}^4C_1 - {}^3C_0 = \frac{7 \times 6 \times 5}{3!} - 10 - 4 - 1 = 35 - 15 = 20$$

Thus, number of ways of selecting 4 balls from the box subjected to the given conditions is 20.

Alternate solution (Using 'case' method):

The 10 balls are RRRR BBB WW Y (where R, B, W, Y represent red, black, white and yellow balls respectively).

The work of selection of the balls from the box can be divided into following categories.

Case 1: All alike

Number of ways of selecting all alike balls = ${}^1C_1 = 1$

Case 2: 3 alike and 1 distinct

Number of ways of selecting 3 alike and 1 distinct balls = ${}^2C_1 \times {}^3C_1 = 6$

Case 3: 2 alike and 2 alike
Number of ways of selecting 2 alike and 2 alike balls = ${}^3C_2 = 3$

Case 4: 2 alike and 2 distinct
Number of ways of selecting 2 alike and 2 distinct balls = ${}^3C_1 \times {}^3C_2 = 9$

Case 5: All distinct
Number of ways of selecting all distinct balls = ${}^4C_4 = 1$

Total number of ways to select 4 balls = 1 + 6 + 3 + 9 + 1 = 20.

Example 133 *There are three papers of* 100 *marks each in an examination. Then find the number of ways in which a student can get* 150 *marks such that he gets atleast* 60% *in two papers.*

Solution: Suppose the student gets atleast 60% marks in first two papers, then he just get atmost 30% marks in the third paper to make a total of 150 marks.
Let, x_1, x_2, x_3 be marks obtained in 3 papers respectively. The total marks to be obtained is 150.

Therefore, Sum of marks obtained = 150

$$\Rightarrow x_1 + x_2 + x_3 = 150 \tag{1}$$

$$60 \le x_1 \le 100;\ 60 \le x_2 \le 100;\ 0 \le x_3 \le 30.$$

The required number of ways = Number of integral solutions of Eq. (1)

$= \text{Coefficient of } x^{150} \text{ in } \{(x^{60} + x^{61} + \cdots + x^{100})^2 (1 + x + x^2 + \cdots + x^{30})\}$
$= \text{Coefficient of } x^{30} \text{ in } \{(1 + x + \cdots + x^{40})^2 (1 + x + \cdots + x^{30})\}$

$$= \text{Coefficient of } x^{30} \text{ in } \left(\frac{1-x^{41}}{1-x}\right)^2 \left(\frac{1-x^{31}}{1-x}\right)$$

$= \text{Coefficient of } x^{30} \text{ in } (1-x)^{-3} = {}^{30+3-1}C_{3-1} = {}^{32}C_2.$

Thus, the student gets atleast 60% marks in first two papers to get 150 marks as total in ${}^{32}C_2$ ways. But the two papers, of atleast 60% marks, can be chosen out of 3 papers in ${}^3C_2$ ways.

Hence, the required number of ways = ${}^3C_2 \times {}^{32}C_2$.

Example 134 *Find the number of ways in which* 30 *marks can be allotted to* 8 *questions if each questions carries atleast* 2 *marks.*

Solution: Let $x_1, x_2, x_3, x_4, \ldots, x_8$ be marks allotted to 8 questions.
As total marks is 30, we can make following integral equation:

$$x_1 + x_2 + x_3 + \cdots + x_8 = 30.$$

It is given that every question should be of atleast 2 marks. It means

$$2 \le x_i \le 16\ \forall i = 1, 2, 3, \ldots, 8$$

The number of solutions of the integral equation is equal to number of ways to divide marks.

Number of solutions

$= \text{Coefficient of } x^{30} \text{ in } (x^2 + x^3 + \cdots + x^{16})^8$
$= \text{Coefficient of } x^{30} \text{ in } x^{16} (1 + x + \cdots + x^{14})^8$

$$= \text{Coefficient of } x^{14} \text{ in } \left(\frac{1-x^{15}}{1-x}\right)^8$$

$= \text{Coefficient of } x^{14} \text{ in } (1-x)^{-8} = {}^{21}C_{14} = 116280.$

Alternate solution:

Let, the marks given in each question be;

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ [where x_i's ≥ 0 (i = 1, 2 … 8)]

and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 30$

Let, $x_1 - 2 = y_1, x_2 - 2 = y_2, x_3 - 2 = y_3, x_4 - 2 = y_4, x_5 - 2 = y_5, x_6 - 2 = y_6, x_7 - 2 = y_7, x_8 - 2 = y_8$.

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = 14$

where $0 \leq y_i$ $i = 1, 2, 3, \ldots, 8$

$\Rightarrow$Number of solutions $= {}^{14+8-1}C_{8-1} = {}^{21}C_7$.

Example 135 *In an examination the maximum marks for each of three papers is n and that for fourth paper is 2n. Find the number of ways in which a candidate can get 3n marks.*

Solution: Let x_1, x_2, x_3 and x_4 be the marks obtained in papers 1, 2, 3, 4 respectively. The total number of marks to be obtained by the candidate is $3n$.

Therefore, sum of marks obtained in various papers = $3n$.

$$\Rightarrow x_1 + x_2 + x_3 + x_4 = 3n \tag{1}$$

The total number of ways of getting $3n$ marks

= Number of solutions of the integral Eq. (1)

$= \text{Coefficient of } x^{3n} \text{ in } (x^0 + x^1 + x^2 + \cdots + x^n)^3 \times (x^0 + x^1 + \cdots + x^{2n})$

$= \text{Coefficient of } x^{3n} \text{ in } \left(\dfrac{1-x^{n+1}}{1-x}\right)^3 \left(\dfrac{1-x^{2n+1}}{1-x}\right)$

$= \text{Coefficient of } x^{3n} \text{ in } (1 - x^{n+1})^3 (1 - x^{2n+1}) (1 - x)^{-4}$

$= \text{Coefficient of } x^{3n} \text{ in } [(1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3}) (1 - x^{2n+1}) (1 - x)^{-4}]$

$= \text{Coefficient of } x^{3n} \text{ in } [(1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2}) (1 - x)^{-4}]$

$= \text{Coefficient of } x^{3n} \text{ in } (1 - x)^{-4} - 3 \text{ Coefficient of } x^{2n-1} \text{ in } (1 - x)^{-4} - \text{Coefficient of } x^{n-1} \text{ in } (1 - x)^{-4} + 3 \text{ Coefficient of } x^{n-2} \text{ in } (1 - x)^{-4}$

$= {}^{3n+4-1}C_{3n} - 3 \times {}^{2n-1+4-1}C_{2n-1} - {}^{n-1+4-1}C_{n-1} + 3 \times {}^{n-2+4-1}C_{n-2}$

$= {}^{3n+3}C_3 - 3 \times {}^{2n+2}C_3 - {}^{n+2}C_3 + 3 \times {}^{n+1}C_3$ [as ${}^nC_r = {}^nC_{n-r}$]

$$= \frac{(3n+3)(3n+2)(3n+1)}{6} - 3\frac{(2n+2)(2n+1)(2n)}{6} - \frac{(n+2)(n+1)(n)}{6} + 3\frac{(n+1)(n)(n-1)}{6}$$

$$= \frac{1}{2}(n+1)(5n^2 + 10n + 6).$$

Example 136 *In a shooting competition a man can score* 5, 4, 3, 2 *or* 0 *points for each shot. Find the number of different ways in which he can score 30 in seven shots.*

Solution: Let $x_1, x_2, x_3, x_4, \ldots, x_7$ be the scores in 7 shots. As total score of 30 is

Sum of scores in 7 shots = 30

$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 30$[where $x_i \in \{0, 2, 3, 4, 5\}$ i = 1, 2, …, 7]

Number of solutions of above equation

Number of ways of making 30 in 7 shots to be taken,

Coefficient of x^{30} in $(x^0 + x^2 + x^3 + x^4 + x^5)^7$.

$\Rightarrow$ Coefficient of x^{30} in $\{(x^0 + x^2 + x^3) + x^4 (x + 1)\}^7$

$\Rightarrow$ Coefficient of x^{30} in $\{x^{28} (x + 1)^7 + {}^7C_1 x^{24}.(x + 1)^6.(1 + x^2 + x^3) + {}^7C_2 x^{20} (x + 1)^5 (x^3 + x + 1)^2 + \cdots\}$ [using Binomial theorem]

= Number of ways to score 30

$\Rightarrow {}^7C_2 + {}^7C_1\,({}^6C_3 + {}^6C_2 + {}^6C_0) + {}^7C_2\,({}^5C_1 + 2)$

$\Rightarrow 21 + 252 + 147 = 420.$

Example 137 *Find the number of non-negative integral solutions of*

$$x_1 + x_2 + x_3 + 4x_4 = 20.$$

Solution: Number of non-negative integral solutions of the given equation

= Coefficient of x^{20} in $(1-x)^{-1}\,(1-x)^{-1}(1-x)^{-1}\,(1-x^4)^{-1}$

= Coefficient of x^{20} in $(1-x)^{-3}(1-x^4)^{-1}$

= Coefficient of x^{20} in $(1 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + {}^6C_4x^4 + \cdots)(1 + x^4 + x^8 + \cdots)$

$= 1 + {}^6C_4 + {}^{10}C_8 + {}^{14}C_{12} + {}^{18}C_{16} + {}^{22}C_{20} = 536.$

Build-up Your Understanding 6

1. Find the number of ways to select 10 balls from an unlimited number of red, white, blue and green balls.
2. Find the number of ordered triples of positive integers which are solutions of the equation $x + y + z = 100$.
3. Find the number of integral solutions of $x_1 + x_2 + x_3 = 0$, with $x_i \geq -5$.
4. Find the number of integral solutions for the equation $x + y + z + t = 20$, where x, y, z, t are all ≥ -1.
5. Find the number of integral solutions of $a + b + c + d + e = 22$, subject to $a \geq -3$, $b \geq 1$, $c, d, e \geq 0$.
6. If a, b, c are three natural numbers in AP and $a + b + c = 21$ then find the possible number of values of the ordered triplet (a, b, c).
7. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$ then then find the number of values of the ordered quadruplet (a, b, c, d).
8. Find the number of non-negative integral solution of the equation, $x + y + 3z = 33$.
9. Find the number of integral solutions of the equation $3x + y + z = 27$, where $x, y, z > 0$.
10. If a, b, c are positive integers such that $a + b + c \leq 8$ then find the number of possible values of the ordered triplet (a, b, c).
11. Find the number of non-negative integral solution of the inequation $x + y + z + w \leq 7$.
12. Find the number of non-negative even integral solutions of $x + y + z = 100$.
13. Find the number of non-negative integral solutions of $x + y + z + w \leq 23$.
14. Find the total number of positive integral solution of $15 < x_1 + x_2 + x_3 \leq 20$.
15. Find the number of non-negative integer solutions of $(a + b + c)(p + q + r + s) = 21$.
16. There are three piles of identical red, blue and green balls and each pile contains at least 10 balls. Find the number of ways of selecting 10 balls if twice as many red balls as green balls are to be selected.
17. Find the number of terms in a complete homogeneous expression of degree n in x, y and z.
18. In how many different ways can 3 persons A, B and C having 6 one rupee coins, 7 one rupee coins and 8 one rupee coins respectively donate 10 one rupee coins collectively.
    (i) If each one giving at least one coin
    (ii) If each one can give '0' or more coin.
    Also answer the above questions for 15 rupees donation.

19. In an examination, the maximum marks for each of the three papers are 50 each. Maximum marks for the fourth paper is 100. Find the number of ways in which a candidate can score 60% marks on the whole.

20. Between two junction stations A and B, there are 12 intermediate stations. Find the number of ways in which a train can be made to stop at 4 of these stations so that no two of these halting stations are consecutive.

21. The minimum marks required for clearing a certain screening paper is 210 out of 300. The screening paper consists of '3' sections each of Physics, Chemistry and Mathematics Each section has 100 as maximum marks. Assuming there is no negative marking and marks obtained in each section are integers, find the number of ways in which a student can qualify the examination (Assuming no subjectwise cut-off limit).

22. Find the number of ways in which the sum of upper faces of four distinct dices can be six.

23. How many integers > 100 and < 10^6 have the digital sum = 5?

24. In how many ways can 14 be scored by tossing a fair die thrice?

25. Find the number of positive integral solutions of $abc = 30$.

26. Find The number of positive integral solutions of the equation $x_1 x_2 x_3 x_4 x_5 = 1050$.

27. Let y be an element of the set $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and x_1, x_2, x_3 be positive integers such that $x_1 x_2 x_3 = y$, then find the number of positive integral solutions of $x_1 x_2 x_3 = y$.

28. Let $x_i \in \mathbb{Z}$ such that $|x_1 x_2 \ldots x_{10}| = 1080000$. Find number of solutions.

29. Let $x_i \in \mathbb{Z}$ such that $x_1 x_2 \ldots x_{10} = 180000$. Find Number of solutions.

30. Let $x_i \in \mathbb{Z}$, such that $|x_1| + |x_2| + \cdots + |x_{10}| = 100$. Find number of solutions.

7.11 Application of Recurrence Relations

Recurrence relation is a way of defining a series in terms of earlier member of the series with a few initial terms. It is complete description and much simpler than explicit formula. Here are some examples for use of recurrence relation.

Example 138 *Let there be n lines in a plane such that no two lines are parallel and no three are concurrent. Find the number of regions in which these lines divide the plane.*

Solution: Let a_n denotes required number of regions

Initial term $a_0 = 1, a_1 = 2, a_2 = 4$

Let number of region by $(n-1)$ lines be a_{n-1}.

Let us assume our plane be vertical and let us rotate it so that none of the $n - 1$ lines are horizontal.

Now draw nth line, horizontally, below all the point of intersections. All previous $n - 1$ lines meet the nth line at $n - 1$ different points. These points divides the nth line into n parts and each part falls in some old region and will divide the old region in two parts which will generate n new region.

n new regions are added to a_{n-1} regions

$$\Rightarrow a_n = a_{n-1} + n \quad \Rightarrow \quad a_n - a_{n-1} = n$$

$$\Rightarrow a_n - a_1 = \sum_{n=2}^{n} n$$

Hence, $$a_n = 1 + \sum n = 1 + \frac{n(n+1)}{2}. \qquad (\text{as } a_1 = 1)$$

Example 139 *Determine the number of regions that are created by n mutually overlapping circles in a plane. Assume that no three circles passing through same points and every two circles intersect in two distinct points.*

Solution: Let number of regions be h_n. Clearly $h_0 = 1; h_1 = 2, h_2 = 4, h_3 = 8$

It is tempting now to think $h_n = 2^n$ but by drawing diagram we see that $h_4 = 14$.

We obtain recurrence relation as follows:

Let $(n - 1)$ mutually overlapping circle creating h_{n-1} regions.

Now draw nth circle. nth circle is intersected by each of $(n-1)$ circles in two points,

$\Rightarrow$ We are getting $2(n-1)$ distinct points, these points divides nth circle into $2(n-1)$ arcs. Each arc falls in some old region and will divide the old region in two parts and thus will generate $2(n-1)$ new regions.

$$\Rightarrow h_n = h_{n-1} + 2(n-1); n \geq 2$$

$$\Rightarrow h_n - h_{n-1} = 2(n-1)$$

$$\Rightarrow h_n = h_1 + 2\sum_{n=2}^{n}(n-1)$$

$$= h_n = 2 + 2\frac{n(n-1)}{2} = n^2 - n + 2. \qquad (\text{as } h_1 = 2)$$

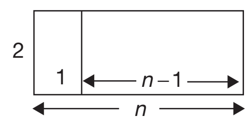
Example 140 *Determine number of ways to perfectly cover a $2 \times n$ board with dominoes (domino means a tile of size 2×1).*

Solution: Let number of ways be h_n. Then $h_0 = 1; h_1 = 1; h_2 = 2$

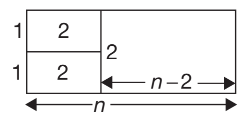
Let $n \geq 2$.

We divided the perfect covers of $2 \times n$ board into two parts A and B depending upon the domino placed at first place.

A: Those perfect covers in which there is a vertical domino at the first place as shown in figure.

(A)


B: Those perfect covers in which there are two horizontal domino at the first place as shown in the figure.

(B)


Now, perfect covers in A = perfect covers in $2 \times (n-1)$ board.

$\Rightarrow |A| = h_{n-1}$

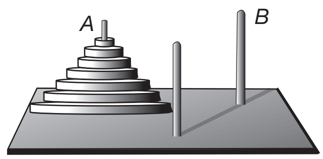
Similarly $|B| = h_{n-2}$

$\Rightarrow h_n = h_{n-1} + h_{n-2}$

This is our famous fibonacci sequence. Its general solution already discussed in the chapter of recurrence relation.

Example 141 *Tower of Brahma (or Tower of Hanoi) is a puzzle consisting of three pegs mounted on a board and n discs of different sizes. Initially all the n discs are stacked on the first peg so that any disc is always above a larger disc. The problem is to transfer all these discs to peg 2, with minimum number of moves, each move consisting of transferring one disc from any peg to another so that on the new peg the transferred disc will be on top of a larger disc (i.e., keeping a disc on a smaller one is not allowed).*

Find the total (minimum) number of moves required to do this.



Solution: Here again we shall give the explanation through four columns representing several number of the move: the positions of discs at each stage in peg 1, peg 2 and peg 3.

When there is just one disc, the problem is trivial, *i.e.*, in 1 move it is transferred directly to peg 2. We shall see the scheme of transfers for $n = 1$, 2 and 3, before finding the formula and proving it $n = 1$. Let name the discs as $d_1, d_2, \ldots, d_n$ with d_{i+1} to be smaller than d_i for all i, $1 \le i \le n - 1$.

Serial No. of the Moves	Peg 1	Peg 2	Peg 3
Initial stage	d_1	–	–
1		d_1	–

So in one move d_1 is transferred to peg 2, when $n = 1$, *i.e.*, total number of moves when $n = 1$ is 1.

$n = 2$, discs are d_1 and d_2, d_2 smaller than d_1.

Serial No. of the Moves	Peg 1	Peg 2	Peg 3
Initial stage	d_1, d_2	–	–
1	d_1		d_2
2	–	d_1	d_2
3	–	d_1, d_2	–

Thus, total no. of moves when $n = 2$ is 3.

$n = 3$, discs are d_1, d_2, d_3 with d_3 smaller than d_2, d_2 smaller than d_1.

Serial No. of the Moves	Peg 1	Peg 2	Peg 3
Initial stage	d_1, d_2, d_3	–	–
1	$d_1 d_2$	d_3	–
2	d_1	d_3	d_2
3	d_1	–	d_2, d_3
4	–	d_1	d_2, d_3
5	d_3	d_1	d_2
6	d_3	d_1, d_2	–
7	–	d_1, d_2, d_3	–

So, when there are 3 discs, *i.e.*, $n = 3$, the minimum number of moves is 7.

Note that here when the biggest disc alone is still in peg 1, all the discs are transferred to peg 3 and peg 2 is empty, so that the biggest one can now occupy peg 2. Then all the discs from peg 3 now can be transferred to peg 2 above the biggest one and it will again take as many times (to be transferred to peg 2), as it took to be transferred from peg 1 to peg 3.

Thus, to transfer two discs d_1 d_2 from peg 1 to peg 2:

d_2 goes to peg 3 in one move in the next move, d_1, goes to peg 2.

Now, disc d_2 takes the same 1 move to go to peg 2. Thus, the required number of moves is $1 + 2(1) = 3$.

Again, when there are 3 discs, as has been seen in the case of two discs, it takes 3 moves to transfer d_2 and d_3 to peg 3 (not peg 2 in this case) and it takes one move to transfer disc d_1 to peg 2 and it takes again another 3 moves to transfer discs d_2 and d_3 to peg 2.

So, the total number of moves $= 1 + 2 \times 3 = 7$. For 1 disc, there is one move; for 2 discs, there are $1 + (2 \times 1)$ moves or $2^2 - 1$; for 3 discs, there are $2\{1 + (2 \times 1)\} + 1$

$$= 2(2^2 - 1) + 1$$
$$= 2^3 - 2 + 1$$
$$= 2^3 - 1 \text{ moves.}$$

So, we can guess that when there are 4 discs, the number of moves is $2(2^3 - 1) + 1 = 2^4 - 2 + 1 = 2^4 - 1$.

Thus, to find the minimum number of moves, we can use the formula, $2^n - 1$, when there are n discs to be transferred from peg 1 to peg 2.

Now, proving this is very simple by using the principle of Mathematical induction.

We have already verified that this formula holds for the number of discs $n = 1, 2$ and 3.

So, let us assume that it holds for $n = k$, *i.e.*, when there are k discs, the minimum number of moves required to transfer the k discs from peg 1 to peg 2 is $2^k - 1$.

When there are $(k + 1)$ discs, we should verify if the number of moves is $2^{k+1} - 1$.

Serial No. of the Moves	Peg 1	Peg 2	Peg 3
After k discs are transferred $2^k - 1$	d_{k+1}	–	$d_1, d_2, \ldots, d_k$
$2^{k\text{th}}$ move	–	d_{k+l}	$d_1, d_2, \ldots, d_k$

Now, by our assumption for $n = k$, it takes $2^k - 1$ moves to transfer $d_1, d_2, \ldots, d_k$ discs (k in all) to peg 2 from peg 3.

So, the total number of moves $= 2^k + 2^k - 1$

$$= 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

Thus, whenever the formula to find the number of moves for $n = k$ (*i.e.*, no. of moves $= 2^k - 1$) is true, the formula is true for $n = k + 1$.

From the fact that the formula is true for $n = 1$, together with the last statement we find, that the formula is true for all $n \in \mathbb{N}$, *i.e.*, the minimum number of moves required to transfer n discs from peg 1 to peg 2, according to the given condition is $2^n - 1$.

Aliter: Let a_n be the minimum number of moves that will transfer n disks from one peg to other peg under given restriction. Then a_1 is obviously 1, and $a_2 = 3$.

Let as think when we can move the largest disk from the first peg? We first transfer the $n - 1$ smaller disk to peg 3 which requires a_{n-1} moves, then move the largest disk to peg 2 requiring one move and finally transfer the $n - 1$ smaller back to peg 2 on top of largest disk which require another a_{n-1} moves thus

$$a_n = a_{n-1} + 1 + a_{n-1}$$

$$\Rightarrow a_n = 2a_{n-1} + 1$$

$$\Rightarrow a_n + 1 = 2(a_{n-1} + 1)$$

$$\Rightarrow a_n + 1 = 2^{n-1}(a_1 + 1)$$

$$= 2^n \qquad (\text{as } a_1 = 1)$$

$$\Rightarrow a_n = 2^n - 1$$

Abraham de Moivre

26 May 1667–27 Nov 1754
Nationality: French

7.12 Principle of Inclusion and Exclusion (PIE)

This principle is used in most counting situations.

The addition principle for counting is stated for disjoint sets as

$|A \cup B| = |A| + |B|$ or $n(A \cup B) = n(A) + n(B)$, where A and B are disjoint sets.

If A and B are not disjoint, then $|A \cup B| = |A| + |B| - (A \cap B)$.

We count the elements of A and B in turn and subtract the common elements of A and B, *i.e.*, the elements in $A \cap B$, as they are counted twice: firstly when we counted the elements of A and secondly, when we counted the elements of B.

For three sets A, B and C, the counting principle states that

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

The general PIE is stated as follows:

For any sets $A_1 A_2, \ldots, A_n$, $n \geq 2$

$$|A_1 \cup A_2 \cup \cdots A_n|$$

$$= \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k}^{n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

In other words, consider properties $P_1, P_2, \ldots, P_n$. Let $n(A_k)$ or $|A_k|$ be the number of objects satisfying the property P_k, $k = 1, 2, \ldots, n$. A commonly asked question is 'how many elements satisfy atleast one of the properties '$P_1, P_2, \ldots, P_n$'?

This question is answered by the inclusion-exclusion principle which is stated below:

If $A_1, A_2, \ldots, A_m$ are m sets and $n(S)$ denotes the number of elements in the set S, then, $n\left(\bigcup_{k=1}^{m} A_k\right)$

$$= \sum_{k=1}^{m} n(A_k) - \sum_{1 \le i < j \le m} n(A_i \cap A_j) + \cdots + (-1)^{s-1} \sum_{1 \le i_1 < i_2 < \cdots < i_s \le m} n\left(\bigcap_{k=1}^{s} A_{i_k}\right)$$
$$+ \cdots + (-1)^{m-1} n\left(\bigcap_{k=1}^{m} A_{i_k}\right)$$

Note that if $x \in \bigcup_{k=1}^{m} A_k$, then x belongs to at least one of $A_k, 1 \le k \le m$.

Note: For notational ease we may use $A_1 + A_2 + \cdots + A_k$ in place of $A_1 \cup A_2 \cup \ldots \cup A_k$ and $A_1 A_2 \ldots A_k$ in place of $A_1 \cap A_2 \cap \ldots \cap A_k$.

7.12.1 A Special Case of PIE

For any set $A_1, A_2, \ldots, A_n$, $n \geq 2$,

$$|A_1 + A_2 + \cdots + A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i A_j| + \sum_{i<j<k} |A_i A_j A_k| - \cdots + (-1)^{n-1} |A_1 A_2 \cdots A_n|$$

We consider here a special case of the principle of inclusion and exclusion.

In some applications we deal with properties, $a_1, a_2, \ldots, a_n$ and numerical values associated with properties, *i.e.*, $n(a_1), n(a_2), \ldots, n(a_n), n(a_1 a_2), \ldots, n(a_{n-1} a_n) \ldots$ and so on.

It is known that the numerical value assigned to a single property is a constant, and numerical values assigned to two properties $a_i a_j$, $i \neq j$ is also a constant and so on.

In other words

1. $n(a_1) = n(a_2) = \cdots = n(a_n)$
2. $n(a_1 a_2) = n(a_1 a_3) = \cdots = n(a_1 a_n) = n(a_2 a_3) = \cdots = n(a_{n-1} a_n)$
3. $n(a_1 a_2 a_3) = n(a_1 a_2 a_4) = \cdots = n(a_i a_j a_k)$, $i \neq j \neq k$
   and so on.

Again we denote by $N(1)$, the common value of the properties $a_1, a_2, \ldots, a_n$ taken one at a time, *i.e.*, $N(1) = n(a_1) = n(a_2) = \cdots = n(a_n)$.

$N(2)$ is the common value of the properties $a_1, a_2, \ldots, a_n$ when taken two at a time, etc. and $N(n)$ the number denoting the value $n(a_1 a_2 \ldots a_n)$, *i.e.*, the number denoting the value of the properties when all of them are taken together and $N(0)$ is the value of $n(a_1' a_2' \ldots a_n')$ where a_i' is the complementary property of the property a_i and N is the value of collection of zero property or atleast one property.

Now,
$$\sum_{i=1}^{n} n(a_i) = \binom{n}{1} N(1)$$
$$\sum_{i,j} n(a_i a_j) = \binom{n}{2} N(2)$$
$$\sum_{i,j,k,\ldots,r} n\Bigg(\underbrace{a_i a_j a_k \ldots a_r}_{\text{taken } r \text{ at time}}\Bigg) = \binom{n}{r} N(r)$$
$$n(a_1 a_2 \ldots a_n) = \binom{n}{n} N(n) = N(n)$$

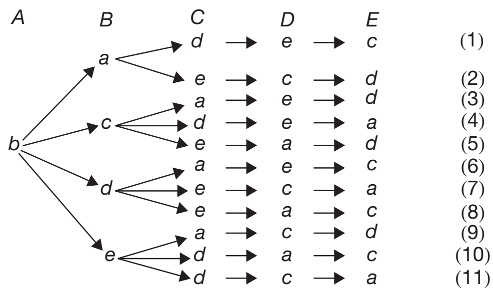
Now, with this explanation, the principle of inclusion and exclusion takes the form
$$N(0) = N - \binom{n}{1} N(1) + \binom{n}{2} N(2) - \binom{n}{3} N(3) + \cdots + (-1)^{n+1} \binom{n}{n} N(n)$$

Example 142 *Five letters are written to five different persons and their addresses are written on five envelopes (one address on each envelope). In how many ways can the letters be placed in the envelopes so that no letter is placed in the correct envelope?*

Solution: Let us name the envelopes A, B, C, D, E and the corresponding letters a, b, c, d, e.

We shall now see, when the letter b is placed in envelope A, in how many ways the other 4 letters a, c, d, e can go to the wrong envelopes.

Envelopes



Thus for placing the letter b in envelope A, we have 11 different ways in which no letter goes to the correct envelope.

But we can also place c, d or e in envelope A, and in each case we get 11 different ways of placing letter in which no letter goes to the correct envelope.

Therefore, there are $11 \times 4 = 44$ different ways in which we can place the five letters, one in each of five envelopes so that no letter goes to the right envelope.

Aliter 1: Let us use special case of PIE

In to our problem of letters and envelopes, we take for each $i = 1, 2, 3, \ldots, 5$, k_i as the property that the letter a_t goes to the envelope A_i.

Here, $n = 5$,

$\therefore \quad N =$ The total number of ways of 5 letters can be put into the envelopes = 5!

$$N(0) = N - \binom{5}{1}N(1) + \binom{5}{2}N(2) - \binom{5}{3}N(3) + \binom{5}{4}N(4) - \binom{5}{5}N(5)$$

$N(i)$ is the number of ways in which i letters go to i correct envelopes, so whatever happens to the other letters is $(5 - i)!$

Thus, $\quad N(1) = 4! = 24,$

because $5 - 1 = 4$ letters can be placed in 4 envelopes in 4! ways and there is first one way of placing the letter in the correct envelope.

$$N(2) = 3! = 6,$$

since $5 - 2 = 3$ letters can be placed in 3 envelopes in $3! = 6$ different ways and again there is just one way of placing the two letters in their corresponding envelopes.

Similarly, $\quad N(3) = (5 - 3)! = 2! = 2$

$$N(4) = (5 - 4)! = 1$$

$$N(5) = (5 - 5)! = 0! = 1.$$

$\therefore \quad N(0) =$ The number of ways that none of the letters go into the correct envelope is

$$5! - 5\times 4! + \frac{5\times 4}{1\cdot 2}\times 3! - \frac{5\times 4\times 3}{1\cdot 2\cdot 3}\times 2! + 5\times 1 - 1\times 1$$

$$= 120 - 120 + 60 - 20 + 5 - 1$$

$$= 44.$$

Aliter 2: See the formula given in derangement section 7.13
By using the given formula for $n = 5$, we get

$$D_5 = 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$$

$$= 5!\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right]$$

$$= 60 - 20 + 5 - 1 = 44.$$

Example 143 *Find the number of positive integers from 1 to 1000, which are divisible by at least one of* 2, 3 *or* 5.

Solution: Let A_k be the set of positive integers from 1 to 1000, which are divisible by k.

Obviously we have to find $n(A_2 \cup A_3 \cup A_5)$.

$$n(A_2) = \left\lfloor \frac{1000}{2} \right\rfloor = 500, n(A_3) = \left\lfloor \frac{1000}{3} \right\rfloor = 333, n(A_5) = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$$n(A_2 \cap A_3) = \left\lfloor \frac{1000}{6} \right\rfloor = 166, \text{similarly } n(A_3 \cap A_5) = 66, n(A_2 \cap A_5)$$

$$= 100, n(A_2 \cap A_3 \cap A_5) = 33.$$

Hence, $n(A_2 \cup A_3 \cup A_5) = 500 + 333 + 200 - 166 - 66 - 100 + 33 = 734.$

Note that number of positive integers from 1 to 1000, which are not divisible by any of 2, 3 or 5 is

$1000 - n\ (A_2 \cup A_3 \cup A_5) = 266.$

Example 144 *Find the number of ways in which two Americans, two Britishers, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.*

Solution: Total = 6!
$n(A)$ = when $A_1 A_2$ together = 5 ! 2 ! = 240
$n(B)$ = when $B_1 B_2$ together = 5 ! 2 ! = 240

$\Rightarrow n(A \cup B) = n(A) + n(B) - n(A \cap B) = 240 + 240 - 96 = 384$

Hence $n(\bar{A} \cap \bar{B}) = \text{Total} - n(A \cup B)$

$= 6! - 384$

$= 720 - 384$

$= 336.$

Example 145 *In how many ways can* 5 *cards be drawn from a complete deck (of* 52 *cards) so that all the suites are present?* (*Do not simplify.*)

Solution: Consider the notation: In a selection of 5 cards,
C: the set of selections in which clubs are absent
D: the set of selections in which diamonds are absent
S: the set of selections in which spades are absent
H: the set of selections in which hearts are absent
We have $|C| = |D| = |S| = |H| = {}^{39}C_5$,
$|C \cap D| = \ldots = {}^{26}C_5$,
$|C \cap D \cap S| = \ldots = {}^{13}C_5$,

and $|C \cap D \cap S \cap H| = 0$
Now $|C \cup D \cup S \cup H| = 4({}^{39}C_5) - 6({}^{26}C_5) + 4({}^{13}C_5) - 0$
Finally, the required number is

$$ {}^{52}C_5 - 4\,{}^{39}C_5 + 6\,{}^{26}C_5 - 4\,{}^{13}C_5. $$

Example 146 *In how many ways can* 6 *distinguishable objects be distributed in four distinguishable boxes such that there is no empty box?*

Solution: The number of distributions such that:

(i) atleast one box is empty, is ${}^4C_1 \cdot 3^6$
(ii) atleast two boxes are empty, is ${}^4C_2 \cdot 2^6$
(iii) atleast three boxes are empty, is ${}^4C_3 \cdot 1^6$

The totality of distributions is 4^6.
Hence the required number is

$$ 4^6 - {}^4C_1\, 3^6 + {}^4C_2\, 2^6 - {}^4C_3\, 1^6 = 2260. $$

Note: If there should be exactly one empty box, then the number of distributions is

$$ {}^4C_1(3^6 - {}^3C_1.\ 2^6 + {}^3C_2.\ 1^6) = 2160. $$

Example 147 *Find the number of ways to choose an ordered pair* (a, b) *of numbers from the set* $\{1, 2, \ldots, 10\}$ *such that* $|a - b| \le 5$.

Solution: Let $A_1 = [(a, b)/a, b \in \{1, 2, 3, \ldots, 10\}, /a - b/ = \{i\}, i = 0, 1, 2, 3, 4, 5.$
$A_0 = \{(i, i) | i = 1, 2, 3, \ldots, 10\}$ and $|A_0| = 10$
$A_1 = \{(i, i+1) | i = 1, 2, 3, \ldots, 9\} \cup \{i, i-1\} | i = 2, 3, \ldots, 10\}$ and $|A_1| = 9 + 9 = 18$

$A_2 = \{(i, i+2) | i = 1, 2, 3, \ldots, 8\} \cup \{(i, i-2) | i = 3, 4, \ldots, 10\}$ and $|A_2| = 8 + 8 = 16$

$A_3 = \{(i, i+3) | i = 1, 2, \ldots, 7\} \cap \{(i, i-3) | i = 4, 5, \ldots, 10\}$ and $|A_4| = 6 + 6 = 12$

$A_4 = \{(i, i+4) | i = 1, 2, 3 \ldots, 6\} \cup \{(i, i-4) | i = 5, 6, \ldots, 10\}$ and $|A_4| = 6 + 6 = 12$

$A_5 = \{(i, i+5) | i = 1, 2, \ldots, 5\} \cup \{(i, i-5) | i = 6, 7, \ldots, 10\}$ and $|A_5| = 5 + 5 = 10$

$\therefore$ The required set of pairs $(a, b) = \bigcup_{i=10}^{5} A_i$ and the number of such pairs, (which are disjoint)

$$\left|\bigcup_{i=10}^{5} A_i\right| = \sum_{i=10}^{5} |A_i| = 10 + 18 + 16 + 14 + 12 + 10 = 80.$$

Alternate: Total ways (without condition) = $10^2 = 100$

Let $b - a \geq 6$

$$1 \leq a < b \leq 10 \quad \Rightarrow \quad 1 \leq a < b - 5 \leq 5 \quad \Rightarrow \quad \binom{5}{2} = 10$$

Similarly for $a - b \geq 6$ we will get 10 ways.

Hence required answer = 100 – 10 – 10 = 80.

Example 148 *Identify the set S by the following information:*

(i) $S \cap \{3, 5, 8, 11\} = \{5, 8\}$
(ii) $S \cup (4, 5, 11, 13\} = \{4, 5, 7, 8, 11, 13\}$
(iii) $\{8, 13\} \subset S$
(iv) $S \subset \{5, 7, 8, 9, 11, 13\}$

Also, show that no three of the conditions suffice to identify S uniquely.

Solution: From (i),

$$5, 8 \in S \tag{1}$$

From (ii),

$$7, 8 \in S \tag{2}$$

From (iii),

$$8, 13 \in S \tag{3}$$

Therefore, from Eqs. (1), (2) and (3), we find that

$$5, 7, 8, 13 \in S \tag{4}$$

$$S \subset \{5, 7, 8, 9, 11, 13\} \qquad \text{(Given)}$$

If at all S contains any other element other than those given in (4), it may be 9 or 11 or both.

But $9 \notin S$. [$\because 9 \notin S \cup \{4, 5, 11, 13\} = \{4, 5, 7, 8, 11, 13\}$]

Again $11 \notin S$, for $11 \notin S \cap \{3, 5, 8, 11\} = \{5, 8\}$

$\therefore \quad S = \{5, 7, 8, 13\}$.

If condition (i) is not given, then S is not unique as S may be $\{7, 8, 13\}$ or $\{5, 7, 8, 13\}$ or $\{5, 7, 8, 11, 13\}$.

Similarly, deleting any other data leads to more than one solution to S (Verify.)

Example 149 *Suppose that in a poll made of 150 people, the following information was obtained: 70 of them read The Hindu, 80 read The Indian Express and 50 read Deccan Herald. 30 read both The Hindu and The Indian Express; 20 read both The Hindu and the Deccan Herald and 25 read both The Indian Express and Deccan Herald. Find at most how many of them read all the three.*

Solution: Let H, I and D be the set of those who read The Hindu, The Indian Express and the Deccan Herald, respectively.
So, the data given in mathematical symbols are as follows:

1. $|H \cup I \cup D| \le 150$
2. $|H| = 70$
3. $|I| = 80$
4. $|D| = 50$
5. $|H \cap I| = 30$
6. $|H \cap D| = 20$
7. $|I \cap D| = 25$

We need to find the maximum possible value of $|H \cap I \cap D|$.

$$150 \ge |H \cup I \cup D| = |H| + |I| + |D| - |H \cap I| - |I \cap D| - |H \cap D| + |H \cap I \cap D|$$
$$\Rightarrow \quad 150 - 70 - 80 - 50 + 30 + 50 + 25 \ge |H \cap I \cap D|$$
$$\therefore \quad |H \cap I \cap D| \le 25$$

$\therefore$ At most 25 of them read all the three. If every one of the 150 people interviewed read at least one of these three newspapers, then exactly 25 of them read all the three.

Example 150 *Lewis Carroll, the famous author of Alice in Wonderland, Through the Looking Glass, The hunting of the Shark and other wonderful works, was a mathematician whose real name was Charles Lutwidge Dodgson* (1832–1898). *Here is a problem from his book 'A Tangled Tale'.*

Let S be the set of pensioners, E the set of those who lost an eye, H those who lost an ear, A those who lost an arm and L those who lost a leg.

Given that $n(E) = 70\%$, $n(H) = 75\%$, $n(A) = 80\%$ *and* $n(L) = 85\%$. *Find what percentage at least must have lost all the four.*

Solution: Let $n(S)$ be 100.

$$\therefore \quad n(S) \ge n(E \cup H) = n(E) + n(H) - n(E \cap H)$$
$$\Rightarrow \quad 100 \ge 70 + 75 - n(E \cap H)$$
$$\Rightarrow \quad n(E \cap H) \ge 45.$$

Similarly $n(S) \ge n(L \cup A) = n(L) + n(A) - n(L \cap A)$

$$= 80 + 85 - n(L \cap A)$$
$$\Rightarrow \quad n(L \cap A) \ge 65.$$

Now, $n(S) = 100 \ge n[(E \cap H) \cup (L \cap A)]$

$$= n[(E \cap H) + n(L \cap A) - n(E \cap H \cap L \cap A)$$
$$\Rightarrow 100 \ge 45 + 65 - n(E \cap H \cap L \cap A)$$
$$\Rightarrow n(E \cap H \cap L \cap A) \ge 110 - 100 = 10.$$

That is at least 10% of the people must have lost all the four.

Example 151 *In the above problem, if those who lost all the four are more than* 10 *and less than* 70, *construct an example.*

Solution: Here we have to find

$$n(E \cap H \cap A \cap L) = 10 + k, \text{ where } 0 < k < 60.$$

We have $n[(E \cap H) \cup (A \cap L)] = n(E \cap H) + n(A \cap L) - n(E \cap H \cap A \cap L)$

But we know that $100 \geq n\,[(E \cap H) \cup (A \cap L)]$

$$\therefore \quad 100 \geq n(E \cap H) + n(A \cap L) - (10 + k)]$$
$$\Rightarrow \quad n(E \cap H) + n\,(A \cap L) \leq 110 + k.$$

$\therefore$ We can have $n(E \cap H)$ to be say $= (45 + k)$ and $n(A \cap L) = 65$.

But, $n(S) = 100 \geq (E \cup H) = n(E) + n(H) - n(E \cap H)$

$$\Rightarrow \quad 100 + n(E \cap H) \geq n(E \cup H) = n(E) + n(H) - n(E \cap H)$$
$$\Rightarrow \quad 145 + k \geq n(E) + n(H).$$

So, we can take $n(E) = 65 + k$, $n(H) = 80$.
Similarly, for $n\,(A \cap L)$

$$100 \geq n(A) + n(L) - n(A \cap L)$$
$$\Rightarrow \quad 100 + n(A \cap L) \geq n(A) + n(L)$$
$$\Rightarrow \quad 165 \geq n(A) + n(L).$$

We can take $n(A) = 75$, $n(L) = 90$

Now, we find

$$n(E) = 65 + k,\ n(H) = 80,\ n(A) = 75,\ n(L) = 90.$$

Let us check if we are correct in our choice of the cardinal number of each of these four.

$$100 \geq n(E \cup H) = n(E) + n(H) - n(E \cap H)$$
$$\Rightarrow \quad n(E \cap H) \geq (65 + k) + 80 - 100 = 45 + k$$

and again,

$$100 \geq n(A \cup L) = n(A) + n(L) - n(A \cap L)$$
$$= 75 + 90 - n(A \cap L)$$
$$\Rightarrow \quad n(A \cap L) \geq 65$$

again, $100 \geq n[(E \cap H) \cup (A \cap L)]$

$$= n(E \cap H) + n(A \cap L) - n(E \cap H \cap A \cap L)$$
$$\geq 45 + k + 65 - n(E \cap H \cap A \cap L)$$
$$\Rightarrow \quad n(E \cap H \cap A \cap L) \geq 10 + k \text{ as desired.}$$

In fact, this is just one solution. You can have yet a number of (only finite! Why don't you find them) other solutions. Once you get the cardinal number of the sets E, H, A and L, you can even combine E, A and H, L or E, L and H, A, as well. You shall get the same result.

For $n(S) = 100 \geq n(E \cup A) = n(E) + n(A) - n(E \cap A)$

$$\Rightarrow \quad n(E \cap A) \geq n(E) + n(A) - 100 = 65 + k + 75 - 100 = 40 + k$$

and Similarly $n(H \cap L) \geq n(H) + n(L) - 100 = 80 + 90 - 100 = 70$

$$\therefore \quad n[(E \cap A) \cap (H \cap L)] \geq n(E \cap A) + n(H \cap L) - 100$$
$$= 40 + k + 70 - 100 = 10 + k.$$

You can verify this by taking the pairs of sets H, A and E, L.

Example 152 *a, b, c, d be integers ≥ 0, $d \leq a$, $d \leq b$, and $a + b = c + d$. Prove that there exist sets A and B satisfying $n(A) = a$, $n(B) = b$, $n(A \cup B) = c$, $n(A \cap B) = d$.*

Solution: $(A \cap B) \subset A$

$$\Rightarrow \quad n(A \cap B) \leq n(A)$$

or, $d \leq a$

Again, $(A \cap B) \leq B$

$$n(A \cap B) \leq n(B)$$

$$d \leq a$$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow \quad n(A \cup B) + n(A \cap B) = n(A) + n(B)$$

$$\Rightarrow \quad c + d = a + b.$$

Example 153 *How many positive integers of n digits exist such that each digit is* 1, 2 *or* 3*? How many of these contain all three of the digits* 1, 2 *and* 3 *at least once?*

Solution: There are three digits 1, 2, 3 and an n-digit number is to be formed, repetitions allowed.

Thus, number of possibilities is $\underbrace{3 \times 3 \times 3 \times \cdots \times 3}_{n\,\text{times}} = 3^n$

For the second part of the question:

In (1), we include the possibility that all the n digits consist of (a) 1 only, (b) 2 only, (c) 3 only and again in (2), we include the possibility that the n digits consist of only (i) 1 and 2 (ii) 2 and 3 (iii) 1 and 3.

The number of n-digit numbers all of whose digits are 1 or 2 or 3 is 3^n.

(i) The number of n-digit numbers all of whose digits are 1 and 2, each of 1 and 2 occurring at least once is $2^n - 2$.

(ii) The number of n-digit numbers all of whose digits are 2 and 3, each of 2 and 3 occurring at least once is again $2^n - 2$.

(iii) The number of n-digit numbers all of whose digits are 1 and 3, each of 1 and 3 occurring at least once is $2^n - 2$.

Thus, the total numbers made up of the digits 1, 2 and 3 is

$$3^n - 3(2^n - 2) - 3 = 3^n - 3 \cdot 2^n + 3.$$

Example 154 *A, B and C are the set of all the positive divisors of* 10^{60}, 20^{50} *and* 30^{40}, *respectively. Find* $n(A \cup B \cup C)$.

Solution: Let $n(A)$ = number of positive divisors of

$$10^{60} = 2^{60} \times 5^{60} \text{ is } 61^2$$

$n(B)$ = number of positive divisors of

$$20^{50} = 2^{100} \times 5^{50} \text{ is } 101 \times 51 \text{ and}$$

$n(C)$ = number of positive divisors of

$$30^{40} = 2^{40} \times 3^{40} \times 5^{40} = 41^3$$

The set of common factors of A and B will be of the form $2^m \cdot 5^n$ where $0 \leq m \leq 60$ and $0 \leq n \leq 50$.

So, $\quad n(A \cap B) = 61 \times 51.$

Similarly, since the common factors of B and C and A and C are also of the form $2^m \times 5^n$,

and in the former case $0 \leq m \leq 40$,

$$0 \leq n \leq 40,$$

and in the latter case $0 \leq m \leq 40$,

$$0 \leq n \leq 40,$$

$\therefore \quad n(B \cap C) = 41^2 \quad \text{also} \quad n(A \cap C) = 41^2$

and, $\quad n(A \cap B \cap C)$ is also 41^2.

$\therefore \quad n(A \cap B \cap C)$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 61^2 + 101 \times 51 + 41^3 - 61 \times 51 - 41^2 - 41^2 + 41^2$$
$$= 61(61 - 51) + 41^2(41 - 1) + 101 \times 51$$
$$= 610 + 1681 \times 40 + 5151 = 73001.$$

Example 155 *Find the number of integer solutions to the equation $x_1 + x_2 + x_3 = 28$ where $3 \le x_1 \le 9$, $0 \le x_2 \le 8$ and $7 \le x_3 \le 17$.*

Solution: Consider three numbered boxes whose contents are denoted as x_1, x_2, x_3, respectively. The problem now reduces to distributing 28 balls in the three boxes such that the first box has at least 3 and not more than 9 balls, the second box has at most 8 balls and the third box has at least 7 and at most 17 balls. At first, put 3 balls in the first box, and 7 balls in the third box. This takes care of the minimum needs of the boxes. So, now the problem reduces to finding the number of distribution of 18 balls in 3 boxes such that the first has at most $(9 - 3) = 6$, the second at most 8 and the third at most $(17 - 7) = 10$. The number of ways of distributing 18 balls in 3 boxes with no condition is $\binom{18+3-1}{3-1} = \binom{20}{2} = 190$.

[See article 7.14: The number of ways of distributing n identical objects in r distinct boxes is $\binom{n+r-1}{r-1}$ where 'r' stands for the numbers of boxes and n for balls.]

Let d_1 be the distribution where the first box gets at least 7; d_2, the distributions where the second box gets at least 9 and d_3, the distributions where the third gets at least 11.

$$|d_1| = \binom{18-7+3-1}{3-1} = \binom{13}{2} = \frac{13 \times 12}{1.2} = 78$$

$$|d_2| = \binom{18-9+3-1}{3-1} = \binom{11}{2} = \frac{11 \times 10}{1.2} = 55$$

$$|d_3| = \binom{18-11+3-1}{3-1} = \binom{9}{2} = \frac{9 \times 8}{1.2} = 36$$

$$\therefore \quad |d_1 \cap d_2| = \binom{18-7-9+3-1}{3-1} = \binom{4}{2} = 6$$

$$|d_2 \cap d_3| = \binom{18-9-11+3-1}{3-1} = \binom{0}{2} = 0,$$

$$|d_3 \cap d_1| = \binom{18-11-7+3-1}{3-1} = \binom{2}{2} = 1.$$

Also, $|d_1 \cap d_2 \cap d_3| = 0$,

$$\Rightarrow \quad |d_1 \cup d_2 \cup d_3| = 78 + 55 + 36 - 6 - 0 - 1 + 0 = 162.$$

So, the required number of solutions $= 190 - 162 = 28$.

Note: The number of ways the first box gets at most 6, the second at most 8 and the third at most 10 = Total number of ways of getting 18 balls distributed in 3 boxes – (the number of ways of getting at least 7 in the first box, or at least 9 in the second box or at least 11 in the third box).

Example 156 *I have six friends and during a certain vacation, I met them during several dinners. I found that I dined with all the six exactly on 1 day, with every five of them on 2 days, with every four of them on 3 days, with every three of them on 4 days and with every two of them on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?*

Solution: For $i = 1, 2, 3, \ldots, 6$, let A_i be the set of days on which ith friend is present at dinner.

Then given $n(A_i)$ or $|A_i| = 7$ and $|A_i'| = 7$.

So, $|A_i \cap A_j| = 5$, $|A_i \cap A_j \cap A_k| = 4$, $|A_i \cap A_j \cap A_k \cap A_l| = 3$, $|A_i \cap A_j \cap A_k \cap A_l \cap A_m| = 2$,

and, $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6| = 1.$

where i, j, k, l, m vary between 1 to 6 and are distinct.

$$|A_1 \cup A_2 \cup A_3 \ldots \cup A_6|$$
$$= \sum_{i=1}^{6} |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \sum |A_i \cap A_j \cap A_k \cap A_l|$$
$$+ \sum |A_i \cap A_j \cap A_k \cap A_l \cap A_m| - |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6|$$
$$= \binom{6}{1}\times 7 - \binom{6}{2}\times 5 + \binom{6}{3}\times 4 - \binom{6}{4}\times 3 + \binom{6}{5}\times 2 - \binom{6}{6}\times 1$$
$$= 42 - 75 + 80 - 45 + 12 - 1 = 13.$$

The total number of dinners $|A_i| + |A_i'| = 7 + 7 = 14$.

The number of dinners in which at least one friend was present $= |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = 13$.

The number of dinners I dine alone $= 14 - 13 = 1$.

Aliter: Let the proposer of the problem be called X, and the friends be denoted as A, B, C, D, E, F. Since X dines with all the 6 friend exactly on one day. We have the combination $XABCDEF$ (1) for one day.

Thus, every five of A, B, C, D, E, F had already dined with X for a day. According to the problem, every five of them should dine on another day. It should happen in ${}^nC_5 = 6$ days. The combination is $XABCDEF$ (2), $XABCDF$ (3), $XABCEF$ (4), $XABDEF$ (5), $XACDEF$ (6), $XBCDEF$ (7).

In (1) and (2) together, X has already dined with every four friends three times, for example, with $ABCD$, he dined on the first day the numbers above the combinations can be taken as the rank of the days X dined with his friends. 2nd and 3rd days, X has dined with every three friends of them on four days, for example, with ABC, 1st, 2nd, 3rd and 4th days, X has dined with every two friends, of them for five days for example, with AB, 1st, 2nd, 3rd, 4th and 5th days,

With just one of them he has dined so far 6 days (with A, 1st, 2nd, 3rd, 4th, 5th and 6th days).

So, he has to dine with every one of them for one more day he should dine with XA, XB, XC, XD, XE and XF for 6 more days. Thus, the total number of days he dined so far with at least one of his friends is 1 + 6 + 6 = 13 days. In this counting, we see that he has dined with every one of them for 7 days. That shows that he has not dined with every one of them for 6 days.

But it is given that every friend was absent for 7 days. Since each one of them has been absent for 6 days already, all of them have to be absent for one more day.

Thus, he dined alone for 1 day and the total number of dinners he had is 13 + 1 = 14.

Example 157 *A student on vacation for d days observed that (a) it rained seven times morning or afternoon; (b) when it rained in the afternoon, it was clear in the morning; (c) there were five clear afternoon and (d) there were six clear mornings. Find d.*

Solution: Let the set of days it rained in the morning be M_r and the set of days it rained in the afternoon be A_r.

Then, clearly the set of days when there were clear morning is M'_r and the set of days when there were clear afternoon is A'_r.

By condition (*b*), we get $M_r \cap A_r = \phi$,

By (*d*), *we* get $M'_r = 6$,

By (*c*), we get $A'_r = 5$,

and by (*a*), we get $M_r \cup A_r = 7$.

M_r and A_r are disjoint sets and $n(M_r) = d - 6$, $n(A_r) = d - 5$.

$\therefore$ Applying the principle of inclusion and exclusion, we get'

$$n(M_r \cup A_r) = n(M_r) + n(A_r) - n(M_r \cap A_r)$$
$$\Rightarrow \quad 7 = (d - 6) + (\mathrm{d} - 5) - 0$$
$$\Rightarrow \quad 2d = 18$$
$$\Rightarrow \quad d = 9.$$

Aliter: Observe the tabular columns for rainy mornings, rainy afternoons, clear mornings and clear afternoons.

	Rainy afternoon	Clear afternoon
Rainy morning	x	y
Clear morning	z	w

Now, by the hypothesis, we have

$$x + y + z + w = d \tag{1}$$
$$x + y + z = 7 \tag{2}$$
$$y + w = 5 \tag{3}$$
$$z + w = 6 \tag{4}$$

By condition (*b*), $x = 0$.

From Eqs. (3) and (4),

$$y + z + 2w = 11 \tag{5}$$

From Eq. (2),

$$y + z = 7 \tag{6}$$

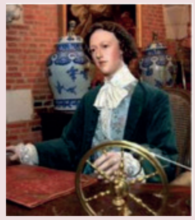
Solving Eqs. (5) and (6), we get

$$2w = 4 \quad \text{or}$$
$$w = 2$$

$$\therefore \quad d = x + y + z + w = 0 + y + z + w$$
$$= 0 + 7 + 2 = 9.$$

7.13 Derangement

Pierre Raymond de Montmort



27 Oct 1678–7 Oct 1719
Nationality: French

A derangement of 1, 2, …, n is a permutation of the numbers such that no number occupies its natural position. Thus (2, 3,1) and (3, 1, 4, 2) are derangements. On the other hand, (2, 4, 3, 5, 1) is not a derangement as 3 is at the 3rd position.

The total number of derangements of 1, 2, …, n will be denoted by D_n.

It is easy to realise that $D_1 = 0$, $D_2 = 1$ and $D_3 = 2$, etc.

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right]$$

Proof: Let A_i be the collection of all ways such that i be at ith position. Now we need to get D_n which is $N(A'_1 A'_2 A'_3 \ldots A'_n)$. Using special inclusion and exclusion formula we get

$$N(A'_1 A'_2 \ldots A'_n) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^r \binom{n}{r}(n-r)! + \cdots$$

$$= n! - \frac{n!}{(n-1)! \times 1!} \times (n-1)! + \frac{n!}{(n-2)! \times 2!} \times (n-2)! - \cdots + (-1)^r \frac{n!}{(n-r)! \times r!}(n-r)! + \cdots + (-1)^n$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^r \frac{n!}{r!} + \cdots + (-1)^n$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^r}{r!} + \cdots + \frac{(-1)^n}{n!}\right].$$

For an alternate proof see the Example 158.

Note that $\lim\limits_{n \to \infty} D_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} = e^{-1}$.

For example, let S_1, S_2, S_3 are three slots where objects A, B, C should be placed. Number of ways to place A, B, C in S_1, S_2, S_3 such that A goes to S_1, B goes to S_2 and C goes to S_3, *i.e.*, all object are placed in there correct places = 1. Number of way to place only one object in a wrong slot is not possible because if A is placed in say S_2, then B, whose correct slot is S_2, would take either S_1 or S_3. It means B is also placed in the wrong slot. So it is not possible to place only one object in wrong slot. To place objects A, B, C in S_1, S_2, S_3 such that all objects are placed in wrong slots we use derangement formulae, *i.e.*,

Number of way to place A, B, C all in wrong slots

$$= \lfloor\!\underline{3}\left[1 - \frac{1}{\lfloor\!\underline{1}} + \frac{1}{\lfloor\!\underline{2}} - \frac{1}{\lfloor\!\underline{3}}\right] = 2 \text{ ways.}$$

Example 158 *On a rainy day n people go to a party. Each of them leaves his raincoat at the counter of the gate. Find the number of ways in which the raincoats are handed over to the guests after the function is over so that no one receives his/her own raincoat.*

Solution: Let us name the guests as $g_1, g_2, \ldots, g_n$ and their raincoats as $r_1, r_2, \ldots, r_n$, respectively.

Let us denote number of ways for the event that no one gets his/her raincoat by D_n.

We shall find a recurrence relation for D_n, as follows:

For g_1 there are $(n-1)$ possible ways of getting the wrong raincoats.

If g_1 is given the raincoat r_2, Case (1) r_1 may be given to g_2 or Case (2) r_1 may not be given to g_2.

In case (1) if g_2 receives r_1 then the remaining $(n-2)$ guests may not get their raincoats in D_{n-2} different ways.

In case (2) if g_2 does not receive the raincoat r_1 then the number of ways in which g_2 does not receive r_1, g_3 does not receive $r_3,\ldots$, g_n does not receive r_n is D_{n-1} as there are $(n-1)$ guests and also $(n-1)$ raincoats.

Thus, the total number of ways in which the remaining $(n-1)$ guests do not receive their raincoats is $D_{n-1}+D_{n-2}$ as the two cases mutually exclusive.

For each one way of giving the wrong raincoat to g_1 there are $D_{n-1}+D_{n-2}$ ways that the remaining $(n-1)$ guests get the wrong raincoats.

But there are $(n-1)$ different ways in which g_1 can get a wrong raincoat.

So, $$D_n=(n-1)[D_{n-1}+D_{n-2}]$$

or $$D_n=nD_{n-1}-D_{n-1}+(n-1)\,D_{n-2}$$

or
$$\begin{aligned}D_n-nD_{n-1}&=-[D_{n-1}-(n-1)D_{n-2}] &(1)\\ &=(-1)^2[D_{n-2}-(n-2)\,D_{n-3}] &(2)\\ &=(-1)^3[D_{n-3}-(n-3)D_{n-4}]\\ &\vdots\quad\vdots\quad\vdots\\ &=(-1)^{n-2}[D_2-2D_1]\end{aligned}$$

[Here replacing n by $(n-1)$ in Eq. (1), we get $D_{n-1}-(n-1)D_{n-2}=-\{D_{n-2}-(n-2)D_{n-3}\}$ and hence from Eq. (1), we get Eq. (2) and so on.]

$\therefore$ We have,

$$D_n-nD_{n-1}=(-1)^{n-2}[D_2-2D_1].$$

Now, $D_1=0$, $D_2=1$, since D_1 stands for just one guest that does not get his/her raincoat, which is clearly zero.

Also $D_2=1$, since there are just two guests, there is only one way of getting their raincoats exchanged so that neither of the two get their raincoat.

$$\therefore\ D_n-nD_{n-1}=(-1)^{n-2}(1-0)=(-1)^{n-2}=(-1)^n$$

$$\therefore\ \frac{D_n}{n!}-\frac{nD_{n-1}}{n!}=\frac{(-1)^n}{n!}$$

$$\Rightarrow\ \frac{D_n}{n!}-\frac{D_{n-1}}{(n-1)!}=\frac{(-1)^n}{n!}.$$

Substituting $n-1$, $n-2$, … for n successively, we get

$$\frac{D_{n-1}}{(n-1)!}-\frac{D_{n-2}}{(n-2)!}=\frac{(-1)^{n-1}}{(n-1)!}$$

$$\frac{D_{n-2}}{(n-2)!}-\frac{D_{n-3}}{(n-3)!}=\frac{(-1)^{n-2}}{(n-2)!}$$

$$\vdots\quad\vdots$$

$$\frac{D_2}{2!}-\frac{D_1}{1!}=\frac{(-1)^2}{2!}.$$

Adding both the sides, we get,

$$\frac{D_n}{n!} - \frac{D_1}{1!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{(-1)^n}{n!}$$

$$\Rightarrow \quad D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right] (\because \; D_1 = 0)$$

Note that $1 - \frac{1}{1!} = 0,$ and thus zero is added to the right hand side to get the formula in the proper format.

Aliter: Use derangement formulae (which was obtained by using the special inclusion and exclusion principle).

Example 159 *Find D_4.*

Solution: The totality of permutations of 1, 2, 3, 4 is 4!
The number of permutations, which leave fixed

(i) atleast one of 1, 2, 3, 4, is ${}^4C_1\ 3!$
(ii) atleast two of 1, 2, 3, 4 is ${}^4C_2\ 2!$
(iii) atleast three of 1, 2, 3, 4 is ${}^4C_3\ 1!$ and, finally,
(iv) all of 1, 2, 3, 4, is 1

By the inclusion-exclusion principle,

$$D_4 = 4! - {}^4C_1\ 3! + {}^4C_2\ 2! - {}^4C_3\ 1! + 1 = 9.$$

Example 160 *Find the number of permutations of* 1, 2, 3, 4, 5 *in which exactly one number occupies its natural position.*

Solution: Choose the number which should occupy its natural position (${}^5C_1$)
The number of arrangements of the others is D_4.
Hence the required number = ${}^5C_1 \cdot D_4 = 45$.

Example 161 *There are 5 boxes of 5 different colours. Also there are 5 balls of colours same as those of the boxes. In how many ways we can place 5 balls in 5 boxes such that*

(i) *all balls are placed in the boxes of colours not same as those of the ball.*
(ii) *at least 2 balls are placed in boxes of the same colour.*

Solution:

(i) All the balls should be placed in the wrong boxes.
That is, boxes not of the colour same as balls.
Using derangement formulae, number of ways in which this can be done.

$$= \lfloor 5 \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5}\right]$$

$$= 120\left[1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right]$$

$$= 60 - 20 + 5 - 1 = 44.$$

(ii) Atleast 2 balls are placed in the correct boxes, *i.e.*, boxes of the colour same as ball = Total number of ways to place balls in boxes – Number of ways to place balls such that all balls are placed in wrong boxes – Number of ways to place balls in boxes such that 1 ball is placed in the correct box (*i.e.*, box of the same colour as balls).

$=\lfloor 5-44-$ Number of ways to select a ball that will be in correct box $\times$ Number of ways in which remaining 4 balls can be placed in 4 boxes such that all balls go in wrong boxes (boxes of colour different from balls).

$$=\lfloor 5-44-{}^5C_1\times\lfloor 4\left[1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}\right]$$

$= 120 - 44 - 5\times 9$ [using answer of (i) part and derangement formulae]
$= 120 - 44 - 45$
$= 31.$

Example 162 *In how many ways 6 letters can be placed in 6 envelopes such that*

(i) *No letter is placed in its corresponding envelope.*
(ii) *at least 4 letters are placed in correct envelopes.*
(iii) *at most 3 letters are placed in wrong envelopes.*

Solution:

(i) Using derangement formulae:

Number of ways to place 6 letters in 6 envelopes such that all are placed in wrong envelopes.

$$=6!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{1}{6!}\right]$$
$$= 360 - 120 + 30 - 6 + 1 = 265.$$

(ii) Number of ways to place letters such that at least 4 letters are placed in correct envelopes

$=$ 4 letters are placed in correct envelopes and 2 are in wrong $+$5 letters are placed in correct envelopes and 1 in wrong $+$ All 6 letters are placed in correct envelopes

$= {}^6C_4\times 1 + 0$ (not possible to place 1 in wrong envelope) $+1=\dfrac{6\times 5}{2}+1=16.$

(iii) Number of ways to place 6 letters in 6 envelopes such that at most 3 letters are placed in wrong envelopes

$=$ 0 letter is wrong envelope and 6 in correct $+$ 1 letter in wrong envelop and 5 in correct $+$ 2 letters in wrong envelopes and 4 are in correct $+$ 3 letters in wrong envelopes and 3 in correct

$=1+0$ (not possible to place 1 in wrong envelope) $+\ {}^6C_4\times 1+\ {}^6C_3\,\lfloor 3\left[1-\dfrac{1}{\lfloor 1}+\dfrac{1}{\lfloor 2}-\dfrac{1}{\lfloor 3}\right]$

$$=1+\frac{6\times 5}{2}+\frac{6\times 5\times 4}{6}\left(\frac{\lfloor 3}{\lfloor 2}-\frac{-\lfloor 3}{\lfloor 3}\right)$$

$= 1 + 15 + 20\times 2 = 56.$

Build-up Your Understanding 7

1. Find the numbers from 1 to 100 which are neither divisible by 2 nor by 3 nor by 7.
2. Find the number of numbers, from amongst 1, 2, 3, …, 500, which are divisible by none of 2, 3, 5.
3. Find the number of 3 element subsets of the set $\{1, 2, \ldots, 10\}$, in which the least element is 3 or the greatest element is 7.
4. Find the number of n digit numbers, which contain the digits 2 and 7, but not the digits 0, 1, 8, 9.
5. How many integers from 1 through 999 do not have any repeated digits?
6. Find the number of natural numbers less than or equal to 10^8 which are neither perfect squares, nor perfect cubes, nor perfect fifth powers.

7. In a certain state, license plates consist of from zero to three letters followed by from zero to four digits, with the provision, however, that a blank plate is not allowed.
(i) How many different license plates can the state produce?
(ii) Suppose 85 letter combinations are not allowed because of their potential for giving offense. How many different license plates can the state produce?

8. If the number of ways of selecting K coupons one by one out of an unlimited number of coupons bearing the letters A, T, M so that they cannot be used to spell the word MAT is 93, then find K.

9. How many positive integers divide 10^{40} or 20^{30}?

10. Find the number of permutations of letters a, b, c, d, e, f, g taken all together if neither 'beg' nor 'cad' pattern appear.

11. Find the number of permutations of the letters of the word HINDUSTAN such that neither the pattern 'HIN' nor 'DUS' nor 'TAN' appears.

12. Find the number of permutations of the 8 letters AABBCCDD, taken all at a time, such that no two adjacent letters are alike.

13. Find the number of non-negative integer solutions of $x_1 + x_2 + x_3 = 15$, subject to $x_1 \le 5$, $x_2 \le 6$, and $x_3 \le 7$.

14. According to the Gregorian calendar, a leap year is defined as a year n such that
(i) n divides 4 but not 100; or (ii) n divides 400.
Find the number of leap years from the year 1000 to the year 3000, inclusive.

15. Find the number of onto functions from a set containing 6 elements to a set containing 3 elements.

16. How many 6-digit numbers contain exactly three different digits?

17. Let D_n be the nth derangement number. Prove that
(i) $D_n = (n-1)(D_{n-1} + D_{n-2}),\ n > 2$;
(ii) $\lim\limits_{n\to\infty} \dfrac{D_n}{n!} = \dfrac{1}{e}$

18. Show that n letters in n corresponding envelopes can be put such that none of the letters goes to the correct envelop is $n!\left(1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\cdots+\dfrac{(-1)^n}{n!}\right)$ ways.

19. Five pairs of hand gloves of different colours are to be distributed to each of five people. Each person must get a left glove and a right glove. Find the number of distributions so that,exactly one person gets a proper pair.

20. Prove (combinatorially) that $\sum\limits_{r=1}^{n} r!r = (n+1)! - 1$.

21. In maths paper there is a question on 'Match the column' in which column A contains 6 entries and each entry of column A corresponds to exactly one of the 6 entries given in column B written randomly. 2 marks are awarded for each correct matching and 1 mark is deducted from each incorrect matching. A student having no subjective knowledge decides to match all the 6 entries randomly. Find the number of ways in which he can answer, to get atleast 25% marks in this question.

22. Ten parabolas are drawn in a plane. Any two parabola intersect in four real, and distinct, points. No three parabola are concurrent. Find the total number of disjoint regions of the plane.

23. In how many ways can a 12 step staircase be climbed taking 1 step or 2 steps at a time?

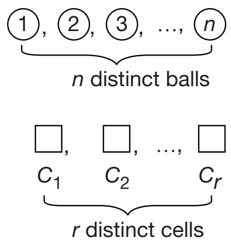
24. A coin is tossed 10 times. Find the number of outcomes in which 2 heads are not successive.

25. Find the number of ways to pave a 1 × 7 rectangle by 1 × 1, 1 × 2, 1 × 3 tiles, if tiles of the same size are indistinguishable.

7.14 Classical Occupancy Problems

The problems of the number of distributions of balls into cells are called occupancy problems. We distinguish several cases as described below:

7.14.1 Distinguishable Balls and Distinguishable Cells



1. Number of ways to divide n non-identical balls in r different cells such that each cell gets 0 or more number of balls (empty cells are allowed) $= r^n$.
2. If no cell is empty, then the number is determined by the inclusion/exclusion principle or by recurrence relation or by generating function method. Using any one of them we can get number of ways to divide n non-identical balls in r different cells such that each cell gets at least one object (empty cells are not allowed)

$$= r^n - {}^rC_1\,(r-1)^n + {}^rC_2\,(r-2)^n - {}^rC_3\,(r-3)^n + \ldots\,(-1)^{r-1}\,{}^rC_{r-1}\,1^n.$$

Example 163 *Find the number of distributions of* 5 *distinguishable balls in* 3 *distinguishable cells, if*

(i) *an empty cell is allowed;*
(ii) *no cell is empty.*

Solution:

(i) $3^5 = 243$.
(ii) Method 1:

The five balls can be distributed in 3 non-identical boxes in the following 2 ways:

Boxes	Box1	Box 2	Box 3
Number of balls	3	1	1
Number of balls	2	2	1

Case 1: 3 in one Box, 1 in another and 1 in third Box (3, 1, 1) (1)
Number of ways to divide balls corresponding to (1)

$$= \frac{5!}{3!\,1!\,1!}\,\frac{1}{2!} = 10$$

But corresponding to each division there are 3! ways of distributing the balls into 3 boxes.

So number of ways of distributing balls corresponding to (1)
= (Number of ways to divide balls) $\times$ 3! = 10 $\times$ 3! = 60

Case 2: 2 in one Box, 2 in another and 1 in third Box (2, 2, 1) (2)
Number of ways to divide balls corresponding to (2)

$$= \frac{5!}{2!\,2!\,1!}\,\frac{1}{2!} = 15$$

But corresponding to each division there are 3! ways of distributing balls into 3 boxes.

So number of ways of distributing balls corresponding to (2)
= (Number of ways to divide balls) $\times$ 3!
= 15 $\times$ 3! = 90
Hence, required number of ways = 60 + 90 = 150.

Method 2:
Let us name the Boxes as A, B and C. Then there are following possibilities of placing the balls.

Box A	Box A	Box A	Number of ways
1	2	2	${}^5C_1 \times {}^4C_2 \times {}^2C_2 = 30$
1	1	3	${}^5C_1 \times {}^4C_1 \times {}^3C_3 = 20$
1	3	1	${}^5C_1 \times {}^4C_3 \times {}^1C_1 = 20$
2	1	2	${}^5C_2 \times {}^3C_1 \times {}^2C_2 = 30$
2	2	1	${}^5C_2 \times {}^3C_2 \times {}^1C_1 = 30$
3	1	1	${}^5C_3 \times {}^2C_1 \times {}^1C_1 = 20$

Therefore required number of ways of placing the balls

$$= 30 + 20 + 20 + 30 + 30 + 20 = 150$$

Method 3:

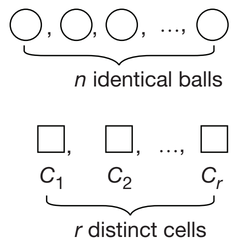
Number of ways of distributing 5 balls in 3 boxes so that no Box is empty

$$r^n - {}^rC_1\,(r-1)^n + {}^rC_2\,(r-2)^n - {}^rC_2\,(r-3)^n + \cdots$$

Put $n = 5$ and $r = 3$ to get:

Number of ways $= 3^5 - {}^3C_1\, 2^5 + {}^3C_1\, 1^5 = 243 - 3 \times 32 + 3 = 246 - 96 = 150$ ways.

7.14.2 Identical Balls and Distinguishable Cells



If an empty cell is allowed, then the number of distributions is $\binom{n+r-1}{r-1}$ (use binary sequences).

In other words the number of ways to divide n identical objects into r groups (different) such that each gets 0 or more objects (empty groups are allowed) $= {}^{n+r-1}C_{r-1}$.

Proof:

Let $x_1, x_2, x_3, \ldots, x_r$ be the number of objects given to groups 1, 2, 3, …, r respectively. As total objects to be divided is n, we can take

Sum of the objects given to all groups $= n$

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + \cdots + x_r = n.$$

This equation is known as integral equation as all variables are integer.

As each group can get 0 or more, following are constraints on integer variables.

$$0 \le x_1 \le n\,;\ 0 \le x_2 \le n, \ldots\ 0 \le x_r \le n,\ i.e.,\ 0 \le x_i \le n\ i = 1, 2, 3, \ldots, r.$$

We can observe that number of integral solutions of the above equation is equal to number of ways to divide n identical objects among r groups such that each gets 0 or more.

$$= {}^{n+r-1}C_n = {}^{n+r-1}C_{r-1}.$$

If no cell is allowed to remain empty, then the number is ${}^{n-1}C_{r-1}$.

In other words the number of ways to divide n identical objects into r groups (different) such that each group receives at least one object (empty groups are not allowed).

$$= {}^{n-1}C_{r-1}.$$

Example 164 *How many terms are there in the expansion of* $(a + b + c + d)^{24}$?

Solution: A typical term is $a^{k_1} \cdot b^{k_2} \cdot c^{k_3} \cdot d^{k_4}$, where k_1, k_2, k_3, k_4 are non-negative integers whose sum $= 24$.

The number of terms is the same as the number of distributions of 24 identical balls in four distinguishable cells, empty cell allowed. This is ${}^{24+4-1}C_{24} = {}^{27}C_{24}$.

Example 165 *Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.*

Solution: Let x_1, x_2 and x_3 be the number of balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.
The number of ways of distributing 5 balls into Boxes 1, 2 and 3 is the number of integral solutions of the equation $x_1 + x_2 + x_3 = 5$ subjected to the following conditions on x_1, x_2, x_3. (1)

Conditions on x_1, x_2 and x_3:
According to the condition that the boxes should contain at least one ball, we can find the range of x_1, x_2 and x_3, *i.e.*,

Min $(x_i) = 1$ and Max $(x_i) = 3$ for $i = 1, 2, 3$ [using: Max $(x_1) = 5 -$ Min $(x_2) -$ Min(x_3)]
or $1 \le x_i \le 3$ for $i = 1, 2, 3$
So, number of ways of distributing balls
= Number of integral solutions of (1)
= Coefficient of x^5 in the expansion of $(x + x^2 + x^3)^3$
= Coefficient of x^5 in $x^3 (1 - x^3)(1 - x)^{-3}$
= Coefficient of x^2 in $(1 - x^3)(1 - x)^{-3}$
= Coefficient of x^2 in $(1 - x)^{-3}$ [as x^3 cannot generate x^2 terms]
$= {}^{3+2-1}C_2 = {}^4C_2 = 6.$

Alternate solution:
The number of ways of dividing n identical objects into r groups so that no group remains empty

$$= {}^{n-1}C_{r-1}$$
$$= {}^{5-1}C_{3-1} = {}^4C_2 = 6.$$

Example 166 *Find the number of ways of distributing 10 identical balls in 3 boxes so that no box contains more than four balls and less than 2 balls.*

Solution: Let x_1, x_2 and x_3 be the number of balls placed in Boxes 1, 2 and 3 respectively.
Number of ways of distributing 10 balls in 3 boxes
= Number of integral solutions of the equation $x_1 + x_2 + x_3 = 10$ (1)

Conditions on x_1, x_2 and x_3:
As the boxes should contain atmost 4 ball and at least 2 balls, we can make

Max $(x_i) = 4$ and Min $(x_i) = 2$ for $i = 1, 2, 3$
or $2 \le x_i \le 4$ for $i = 1, 2, 3$
So the number of ways of distributing balls in boxes
= Number of integral solutions of equation (i)
= Coefficient of x^{10} in the expansion of $(x^2 + x^3 + x^4)^3$
= Coefficient of x^{10} in $x^6 (1 - x^3)^3 (1 - x)^{-3}$
= Coefficient of x^4 in $(1 - x^3)^3 (1 - x)^{-3}$
= Coefficient of x^4 in $(1 - {}^3C_1 x^3 + {}^3C_2 x^6 + \cdots)(1 - x)^{-3}$
= Coefficient of x^4 in $(1 - x)^{-3}$ – Coefficient of x in ${}^3C_1 (1 - x)^{-3}$
$= {}^{4+3-1}C_4 - 3 \times {}^{3+1-1}C_1 = {}^6C_4 - 3 \times {}^3C_1 = 15 - 9 = 6.$

Example 167 *Find the number of ways in which* 14 *identical toys can be distributed among three boys so that each one gets atleast one toy and no two boys get equal number of toys.*

Solution: Let the boys get a, $a + b$ and $a + b + c$ toys respectively.

$$a + (a + b) + (a + b + c) = 14, a \geq 1, b \geq 1, c \geq 1$$

$$\Rightarrow 3a + 2b + c = 14, a \geq 1, b \geq 1, c \geq 1$$

$\therefore$ The number of solutions

$= \text{Coefficient of } t^{14} \text{ in } \{(t^3 + t^6 + t^9 + \ldots)(t^2 + t^4 + \cdots)(t + t^2 + \cdots)\}$

$= \text{Coefficient of } t^8 \text{ in } \{(1 + t^3 + t^6 + \cdots)(1 + t^2 + t^4 + \ldots)(1 + t + t^2 + \cdots)\}$

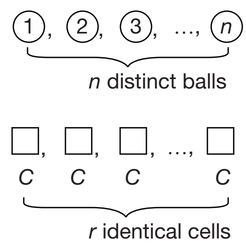
$= \text{Coefficient of } t^8 \text{ in } \{(1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8)(1 + t + t^2 + \cdots + t^8)\}$

$= 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 = 10.$

Since, three distinct numbers can be assigned to three boys in 3! ways.

So, total number of ways $= 10 \times 3! = 60$.

7.14.3 Distinguishable Balls and Identical Cells



Label the balls by the natural numbers 1, 2, …, n. A partition of $\{1, 2, \ldots, n\}$ in r part is a set of r non-empty subsets, $A_1, A_2, \ldots, A_r$ of $\{1, 2, \ldots n\}$ such that $A_1 \cup A_2 \cup \ldots \cup A_r = \{1, 2, \ldots, n\}$ and any two of $A_1, \ldots A_r$ are disjoint.

For example, $\{\{1\}, \{2, 3\}, \{4\}\}$ is a 3 partition of $\{1, 2, 3, 4\}$.

Denote the number of r partitions of $\{1, 2, \ldots, n\}$ by $S(n, r)$.

$S(n, r)$ is called a Stirling number of the second kind**.**

It is easy to see that:

$S(n, 1) = 1, S(n, n) = 1, S(n, r) = 0$, if $r > n$.

To determine $S(n, r)$ for $1 < r < n$.

There are two possibilities:

1. The number n is by itself is a partition.
   $\Rightarrow$ The numbers 1, 2, …, $n - 1$ must form a $r - 1$ partition.
   The number of such partitions $= S(n - 1, r - 1)$.
2. The number n is along with atleast one of 1, 2, …, $n - 1$ in a partition.
   $\Rightarrow$ The numbers 1, 2, …, $n - 1$ must form a r partition and n must be inserted in any one of the r subsets. So n can be put in r ways.

The number of such partitions $= r\,S(n - 1, r)$

Hence $S(n, r) = S(n - 1, r - 1) + r\,S(n - 1, r), 1 < r < n$

Use this to show that $S(n, 2) = 2^{n-1} - 1$

In general, we can easily get

$$S(n, r) = \frac{1}{r!}\left[r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \cdots + (-1)^{r-1}\binom{r}{r-1}1^n\right]$$

Note: If n distinguishable balls are to be distributed in r identical cells, an empty cell allowed, then the number of distributions is $\sum_{k=1}^{r} S(n, k)$.

Example 168 *Find the number of distributions of* 5 *distinguishable balls in* 3 *identical cells, an empty cell allowed.*

Solution: The sought after number is $S(5, 1) + S(5, 2) + S(5, 3)$.

Now $S(5, 1) = 1$, $S(5, 2) = 2^{5-1} - 1 = 15$, and

$$\begin{aligned} S(5, 3) &= S(4, 2) + 3S(4, 3) \\ &= (2^3 - 1) + 3(S(3, 2) + 3\,S(3, 3)) \\ &= 7 + 3((2^2 - 1) + 3) \\ &= 25 \end{aligned}$$

Hence, the answer is $1 + 15 + 25 = 41$.

Leonhard Euler

15 Apr 1707–18 Sep 1783
Nationality: Swiss

7.14.4 Identical Balls and Identical Cells

Consider the problem of distributing n identical balls in k identical cells, no cell remaining empty.

The number of distributions = The number of ways of writing n as the sum $\underbrace{x_1 + x_2 + \cdots + x_k}_{\text{positive integers}}$, the order of terms being ignored = number of **Partition** of n in k parts.

This is equivalent to number of integral solution of $x_1 + x_2 + x_3 + \cdots + x_k = n$ with

$1 \le x_1 \le x_2 \le x_3 \le \cdots \le x_k$ which is equal to $[x^n]$ in $\dfrac{x^k}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^k)}$

Aleternatively denote this number by $P_k(n)$.

Clearly, $P_1(n) = P_n(n) = 1$, $P_2(n) = \left\lfloor \dfrac{n}{2} \right\rfloor$, $P_k(n) = 0$, $k > n$

For example, $\left.\begin{aligned} 5 &= 2+2+1 \\ &= 3+1+1 \end{aligned}\right\} \Rightarrow P_3(5) = 2$

To determine $P_k(n)$, $1 < k < n$

Let us divide all partitions in two types:

(A) Atleaset one partition of size 1

(B) No partition of size 1

Number of partitions of type A is $p_{k-1}(n - 1)$ (As make one partition of size 1 and remaining $n - 1$ in $k - 1$ parts). Number of partitions of type B is $p_k(n - k)$ (As first remove k objects and divide $n - k$ objects in k parts). Now add one object in each part so that each part will be of size atleast 2.

Hence, $P_k(n) = P_{k-1}(n - 1) + P_k(n - k)$, $1 < k \le \left\lfloor \dfrac{n}{2} \right\rfloor$

Using the above reccurence we can easily prove $P_3(n) = \left\langle \dfrac{n^2}{12} \right\rangle$. Read it "nearest integer" (see the Example 169).

Note: If n identical balls are to be distributed in r identical cells, an empty cell allowed, then the number is $\sum_{k=1}^{r} P_k(n)$.

Example 169 *What is the number of necklaces that can be made from 6n identical blue beads and 3 identical red beads?*

Solution: The sought after number is $P_3(6n) + P_2(6n) + P_1(6n)$.

We have

$$P_k(n) - P_k(n - k) = P_{k-1}(n - 1)$$

$$\Rightarrow \quad P_3(6n) - P_3(6n - 3) = P_2(6n - 1) = \left\lfloor \frac{6n-1}{2} \right\rfloor = 3n - 1 \qquad (1)$$

and $P_3(6n-3) - P_3(6n-6) = P_2(6n-4) = \left\lfloor \frac{6n-4}{2} \right\rfloor = 3n-2$ (2)

Adding (1) and (2), we get, $P_3(6n) - P_3(6(n-1)) = 3(2n-1)$ (3)

Let $P_3(6n) = a_n$, then the Eq. (3) becomes $a_n - a_{n-1} = 3(2n-1)$ (4)

Now plugging $n = 2, 3, \ldots, n$ in Eq. (4) and adding all, we get $a_n - a_1 = 3(n^2 - 1)$

As, $a_1 = P_3(6) = 3$

$\Rightarrow \quad a_n = 3n^2$

$\Rightarrow \quad P_3(6n) = 3n^2$

Also $P_2(6n) = \left\lfloor \frac{6n}{2} \right\rfloor = 3n$

and $P_1(6n)$

$\therefore$ The required number is $3n^2 + 3n + 1$.

Build-up Your Understanding 8

1. Find the number of ways in which n distinct objects can be put into two different boxes so that no box remains empty.
2. Find the number of ways in which n distinct objects can be kept into two identical boxes so that no box remains empty.
3. 10 identical balls are to be distributed in 5 different boxes kept in a row and labeled A, B, C, D and E. Find the number of ways in which the balls can be distributed in the boxes if no two adjacent boxes remain empty.
4. Find the number of distributions of 6 distinguishable objects in three distinguishable boxes such that each box contains an object.
5. Find the number of ways in which 12 identical coins can be distributed in 6 different purses, if not more than 3 and not less than 1 coin goes in each purse.
6. Find the number of ways in which 30 coins of one rupee each be given to six persons so that none of them receive less than 4 rupees.
7. Find the number of ways of wearing 8 distinguishable rings on 5 fingers of right hand.
8. 15 identical balls have to be put in 5 different boxes. Each box can contain any number of balls. Find total number of ways of putting the balls into box so that each box contains atleast 2 balls.
9. In how many ways can 3 blue, 4 red and 2 green balls be distributed in 4 distinct boxes? (Balls of the same colour are identical)
10. How many different ways can 15 Candy bars be distributed to Tanya, Manya, Shashwat and Adwik, if Tanya cannot have more than 5 candy bars and Manya must have at least two. Assume all Candy bars to be alike.
11. In how many ways, 16 identical coins can be distributed to 4 beggars when
    (i) any beggar may get any number of coins?
    (ii) every beggar gets atleast one coin?
    (iii) every beggar gets atleast two coins?
    (iv) every beggar gets atleast three coins?
12. Prove that the number of n digit quaternary sequences (whose digits are 0, 1, 2, and 3), in which each of the digits 2 and 3 appear atleast once, is $4^n - 2 \cdot 3^n + 2^n$.
13. Shivank has 15 ping-pong balls each uniquely numbered from 1 to 15. He also has a red box, a blue box, and a green box.
    (i) How many ways can Shivank place the 15 distinct balls into the three boxes so that no box is empty?
    (ii) Suppose now that Shivank has placed 5 ping-pong balls in each box. How many ways can he choose 5 balls from the three boxes so that he chooses at least one from each box?

14. In how many ways we can place 9 different balls in 3 different boxes such that in every box at least 2 balls are placed?

15. In how many ways can we put 12 different balls in three different boxes such that first box contains exactly 5 balls.

16. Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty, if
  (i) balls and boxes are all different?
  (ii) balls are identical but boxes are different?
  (iii) balls are different but boxes are identical?
  (iv) balls as well as boxes are identical?

17. A man has 3 daughters. He wants to bequeath his fortune of 101 identical gold coins to them such that no daughter gets more share than the combined share of the other two. Find the number of ways of accomplishing this task.

18. There are six gates in an auditorium. suppose 20 delegates arrive. How many records could be there?

19. A man has to move 9 steps. He can move in 4 directions: left, light, forward, backward.
  (i) In how many ways he can take 9 steps in 4 direction?
  (ii) In how many ways he can move 9 steps if he has to take atleast one step in every direction.
  (iii) In how many ways he can move 9 steps such that he finish his journey one step away (either left or right or forward or backward) from the starting position.

Johann Peter Gustav Lejeune Dirichlet

13 Feb 1805–5 May 1859
Nationality: German

7.15 Dirichlet's (Or Pigeon Hole) Principle (PHP)

Let $k, n \in \mathbb{N}$. If at least $kn + 1$ objects are distributed among k boxes, then atleast one of the box, must contain atleast $(n + 1)$ objects. In particular, if atleast $(n + 1)$ objects are put into n boxes, then atleast one of the box must contain atleast two objects. For arbitrary n objects and m boxes this generalizes to atleast one box will contain atleast $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ objects.

Example 170 *Divide the numbers* 1, 2, 3, 4, 5 *into two arbitrarily chosen sets. Prove that one of the sets contains two numbers and their difference.*

Solution: Let us try to divide 1, 2, 3, 4, 5 into two sets in such a way that neither set contains the difference of two of its numbers.

2 cannot be in the same set as 1, 4, because if 2 and 1 are in the same sets 2 – 1 = 1 belongs to the set; again if 2 and 4 are in the same set then $4 - 2 = 2$ belongs to the set and hence, if we name the sets as A and B, and if $2 \in A$, then 1, 4 both belong to B.

$$\begin{array}{cc} A & B \\ \{2, \square, \square\} & \{1, 4, \square\} \end{array}$$

We cannot put 3 in set B as $4 - 3 = 1$ belongs to B, so 3 belongs to A.

$$A = \{2, 3, \square\} \; B = \{1, 4\, \square\}$$

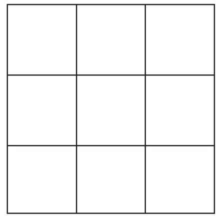
Now, 5 is the only number left out. Either 5 should be in set A or in B, but then if $5 \in A \Rightarrow 5 - 3 = 2 \in A$.

So, 5 cannot be in A.

However, if 5 is put in set B, then $5 - 4 = 1 \in B$. So, 5 cannot be in set B.

Thus, we cannot put 5 in either set and hence, the result.

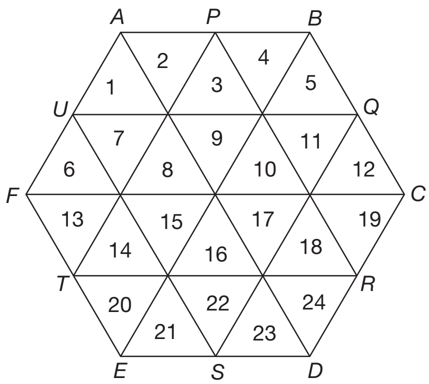
Example 171 *Show that for any set of* 10 *points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance is at most* $\sqrt{2}$.



Solution: Divide the square into 9 unit squares as given in the figure. Out of the 10 points distributed in the big square, at least one of the small squares must have at least two points by the Pigeon hole principle. These two points being in a unit square, are at the most $\sqrt{2}$ units distance apart as $\sqrt{2}$ is the length of the diagonal of the unit square.

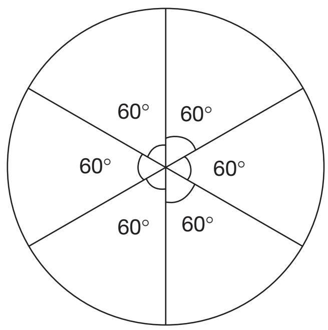
Example 172 *Show that given a regular hexagon of side* 2 *cm and* 25 *points inside it, there are at least two points among them which are at most 1 cm distance apart.*

Solution: If *ABCDE* is the regular hexagon of side 2 cm and *P*, *Q*, *R*, *S*, *T* and *U* are respectively the midpoints of *AB*, *BC*, *CD*, *DE*, *EF* and *FA*, respectively, then by joining the opposite vertices, and joining *PR*, *RT*, *TP*, *UQ*, *QS* and *SU*, we get in all 24 equilateral triangles of side 1 cm.



We have 25 points. So, of these 25 points inside the hexagon *ABCDEF*, at least 2 points lie inside any one triangle whose sides are 1 cm long. So, at least two points among them, will be at most 1 cm apart.

Example 173 *If* 7 *points are chosen on the circumference or in the interior of a unit circle, such that their mutual distance apart is greater than or equal to* 1, *then one of them must be the centre.*



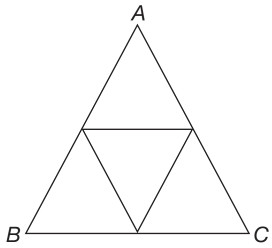
Solution: Divide the circle into six equal parts by drawing radii with two adjacent radii making an angle of 60°. Then, two of the seven points cannot lie in the interior of any one of the six sectors, since the distance between any two points is greater than or equal to 1.

If at all, in any sector, with boundaries included, two of the points may lie on the circular arc as end points (of the arc of any one of these sectors) or one on the arc and one at the centre of the circle.

Even if two lie on the ends of each circular arc, we have only 6 points satisfying the condition, thus forcing the seventh point to lie at the centre.

Example 174 $4^n + 1$ *points lie within an equilateral triangle of side* 1 *cm. Show that it is possible to choose out of them, at least two, such that the distance between them is at most* $\frac{1}{2^n}$ *cm.*

Solution: *ABC* is an equilateral triangle of side 1 cm. If the sides are divided into two equal parts, we get 4 equilateral triangles with side $\frac{1}{2}$ cm.



Again, if each of these four triangles is subjected to the above method, we get 4×4 triangles of side $\frac{1}{2} \times \frac{1}{2} = \frac{1}{2^2}$ cm.

Thus, after n steps we get, 4^n triangles of side $\frac{1}{2^n}$ cm.

Now, if we take $4^n + 1$ points inside the original equilateral ΔABC, then at least two of the points lie on the same triangle out of 4^n triangles by Pigeon hole principle. Hence, the distance between them is less than or at the most equal to the length of the side of the triangle, in which they lie, *i.e.*, they are $\frac{1}{2^n}$ cm apart or they are less than $\frac{1}{2^n}$ cm apart.

Example 175 *Let A be any set of* 19 *distinct integers chosen from the Arithmetic Progression* 1, 4, 7, …, 100. *Prove that there must be two distinct integers in A, whose sum is* 104.

Solution: There are $\frac{(100-1)}{3} + 1 = 34$ elements in the progression.

1, 4, 7, …, 100. Consider the following pairs:

$$(4, 100), (7, 97), (10, 94), \ldots, (49, 55).$$

There are in all $\frac{49-4}{3} + 1 = 16$ pairs $\left(\text{or } \frac{100-55}{3} + 1\right)$.

Now, we shall show that we can choose eighteen distinct numbers from the AP, such that no two of them add up to 104. In the above 16 pairings of the AP the numbers 1 and 52 are left out.

Now, taking one of the numbers from each of the pairs, we can have 16 numbers and including 1 and 52 with these 16 numbers, we now have 18 numbers.

But, no pair of numbers from these 18 numbers can sum up to 104, since just one number is selected from each pair and the other number of the pair (not selected) is 104, the number chosen.

Also $1 + 52 \neq 104$. Thus, we can choose 18 numbers, so that no two of them sum up to 104.

For getting 19 numbers (all these should be distinct), we should choose one of the 16 not chosen numbers, but then this number chosen is the 104 complement of one of the 16 numbers chosen already (among the 18 number). Thus, if a set of 19 distinct elements are chosen, then we must have at least one pair whose sum is 104.

Example 176 *Let* $X \subset \{1, 2, 3, \ldots, 99\}$ *and* $n(X) = 10$. *Show that it is possible to choose two disjoint non-empty proper subsets Y, Z of X such that* $\sum_{y \in Y} y = \sum_{z \in Z} z$.

Solution: Since $n(X) = 10$, the number of non-empty, proper subsets of X is $2^{10} - 2 = 1022$.

The sum of the elements of the proper subsets of X can possibly range from 1 to $\sum_{i=1}^{9}(90+i)$. That is 1 to $(91 + 92 + \cdots + 99)$, *i.e.*, 1 to 855.

$i = l$

That is, the 1022 subsets can have sums from 1 to 855.

By Pigeon hole principle, at least two distinct subsets B and C will have the same sum.

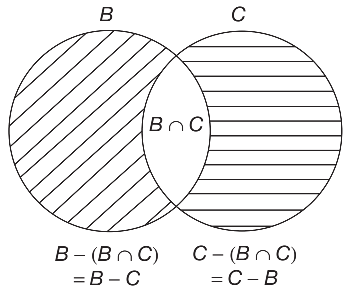
($\because$ There are 855 different sums, and so if we have more than 855 subsets, then at least two of them have the same sum.)

If B and C are not disjoint, then let

$$X = B - (B \cap C)$$

and, $$Y = C - (B \cap C).$$

Clearly, X and Y are disjoint and non-empty and have the same sum of their elements.



Define $s(A)$ = sum of the elements of A. We have B and C not necessarily disjoint such that $s(B) = s(C)$.

Now, $$s(X) = s(B) - s(B \cap C)$$

$$s(Y) = s(C) - s(B \cap C)$$

but, $$s(B) = s(C).$$

Hence, $s(X) = s(Y)$.

Also $X \neq \phi$. For if X is empty, then $B \subset C$ which implies $s(B) < s(C)$ (a contradiction). Thus, X and Y are non-empty and $s(X) = s(Y)$.

Example 177 *If repetition of digits is not allowed in any number* (*in base* 10)*, show that among three four-digit numbers, two have a common digit occurring in them.*

Also show that in base 7 *system any two four-digit numbers without repetition of digits will have a common number occurring in their digits.*

Solution: In base 10, we have ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Thus, for 3 four-digit numbers without repetition of digits, we have to use in all 12 digits, but in base 10 we have just 10 digits. Thus, at least any two of the three four-digit numbers have a common number occurring in their digits by Pigeon hole principle. Again for base 7 system, we have seven digits 0, 1, 2, 3, 4, 5, 6. For two four-digit numbers without repetition we have to use eight digits and again by Pigeon hole Principle, they have atleast one common number in their digits.

Example 178 *In base* $2k$*,* $k \geq 1$ *number system, any* 3 *non-zero, k-digit numbers are written without repetition of digits. Show that two of them have a common digit among them.*

In base $2k + 1$, $k \geq 1$ *among any* $3k + 1$ *digit non-zero numbers, there is a common number occurring in any two digits.*

Solution:

Case 1: In case $k' = 1$, we have the digits 0, 1 and the k-digit non-zero number(s) is 1 only. Thus, all the three numbers in this case are trivially the same 1.

For $k > 1$: Three k-digit (non-zero) numbers will have altogether $3k$ digits and the total number of digits in base $2k$ system is $2k$. Since repetition of digits is not allowed and $3k > 2k$ implies that among the digits of at least two of the numbers, there is at least one digit common among them (by Pigeon hole principle).

Case 2: In the case of $k = 1$, $2k + 1 = 3$, the three digits in base $2k + 1 = 3$ systems are 0, 1 and 2.

$k + 1 = 1 + 1 = 2$ and the digits non-zero numbers here are 10, 20, 12, 21.

So, we can pick up 10, 20 and 12, or 10, 20, 21, …. In each of the cases there is a common digit among two of them. (In fact, any two numbers will have a common digit 1.)

In general case, $3(k + 1)$ digit numbers will have $3k + 3$ digits in all. But it is a base $(2k + 1)$ system.

The numbers are written without repetition of digits, since $3k + 3 > 2k + 1$ (In fact, any two $k + 1$ digit numbers could also have the same property as $2k + 2 > 2k + 1$, again by the Pigeon hole principle at least two of the numbers, will have at least one common number in their digits.

Example 179 *Let A denote the subset of the set* $S = \{a, a + d, \ldots, a + 2nd\}$ *having the property that no two distinct elements of A add up to* $2(a + nd)$. *Prove that A cannot have more than* $(n + 1)$ *elements. If in the set S, 2nd is changed to* $a + (2n + 1)d$, *what is the maximum number of elements in A if in this case no two elements of A add up to* $2a + (2n + 1)d$?

Solution: Pair of the elements of S as $[a, a + 2nd]$, $[a + d, a + (2n - 1)d]$, …, $[a + (n - 1)d, a + (n + 1)d]$ and one term $a + nd$ is left out.

Now, sum of the terms in each of the pairs is $2(a + nd)$. Thus, each term of the pair is $2(a + nd)$ complement of the other term.

Now, there are n pairs. If we choose one term from each pair, we get n term. To this collection of terms include $(a + nd)$ also.

Now, we have $(n + 1)$ numbers. Thus, set A can be taken as the set of the above $(n+1)$ numbers. Here no two elements of the set A add up to $2(a+nd)$ as no element has its $2(a + nd)$ complement in A except $a + nd$, but then, we should take two distinct elements.

If we add any more terms to A so that A contains more than $(n + 1)$ elements, then some of the elements will now have then $2(a + nd)$ complement in A, so that sum of these two elements will be $2(a + nd)$, and hence, the result.

In the second case, we have

$$S = \{a, a + d, \cdots, a + \{2n + 1)\, d\}$$

There are $2(n + 1)$ elements. So, pairing them as before gives $(n + 1)$ pairs, *i.e.*, $[a, a + (2n + 1)\, d]$, $[a + d, a + 2nd]$, …, $[a + nd, a + (n + 1)d]$.

Now, we can pick exactly one term from each of these $(n + 1)$ pairs.

We get a set A of $(n + 1)$ elements where no two of which add up to $[2a + 2(n + 1)d]$.

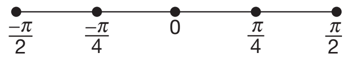
Note: Here we need not use distinct numbers, even if the same number is added to itself, the sum will not be $[2a + 2(n + 1)d]$. Here again, even choosing one more term from the numbers left out and adding it to A; A will have a pair which adds up to $[2a + 2(n + 1)d]$. Thus, the maximum number of elements in A satisfying the given condition is $(n + 1)$.

Example 180 *Given any five distinct real numbers, prove that there are two of them, say x and y, such that* $0 < \dfrac{(x-y)}{(1+xy)} \le 1$.

Solution: Here we are using the property of tangent functions of trigonometry.

Given a real number a, we can find a unique real number A, lying between $\dfrac{-\pi}{2}$ and $\dfrac{\pi}{2}$, *i.e.*, lying in the real interval $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ such that $\tan A = a$, as the tangent function in the open interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is continuous and strictly increasing and covers R completely. Therefore, corresponding to the five given real numbers $a_i (i = 1, 2, 3, 4, 5)$, we can find five distinct real numbers A_i $(i = 1, 2, 3, 4, 5)$ lying between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ such that $\tan A_i = a_i$.

Divide the open interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ into four equal intervals, each of length $\frac{\pi}{4}$. Now, by Pigeon hole principle at least two of the A_i's must lie in one of the four intervals. Suppose A_k and A_l with $A_k > A_l$ lie in the same interval, then



$$0 < A_k - A_l \le \frac{\pi}{4}.$$

$\Rightarrow \tan 0 < \tan (A_k - A_l) < \tan \frac{\pi}{4}$

[It is because tan function increases in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$]

i.e.,
$$0 < \frac{\tan A_k - \tan A_l}{1 + \tan A_k \tan A_l} < 1$$

$$0 < \frac{a_k - a_l}{1 + a_k a_l} < 1.$$

Hence, there are two real numbers $x = a_k$, $y = a_1$ such that $0 < \frac{x - y}{1 + xy} \le 1$.

Build-up Your Understanding 9

1. Prove that, among any 52 integers, two can always be found, such that the difference of their squares, is divisible by 100.
2. Show that, for any set of 10 points, chosen within a square, whose side is 3 units, there are two points, in the set, whose distance is at most $\sqrt{2}$.
3. There are 7 persons in a group, show that, some two of them, have the same number of acquaintances among them.
4. 51 points are scattered inside a square, with a side of one metre. Prove that some set of three of these points can be covered by a square, with side 20 cm.
5. Let $1 < a_1 < a_2 < a_3 < \ldots < a_{51} < 142$. Prove that, among the 50 consecutive differences $(a_i - a_{i-1})$ where $i = 1, 2, 3, \ldots, 51$, some value, must occur at least twelve times.
6. You are given 10 segments, such that, every segment is larger than 1 cm but shorter than 55 cm. Prove that, you can select three sides of a triangle, among these segments.
7. There are 9 cells in a 3×3 square. When these cells are filled by numbers 1, 2, 3 only, prove that, of the eight sums obtained, at least, two sums are equal.
8. Let there be given 9 lattice points in a 3-D Euclidean space. Show that, there is a lattice point, on the interior of one of the line segments joining two of these nine points.
9. Consider seven distinct positive integers, not exceeding 1706. Prove that, there are three of them, say a, b, c such that, $a < b + c < 4a$.

10. One million pine trees grow in a forest. It is known that, no pine tree, has more than 60000 pine needles in it. Show that, two pine trees in the forest must have the same number of pine needles.

11. In a circle of radius 16, there are placed 650 points; Prove that there exists a ring (annulus) of inner radius 2 and outer radius 3, which contains not less than 10 of the given points.

12. On a rectangular table of dimensions 120″ by 150″, we set 14001 marbles of size 1″ by 1″. Prove that, no matter how these are arranged, one can place a cylindrical glass with diameter of 5″ over atleast 8 marbles.

13. Let A be the set of 19 distinct integers, chosen from the AP 1, 4, 7, 10, …, 100. Prove that, there should be two distinct integers in A, such that, their sum is 104.

14. If a line is coloured in 11 colours, show that, there exist two points, whose distance apart, is an integer, which have the same colour.

15. Show that, given 12 integers, there exists two of them whose difference is divisible by 11.

16. Given eleven triangles, show that, some three of them belong to the same type (such as equilateral, isosceles, etc.)

17. A is a subset of the AP 2, 7, 12, …, 152. Prove that, there are two distinct elements of A whose sum is 159. What can you conclude if A has only 14 elements?

18. Given three points, in the interior of a right angled triangle, show that, two of them are at a distance not greater than the maximum of the lengths of the sides containing the right angle.

19. There are 90 cards numbered 10 to 99. A card is drawn and the sum of the digits of the number in the card is noted; show that if 35 cards are drawn, then, there are some three cards, whose sum of the digits are identical.

20. If in a class of 15 students, the total of the marks in a subject is 600, then show that, there is a group of 3 students, the total of whose marks is at least 120.

21. Let $ABCD$ be a square of side 20. Let T_i $(i = 1, 2, \dots, 2000)$ be points in the interior of the square, such that, no three points from the set $S = \{A, B, C, D\} \subset T_i \ \forall\ i = 1, 2, 3, \dots, 2000$ are collinear, Prove that, at least one triangle, with the vertices in S has area less than $\frac{1}{10}$.

22. 5 points are plotted inside a circle. Prove that, there exist two points, which form an acute angle with the centre of the circle.

23. Let A denote a subset of $\{1, 11, 21, 31, \dots, 551\}$ having the property that, no two elements of A, add up to 552. Prove that A cannot have more than 28 elements.

24. Prove that, there exist two powers 3, which differ by a multiple of 2005.

25. All the points in the plane are coloured, using three colours. Prove that, there exists a triangle with vertices, having the same colour, such that, either it is isosceles or its angles are in geometric progression.

Solved Problems

Problem 1 *In how many ways can a pack of 52 cards be*

(i) *distributed equally among four players in order?*

(ii) *divided into 4 groups of 13 cards each?*

(iii) *divided into four sets of 20, 15, 10, 7 cards?*

(iv) *divided into four sets, three of them having 15 cards each and the fourth having 7 cards?*

Solution:

(i) From 52 cards of the pack, 13 cards can be given to the first player in ${}^{52}C_{13}$ ways.
From the remaining 39 cards, 13 cards can be given to the second player in ${}^{39}C_{13}$ ways.
From the remaining 26 cards, 13 cards can be given to the third player in ${}^{26}C_{13}$ ways.
The remaining 13 cards can be given to the fourth player in ${}^{13}C_{13} = 1$ way.
By fundamental theorem, the number of ways of dividing 52 cards equally among four players $= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13} = \dfrac{52!}{13!39!} \times \dfrac{39!}{13!26!} \times \dfrac{26!}{13!13!} \times 1 = \dfrac{52!}{(13!)^4}$.

(ii) By standard result, the number of ways of forming 4 groups, each of 13 cards $= \dfrac{52!}{4!(13!)^4}$.

(iii) Here the sets have unequal number of cards, hence the required number of ways

$$= {}^{52}C_{20} \times {}^{32}C_{15} \times {}^{17}C_{10} \times {}^{7}C_{7} = \frac{52!}{20!32!} \times \frac{32!}{15!17!} \times \frac{17!}{10!7!} \times 1 = \frac{52!}{20!15!10!7!}.$$

(iv) By standard result, the required number of ways $= \dfrac{52!}{15!15!15!7!3!} = \dfrac{52!}{(15!)^3 .3!7!}$.

Problem 2 *Find the number of ways of filling three boxes (named A, B and C) by 12 or less number of identical balls, if no box is empty, box B has at least 3 balls and box C has at most 5 balls.*

Solution: Suppose box A has x_1 balls, box B has x_2 balls and box C has x_3 balls. Then,

$$x_1 + x_2 + x_3 \le 12,\ x_1 \ge 1,\ x_2 \ge 3,\ 1 \le x_3 \le 5$$

Let $x_4 = 12 - (x_1 + x_2 + x_3)$. Then

$$x_1 + x_2 + x_3 + x_4 = 12\ (1 \le x_1 \le 8,\ 3 \le x_2 \le 10,\ 1 \le x_3 \le 5 \text{ and } 0 \le x_4 \le 7)$$

The required number = Coefficient of x^{12} in

$$(x^1 + x^2 + \cdots + x^8)(x^3 + x^4 + \cdots + x^{10})(x^1 + x^2 + \cdots + x^5)(x^0 + x^1 + \cdots + x^7)$$

$=$ Coefficient of x^{12} in $(x + x^2 + x^3 + \cdots)(x^3 + x^4 + x^5 + \cdots)(x + x^2 + \cdots + x^5)(1 + x + x^2 + \cdots)$

$=$ Coefficient of x^7 in $(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + \cdots)$

$=$ Coefficient of x^7 in $(1 - x)^{-4}(1 - x^5)$
$=$ Coefficient of x^7 in $(1 - x^5)(1 + {}^4C_1 x + {}^5C_2 x^2 + {}^6C_3 x^3 + \cdots)$
$= {}^{10}C_7 - {}^5C_2 = 110$.

Problem 3 *A person writes letters to six friends and address the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that*

(i) *at least two of them are in the wrong envelopes?*
(ii) *all the letters are in the wrong envelopes?*

Solution:

(i) The number of all the possible ways of putting 6 letters into 6 envelopes is 6!.
There is only one way of putting all the letters correctly into the corresponding envelopes.
Hence if there is a mistake, at least 2 letters will be in the wrong envelope.
Hence the required answer is $6! - 1 = 719$.

(ii) Using the result of derangements, the required number of ways

$$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 720\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$$

$$= 360 - 120 + 30 - 6 + 1 = 265.$$

Problem 4 *Find the number of integers which lie between* 1 *and* 10^6 *and which have the sum of the digits equal to* 12.

Solution: Consider the product $(x^0 + x^1 + x^2 + \cdots + x^9)\,(x^0 + x^1 + x^2 + \cdots + x^9)\ldots 6$ factors. The number of ways in which the sum of the digits will be equal to 12 is equal to the coefficient of x^{12} in the above product. So, required number of ways = Coefficient of x^{12} in $(x^0 + x^1 + x^2 + \cdots + x^9)^6$.

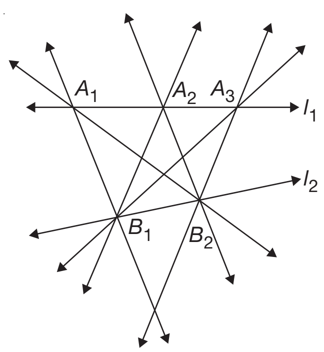
$$= \text{Coefficient of } x^{12} \text{ in } (1 - x^{10})^6\,(1 - x)^{-6}$$

$$= \text{Coefficient of } x^{12} \text{ in } (1 - x)^{-6}\,(1 - {}^6C_1\, x^{10} + \cdots)$$

$$= \text{Coefficient of } x^{12} \text{ in } (1 - x)^{-6} - {}^6C_1 \cdot \text{Coefficient of } x^2 \text{ in } (1 - x)^{-6}$$

$$= {}^{12+6-1}C_{6-1} - {}^6C_1 \times {}^{2+6-1}C_{6-1} = {}^{17}C_5 - 6 \times {}^7C_5 = 6062.$$

Problem 5 *Straight lines are drawn by joining m points on a straight line to n points on another line. Then excluding the given points, prove that the lines drawn will intersect at* $\frac{1}{2}mn(m-1)(n-1)$ *points.* (*No two lines drawn are parallel and no three lines are concurrent.*)



Solution: Let $A_1, A_2, \ldots, A_m$ be the points on the first line (say l_1) and let $B_1, B_2, \ldots, B_n$ be the points on the second line (say l_2). Now any point on l_1 can be chosen in m ways and any point on l_2 can be chosen in n ways. Hence number of ways of choosing a point l_1 and a point on l_2 is mn.

Hence number of lines obtained on joining a point on l_1 and a point on l_2 is mn. Now any point of intersection of these lines, which can be done in ${}^{mn}C_2$ ways. Hence number of point is ${}^{mn}C_2$. But some of these points are the given points and counted many times. For example, the point A_1 has been counted ${}^nC_2$ times. Hence required number of points is

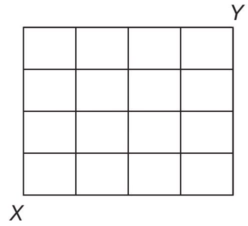
$${}^{mn}C_2 - m \cdot {}^nC_2 - n \cdot {}^mC_2 = \frac{1}{2}mn(m-1)(n-1)$$

Aliter: If we select two points from first line and two from the second line then we will have 2 required points from every such selection

Hence number of such points $= 2 \times \binom{m}{2} \times \binom{n}{2} = \frac{1}{2}mn(m-1)(n-1).$

Problem 6 *In the figure you have the road plan of a city. A man standing at X wants to reach the cinema hall at Y by the shortest path. What is the number of different paths that he can take?*

Solution: A path from X to Y is shown by dark line segments which corresponds $y\,x\,x\,x\,y\,y\,x\,y$. It is easy to see that any path of required type corresponds to an arrangement of x, x, x, x, y, y, y and y and vice versa. Hence required number of ways = number of arrangements of $4x$'s and $4y$'s, which is $\dfrac{\lfloor 8}{\lfloor 4\, \lfloor 4}$.



Problem 7 *Show that the number of combinations of n letters out of 3n letters of which n are a's, n are b's and the rest are unequal is* $(n + 2) \cdot 2^{n-1}$.

Solution: From n we have 0, 1, 2, 3 . . ., n From n we may have 0, 1, 2, 3 …, n, while for each of the rest n letters we may have 2 combinations 0 or 1. Thus the required number of combinations is thus

= Coefficient of x^n in

$$(1 + x + x^2 + \cdots + x^n)\,(1 + x + x^2 + \cdots + x^n)\,(1 + x)\,(1 + x) + \cdots (1 + x)$$

= Coefficient of x^n in $\dfrac{(1-x^{n+1})^2}{(1-x)^2}\cdot(1+x)^n$

= Coefficient of x^n in $(1 - x^{n+1})^2\,(1 + x)^n\,(1 - x)^{-2}$

Since $(1 - x^{n+1})^2$ will not contain x^n, we have required number of combinations

= Coefficient of x^n in $(1 + x)^n \cdot (1 - x)^{-2}$

= Coefficient of x^n in $[2 - (1 - x)]^n\,(1 - x)^{-2}$

= Coefficient of x^n in $2^n\,(1 - x)^{-2} - {}^nC_1\, 2^{n-1}\,(1 - x)^{-1} + {}^nC_1.\, 2^{n-2} \cdot (1 - x)^0$

$- {}^nC_3 \cdot 2^{n-3}\,(1 - x) + \cdots + (-1)^n \cdot {}^nC_n\,(1 - x)^{n-2}$

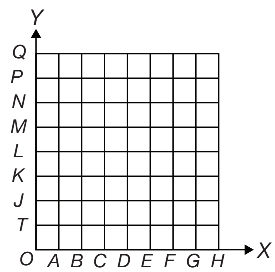
= Coefficient of x^n in $2^n\,(1 - x)^{-2} - n \cdot 2^{n-1} \cdot (1 - x)^{-1}$

$= 2^n\dfrac{(n+1)!}{n!} - n\cdot 2^{n-1} = 2^n\cdot(n+1) - n\cdot 2^{n-1} = 2^{n-1}\cdot(n+2).$

Problem 8 *Show that the number of rectangles of any size on a chess board is* $\sum_{k=1}^{8} k^3$.

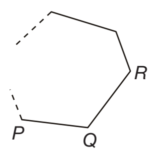
Solution: A rectangle can be fixed on the chess board if and only if we fix two points on x-axis and two points on y-axis. For example, in order to fix the rectangle $RSTU$, we fix B and G on x-axis and K and M on y-axis and vice-versa.

Hence total number of rectangles on the chess board is the number of ways of choosing two points on x-axis (which can be done in ${}^9C_2$ ways) and two points on y-axis (which can also be done is ${}^9C_2$ ways). Hence require number is $({}^9C_2)^2 = \sum_{k=1}^{8} k^3$.



Problem 9 *Find the number of triangles whose angular points are at the angular points of a given polygon of n sides. but none of whose sides are the sides of the polygon.*

Solution: A n-sided polygon has n angular points. Number of triangles formed from these n angular points = ${}^nC_3$. But it also includes the triangles with sides on the polygon.



Let us consider a side PQ. If each angular point of the remaining $(n - 2)$ points is joined with PQ, we get a triangle with one side PQ.

$\therefore$ Number of triangles with PQ as one side = $n - 2$. In similar ways n sides like QR can be considered. Hence number of triangle = $n\,(n - 2)$. But some triangles have been counted twice. For example, PQ side with R gives ΔPQR. and QR side with P gives same ΔPQR.

Number of such triangles = n

[As for each side, one triangle is repeated. Hence for n sides, n triangle's have been counted more.]

Hence, the number of triangles of which one side is the side of the triangle $= n(n-2) - n = n(n-3)$

Hence number of required triangles

$$= {}^nC_3 - n(n-3) = \frac{n(n-1)(n-2)}{6} - n(n-3) = \frac{n}{6}(n^2 - 9n + 20) = \frac{n}{6}(n-4)(n-5).$$

Problem 10 *Find the number of all whole numbers formed on the screen of a calculator which can be recognized as numbers with (unique) correct digits when they are read inverted. The greatest number formed on its screen is 999999.*

Solution: The digits 0, 1, 2, 5, 6, 8 and 9 can be recognized as digits when they are seen inverted hence number can contain these digits only.

Note that number can be of 1 digit to 6 digit number. But in more than one digit numbers, 0 cannot come in first place and also in unit place (Imagine inverted case).

Number of digits	Total numbers
1	7
2	$6 \times 6 = 36$
3	$6 \times 7 \times 6 = 252$
4	$6 \times 7^2 \times 6 = 1764$
5	$6 \times 7^3 \times 6 = 12348$
6	$6 \times 7^4 \times 6 = 86436$
	Total = 100843

Problem 11 *Find the number of positive integral solutions of* $x + y + z + w = 20$ *under the following conditions:*

(i) *Zero values of x, y, z, w are included.*
(ii) *Zero values are excluded.*
(iii) *No variable may exceed 10; zero values excluded*
(iv) *Each variable is an odd number.*
(v) $0 < x < y < z < w$.

Solution:

(i) $x + y + z + w = 20; x \geq 0, y \geq 0, z \geq 0, w \geq 0$
Coefficient of a^{20} in $(a^0 + a^1 + a^2 + \cdots)^4$
$= (1 - a)^{-4} = {}^{20+4-1}C_{20}$
$= {}^{23}C_3 = 1771$

Note: You can directly use the result ${}^{n+r-1}C_{r-1}$ or ${}^{n+r-1}C_n$

(ii) Number of ways = Coefficient of a^{20} in $(a + a^2 + a^3 + \cdots)^4$
= Coefficient of a^{20} in $a^4 (1 - a)^{-4}$
= Coefficient of a^{16} in $(1 - a)^{-4} = {}^{19}C_{16}$
$= 969$.

Note: that you can directly use ${}^{n-1}C_{r-1}$

(iii) If no variable exceeds 10, then sum of rest should be less than or equal to 10 [as $20 - 10 = 10$]

Let $x \le 10$, then $y + z + w \ge 10$

and $\quad \max(y + z + w) = 20 - \min(x)$

$\max(y + z + w) = 20 - 1 = 19$

$\therefore\ 10 \le y + z + w \le 19$ [where $y \ge 1, z \ge 1, w \ge 1$]

$\Rightarrow 10 \le (y_1 + 1) + (z_1 + 1) + (w_1 + 1) \le 19$

$\Rightarrow 7 \le y_1 + z_1 + w_1 \le 16;\ 0 \le y_1 \le 9,\ 0 \le z_1 \le 9,\ 0 \le w_1 \le 9$

Number of solutions = (Number of solutions of $y_1 + z_1 + w_1 < 16$) – (Number of solutions of $y_1 + z_1 + w_1 \le 6$)

Now,

Number of solutions of $y_1 + z_1 + w_1 \le 16$ can be obtained by adding a dummy variable x_1 ($x_1 \ge 0$) such that $x_1 + y_1 + z_1 + w_1 = 16$.

Number of solutions = Coefficient of x^{16} in $(1 - x^{10})^3 (1 - x)^{-4} = {}^{19}C_4 - 3\,{}^{9}C_3$

Again,

Number of solutions of $y_1 + z_1 + w_1 \le 6$ can be obtained by adding a dummy variable l_1 ($l_1 \ge 0$) such that $l_1 + y_1 + z_1 + w_1 = 6$

Number of solutions = Coefficient of x^6 in $(1 - x^{10})^3 (1 - x)^{-4} = {}^{9}C_3$

Hence, Total number of solutions = ${}^{19}C_3 - 4\,{}^{9}C_3 = 633$.

(iv) Each variable is an odd number.

$\therefore\ x = 2x_1 + 1 \quad y = 2y_1 + 1$

$z = 2z_1 + 1 \quad w = 2w_1 + 1$ [where $x_1, y_1, z_1, w_1 \ge 0$]

$x + y + z + w = 20$

$\Rightarrow (2x_1 + 1) + (2y_1 + 1) + (2z_1 + 1) + (2w_1 + 1) = 20$

$2x_1 + 2y_1 + 2z_1 + 2w_1 = 16$

$\Rightarrow x_1 + y_1 + z_1 + w_1 = 8$ [where $x_1, y_1, z_1, w_1 \ge 0$]

Number of solutions $= {}^{8+4-1}C_{4-1}$

$= {}^{11}C_3 = 165$

(v) Assume $0 < x < y < z < w$

Let $x = x_1$

$y = x + x_2 = (x_1) + x_2$

$z = y + x_3 = (x_1 + x_2) + x_3$

$w = z + x_4 = (x_1 + x_2 + x_3) + x_4$ [where $x_1, x_2, x_3, \ge 1$]

$x + y + z + w = 20$

$\Rightarrow x_1 + (x_1 + x_2) + (x_1 + x_2 + x_3) + (x_1 + x_2 + x_3 + x_4) = 20$

$4x_1 + 3x_2 + 2x_3 + x_4 = 20$ (1) [where $x_1, x_2, x_3, x_4 \ge 1$]

Let us again change the variables

$x_1 = y_1 + 1;\ x_2 = y_2 + 1;\ x_3 = y_3 + 1;\ x_4 = y_4 + 1$ [where $y_1, y_2, y_3, y_4 \ge 0$]

Substituting above values in Eq. (1), we get

$4(y_1 + 1) + 3(y_2 + 1) + 2(y_3 + 1) + (y_4 + 1) = 20.$

$\Rightarrow \quad 4y_1 + 3y_2 + 2y_3 + y_4 = 10$ [where $y_1, y_2, y_3, y_4 \ge 0$]

Y_1	$3y_2 + 2y_3 + y_4$	Number of solutions
0	10	14 (Use Table-1)
1	6	7 (Use Table-2)
2	2	2 (Use Table-3)

Total Number of solutions = 23

Table 1

$3y_2 + 2y_3 + y_4 = 10$		
y_2	$2y_3 + y_4$	Number of solutions
0	10	6
1	7	4
2	4	3
3	1	1
		14

Table 2

$3y_2 + 2y_3 + y_4 = 6$		
y_2	$2y_3 + y_4$	Number of solutions
0	6	4
1	3	2
2	0	1
		7

Table 3

$3y_2 + 2y_3 + y_4 = 6$		
y_2	$2y_3 + y_4$	Number of solutions
0	2	2
		2

Problem 12 *There are* 12 *seats in the first row of a theater of which* 4 *are to be occupied. Find the number of ways of arranging 4 persons so that:*

(i) *no two persons sit side by side.*
(ii) *there should be atleast 2 empty seats between any two persons.*
(iii) *each person has exactly one neighbour.*

Solution:

(i) We have to select 4 seats for 4 persons so that no two persons are together. It means that there should be atleast one empty seat vacant between any two persons.
To place 4 persons we have to put 4 seats between the remaining 8 empty seats so that all persons should be separated.
Between 8 empty seats 9 gaps are available for 4 seats to put.
We can select 4 gaps in ${}^9C_4$ ways.
Now we can arrange 4 persons on these 4 seats in 4! ways. So total number of ways to give seats to 4 persons so that no two of them are together
$= {}^9C_4 \times 4! = {}^9P_4 = 3024.$

(ii) Let x_0 denotes the empty seats to the left of the first person, x_i (i = 1, 2, 3) be the number of empty seats between ith and $(i + 1)$th person and x_4 be the number of empty seats to the right of 4th person.
Total number seats are 12. So we can make this equation :

$$x_0 + x_1 + x_2 + x_3 + x_4 = 8 \qquad (1)$$

Number of ways to give seats to 4 persons so that there should be two empty seats between any two persons is same as the number of integral solutions of the Eq. (1) subjected to the following conditions.

Conditions on x_1, x_2, x_3, x_4:
According to the given condition, these should be two empty seats between any two persons. That is,
Min $(x_i) = 2$ for $i = 1, 2, 3$ and Min $(x_0) = 0$
Max $(x_0) = 8 -$ Min $(x_1 + x_2 + x_3 + x_4) = 8 - (2 + 2 + 2 - 0) = 2$
Max $(x_4) = 8 -$ Min $(x_0 + x_1 + x_2 + x_3) = 8 - (2 + 2 + 2 - 0) = 2$
Similarly,
Max $(x_i) = 4$ for $i = 1, 2, 3$
Number of integral solutions of the equation (i) subjected to the above condition
= Coefficient of x^8 in the expansion of $(1 + x + x^2)^2\,(x^2 + x^3 + x^4)^3$
= Coefficient of x^8 in $x^6\,(1 + x + x^2)^5$
= Coefficient of x^2 in $(1 - x^3)^5\,(1 - x)^{-5}$
= Coefficient of x^2 in $(1 - x)^{-5}$
$= {}^{5+2-1}C_2 = {}^6C_2 = 15.$

Number of ways to select 4 seats so that there should be atleast two empty seats between any two persons = 15. But 4 persons can be arranged in 4 seats in 4! ways. So total number of ways to arrange 4 persons in 12 seats according to the given condition = $15 \times 4! = 360$.

(iii) As every person should have exactly one neighbour, divide 4 persons into groups consisting two persons in each group.

Let G_1 and G_2 be the groups in which 4 persons are divided.

According to the given condition G_1 and G_2 should be separated from each other.

Number of ways to select seats so that G_1 and G_2 are separated = ${}^{8+1}C_2$

But 4 persons can be arranged in 4 seats in 4! ways.

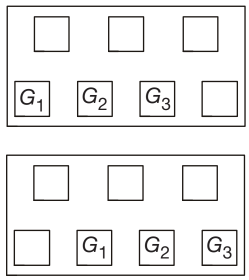
So total number of ways to arrange 4 persons so that every person has exactly one neighbour = ${}^{9}C_2 \times 4! = 864$

Problem 13 *In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats?*

Solution:

(i) Out of 14 seats (7 in each Van), we have to select 12 seats for 3 girls and 9 boys. 12 seats from 14 available seats can be selected in ${}^{14}C_{12}$ ways.
Now on these 12 seats we can arrange 3 girls and 9 boys in 12! ways.
So total number of ways ${}^{14}C_{12} \times 12! = 91 \times 12!$

(ii) One van out of two available can be selected in ${}^{2}C_1$ ways.



Out of two possible arrangements (see figure) of adjacent seats, select one in ${}^{2}C_1$ ways. Out of remaining 11 seats, select 9 seats for 9 boys in ${}^{11}C_9$ ways.

Arrange 3 girls on 3 seats in 3! ways and 9 boys on 9 seats 9! ways.

So possible arrangement of sitting (for 3 girls and 9 boys in 2 vans) is:
${}^{2}C_1 \times {}^{2}C_1 \times {}^{11}C_9 \times 3! \times 9! = 12!$ ways.

Problem 14 *How many seven-letters words can be formed by using the letter of the word SUCCESS so that:*

(i) *the two C are together but not two S are together?*
(ii) *no two C and no two S are together?*

Solution:

(i) Considering CC as single object, U, CC, E can be arranged in 3! ways.
X U X C C X E X
Now the three S are to be placed in the 4 available places (X) so that C C are not separated but S are separated.
Number of ways to place S S S = (No of ways to select 3 places) $\times 1 = {}^{4}C_3 \times 1 = 4$
$\Rightarrow$ Number of words = $3! \times 4 = 24$.

(ii) Let us first find the words in which no two S are together. To achieve this, we have to do following operations.

(a) Arrange the remaining letter U C C E in $\dfrac{4!}{2!}$ ways.

(b) Place the three S S S in any arrangement from (a)

X U X C X C X E X

There are five available places for three S S S.

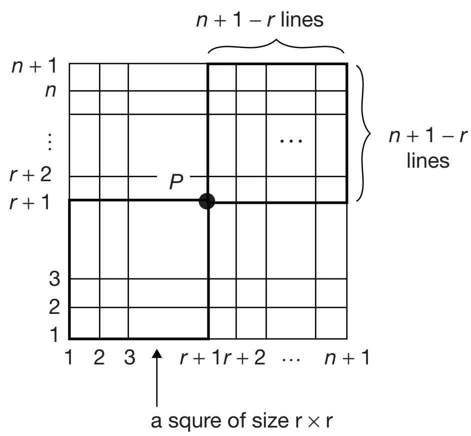
Number of placements = ${}^5C_3$

Hence total number of words with no two S together $= \dfrac{4!}{2!}\,{}^5C_3 = 120.$

Number of words having C C separated and S S S separated = (Number of words having S S S separated) – (Number of words having S S S separated but C C together) = 120 – 24 = 96 [using result of part (i)].

Problem 15 *A square of n units by n units is divided into n^2 squares each of area 1 sq. units. Find the number of ways in which 4 points (out of $(n + 1)^2$ vertices of unit squares) can be chosen so that they form the vertices of a square.*

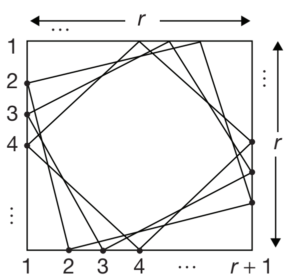
Solution:



We can easily see that number of squares of size $r \times r$ with its sides along the horizontal and vertical lines is equal to number of positions of P on the lattice points formed by $(n + 1 - r)$ horizontal and $(n + 1 - r)$ vertical lines which is $(n + 1 - r) \times (n + 1 - r)$.

$\Rightarrow$ Number of squares of size $r \times r = (n + 1 - r)^2$

In addition to these squares there are squares whose sides are not parallel to horizontal/vertical lines. Each of these squares is inscribed in some previously counted squares. So we will first count how many are inscribed in our $r \times r$ size square. Then we will sum over 'r'.



From the adjacent figure we can see that these are r inscribed squares, including the $r \times r$ square itself.

Now total number of squares

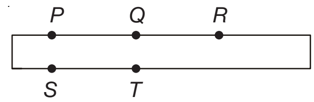
$$= \sum_{r=1}^{n} r(n+1-r)^2$$

$$= \sum_{r=1}^{n} (n+1-r)\cdot r^2 = (n+1)\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r^3$$

$$= \frac{(n+1)n(n+1)(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2$$

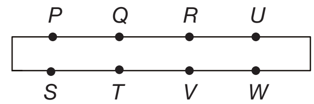
$$= \frac{n(n+1)^2(n+2)}{12}$$

Problem 16 *A boat's crew consists of 8 men, 3 of whom can only row on one side and 2 only on the other. Find the number of ways in which the crew can be arranged.*



Solution: Let the man P, Q, R, S, T, U, V, W and suppose P, Q, R can row only on one side and S, T on the other as represented in the figure.
Then, since 4 men must row on each side, of the remaining 3, one must be placed on the side of P, Q, R and the other two on the side S, T; and this can evidently be done in 3 ways, for we can place any one of the three side of P, Q, R.

Now 3 ways of distributing the crew let us first consider one, say that in which *U* is on the side of P, Q, R as shown in the figure.



Now, P, Q, R, U can be arranged in 4! ways and S, T, V, W can be arranged in 4! ways.

Hence total number of ways arranging the men = 4! × 4! = 576

Hence the number of ways of arranging the crew = 3 × 576 = 1728.

Problem 17 *How many integers between* 1 *and* 1000000 *have the sum of the digits equal to* 18.

Solution: Integers between 1 and 1000000 will be, 1,2, 3, 4, 5 or 6-digits numbers, and given sum of digits = 18
Thus we need to obtain the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18 \qquad (1)$$

Where $0 \leq x_i \leq 9$, i = 1, 2, 3, 4, 5, 6

Therefore, the number of solutions of the Eq. (1), will be

= Coefficient of x^{18} in $(x^0 + x^1 + x^2 + x^3 + \cdots + x^9)$

= Coefficient of x^{18} in $\left(\dfrac{1-x^{10}}{1-x}\right)^6$

= Coefficient of x^{18} in $(1 - x^{10})^6\,(1 - x)^{-6}$

= Coefficient of x^{18} in $(1 - 6x^{10})\,(1 - x)^{-6}$

$= {}^{6+18-1}C_{18} - 6 \cdot {}^{6+8-1}C_8$

$= {}^{23}C_{18} - 6 \cdot {}^{13}C_8 = {}^{23}C_5 - 6 \cdot {}^{13}C_5$

= 33649 – 7722 = 25927.

Problem 18 *How many three digit numbers are of the form xyz with x < y; z < y and* $x \neq 0$.

Solution: Since, $x \geq 1$, then $y \geq 2$ ($\therefore x < y$)
If $y = n$ then n take the values from 1 to $n - 1$ and z can take the value from 0 to $n - 1$ (*i.e.*, n values) thus for each value of y $(2 < y < 9)$, x and z take $n(n - 1)$ values.

Hence, the 3-digit numbers are of the from *xyz*

$$= \sum_{n=2}^{9} n(n-1) = \sum_{n=1}^{9} n(n-1)\left\{\sum 1 \times (1-1) = 0\right\}$$

$$= \sum_{n=1}^{9} n^2 - \sum_{n=1}^{9} n == \frac{9(9+1)\,(18+1)}{6} - \frac{9(9+1)}{2}.$$

$$= 285 - 45 = 240.$$

Problem 19 *Find the number of polynomials of the form* $x^3 + ax^2 + bx + c$ *which are divisible by* $x^2 + 1$ *and where a, b, c belong to* (1, 2,…, *n*).

Solution: Let $f(x) = x^3 + ax^2 + bx + c$ be the polynomial divisible by $x^2 + 1$ or $(x + i)(x - i)$.

$$f(i) = 0 \quad \Rightarrow \quad i^3 + ai^2 + bi + c = 0$$
$$(b - 1)\, i + (c - a) = 0$$
$$b - 1 = 0 \text{ and } c - a = 0$$
$$b = 1,\ c = a$$

Hence, number of polynomials = Number of values which a or c can take.

As a or c can takes n values, therefore number of polynomials = n.

Problem 20 *John has x children by his first wife. Mary has* $(x + 1)$ *children by her first husband. They marry and have children of their own. The whole family has 24 children. Assuming that two children of the same parents do not fight. Prove that the maximum number of fights thats can take place in* 191.

Solution: Let number of children of John and Mary are y and No. of children of John and his first wife is x. Hence, number of children of Mary from his first husband are $(x + 1)$.

$$x + x + 1 + y = 24 \tag{1}$$

Total number of fights between two children subject to the condition that any children of same parents do not flight.

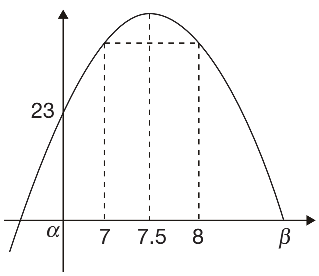
$$N(x) = {}^{24}C_2 - \left[{}^{x}C_2 + {}^{x+1}C_2 + {}^{y}C_2\right]$$

$$N(x) = 276 - \left[\frac{x(x-1) + (x+1)x}{2} + {}^{y}C_2\right]$$

$$= 276 - \left[x^2 + \frac{y(y-1)}{2}\right]$$

$$= 276 - \left[x^2 - \frac{(23-2x)(22-2x)}{2}\right] \quad \text{[using Eq. (1)]}$$

$$N(x) = 276 - (3x^2 - 45x + 253) = -3x^2 + 45x + 23$$



Maximum value of $N(x)$ can occur at $x = -\dfrac{(45)}{2(-3)} = 7.5$

But $x \in I$ hence $x = 7$ or 8 [as Graph is symmetrical about $x = 7.5$]

Maximum value $= 23 - 3(7)^2 + 45(7)$

$= 191.$

Problem 21 *There are 2n guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must not be placed next to one another, find the number of ways in which the company can be placed.*

Solution: Let the M and M' represent seats of the master and mistress respectively, and let $a_1, a_2 \ldots, a_{2n}$ represent the $2n$ seats.

Let the guests who must not be placed next to one another be called P and Q.

Now put P at a_1, and Q at any position, other than a_2, say at a_3; then remaining $2n - 2$ guests can be positioned in $(2n - 2)!$ ways. Hence there will be altogether $(2n - 2)(2n - 2)!$ arrangements of the guests when P is at a_1.

The same number of arrangements when P is at a_n or a_{n+1} or a_{2n}.

Hence, for these position $(a_1, a_n, a_{n+1}, a_{2n})$ of P, there are altogether in 4 $(2n - 2)$ $(2n - 2)!$ ways. (1)

If P is at a_2 there are altogether $(2n - 3)$ positions for Q.

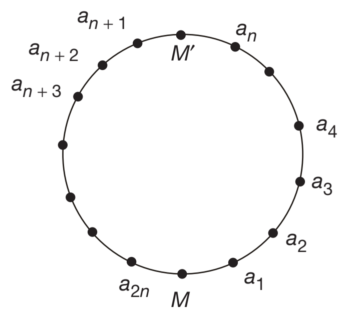
Hence, there will be altogether $(2n - 3)$ $(2n - 2)!$ arrangements of the guests when P is at a_2.

The same number of arrangements can be made when P is at any other position except the four position $a_1, a_n, a_{n+1}, a_{2n}$.

Hence, for these $(2n - 4)$ positions of P there will be altogether in $(2n - 4)$ $(2n - 3)$ $(2n - 2)!$ arrangements of the guests. (1)

Hence, from Eqs. (1) and (2), the total number of ways of arranging the guests

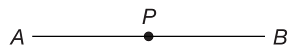
$$= 4(2n - 2)\,(2n - 2)! + (2n - 4)\,(2n - 3)\,(2n - 2)!$$
$$= (4n^2 - 6n + 4)\,(2n - 2)!$$



Problem 22 *There are n straight lines in a plane, no two of which are parallel and no three passes through the same point. Their point of intersection are joined. Show that the number of fresh lines thus introduced is:*

$$\frac{1}{8}n(n-1)(n-2)(n-3)$$

Solution: Let AB be any one of the n straight lines and suppose it is intersected by some other straight line CD at P.



Then it is clear that AB contains $(n - 1)$ points of intersection because it is intersected by the remaining $(n - 1)$ straight lines in $(n - 1)$ different points. Hence, the aggregate number of points contained in the n straight lines $= n\,(n - 1)$. But in making up this aggregate each point has evidently been counted twice. For instance, the point P has been counted one among the points situated on AB and again among those on CD.

Hence, the actual number of points $= \dfrac{n(n-1)}{2}$

Now we have to find the number of new lines formed by joining these points. The number of new lines passing through P is evidently equal to the number of points lying outside the lines AB and CD for we get a new lines joining P with each of these points only.

Now, since, each of the lines AB and CD contained $(n - 2)$ points besides the point P, the number of points situated on AB and CD.

$$= 2(n - 2) + 1 = (2n - 3)$$

Thus, the number of points outside AB and CD are $\dfrac{n(n-1)}{2}(2n-3)$ = The number of new lines passing through P and similarly through each other points.

So, the aggregate number of new lines passing through the points.

$$= \frac{n(n-1)}{2}\left\{\frac{n(n-1)}{2} - (2n-3)\right\}$$

But in the making up this aggregate every new line is counted twice; for instance if Q be one of the points outside AB and CD, the line PQ is counted once among the lines passing through P and again among these passing through Q.

Hence, actual number of fresh lines introduced

$$= \frac{1}{2}\left[\frac{n(n-1)}{2}\left\{\frac{n(n-1)}{2} - (2n-3)\right\}\right]$$

$$= \frac{1}{8}n(n-1)(n-2)(n-3).$$

Problem 23 *Let set $S = \{a_1, a_2, a_3, \ldots, a_{12}\}$ where all twelve elements are distinct, we want to form sets each of which contains one or more of the elements of set S (including the possibility of using all the elements of S). The only restriction is that the subscript of each element in a specific set must be an integral multiple of the smallest subscript in the set. For example, $\{a_2, a_6, a_8\}$ is one acceptable set, as is $\{a_6\}$. How many such sets can be formed? Can you generalize the result?*

Solution: Every (positive) integer is a multiple of 1.

So, we will first see a set consisting of a_1 and other elements:

There are 11 elements other than a_1. So the set with a_1 and another element, with one other element, 2 other elements, and all the 11 other elements, … and all the 11 other elements, *i.e.*, we have to choose a_1 and 0, 1, 2,…, 11 other elements out of a_2, a_3, …, a_{12}. This could be done in $\binom{11}{0}+\binom{11}{1}+\cdots+\binom{11}{11} = 2^{11}$ ways.

If a set contains a_2, as the element with the least subscript, then besides a_2, the set can have a_4, a_6, a_8, a_{10}, a_{12} elements, none or one or more of them. This could be done in $\binom{5}{0}+\binom{5}{1}+\cdots+\binom{5}{5} = 2^5$ ways.

Similarly, for having a_3 as the element with the least subscript 3, we have a_6, a_9, a_{12} to be the elements such that the subscripts (6, 9, 12) are divisible by 3.

So, the number of subsets with a_3 as one element is ${}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3$.

For a_4, one of the elements, the number of subsets (other elements being a_8 and a_{12}) is 2^2.

For a_5 it is 2^1 (there is just an element a_{10} such that 10 is a multiple of 5).

For a_6, it is again 2^1 (as 6/12)

For a_7, a_8, a_9, a_{10}, a_{11} and a_{12}, there is just one subset, namely, the set with these elements. This is total up to 6.

So, the total number of acceptable set according to the condition is

$$2^{11} + 2^5 + 2^3 + 2^2 + 2^1 + 2^1 + 6$$
$$= 2048 + 32 + 8 + 4 + 2 + 2 + 6 = 2102$$

If there are n elements in the set a_1, a_2, a_3, …, a_n then there are n multiples of 1.

$\left\lfloor \frac{n}{2} \right\rfloor$ multiples of 2

$\left\lfloor \frac{n}{3} \right\rfloor$ multiples of 3

… … … … … … …

… … … … … … …

$\left\lfloor \frac{n}{n} \right\rfloor$ multiples of n

So that the total number of such sets is given by

$$2^{n-1}+2^{\left\lfloor \frac{n}{2}\right\rfloor -1}+2^{\left\lfloor \frac{n}{3}\right\rfloor -1}+\cdots+2^{\left\lfloor \frac{n}{n}\right\rfloor -1}.$$

Problem 24 *Find the number of* 6*-digit natural numbers where each digit appears at least twice.*

Solution: We consider numbers like 222222 or 233200 but not 212222, since the digit 1 occurs only once.

The set of all such 6-digits can be divided into the following classes.

S_1 = the set of all 6-digit numbers where a single digit is repeated six times.

$n(S_1) = 9$, since '0' cannot be a significant number when all its digits are zero.

Let S_2 be the set of all 6-digit numbers, made up of three distinct digits.

Here we should have two cases: $S_2(a)$ one with the exclusion of zero as a digit and other $S_2(b)$ with the inclusion of zero as a digit.

$\boldsymbol{S_2(a)}$**:** The number of ways, three digits could be chosen from 1, 2, …, 9 is ${}^9C_3$. Each of these three digits occurs twice. So, the number of 6-digit numbers in this case is

$$= {}^9C_3 \times \frac{6!}{2!\times 2!\times 2!} = \frac{9\times 8\times 7}{1\times 2\times 3}\times\frac{720}{8} = 9\times 8\times 7\times 15 = 7560.$$

$\boldsymbol{S_2(b)}$**:** The three digits used include one zero, implying, we have to choose the other two digits from the 9 non-zero digits.

This could be done in ${}^9C_2 = \dfrac{9\times 8}{1.2} = 36.$ Since zero cannot be the leading digit, so let us fix one of the fixed non-zero numbers in the extreme left. Then the other five digits are made up of two zeroes, two fixed non-zero numbers and another non-zero number, one of which is put in the extreme left.

In this case the number of 6-digit numbers that could be formed is $\dfrac{5!}{2!\times 2!\times 1!}\times 2$ (since from either of the pairs of fixed non-zero numbers, one can occupy the extreme left digit) = 60.

So, the total number in this case = 36 × 60 = 2160.

$\therefore \quad n(S_2) = n(S_2a) + n(S_2b) = 7560 + 2160 = 9720.$

Now, let S_3 be the set of 6-digit numbers whose digits are made up of two distinct digits each of which occurs thrice. Here again, there are two cases: $S_3(a)$ excluding the digit zero and $S_3(b)$ including the digit zero.

$\boldsymbol{S_3(a)}$ is the set of 6-digit numbers, each of whose digits are made up of two non-zero digits each occurring thrice.

$$\therefore\; n[S_3(a)] = {}^9C_2 \times \frac{6!}{3!\times 3!} = 36\times 20 = 720.$$

$\mathbf{S_3}(\boldsymbol{b})$ consists of 6-digit numbers whose digits are made up of three zeroes and one of non-zero digit, occurring thrice. If you fix one of the nine non-zero digit, use that digit in the extreme left. This digit should be used thrice. So in the remaining 5 digits, this fixed non-zero digit is used twice and the digit zero occurs thrice.

So, the number of 6-digit numbers formed in this case is

$$9\times\frac{5!}{3!\times 2!} = 90.$$

$\therefore \quad n(S_3) = nS_3(a) + nS_3(b) = 720 + 90 = 810.$

Now, let us take S_4, the case where the 6-digit number consists of exactly two digits, one of which occurs twice and the other four times.

Here again, there are two cases: $S_4(a)$ excluding zero and $S_4(b)$ including zero.

***$S_4(a)$*:** If a and b are the two non-zero numbers, then when a is used twice and b is four times, we get $\dfrac{6!}{2! \times 4!}$ and when a is used four times and b is used twice, we again get $\dfrac{6!}{4! \times 2!}$.

So, when two of the nine non-zero digits are used to form the 6-digit number in this case, the total numbers formed is

$$^9C_2 \times 2 \times \frac{6!}{4! \times 2!} = 36 \times 5 \times 6 = 1080.$$

Thus, $\quad n[S_4(a)] = 1080.$

***$S_4(b)$*:** In this case we may use four zeroes and a non-zero number twice or two zeroes and a non-zero number four times.

In the former case, assuming the one of the fixed non-zero digit occupying the extreme left, we get the other five digits consisting of four zeroes and one non-zero number.

This results in $9 \times \dfrac{5!}{4! \times 1!}$ 6-digit numbers.

When we use the fixed non-zero digit four times and use zero twice, then we get $9 \times \dfrac{5!}{3! \times 2!} = 90$ six-digit numbers, as the fixed number occupies the extreme left and for the remaining three times it occupies 3 of the remaining digits, other digits being occupied by the two zeroes.

So, $\quad n(S_4) = n[S_4(a)] + n[S_4(b)]$

$$= 1080 + 45 + 90 = 1215.$$

Hence, the total number of 6-digit numbers satisfying the given condition

$$= n(S_1) + n(S_2) + n(S_3) + n(S_4)$$
$$= 9 + 720 + 810 + 1215$$
$$= 2754.$$

Problem 25 *Let $X = \{1, 2, 3, \ldots, n\}$, where $n \in N$. Show that the number of r combinations of X which contain no consecutive integers is given by*

$$\binom{n-r+1}{r} \textit{ where } 0 \le r \le \frac{n+1}{2}.$$

Solution: Each such r combination can be represented by a binary sequence b_1, b_2, b_3, …, b_n where $b_i = 1$, if i is a member of the r combination and 0, otherwise with no consecutive b_i's = 1 (the above r combinations contain no consecutive integers). The number of 1's in the sequence is r.

Now, this amounts to counting such binary sequences.

Now, look at the arrangement of the following boxes and the balls in them.

1	2	3	4	5	6	7
00	000	00	0000	0	0	000

00	000	00	0000	0	0	000

Here, the balls stand for the binary digits zero, and the boundaries on the left and right of each box can be taken as the binary digit one. In this display of boxes and balls as interpreted gives previously how we want the binary numbers. Here, there are 7 boxes, and 6 left/right boundary for the boxes (stating from 2 to 6). So, this is an illustration of 6 combinations of non-consecutive numbers.

The reason for zeroes in the front and at the end is that we can have leading zeroes and trailing zeroes in the binary sequence $b_1, b_2, \ldots, b_n$.

Now, clearly finding the r combination amounts to distribution of $(n-r)$ balls into $(r+1)$ distinct boxes [$(n-r)$ balls $= (n-r)$ zeroes as these are r ones, in the n number sequence] such that the 2nd, 3rd, …, rth boxes are non-empty. (The first and the last boxes may or may not be empty—in the illustration 1st and the 7th may have zeroes or may not have balls as we have already had six combinations!). Put $(r-1)$ balls one in each of 2nd, 3rd, …, rth boxes, (so that no two 1's occur consecutively). Now we have $(n-r)-(r-1)$ balls to be distributed in $(r+1)$ distinct boxes.

This could be done in $\binom{[(n-r)-(r-1)+(r-1)+1]}{[(n-r)-(r-1)]}$ ways,

i.e., $\binom{n-r+1}{n-2r+1}$ ways which is equal to

$$\binom{n-r+1}{(n-r+1)-(n-2r+1)} = \binom{n-r+1}{r} \text{ ways.}$$

Here $(n-2r+1)$ is the number of that of identical objects (zeroes of the binary representation) and (the distinct boxes is $(r+1-1) = r$. Thus, we apply the formula for distributing r identical objects in n distinct boxes as given by $\binom{n-r+1}{r}$.

[Distribution formula]

Problem 26 *Let $S = \{1, 2, 3, \ldots, (n+1)\}$, where $n \geq 2$ and let $T = \{(x, y, z) / x, y, z \in S, x < z, y < z\}$. By counting the members of T in two different ways, prove that* $\sum_{k=1}^{n} k^2 = \binom{n+2}{2} + 2\binom{n+1}{3}$.

Solution: T can be written as $T = T_1 \cup T_2$, $T_1 = \{(x, y, z) \mid x, y, z \in S, x = y < z)$ and $T_2 = \{(x, y, z) \mid x, y, z \in S, x \neq y, x, y < z\}$.

The number of elements in T_1 is the same as choosing two elements from the set S, where $n(S) = (n+1)$, *i.e.*, $n(T_1) = \binom{n+1}{2}$, (as every subset of two elements, the larger element will be z and the smaller will be x and y.)

In T_2 we have $2\binom{n+1}{3}$ elements, after choosing three elements from the set S, two of the smaller elements will be x and y and they may be either taken as (x, y, z) or as (y, x, z) or in other words, every three element subset of S, say $\{a, b, c\}$, the greatest is z, and the other two can be placed in two different ways in the first two positions,

$$\therefore\ n(T)(\text{or}\,|T|) = \binom{n+1}{2} + 2\binom{n+1}{3}$$

T, can also be considered as $\bigcup_{i=2}^{n+1} S_i$, where $S_i = \{(x, y, i) / x, y < i, x, y \in S\}$. All these sets are pair-wise disjoint as for different i, we get different ordered triplets (x, y, i).

Now in S_i the first two components of (x, y, i) namely (x, y), can be any element from me set $1, 2, 3, \ldots, i - 1$.

x and y can be any member from $1, 2, 3, \ldots, (i - 1)$, equal or distinct.

$\therefore$ The number of ways of selecting (x, y), $x, y \in \{1, 2, 3, \ldots, (i - 1)\}$ is $(i - 1)^2$.

Thus, $n(S_i)$ for each i is $(i-1)^2$, $i > 2$. For example, $n(S_2) = 1$, $n(S_3) = 2^2 = 4$ and so on.

Now,
$$n(T) = n\left(\bigcup_{i=2}^{n+1} S_1\right)$$
$$= \sum_{i=2}^{n+1} n(S_i)$$

(because all S_i's are pair-wise disjoint)

$$= \sum_{i=2}^{n+1} (i-1)^2 = \sum_{i=1}^{n} i^2$$

and hence,
$$\binom{n+1}{2} + 2\binom{n+1}{3} = \sum_{k=1}^{n} k^2.$$

Problem 27 *Show that the number of ways in which three numbers in AP can be selected from 1, 2, 3,…, n is* $\frac{1}{4}(n-1)^2$ *or* $\frac{1}{4}n(n-2)$ *accordingly as n is odd or n is even.*

Solution: Let three numbers be a, b, c with common diffrence 'd', Now $c - a = 2d$ $\Rightarrow c \equiv a(\text{mod } 2) \Rightarrow c, a$ both ever or odd.

Let $n = 2m$ then there are m even numbers and m odd numbers. For c, a both even $\binom{m}{2}$ choices and for both odd $\binom{m}{2}$ choices. Hence for $n = 2m$, $2\binom{m}{2}$ AP's. For n even, $2 \cdot \frac{m(m-1)}{2} = \frac{n}{4}(n-2)$ AP's.

Similarly for $n = 2m + 1$, $\binom{m}{2} + \binom{m+1}{2} = m^2 = \left(\frac{n-1}{2}\right)^2$ AP's.

Problem 28 *A train going from station X to station Y, has 11 stations in between, as halts. 9 persons enter the train during the journey with 9 different tickets of the same class. How many different sorts of tickets they may have had?*

Solution: 9 people enter the train during the journey, that is, they enter possibly from halt 1 to halt 11. But they can have tickets from halt i to halt j, $1 \le i \le j < 12$ (where 12th station is Y).

$\therefore$ The total number of different tickets

$$= {}^{12}C_2 = \frac{12 \times 11}{2} = 66.$$

So, the total number of different sort of available tickets is

$$^{12}C_2 = \frac{12 \times 11}{1.2} = 66.$$

From these 66, we have to choose 9 tickets.

This can be done in $^{66}C_9$ ways.

Aliter: Halt 1 issues 11 different tickets.

Halt 2 issues 10 different tickets.

$$\cdots \cdots \cdots \cdots \cdots$$

Halt 11 issues 1 ticket.

As the travellers might have got into the train from Halt 1 to 11. So, the total number of different types of available tickets is

$$1 + 2 + 3 + \cdots + 10 + 11 = \frac{11 \times 12}{1.2} = 66.$$

So, there are 66 possible types of tickets to be issued to 9 persons. This could be done in $^{66}C_9$ ways.

Problem 29 *There are two bags, each containing m numbered balls. A person has to select an equal number of balls from both the bags. Find the number of ways in which he can select at least one ball from each bag.*

Solution: He may choose one ball or two balls or m balls from each bag.

Choosing one ball from one of the bags can be done in $^mC_1$ ways. Then, choosing one ball from the other bag also can be done in $^mC_1$ ways.

Thus, there are $^mC_1 \times {}^mC_1$ ways of choosing one ball from each bag. Similarly, if r balls, $1 \times r \times m$ are chosen from each of the two bags, the number of ways of doing this is

$$({}^mC_r) \cdot ({}^mC_r) = ({}^mC_r)^2$$

Thus, the total number of ways of choosing at least one ball from both the bags is

$$\sum_{r=1}^{m} ({}^mC_r)^2 = \sum_{r=0}^{m} ({}^mC_r)^2 + ({}^mC_0)^2 = {}^{2n}C_n - 1$$

$$= \frac{(2n)!}{n! \cdot n!} - 1 \text{ as } {}^mC_0 = 1 \left[\sum_{r=0}^{m} ({}^mC_r)^2 = {}^{2m}C_m \right].$$

Problem 30 *If n points (no three of which are collinear) in a plane be joined in all possible ways by straight lines and if no two of the straight lines coincide or are parallel and no three lines pass through the same point (with the exception of the n original points), then prove that the number of points of intersection, exclusive of these n points is*

$$\frac{1}{8} n(n-1)(n-2)(n-3).$$

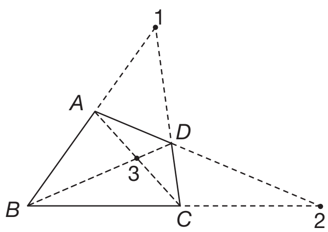
Solution: Every pair of distinct points determines a straight line. Given n points, no three of which are collinear, we get $^nC_2$ lines, *i.e.*, the number of lines determined by n distinct points, no three of which are collinear is $^nC_2 = \dfrac{n(n-1)}{2}$.

In turn these lines, taken two at a time, intersect. However, through joining each one of points to the other $(n-1)$ points, we see that there are $(n-1)$ lines passing through each one of these original points. Thus, each of these original points will be counted ${}^{n-1}C_2$ times and all the original points will be counted as $n \times {}^{n-1}C_2$ points.

The total number of points of intersection of the lines $\frac{n(n-1)}{2}$ including these n original points counted $n \cdot {}^{n-1}C_2$ times is, thus, ${}^{\frac{n(n-1)}{2}}C_2$.

So, the points of intersection other than the original points is thus

$$
\begin{aligned}
&{}^{\frac{n(n-1)}{2}}C_2 - n \times {}^{n-1}C_2 \\
&= \frac{\frac{n(n-1)}{2}\left[\frac{n(n-1)}{2}-1\right]}{2} - \frac{n(n-1)(n-2)}{1.2} \\
&= \frac{n(n-1)[n(n-1)-2]}{8} - \frac{n(n-1)(n-2)}{2} \\
&= \frac{n(n-1)}{8}[n^2-n-2-4(n-2)] \\
&= \frac{n(n-1)}{8}[n^2-5n+6] = \frac{n(n-1)(n-2)(n-3)}{8}.
\end{aligned}
$$



Aliter: Selection of any four points out of n points corresponds to a complete quadrilateral for a complete quadrilateral we get three new points of intersection as shown in the figure.

Hence $3 \cdot \binom{n}{4}$ points $= 3\frac{n(n-1)(n-2)(n-3)}{4\times 3\times 2\times 1} = \frac{n(n-1)(n-2)(n-3)}{8}$.

Problem 31 *You have n objects, each of weight w. When they are weighed in pairs, the sum of the weights of all the possible pairs is* 120. *When they are weighed in triplets, the sum of the weights of all possible triplets is* 480. *Find n.*

Solution: The number of all possible pairs of objects that could be obtained from n objects is

$$ {}^nC_2 = \frac{n(n-1)}{2} $$

Total weight of $\frac{n(n-1)}{2}$ pairs $= \frac{n(n-1)}{2} \times 2 \times w$

$$ = n(n-1)w \text{ units} = 120 \quad (1) $$

The number of all possible triplets of objects that could be obtained from n objects $= {}^nC_3 = \frac{n(n-1)(n-2)}{6}$.

The total weight of all these triplets $= \frac{n(n-1)(n-2)}{6} \times 3w$

$$ = \frac{n(n-1)(n-2)\times w}{2} = 480 \quad (2) $$

Dividing Eq. (2) by (1), we get

$$\frac{n-2}{2}=\frac{480}{120}=4$$

$\Rightarrow$ $n-2=8$ or $n=10$.

Problem 32 *Find the number of permutations* $(p_1, p_2, p_3, p_4, p_5, p_6)$ *of* (1, 2, 3, 4, 5, 6) *such that for any k,* $1 \le k \le 5$ $(p_1, p_2, p_3, \ldots, p_k)$ *does not form a permutation of* 1, 2, 3, …, *k, i.e.,* $p_1 \neq 1$, (p_1, p_2) *is not a permutation of* (1, 2) (p_1, p_2, p_3) *is not a permutation of* (1, 2, 3)*, etc.* **[INMO, 1992]**

Solution For each positive integer k, $1 \le k \le 5$, let N_k denote the number of permutations $(p_1, p_2, \ldots, p_6)$ such that $p_1 \neq 1$, (p_1, p_2) is not a permutation of (l, 2), … $(p_1, p_2, \ldots, p_k)$ is not a permutation of $(1, 2, \ldots, k)$. We are required to find N_5.

We shall start with N_1.

The total number of permutations of (1, 2, 3, 4, 5, 6) is 6! and the permutations of (2, 3, 4, 5, 6) is 5!. Thus, the number of permutations in which $p_1 = 1$ is 5!.

So, the permutation in which $p_1 \neq 1$ is $6! - 5! = 720 - 120 = 600$. Now, we have to remove all the permutations with (1, 2) and (2, 1) as the first two elements to get N_2. Of these, we have already taken into account (1, 2) in considering N_1 and subtracted all the permutations starting with 1. So, we should consider the permutation beginning with (2, 1). When $p_1 = 2$, $p_2 = 1$ (p_3, p_4, p_5 and p_6) can be permuted in 4! ways.

So, $N_2 = N_1 - 4! = 600 - 24 = 576$.

Having removed the permutations beginning with (1, 2), we should now remove those beginning with (1, 2, 3). But, corresponding to the first two places (1, 2) and (2, 1), we have removed all the permutations. So, we should now remove the permutations with first three places (3, 2, 1), (3, 1, 2), (2, 3, 1).

Note that the first 3 positions being 1, 2, 3 is included in the permutations beginning with 1.

For each of these three arrangements, there are 3! ways of arranging 4th, 5th and 6th places and hence,

$N_3 = N_2 - 3 \times 3! = 576 - 18 = 558$. To get N_4, we should remove all the permutations beginning with the permutations of (1, 2, 3, 4), of which the arrangement of (1, 2, 3) in the first three places have already been removed. We have to account for the rest. So, when 4 is in the first place, 3! arrangements of 1, 2, 3 in the 2nd, 3rd and 4th places are possible. Also, when 4 is in the second place, since we have removed the permutation when 1 occupies the first place, there are two choices for the first place with 2 or 3 and for each of these there are 2 arrangements, *i.e.*, (2, 4, 1, 3), (2, 4, 3, 1), (3, 4, 2, 1), (3, 4, 1, 2). When 4 is in the third place, then there are first 3 arrangements (2, 3, 4, 1), (3, 2, 4, 1) and (3, 1, 4, 2).

So, the total permutations of (1, 2, 3, 4) to be removed from N_3 to get N_4 is $(6 + 4 + 3) \times 2 = 26$, because there are 2 ways of arranging the 5th and 6th places for each one of the above permutations of (1, 2, 3, 4).

$$\therefore N_4 = N_3 - 26$$
$$= 558 - 26 = 532.$$

To get N_5, we should remove from N_4 all the permutations of (1, 2, 3, 4, 5) which have not been removed until now. When 5 occupies the first position, there are $4! = 24$ ways of getting 2nd, 3rd, 4th and 5th places which have not been removed so far. When $p_2 = 5$, p_1 cannot be 1, so p_1 can be chosen from the other 3, *viz.*, 2, 3 and 4, in 3 ways and 3rd, 4th and 5th places can be filled for each of the first place choice in $3 \times 2 \times 1 = 6$ ways.

So, when $p_2 = 5$, there 18 different arrangements to be removed.

When $p_3 = 5$, the first two places cannot be (1, 2) so that they can be filled in (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3) and for the fourth and fifth places there are exactly two choices for each of the above arrangements for first and second place.

So, when $p_3 = 5$, the number of arrangements to be removed is $8 \times 2 = 16$. When $p_4 = 5$, $p_1\, p_2\, p_3$ can be removed (241, 412, 421, 234, 243, 342, 324, 423, 432, 314, 341, 413, 431) and since there is only one choice left, we have now to remove 13 arrangements when

$$p_4 = 5.$$

When $p_5 = 5$, we have already removed the permutations of (1, 2, 3, 4) of the first four places to find S_4.

So, now
$$S_5 = S_4 - (24 + 18 + 16 + 13)$$
$$= 534 - 71 = 463.$$

So, 463 is the desired number of permutations.

Problem 33 *Consider the collection of all three element subsets drawn from the set* $\{1, 2, 3, 4, \ldots, 299, 300\}$. *Determine the number of subsets for which, the sum of the elements is a multiple of 3.*

Solution: The given set $S = \{1, 2, 3, 4, \ldots, 299, 300\}$ can be realised as the union of the three disjoint sets S_1, S_2 and S_3 with

$$S_1 = \{x \in S : x = 3n + 1, 0 \le n \le 99\},$$
$$S_2 = \{x \in S : x = 3n + 2, 0 \le n \le 99\},$$
$$S_3 = \{x \in S : x = 3n, 1 \le n \le 100\}.$$

Now, to get the set of all three element subsets of S, with the sum of the elements of the subset a multiple of 3, we should choose all three elements from the same set S_1, S_2 or S_3 or we should choose one element from each of the set S_1, S_2 and S_3.

We see that, $\quad n(S_1) = n(S_2) = n(S_3) = 100.$

Choosing all the three elements from either S_1, S_2 or S_3 will give $3 \times {}^{100}C_3$ triplets and its sum is also divisible by 3.

Choosing the three elements, one each from S_1, S_2 and S_3 will give ${}^{100}C_1 \times {}^{100}C_1 \times {}^{100}C_1$ triplets and its sum is also divisible by 3.

So, the total number of 3 element subsets with the required property is

$$3 \times {}^{100}C_3 + {}^{100}C_1 \times {}^{100}C_1 \times {}^{100}C_1$$
$$= \frac{3 \times 100 \times 99 \times 98}{1 \times 2 \times 3} + 100^3$$
$$= 100 \times 99 \times 49 + 1000000$$
$$= 485100 + 1000000$$
$$= 14{,}85{,}100.$$

Problem 34 *A normal die bearing the numbers* 1, 2, 3, 4, 5, 6 *on its faces is thrown repeatedly until the running total first exceeds* 12. *What is the most likely total that will be obtained?*

Solution: Consider the throws before the last one. After this penultimate throw, the running total 's' should be such that $7 \le s \le 12$; since, if we take the least value of s, *i.e.*,

$s = 7$, then we would just cross 12, if the final throw gives 6, and the maximum value of s is 12; in the final throw by getting any number 1 to 6, the running total exceeds 12. Thus, the possible values of the running total in the penultimate throw is 7, 8, 9, 10, 11 and 12.

Let us tabulate the possible running totals after the final throw.

Possible Running totals after the penultimate throw	Possible running totals after the final throw					
7	13	14	15	16	17	18
8	13	14	15	16	17	
9	13	14	15	16		
10	13	14	15			
11	13	14				
12	13					

Thus, the number that occurs most number of times in the possible running total after the final throw is 13.
[Since, the die is a fair die and so getting any one of 1 to 6 is equally likely and hence, the possible running totals 7, 8, 9, 10, 11 and 12 in the penultimate throw is also equally likely.]

Problem 35 *Create two fair dice which when rolled together have an equal probability of getting any sum from* 1 *to* 12.

Solution: The only sums that we want are from 1 to 12, using two dice with faces marked, say $a_1, a_2, \ldots, a_6$ and $b_1, b_2, b_3, \ldots, b_6$. We have totally $6 \times 6 = 36$ outcomes.

So, each number from 1 to 12 should occur $\frac{36}{12} = 3$ times.

If one die has numbers 1, 2, 3, 4, 5, 6 on its faces, then for 1 to 6 occur thrice, there should be three zeroes on the three faces of the second die. For each of 7, 8, …, 12 to occur thrice, three should be 3 sixes on the other three faces, so that (1, 6), (2, 6), (3, 6), …, (6, 6) can occur thrice.

Thus, the probability of getting 1 from the first die is $\frac{1}{6}$ and the probability of getting zero from the second die is $\frac{3}{6} = \frac{1}{2}$. So, probability of getting the pair (1, 0) is $\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$ and similarly for each of numbers from 1 to 12 [$1 = 1 + 0, 2 = 2 + 0, \ldots, 6 = 6 + 0, 7 = 1 + 6, 8 = 2 + 6, \ldots, 12 = 6 + 6$].

Problem 36 *If the numbers x, y are chosen at random from* 1, 2,…, *n with replacement, $n \geq 3$, show that $P(x^3 + y^3$ is a multiple of 3) is less than $P(x^3 + y^3$ is a multiple of 7).*

Solution: Let $S = \{1, 2, 3, \ldots, n\}$.

We shall first take $n = 2$, $n = 3$ and $n = 4$ and find, in how many ways we get $(x^3 + y^3)$ and how many of them are divisible by (*a*) 3; (*b*) 7.

For $$n = 2,$$
$$(x, y) = (1, 1), (1, 2), (2, 1), (2, 2),$$
$$(x^3, y^3) = (1, 1), (1, 8), (8, 1), (8, 8)$$

and $(x^3 + y^3)$ is divisible by 3 for $x^3 = 1, y^3 = 8$ and $x^3 = 8, y^3 = 1$.

Thus, $P[(x^3 + y^3)$ is a multiple of 3] in this case is $\frac{2}{4} = \frac{1}{2}$ and $P[(x^3 + y^3)$ is a multiple of 7] is an impossible event. Therefore, the statement does not hold for $n = 2$.

For $n = 3$, $\{(x, y) \mid (x, y) \in S) = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

and, $\{(x^3, y^3) \mid (x, y) \in S) = \{(1, 1), (1, 8), (1, 27), (8, 1), (8, 8), (8, 27), (27, 1), (27, 8), (27, 27)\}$.

Of these set of ordered pairs, we get $(x^3 + y^3)$ divisible by 3 as $(1 + 8)$, $(8 + 1)$, $(27 + 27) = 3$.

So, here $P[(x^3 + y^3)$ is a multiple of 3] $= \frac{3}{9} = \frac{1}{3}$ and the set of ordered pairs we get set $(x^3 + y^3)$ is divisible by 7 is $(1 + 27)$, $(8 + 27)$, $(27 + 1)$, $(27 + 8) = 4$.

$\therefore$ In this case, $P[(x^3 + y^3)$ is a multiple of 7] $= \frac{4}{9}$,

and clearly, $P[(x^3 + y^3)$ is a multiple of 7] $>$ $P[(x^3 + y^3)$ is a multiple of 3].

Now, we shall pass on to the general case where $n > 3$.

For any number, the possible remainders when n is divided by 3 is 0, 1 or 2.

So, the possible ordered pairs (x, y) (mod 3) is $\{(0, 0), (0, 1), (1, 0), (0, 2), (2, 6), (1, 1), (1, 2), (2, 1), (2,2)\}$.

Here $P\{(x^3 + y^3)$ is a multiple of 3$\}$ $= \frac{1}{3}$ as has already been seen.

$$T = \{(x^3 + y^3) \mid (x, y) \in N \text{ (mod 3)}\}$$
$$= \{(0^3 + 0^3), (0^3 + 1^3), (1^3 + 0^3), (0^3 + 2^3), (2^3 + 0^3),$$
$$(1^3 + 1^3), (1^3 + 2^3), (2^3 + 1^3), (2^3 + 2^3)\}.$$

The subset of T which contains elements $x^3 + y^3$ is a multiple of 3 is $\{(0^3 + 1^3), (1^3 + 2^3), (2^3 + 1^3)\}$ and hence, the probability is $\frac{1}{3}$.

Again, when S is listed so that the elements are written in mod 7, we get $S_7 = (0, 1, 2, 3, 4, 5, 6)$.

Now, the set of the cubes of the elements of S_7 is

$$S_c = \{0, 1, 8, 27, 64, 125, 216\}.$$

The pairs (x^3, y^3) such that $(x^3 + y^3)$ is a multiple of 7 are $\{(0, 0), (1, 27), (27, 1), (1, 125), (125, 1)$ $(1, 216), (216, 1), (8, 27), (27, 8), (8, 125), (125, 8), (8, 216), (216, 8), (64, 27), (27, 64), (64, 125), (125, 64), (64, 216), (216, 64)\}$.

Thus, this set of ordered pairs (x^3, y^3) contains 19 elements such that $(x^3 + y^3)$ is a multiple of 7.

So, $P[(x^3 + y^3)$ is a multiple of 7] in this case is $\frac{19}{7 \times 7} = \frac{19}{49}$.

$$[\because\ n(S_c \times S_c) = n(S_c) \times n(S_c) = 7 \times 7 = 49]$$

$P[(x^3 + y^3)$ is a multiple of 3] $= \frac{1}{3}$

and hence, $P[(x^3 + y^3)$ is a multiple of 3] $<$ $P[(x^3 + y^3)$ is a multiple of 7] $= \frac{19}{49}$.

$$\left[\frac{1}{3} < \frac{19}{49} \quad \text{as} \quad \frac{1 \times 49}{3 \times 49} < \frac{3 \times 19}{3 \times 49}\right]$$

Notes:

1. Here we have assumed that n is both a multiple of 3 as well as 7. Actually, we need to prove it for the general case where n need not be either a multiple of 3 or 7. But this can also be enumerated and verified.
2. S_c can be considered as the set of possible remainders as $\{0, 1, 1, -1, 1, -1, -1\}$ in the case of mod 7 and to get $(x^3 + y^3)$ to be divisible by 7, we can choose $(1, -1)$, $(0, 0)$.

Probability of choosing 1 is $\frac{3}{7}$ and probability of choosing -1 is also $\frac{3}{7}$.

$\therefore$ Probability of choosing $(1, -1)$ or $(-1, 1)$ is

$$2\times\frac{3}{7}\times\frac{3}{7}=\frac{18}{49}.$$

Probability of choosing $(0, 0)$ is $\frac{1}{7}\times\frac{1}{7}=\frac{1}{49}$

$\therefore$ Probability of $(x^3 + y^3)$ is a multiple of 7 is $\frac{18}{49}+\frac{1}{49}=\frac{19}{49}$.

In the case of mod 3, also we have the set of possible remainders of x^3 or y^3 on dividing by 3 to be $\{0, 1, -1\}$.

For $(x^3 + y^3)$ to be a multiple of 3, we should choose $x^3 = 0 = y^3$ and $x^3 = 1$ and $y^3 = -1$ or $x^3 = -1$ and $y^3 = 1$.

0 can be chosen in $\frac{1}{3}$ ways.

So, probability of choosing a zero and again a zero is $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$.

Probability of choosing $(1, -1)$ or $(-1, 1)$ is

$$\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}+\frac{1}{9}=\frac{2}{9}.$$

$\therefore$ $P[(x^3 + y^3)$ is divisible by $3]=\frac{1}{9}+\frac{2}{9}=\frac{3}{9}=\frac{1}{3}$ and hence, the result.

Problem 37 *Show that the number of triplets (a, b, c) with $(a + b + c) < 95$ is less than the number of those with $(a + b + c) > 95$. where $a, b, c \in S = \{1, 2, 3, \ldots, 63\}$.*

Solution: Let $S = \{1, 2, 3, \ldots, 63\}$

Let A be the set of all triplets of S such that $(a + b + c) < 95$, *i.e.*,

$$A = \{(a, b, c):(a + b + c) < 95; a, b, c \in S\}.$$

Similarly, let B be the set of all triplets of S such that $(a + b + c) > 95$, where $\{a, b, c\} \in S$,

i.e., $B = \{(a, b, c): (a + b + c) > 95; a, b, c \in S\}$

and $C = \{(a, b, c): (a + b + c) > 97; a, b, c \in S\}$.

Clearly, C is a proper subset of B because $a, b, c \in S$, if $(a + b + c) = 96$ then $(a, b, c) \in B$ and $(a, b, c) \notin C$.

However, every element of $C \in B$,

as, $$a + b + c > 97 \quad\Rightarrow\quad a + b + c > 95$$

hence, $$(a, b, c) \in C \quad\Rightarrow\quad (a, b, c) \in B.$$

Now, it is enough if we show that $n(A) = n(C)$ as $n(C) < n(B)$ and $n(A) = n(C)$

$$\Rightarrow \quad n(A) < n(B).$$

If $(a, b, c) \in A$, then $1 \leq a + b + c < 95$ and also $1 \leq a, b, c \leq 63$.
Therefore, $1 \leq (64 - a), (64 - b), (64 - c) \leq 63$ and as $(a + b + c) < 95$,
$(64 - a) + (64 - b) + (64 - c) = 192 - (a + b + c) > 192 - 95 = 97$.
Thus to each element of A, there is a unique element in C.
Conversely, if $(a, b, c) \in C$, then $((64 - a), (64 - b), (64 - c)) \in A$ for
$(64 - a) + (64 - b)+ (64 - c) = 192 - (a + b + c)$,
and since $(a, b, c) \in C$, $(a + b + c) > 97$

$\therefore$ $192 - (a + b + c) < 192 - 97 = 95$

and thus $((64 - a), (64 - b), (64 - c)) \in A$, which shows that for every element of C there corresponds a unique element in A.

Thus, there is a $1 - 1$ correspondence between the sets A and C.

$\therefore$ $n(A) = n(C) < n(B)$.

Problem 38 *Prove that it is impossible to load a pair of dice (each die has numbers 1 to 6 on their 6 faces) so that every sum 2, 3, …, 12 is equally likely. As customary, assume that the dice are distinguishable (For example, a 2 on the first die with a 4 on the second is different from a 4 on the first die and a 2 on the second, even though the same total 6 is obtained).*

Solution: Let p_i denote the probability of i coming up on the first die and q_i, the probability of i on the second die where $i = 1, 2, \ldots, 6$. The probability of getting the sum 2 is p_1q_1.
The probability of getting the sum 12 is p_6q_6.

If the probability of getting all the 11 sums are same, then probability of each would be $\frac{1}{11}$.

The probability of getting a 7 is also $\frac{1}{11}$ and is equal to

$$\begin{aligned}
\frac{1}{11} &= p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1 \\
&\geq p_1q_6 + p_6q_1 \\
&= p_1q_6\left(\frac{q_1}{q_1}\right) + p_6q_1\left(\frac{q_6}{q_6}\right) \\
&= p_1q_1\left(\frac{q_6}{q_1}\right) + p_6q_6\left(\frac{q_1}{q_6}\right) \\
&= \frac{1}{11}\left(\frac{q_6}{q_1}\right) + \frac{1}{11}\left(\frac{q_1}{q_6}\right) = \frac{1}{11}\left(\frac{q_6}{q_1} + \frac{q_1}{q_6}\right)
\end{aligned}$$

$$\Rightarrow \quad 1 \geq \frac{q_6}{q_1} + \frac{q_1}{q_6}.$$

But $\frac{q_6}{q_1}$ and $\frac{q_1}{q_6}$ are reciprocals of one another and hence their sum should be ≥ 2.

i.e., $\frac{q_6}{q_1} + \frac{q_1}{q_6}$ cannot be less than 1.

It is a contradiction and hence, the result.

Aliter: The probability mass function of the first die can be written as a probability generating function (*pgf*) as

$$p_1x + p_2x^2 + p_3x^3 + p_4x^4 + p_5x^5 + p_6x^6.$$

For the second die, the *pgf* is

$$q_1x + q_2x^2 + q_3x^3 + q_4x^4 + q_5x^5 + q_6x^6.$$

Now, the *pgf* of the sum is given by $\frac{1}{11}(x^2 + x^3 + \cdots + x^{12})$

$$\left(\sum_{i=1}^{6} p_i x^i\right)\left(\sum_{i=1}^{6} q_i x^i\right) \equiv \frac{1}{11}\left(\sum_{i=2}^{12} x^i\right).$$

Cancelling x^2 on both sides, we get

$$\left(\sum_{i=1}^{6} p_i x^{i-1}\right)\left(\sum_{i=1}^{6} q_i x^{i-1}\right) \equiv \frac{1}{11}\left(\sum_{i=0}^{10} x^i\right)$$

The RHS is the product $\frac{1}{11}$ $(x - \omega)$ $(x - \omega^2)$ … $(x - \omega^{10})$, where ω is the 11th roots of unity. All the roots of the RHS are complex and they occur in conjugate pairs. On the LHS we have two real polynomial factors each of degree 5. This is impossible. We cannot have a real 5th degree polynomial factor for $1 + x + x^2 + \cdots + x^{10}$.

Hence, such dice do not exist.

Problem 39 *There are 6 red balls and 8 green balls in a bag. Five balls are drawn out at random and placed in a red box. The remaining 9 balls are put in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number?*

Solution: Let g denote the number of green balls in the red box.

So, the red box contains $(5 - g)$ red balls.

There are 8 green balls in all. So, the number of green balls in the green box

$$= (8 - g)$$

There are 6 red balls in all.

So, the number of red balls in the green box

$$= 6 - (5 - g) = (1 + g)$$

So, the number of red balls in the green box + the number of green balls in the red box $= (1 + g) + g = (2g + 1)$.

Here $(2g + 1)$ is an odd number.

Now, g cannot exceed 5, because only 5 balls are put in red box and it is taken that g green balls are put in red box.

So, $2g + 1$ cannot be greater than $2 \times 5 + 1 = 11$.

Even if $\quad g = 0, 2g + 1 = 1$

and hence, $\quad 1 \le 2g + 1 \le 11$.

The odd primes from 10 to 11 are 3, 5, 7 and 11.

So, the only composite odd number less than 11 is 9, since 1 is neither composite nor prime, $2g + 1$ can either be 9 or 1.

Green Box

5 balls

g green balls

$(5 - g)$

red balls

Red Box

9 balls

$(8 - g)$ green balls

$\left.\begin{array}{l} 6 - (5 - g) \\ = 1 + g \end{array}\right\}$ (red balls)

So, $\qquad 2g+1=1 \quad \Rightarrow \quad g=0$

and $\qquad 2g+1=9 \quad \Rightarrow \quad g=4$

Only for the value of $g = 0$ or 4, we get the number $2g + 1$ to be non-prime.

Thus, it implies that we should find the number of ways of drawing all 5 red (to put in red box) or 4 green and 1 red in the draw.

The number of ways of drawing 5 red out of 6 red and 0 green out of 8 green is

$$= {}^6C_5 - {}^8C_0.$$

The number of ways of drawing 4 green and 1 red balls is

$$= {}^8C_4 \times {}^6C_1$$

Total number of drawing 5 balls is ${}^{14}C_5$ and hence, the required probability is

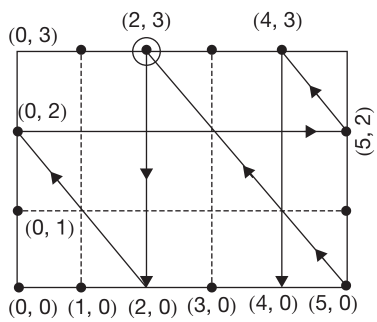
$$\frac{{}^6C_5 \times {}^8C_0 + {}^8C_4 \times {}^6C_1}{{}^{14}C_5}$$

$$= \frac{6\times 1 + \dfrac{8\times 7\times 6\times 5}{1\times 2\times 3\times 4}\cdot 6}{\dfrac{14\times 13\times 12\times 11\times 10}{1\times 2\times 3\times 4\times 5}}$$

$$= \frac{6+420}{14\times 13\times 11} = \frac{426}{14\times 13\times 11} = \frac{213}{1001}.$$

Problem 40 *An oil vendor has three different measuring vessels A, B and C with capacities* 8 *litres,* 5 *litres and* 3 *litres. The vessel A is filled with oil, he wants to divide the oil into two equal parts, by pouring it from one container to another, without using any other measuring vessels other than the three. How can he do it?*

Solution: It is clear that, after pouring the oil several times into the different containers A, B and C, finally he should have 4 litres in vessel A and 4 litres in vessel B. Since C can hold a maximum of 3 litres only, this can be done by using a rectangular coordinate system. B can hold 0, 1, 2, 3, 4 and 5 litres and C can hold only 0, 1, 2 and 3 litres.



We represent the contents of B and C in a rectangular coordinate system using a 5×3 grid. Since no fraction is involved, we take only the 24 lattice points (i, j).

Here $i = 0, 1, 2, 3, 4, 5$; $j = 0, 1, 2, 3$ are used as follows:

In the horizontal lines (x-axis) are plotted (0, 0) to (5, 0) to represent the possibilities of different measures of oil that B can hold, and in the vertical line (y-axis), the points (0, 0) to (0, 3) are plotted to represent the possibilities of different measures of oil that C can hold.

We do not fill both the vessels B and C with 5 litres and 3 litres, respectively (5, 3) at any stage, as this forces us to use vessel A again. Vessel A is filled with 8 litres in the beginning.

To start with, filling the oil in vessel B from vessel A represents the point (5, 0). This is shown by the arrow from (0, 0) to (5, 0) and this is followed by (2, 3) (by pouring oil from B to C, B now has 2 litres and C has 3 litres). This is followed by (2, 0) (by pouring oil from C to A, C is empty and A has $3 + 3 = 6$ litres). Now, (follow the arrows) (0, 2) (by pouring oil from B to C). This is followed by (5, 2) (by pouring 5 litres from A into B) and (5, 2) is followed by (4, 3) [by pouring 1 litre from B to C, as C can hold one more litre and hence $(5 - 1, 2 + 1) = (4, 3)$ is reached].

Now, we finally get (4, 0) from (4, 3) by pouring 3 litres of oil from C into A.

Now, B has 4 litres and A has 4 litres.

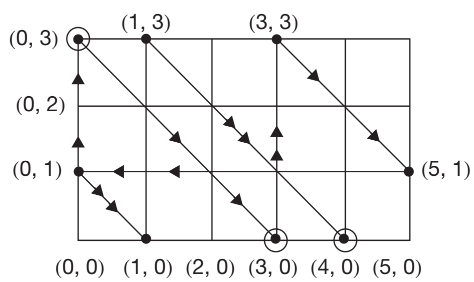
Thus in seven stages (minimum), we accomplish this task.

The above schematic representation can be given in a tabular column also as follows:

Stage	8 L Vessel	5 L Vessel	3 L Vessel
0 (initial)	8	0	0
1	3	5	0
2	3	2	3
3	6	2	0
4	6	0	2
5	1	5	2
6	1	4	3
7	4	4	0

We have several other methods, but the one given above is the best solution. Since in this case, we accomplish the task in the minimum number of steps. We give here a diagrammatic representation as well as a tabular column for yet another solution.

Here we have



(1) – (0, 3)
→ (2) – (3, 0)
→ (3) – (3, 3)
→ (4) – (5, 1)
→ (5) – (0, 1)
→ (6) – (1, 0)
→ (7) – (1, 3)
→ (8) – (4, 0)

In this case we accomplish the task in 8 stages (8 > 7!).

Stage	8 L	5 L	3 L
(Initial) 0	8	0	0
1	5	0	3
2	5	3	0
3	2	3	3
4	2	5	1
5	7	0	1
6	7	**1**	0
7	4	**1**	3
8	4	4	0

Problem 41 *Consider a square array of dots, coloured red or blue, with* 20 *rows and* 20 *columns. Whenever two dots of the same colour are adjacent in the same row or column; they are joined by a segment of their common colour. Adjacent dots of unlike colours are joined by a black segment. There are* 219 *red dots,* 39 *of them on the border of the array, not at the corners. There are* 237 *black segments. How many blue segments are there?*

Solution: In each row, there are 19 segments (Since there are 20 points in each row).

There are 20 rows and hence there are $20 \times 19 = 380$ horizontal segments.

Similarly, there are $20 \times 19 = 380$ vertical segments (There are 20 columns with 19 segments in each column).

Therefore, the total number of segments = 760.

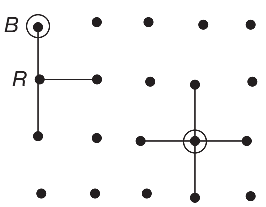
Number of black segments = 237.

Number of segments which are either blue or red = 523.

Let r denote the number of red segments and each red segment has 2 red points as the end point of the segment and each black segment has one end point coloured blue and the other end point coloured red.

So, the total number of times a red dot becomes an end point of a segment is

$$= 2 \times r + 237 = 2r + 237 \tag{1}$$



There are altogether 219 red dots and of these, 39 are on the border.

So, the number of red dots in the interior is 180.

Each red dot on the border accounts for 3 segments (Since none of the red dots is on the corner).

So, the number of segments for which each red point on the border becomes the end points 3.

So, the total number of segments to which the 39 border red dots are end points $39 \times 3 = 117$.

Each of the 180 red points on the interior becomes the end point for 4 segments.

So, the total number of segments for which the 180 red points are the end points $= 180 \times 4 = 720$.

So the total number of times a red dot becomes an end point, *i.e.*, total number of red ends

$$= 117 + 720 = 837 \tag{2}$$

Hence, Eqs. (1) and (2) represent the same number, the result,

$$\therefore\ 2r + 237 = 837$$
$$\therefore\ r = 300.$$

i.e., the number of red segments = 300

and the number of blue segments = 523 − 300 = 223.

Problem 42 *Suppose on a certain island there are* 13 *grey,* 15 *brown and* 17 *crimson chameleons. If two chameleons of two different colours meet, they both change to the third colour.* (*For example, when a grey and brown pair meet, then both would change to crimson*)*. This is the only time they change colour. Is it possible for all chameleons eventually to be of the same colour?*

Solution: We will write the number of grey, brown and crimson chameleons as triples (g, b, c). An encounter of grey and brown changes the count (g, b, c) to $(g, b, c) + (-1, -1, 2)$.

Similarly, the other encounters will lead to changes $(-1, 2, -1)$ and $(2, -1, -1)$ in the count of grey, brown and crimson chameleons. Let there be m encounters of $(-1, -1, 2)$ kind, n encounters of $(-1, 2, -1)$ kind and l encounters of $(2, -1, -1)$ kind leading to all chameleons of the same colour, *i.e.*, the final triples will be either (45, 0, 0) or (0, 45, 0) or (0, 0, 45). Hence, we get the following equations if we end up in the triple (45, 0, 0), *i.e.*, all grey chameleons.

$$(13, 15, 17) + m(-1, -1, 2) + n(-1, 2, -1) + l(2, -1, -1) = (45, 0, 0)$$
$$\therefore\ -m - n + 2l = 32$$
$$-m + 2n - l = -15$$
$$2m - n - l = -17$$

These three equations are consistent, but of rank < 3. Hence, they have infinity of solutions given by

$$m = l - \frac{49}{3} \quad \text{and} \quad n = l - \frac{47}{3}$$

Note that we will never get all the three m, n, l to be integers in these solutions. Hence, the equations even though they are consistent, they are of no use to us as we want l, m, n to be positive integers.

Similarly, when the terminal triple is either (0, 45, 0) or (0, 0, 45), we get systems of equations which do have an infinity of solutions but which do not provide integer solutions. Hence, no sequence of encounters will even lead to all chameleons to be of the same colour.

Aliter 1: For this solution we use very elementary modulo arithmetic. Note that our initial configuration (13, 15, 17) when taken modulo 3 is (1, 0, 2). Let us see the effect of each of the encounters modulo 3 on (1, 0, 2). Consider encounter 1 leading to the change $(-1, -1, 2)$. This leads to the new configuration $(1, 0, 2) + (-1, -1, 2)$ (modulo 3) = $(0, -1, 4)$ (modulo 3) = (0, 2, 1) (modulo 3). Note that one of the components of the triple (original as well as the resultant) was divisible by 3, one left a remainder of 1, and the third left a remainder of 2 when divided by 3. Similarly, using encounters $(-1, 2, -1)$, we get (0, 2, 1) modulo 3 and using $(2, -1, -1)$, we get $(3, -1, 1)$(modulo 3) = (0, 2, 1) (modulo 3). Whatever be the encounter, the resultant triple has the same configuration, one component divisible by 3, one leaves a remainder of 1 and the other leaves a remainder of 2 when divided by 3. So, the successive encounters lead to the triples (0, 2, 1), (2, 1, 0), (1, 0, 2), (0, 2, 1) and so on. But if all chameleons must be of the same colour, we must end with (45, 0, 0) or (0, 45, 0) or (0, 0, 45). Taking modulo 3, this implies that we have to arrive at (0, 0, 0) modulo 3. But we will never arrive at a triple where every component is divisible by 3 by our above discussion. Hence, the chameleons can never be of the same colour.

Aliter 2: Let us use weights for each colour; 0 for grey, 1 for brown and 2 for crimson. The value of a triple (g, b, c) is calculated as $(0 \times g + 1 \times b + 2 \times \mathrm{c})$ modulo 3. For the initial configuration the value is $(0 \times 13 + 1 \times 15 + 2 \times 17)$ modulo 3 = 1 (modulo 3). Let us now see how each of the encounters affects the value. In the case (–1, –1, 2) the value is changed by $-1 \times 0 + (-1) \times 1 + 2 \times 2 = 3 \pmod 3 = 0 \pmod 3$, *i.e.*, no change. Similarly, for the other two encounters (–1, 2, –1) and (2, –1, –1), the value is changed by 0 (mod 3) only. Hence, the value remains the same after any number of encounters in any order. But the value of the final required configurations namely, (45, 0, 0), (0, 45, 0) or (0, 0, 45) is 0 (mod 3). But the original value, namely, 1 (mod 3) does not change by the encounters and hence, can never reach 0 (mod 3). Hence, the chameleons cannot all end up with the same colour.

Aliter 3: We will enumerate all possible triples that we can arrive at due to these encounters and check whether we can ever arrive at (45, 0, 0), (0, 45, 0) or (0, 0, 45). Instead of 1 grey and 1 brown becoming 2 crimson, we will take the general case of r grey and r brown becoming $2r$ crimsons. Similarly for the other encounters as follows:

	G	B	C	Changes	Due to Encounters	
Initial stage	13	15	17	–13	–13	+ 26
1	0	2	43	4	–2	–2
2	4	0	41	–4	8	–4
3	0	8	37	+6	–8	–8
4	16	0	29	–16	32	–16
5	0	32	13	26	–13	–13
6	26	19	0	–19	–19	38
7	7	0	38	–7	14	–7
8	0	14	31	28	–14	–14
9	28	0	17	–17	34	–17
10	11	34	0	–11	–11	22
11	0	23	22	44	–22	–22
12	44	1	0	–1	–1	2
13	43	0	2	–2	4	–2
14	41	4	0	–4	–4	8
15	37	0	8	–8	16	–8
16	29	16	0	–16	–16	32
17	13	0	32	–13	26	–13
18	0	26	19	38	–19	–19

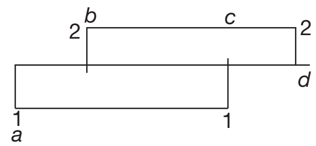
19	38	7	0	–7	–7	14
20	31	0	14	–14	28	–14
21	17	28	0	–17	–17	34
22	0	11	34	34	–11	–11
23	22	0	23	–22	44	–22
24	0	44	1	2	–1	–1
25	2	43	0	–2	–2	4
26	0	41	4	8	–4	.4
27	8	37	0	–8	–8	16
28	0	29	16	32	–16	–16
29	32	13	0	–13	–13	26
30	19	0	26	–19	38	7
31	0	38	7	14	–7	–7
32	14	31	0	–14	–14	28
33	0	17	28	34	–17	–17
34	34	0	11	–11	22	–11
35	23	22	0	–22	–22	44
36	1	0	44	–1	2	–1
37	0	2	43			

In the 37th stage we get back to (0, 2, 43), the same as we got in the first stage. Note that, at no stage did we get 2 components to be equal. Thus, it starts recurring and we will never reach the configurations (0, 0, 45), (0, 45, 0) or (45, 0, 0). Hence, the result.

Problem 43 *During a certain lecture each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that at some moment, any three of them were sleeping simultaneously. Assume that no one was sleeping before the lecture.* **[USA MO, 1986]**

Solution: Here we use proof by contradiction.
That is, we assume that there is no moment when any three of the mathematicians were sleeping simultaneously. Since every pair of mathematicians had some common time interval when both of them were sleeping, there are ${}^5C_2 = 10$ non-overlapping time intervals, (Non-overlapping because at no point of time did three of them sleep simultaneously by our assumption) one interval of common dozing for each of the ten pairs. Each such interval is started by a moment when one of the mathematicians in the pair fell asleep. Each of the 5 mathematicians fell asleep twice.

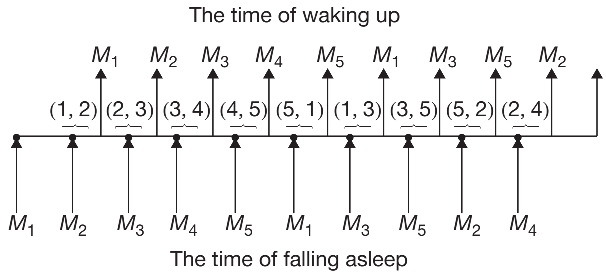
$\therefore$ There are exactly 10 such moments such that each moment initiated a different interval (as we have to account for 10 non-overlapping intervals). Let us now consider the first common dozing interval, say, that of mathematicians 1 and 2. The moment b starts the common interval. But note that moment a is already used up and does not start any other common dozing interval.



$\therefore$ We are left with 8 moments and 9 common dozing intervals which have to start at these 8 moments which is impossible. Hence it is not possible that all the 10 intervals are non-overlapping and hence, in an interval, there will be 3 mathematicians sleeping simultaneously.

Aliter: Let the 5 mathematicians be m_1, m_2, m_3, m_4 and m_5. Let the 10 pairs be (m_1, m_2), (m_1, m_3), (m_1, m_4), (m_1, m_5), (m_2, m_3), (m_2, m_4), (m_2, m_5), (m_3, m_4), (m_3, m_5) and (m_4, m_5).

If these pairs have 10 non-overlapping time intervals when each pair sleeps, then each mathematician sleeps with 4 of his colleagues in turn. But each mathematician can sleep for only 2 stretches. Therefore, we form the time interval as follows: We will represent the mathematicians m_1, m_2, m_3, m_4, m_5 on a line segment showing the moment they fall asleep and the moment they wake up. We will show that the hypothesis is not satisfied (each pair sleeping in a common interval), if we do not allow three of them to sleep during one time interval.



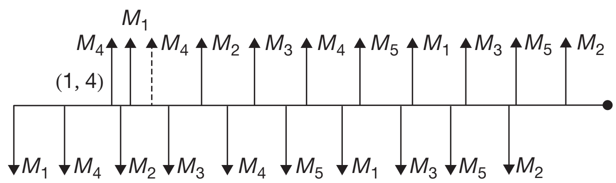
Explanation of the diagram: After representing the mathematicians M_1, M_2, M_3, M_4, M_5 and showing the time of their falling asleep, after the 5th mathematicians falls asleep, M_1 goes to sleep for his second nap. After M_1 starts sleeping for the second time, M_2 cannot come for his second nap, as every pair should occur exactly once and we had M_1 and M_2 sleeping simultaneously at the initial stage itself. So, the points, showing the other four mathematicians to follow M_1 for their second nap, should be M_3, M_5, M_2 and M_4 in that order.

Now each mathematicians appears twice, and we have the pairs (M_1, M_2), (M_2, M_3), (M_3, M_4), (M_4, M_5) (M_5, M_1), (M_1, M_3), (M_3, M_5), (M_5, M_2) and (M_2, M_4).

Here these pairs common sleep period is shown as the ordered pairs of their subscripts (1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 3), (3, 5), (5, 2) and (2, 4).

Thus, we have just nine pairs, sleeping simultaneously and the pair (1, 4) did not sleep simultaneously.

In the diagram, when M_4 appears for the second time, he sleeps along with M_2. So, if we replace M_4 between M_1 and M_2 in the figure, so that M_4's waking up moment is shown after M_2 starts sleeping but before M_3 starts sleeping as in the following figure.



Since both M_1 and M_4 wake up after M_2 falls asleep, both M_1 and M_4 sleep simultaneously with M_2 and the time interval between M_2 falling asleep and M_4 getting up (or M_2 getting up as M_4 may get up after M_2 gets up but before M_3 falls asleep shown by the dotted arrow) shown as (1, 4, 2) is the moment, when all the three M_1, M_4 and M_2 sleep simultaneously. Hence, the statement is proved.

Problem 44 *A difficult mathematical competition consisted of a Part I and a Part II within combined total of 28 problems. Each contestant solved 7 problems altogether. For each pair of problems there were exactly two contestants who solved both of them. Prove that there was a contestant who in Part I solved either no problem or at least 4 problems.*

Solution: We will find the total number of contestants.

Since for each pair of problems there were exactly two contestants, let us assume that an arbitrary problem p_1 was solved by r contestants. Each of these r contestants solved 6 more problems, solving $6r$ more problems in all counting multiplicities. Since every problem, other than p_1 was paired with p_2 and was solved by exactly two contestants, each of the remaining 27 problems (*i.e.*, other than p_1) is counted twice among the problems solved by the r contestants, *i.e.*,

$$6r = 2 \times 27$$

or
$$r = 9.$$

Therefore, an arbitrary problem p_1 is solved by 9 contestants. Hence, in all we have $\frac{9 \times 28}{7} = 36$ contestants, as each contestant solves 7 problems.

For the rest of the proof, let us assume the contrary, that is, every contestant solved either 1, 2 or 3 problems in Part I.

Let us assume that there are n problems in Part I and let x, y, z be the number of contestants who solved 1, 2 and 3 problems in Part I.

Since every one of the contestants solves either 1, 2 or 3 problems in Part I, we get

$$x + y + z = 36 \tag{1}$$

$$x + 2y + 3z = 9n \tag{2}$$

(Since each problem was solved by 9 contestants.)

Since every contestant among y solves a pair of problems in Part I and every contestant among z solves 3 pairs of problems in Part I and as each pair of problems was solved by exactly two contestants, we get the following equations:

$$y + 3z = 2.{}^nC_2 = 2 \cdot \frac{n(n-1)}{2} = n(n-1) \tag{3}$$

From Eqs. (1), (2) and (3), we get

$$z = n^2 - 10n + 36$$

and, $$y = -2n^2 + 29n - 108 = -2\left(n - \frac{29}{4}\right)^2 - \frac{23}{8} < 0.$$

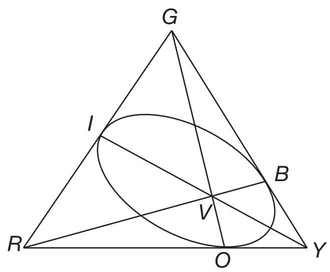
As $y < 0$ is not an acceptable result, our assumption is wrong.

Hence, there is at least one contestant who solved either no problem from Part I or solved at least 4 problems from Part I.

Problem 45 *There are certain number of balls and they are painted with the following conditions:*

(i) *Every two colours appear on exactly one ball.*
(ii) *Every two balls have exactly one colour in common.*
(iii) *There are four colours such that any three of them appear on one ball.*
(iv) *Each ball has three colours.*

Find the number of balls and the number of colours used.



Solution: Let us represent each of the balls by a line segment with three points to show the three colours.

Thus, ROY is a ball with three colours red, orange and yellow. We have to have three more balls such that on each of them one of the colours should be red, orange or yellow. So, next draw lines through R, O, Y to meet at a common point G standing for green colour. But the balls with colours RG, OG and YG must have a third colour in them say, Indigo (I), Violet (V) and Blue (B). Thus, we have 7 balls and 7 colours, in all.

7 colours R, O, Y, G, I, V, B and 7 balls

1. ROY, 2. RIG, 3. RVB, 4. OVG, 5. YBG, 6. YVI, 7. IBO.

Clearly, any pair of the above 7 balls have exactly one colour in common (satisfying condition 2). Each of the balls contribute 3 pairs of colours. In all, we have 21 pairs of colours in all the 7 balls. Now, 7 colours lead to $\frac{7\times 6}{2} = 21$ pairs of colours and each pair of colours is found in exactly one ball (satisfying condition 1). Each ball has 3 colours (condition 4 satisfied). Now, consider the four colours G, R, Y, V. No three of these colours are found on a ball (condition 3 is satisfied).

Thus, the total number of colours is 7 and the total number of balls is also 7.

Problem 46 *It is proposed to partition the set of positive integers into two disjoint subsets A and B. Subject to the following conditions:*

(i) 1 *is in A.*

(ii) *No two distinct members of A have a sum of the form*

$$2^k + 2 (k = 0, 1, 2, \ldots).$$

(iii) *No two distinct members of B have a sum of the form*

$$2^k + 2 (k = 0, 1, 2, \ldots).$$

Show that this partitioning can be carried out in a unique manner and determine the subsets to which 1987, 1988, 1989, 1997, 1998 *belong.*

Solution: Since it is given that $1 \in A$, $2 \notin A$. For if $2 \in A$, then $2^0 + 2 = 3$ is generated by 2 members of A violating the condition for the partitioning.

$$\therefore\ 2 \in B$$

Similarly, $3 \notin A$ as $1 + 3 = 4 = 2^1 + 2$

$$\therefore\ 3 \in B$$

But $4 \notin B$. For if $4 \in B$, then $2^2 + 2 = 4 + 2 = 6$ is generated by two members of B.

$\therefore$ The partitioning for the first few positive integers is

$A = \{1, 4, 7, 8, 12, 13, 15, 16, 20, 23, \ldots\}$

$B = \{2, 3, 5, 6, 9, 10, 11, 14, 17, 18, 19, 21, 22, \ldots\}$

Suppose $1, 2, \ldots, n - 1$ (for $n \geq 3$) have already been assigned to $A \cap B$ in such a way that no two distinct members of A or B have a sum $= 2^l + 2$ $(l = 0, 1, 2, \ldots)$

Now, we need to assign n to A or B.

Let k be a positive integer such that $2^{k-1} + 2 \leq n < 2^k + 2$. Then, assign '$n$' to the complement of the set to which $2^k + 2 - n$ belongs. But for this, we need to check that whether $2^k + 2 - n$ has already been assigned or not. Now as $n \geq 2^{k-1} + 2 > 2^{k-1} + 1$

$$2n > 2^k + 2$$

$$\therefore\ n > 2^k + 2 - n$$

Since all numbers below n have been assumed to be assigned to either A or B, $2^k + 2 - n$ has already been assigned and hence n is also assigned uniquely.

For example, consider $k = 1$

$$3 = 2^0 + 2 \le n < 2^1 + 2 = 4.$$

Consider $n = 3$, $4 - n = 1$ Now $1 \in A$ (given)

$$\therefore 3 \in B$$

Consider $k = 2$

$$\therefore 2^{2-1} + 2 \le n < 2^2 + 2 = 6$$

$$4 \le n < 6$$

When $n = 4$, as $6 - n = 2 \in B$, we assign 4 to A.

When $n = 5$ as $6 - 5 = 1 \in A$, we assign 5 to B.

Since the set to which n gets assigned is uniquely determined by the set to which $2^k + 2 - n$ belongs, the partitioning is unique.

Looking at the pattern of the partitioning of the initial set of positive integers, we conjecture the following:

1. $n \in A$ if $4 \mid n$
2. $n \in B$ if $2 \mid n$ but $4 \nmid n$
3. If $n = 2^r \cdot k + 1$, $r \ge 1$, k odd, then $n \in A$ if k is of the form $4m - 1$ and $n \in B$ if k is of the form $4m + 1$.

Proof of the conjecture: We note that $1, 4 \in A$ and $2, 3 \in B$. If $2^{k-1} + 2 \le n < 2^k + 2$ and all numbers less than n have been assigned to A or B and satisfy the above conjectures, then if $4 \mid n$, as $2^k + 2 - n$ is divisible by 2 but not by 4, $2^k + 2 - n \in B$. Hence, $n \in A$. Similarly, if 2 divides n but not 4, then $2^k + 2 - n$ is divisible by 4 and hence, is in A.

$$\therefore n \in B.$$

If $n = 2^r \cdot k + 1$ where $r \ge 1$, k odd and $k = 4m - 1$, then

$$2^k + 2 - n = 2^k - 2^r \cdot k + 1 = 2^r(2^{k-r} - k) + 1$$

where clearly $2^{k-r} - k$ is odd and equals 1 (mod 4).

$$\therefore 2^k + 2 - n \in B.$$

Hence, $n \in A$. Similarly, it can be shown that if $n = 2^r \cdot k + 1$, where $k \equiv 1 \pmod 4$, then $n \in B$. Thus, the conjecture is proved.

Now, 1988 is divisible by 4.

$$\therefore 1988 \in A$$

$$1987 = 2^1 \cdot 993 + 1 \quad \text{where } 993 = 1 \pmod 4$$

$$\therefore 1987 \in B$$

$$1989 = 2^2 \cdot 497 + 1 \quad \text{where } 497 = 1 \pmod 4$$

$$\therefore 1989 \in B$$

$$2 \mid 1998 \text{ but } 4 \nmid 1998$$

$$\therefore 1998 \in B$$

$$1997 = 2^2 \cdot 499 + 1 \quad \text{where } 499 = 3 \pmod 4$$

$$\therefore 1997 \in A.$$

Check Your Understanding

1. Given $p, q \in \mathbb{N}$, prove that $\sum_{k=1}^{q-1}\left\lfloor \frac{kp}{q} \right\rfloor = \sum_{k=1}^{p-1}\left\lfloor \frac{kq}{p} \right\rfloor$

2. Prove that $\sum_{d|n}\phi(d) = n$, where $\phi(d)$ = number of positive integers coprime with d and less than or equal to d.

3. Prove that $\sum_{k=1}^{n}\tau(k) = \sum_{k=1}^{n}\left\lfloor \frac{n}{k} \right\rfloor$ where $\tau(k)$ is number of divisors of k.

4. Prove that $\frac{{}^{2n}C_n}{n+1} = {}^{2n}C_n - {}^{2n}C_{n-1}$ and hence or otherwise, deduce that ${}^{2n}C_n$ is always divisible by $(n + 1)$.

5. Prove that $\sum_{P\subseteq X}\sum_{Q\subseteq X}|P \cap Q| = n4^{n-1}$,where X is a set of n elements.

6. Let n and r be integers with $0 \le r \le n$. Find a simple expression for $S_r = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^r\binom{n}{r}$.

7. Let n be positive integer not less than 3. Find a direct combinational interpretation of the identity $\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$.

8. Find the number of functions $f: \{1, 2, 3, \ldots, n\} \to \{1947, 1951, 2018, 2020\}$ such that $f(1) + f(2) + \cdots + f(n)$ is odd.

9. Let n be a positive integer. Prove that the binomial coefficients $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \cdots, \binom{n}{n-1}$ are all even if and only if n is a power of 2.

10. Find all $n \in \mathbb{N}$, such that $\binom{n}{r}$ is odd $\forall r = 0, 1, 2, \ldots, n$.

11. Delete 1 0 1 digits from the number 1 3 5 7 9 11 13 15 17 19 … 109 111 in such a way that the remaining number is
    (i) as small as possible,
    (ii) as big as possible.

12. You are given 7 sheets of paper and you cut any number of these into 7 small pieces. Out of the total sheets you get, you again cut some into 7 pieces and you continue the process. At every stage you count the total number of sheets you have. Show that you will never get 605 pieces.

13. During election campaign, n different kinds of promises are made by various political parties, $n > 0$. No two parties have exactly the same set of promises. While several parties may make the same promise, every pair of parties have atleast one promise in common. Prove that there can be at most 2^{n-1} parties.

14. The number 3 can be expressed as an ordered sum of one or more positive integers in four ways as follows:
$3, 1 + 2, 2 + 1, 1 + 1 + 1$.
Show that the positive integer n can be so expressed in 2^{n-1} ways.

15. Let n be any natural number. Find the sum of the digits appearing in the integers $1, 2, 3, \ldots, 10^n - 2, 10^n - 1$.

16. Let $f(n)$ denote the number of solutions (x, y) of $x + 2y = n$ for which both x and y are non-negative integers. Show that $f(0) = f(1) = 1, f(n) = f(n-2) + 1, n = 2, 3, 4, \ldots$. Find a simple explicit formula for $f(n)$.

17. At a party, there are more than 3 people. Every four of the people have the property that one of the four is acquainted with the other three. Show that with the possible exception of three of the people, every one at the party is acquainted with all of the others at the party.

18. What is the least number of plane cuts required to cut a block of size $a \times b \times c$ into abc unit cubes if piling is permitted?

19. In a mathematical competition, a contestant can score 5, 4, 3, 2, 1, or 0 points for each problem. Find the number of ways he can score a total of 30 points for 7 problems.

20. Every person, who has ever lived has upto this moment, made a certain number of hand-shakes. Prove that the number of people who have made an odd number of handshakes is even.

21. Show that among any seven distinct positive integers not greater than 126, one can find two of them, say, x and y satisfying the inequalities $1 < \frac{x}{y} \leq 2$.

22. Given a set of $(n + 1)$ positive integers none of which exceeds $2n$, show that atleast one member of the set must divide another member of the set.

23. There are six closed discs in a plane such that none contains the centre of any other disc (even on the boundary). Show that they do not have a common point.

24. Prove that if 5 pins are stuck on to a piece of cardboard in the shape of an equilateral triangle of side length 2, then some pair of pins must be within distance 1 of each other.

25. Given any $(n + 2)$ integers show that for some pair of them either their sum or their difference is divisible by $2n$.

26. Two players, play the game. The first player selects any integer from 1 to 11 inclusive. The second player adds any positive integer from 1 to 11 inclusive to the number selected by the first player. They continue in this manner alternatively. The player who reaches 56 wins the game. Which player has the advantage?

27. You are given 6 congruent balls two each of colours red, white and blue and informed that one ball of each colour weighs 15 gram, while the other weighs 16 grams. Using an equal arm balance only twice, determine which three are the 16 gram balls.

28. Find the number of integers in the set $\{1, 2, \ldots, 10^3\}$ which are not divisible by 5 nor by 7 but are divisible by 3.

29. Find the number of integers in the set $\{1, 2, \ldots, 120\}$ which are divisible by exactly m of the integers 2, 3, 5, 7 where $m = 0, 1, 2, 3, 4$.

30. For how many paths consisting of a sequence of horizontal and/or vertical line segments with each segment connecting a pair of adjacent letters in the diagram below is the word MATHEMATICS spelled out as the path is traversed from beginning to end.

M
M A M
M A T A M
M A T H T A M
M A T H E H T A M
M A T H E M E H T A M
M A T H E M A M E H T A M
M A T H E M A T A M E H T A M
M A T H E M A T I T A M E H T A M
M A T H E M A T I C I T A M E H T A M
M A T H E M A T I C S C I T A M E H T A M

31. A group of 100 students took examination in English, Science and Mathematics. Among them, 92 passed in English, 75 in Science and 63 in Mathematics; at most 65 passed in English and Science, at most 54 in English and Mathematics and at most 48 in Science and Mathematics. Find the largest possible number of the students that could have passed in all the three subjects.

32. Lines $L_1, L_2, \ldots, L_{100}$ are distinct. All lines L_{4n}, n being positive integer are parallel to each other. All lines L_{4n-3}, n a positive integer pass through a given point A. Find the maximum number of points of intersection of pairs of lines from the complete set $(L_1, L_2, \ldots, L_{100})$.

33. How many integers with four different digits are there between 1,000 and 9999 such that the absolute value of the difference between the first digit and the last digit is 2?

34. A multi set is an ordered collection of elements, where elements can repeat. For example, $\{a, a, b, c, c\}$ is a multiset of size five. Discover the number of multisets of size four, which can be constructed from the given 10 distinct elements.

35. Find the number of numbers from 1 to 10^{100}, having the sum of their digits equal to 3.

36. Two students from Standard XI and several students from Standard XII participated in a chess tournament. Each participant played with every other once only. In each game, the winner has received one point, the loser zero and for the game drawn, both the players got 0.5 points each. The two students from Standard XI together scored 8 points and the scores of all the participants of Standard XII are equal.
(i) How many students of Standard XII participated in the tournament?
(ii) What was the equal score in Standard XII?

37. Show that an equilateral triangle, cannot be covered completely by two smaller equilateral triangles.

38. The diagonal connecting two opposite vertices of a rectangular parallelepiped is $\sqrt{73}$ units. Prove that if the squares of the edges of the parallelepiped are integers, then its volume cannot exceed 120.

39. In a group of 7 people, the sum of the ages of the members is 332 years. Prove that three members can be chosen, so that the sum of their ages, is not less than 142 years.

40. Ten students solved a total of 35 problems in a Mathematics contest; each problem was solved by exactly one student. There is one student who solved exactly one problem, at least one student who solved exactly two problems and at least one student who solved exactly three problems. Prove that, there is also at least one student, who has solved at least 5 problems.

41. Let T be the set of triplets (a, b, c) of integers, such that $1 < a < b < c < 6$. For each triplet (a, b, c) consider the number $a \times b \times c$. Add all these numbers corresponding to the triplets in T. Prove that the resulting sum is a multiple of seven.

42. There are 9 cells in a 3×3 square, when these cells are filled by numbers -1, 0, 1. Prove that, of the 8 sums obtained, at least two sums are equal.

43. How many 6-digit numbers are there such that

(i) The digits of each number are all from the set {1, 2, 3, 4, 5}

(ii) Any digit that appears in the number appears at least twice.

(Example: 225252 is admissible while 222133 is not).

44. Show that, in any group of 5 students there are two students who have identical number of friends within the group.

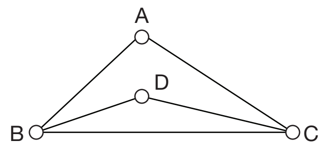
45. Given 11 different natural numbers, none greater than 20. Prove that, two of these can be chosen, one of which divides the other.

46. Find the number of 6-digit natural numbers, such that the sum of their digits is 10 and each of the digits 0, 1, 2, 3, occurs at least once in them.

47. Prove that, among 18 consecutive 3-digit numbers, there is at least one number, which is divisible by the sum of the digits.

48. A rectangle with sides $2m-1$ and $2n-1$ is divided into squares of unit length by drawing parallel lines to the sides. Find the number of rectangles possible with odd side lengths.

49. A road network as shown in the figure connect four cities. In how many ways can you start from any city (say A) and come back to it without travelling on the same road more than once?



50. Consider the lines $x=k$ and $y=k$, $k\in\{1, 2, \ldots, 9\}$. The number of non-congruent rectangles, whose sides are along these lines, is ______.

51. A point P, is at a distance of 12 cm from the centre of a circle of radius 13 cm. Find the number of chords of the circle passing through P which have integral lengths.

52. Let P_n denotes the number of ways of selecting 3 people out of 'n' sitting in a row, if no two of them are consecutive and Q_n is the corresponding figure when they are in a circle. If $P_n - Q_n = 6$, then find the value of n.

53. Take a ΔABC. Take n points of sub-division on side AB and join each of them to C. Likewise, take n points of sub-division on side AC and join each of them to B. Into how many parts is ΔABC divided?

54. Each side of an equilateral ΔABC is divided into 6 equal parts. The corresponding points of subdivision are joined. Find the number of equilateral triangles oriented the same way as ΔABC.

55. Let $n = 10^6$. Evaluate $\sum\limits_{d/n} \log_{10} d$.

56. Let $n = 180$. Find the number of positive integral divisors of n^2, which do not divide n.

57. Show that the number of positive integral divisors of 111 … 1(2010 times) is even.

58. How many unordered pairs $\{a, b\}$ of positive integers a and b are there such that LCM (a, b)= 1,26,000?

(***Note:*** An unordered pair $\{a, b\}$ means $\{a, b\} = \{b, a\}$)

59. The sum of the factors of 7!, which are odd and are of the form $3t+1$ where t is a whole number, is ______.

60. Consider a set {1, 2, 3,…, 100}. Find the number of ways in which a number can be selected from the set so that it is of the form x^y, where $x, y \in N$ and ≥ 2, is ______.

Challenge Your Understanding

1. Let A, B be disjoint finite sets of integers with the following property.
If $x \in (A \cup B)$, then either $x + 1 \in A$ or $x - 2 \in B$.
Prove that $n(A) = 2n(B)$ [*i.e.*, $|A| = 2|B|$].

2. Find all positive integers k, for which the set $A = \{1996, 1996 + 1, 1996 + 2, \ldots, 1996 + k\}$ with $k + 1$ elements can be partitioned into two subsets B and C such that the sum of the elements of B = sum of the elements of C.

3. Suppose you and your husband attended a party with three other married couples. Several hand-shakes took place. No one shook hands with himself or (herself) or with his (or her) spouse, and no one shook hands with other more than once. After all the hand-shaking was completed, suppose you asked each person including your husband, how many hands he or she had shaken? Each person gave a different answer.
(i) How many hands did you shake?
(ii) How many hands did your husband shake?

4. Let $S = \{1, 2, \ldots, 100\}$ and A be any subset of S containing 53 members. Show that A has two numbers a, b such that $a - b = 12$. Construct a subset B of S with 52 numbers such that for any two numbers a, b of B, $|a - b| \neq 12$.

5. Let A be any set of 19 distinct integers chosen from the AP 1, 4, 7, 10, …, 100. Show that A must contain at least two distinct integers whose sum is 104. Find a set of 18 distinct integers from the same progression such that the sum of no two distinct integers from the set equals 104.

6. In a room containing N people $N > 3$, at least one person has not shaken hands with every one else in the room. What is the maximum number of people in the room that could have shaken hands with every one else?

7. A positive integer n has the decimal representation $n = d_1 d_2 \ldots d_m$.
(i) n is called ascending if $0 < d_1 \leq d_2 \leq \ldots \leq d_m$
(ii) n is called strictly ascending if $0 < d_1 < d_2 < \ldots < d_m$.
Find the total number of type (i) and type (ii) numbers, which are less than 10^9.

8. Let $N(k) = \{1, 2, \ldots, k\}$. Find the number of:
(i) functions from $N(n)$ to $N(m)$.
(ii) one-to-one functions from $N(n)$ to $N(m)$, $n \leq m$.
(iii) strictly increasing functions from $N(n)$ to $N(m)$, $n \leq m$.
(iv) non- decreasing functions from $N(n)$ to $N(m)$.

9. Let $n = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^4$. Find the number of positive integral divisors of n which are greater than $\sqrt{n}$.

10. Let $m = \sum_{i=0}^{k} m_i p^i, n = \sum_{i=0}^{k} n_i p^i$; $m_i, n_i \in \{0, 1, 2, \ldots, p-1\}$ and p is a prime number, prove that $\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p}$.

11. Let $T(n)$ denote the number of non-congruent triangles with integer side lengths and perimeter n.
Thus $T(1) = T(2) = T(3) = T(4) = 0$, while $T(5) = 1$. Prove that
(i) $T(2006) < T(2009)$
(ii) $T(2005) = T(2008)$.

12. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be distinct points in a plane. Let D and d be the longest and the shortest distances respectively between pairs of points among them. Prove that, $\frac{D}{d} \geq \sqrt{3}$.

13. Several football teams enter a tournament, in which, each team play every other team exactly once. Show that, at any moment, during the tournament, there will be two teams, which have played up to that moment, an identical number of games.

14. Given 7-element of set $A = \{a, b, c, d, e, f, g\}$. Find a collection T of 3-element subsets of A, such that each pair of elements from A, occurs exactly in one of the subsets of T.

15. In how many different ways, can the digits 1 through 5, be arranged to form a five digit number, in which, the digits, alternately rise and fall? These numbers are called Mountain Numbers; for example, 13254 is a Mountain Number while 12354 is not.

16. If A is a 50 element subset of the set $\{1, 2, 3, \ldots, 100\}$ such that, no two numbers from A, add upto 100, show that A contains a square.

17. Show that, there exist two powers of 1999, whose difference is divisible by 1998.

18. If the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are divided into three groups, show that, the product of the numbers in one of the groups, exceeds 71.

19. Show that, there exists a power of 3 which ends in the digits 001.

20. If 181 square integers are given, prove that, one can find a subset of 19 numbers among these such that, the sum of these elements is divisible by 19.

21. Given any 13 distinct real numbers, prove that, there are two of them, say x and y, such that, $0 < \dfrac{x-y}{1+xy} < 2-\sqrt{3}$.

22. Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person A phones to person B and tells B everything what he knows, while B tells A nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything?

23. Consider a rectangular array of dots with an even number of rows and an even number of columns. Colour the dots, each one red or blue, subject to the condition that a each row, half the dots are red and the other half are blue and in each column also, half the, dots are red and the other half are blue. Now, if two points are adjacent and like coloured, join them by an edge of their colour. Show that the number of blue segments is equal to the number of red segments.

24. Teams $T_1, T_2, \ldots, T_n$ take part in a tournament in which every team plays every other team just once. One point is awarded for each win and it is assumed that there are no draws. Let $s_1, s_2, \ldots, s_n$ denote the total scores of $T_1, T_2, \ldots, T_n$ respectively. Show that for $1 < k < n$, $s_1 + s_2 + \cdots + s_k \le n_k - \dfrac{1}{2}\, k(k+1)$.

25. Seventeen people correspond by mail with one another each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of the topics. Prove that there are atleast three people who write to one another about the same topic.

26. No matter which 55 positive integers one may select from 1, 2, 3, …, 100. Prove that there will be some two that differ by 9, some two that differ by 10, some two that differ by 12, some two that differ by 13, but surprisingly their need not be any two that differ by 11.

27. There is a $2n \times 2n$ array (matrix) consisting of 0's and 1's and there are exactly $3n$ zeroes. Show that it is possible to remove all the zeroes by deleting some n rows and some n columns.

28. Let $a(n)$ denote the number of ways of expressing the positive integer n as an ordered sum of l's and 2's, e.g., $a(5) = 8$ because $5 = 1 + 1 + 1 + 1 + 1 = 2 + 1 + 1 + 1 = 1 + 2 + 1 + 1 = 1 + 1 + 2 + 1 = 1 + 1 + 1 + 2 = 2 + 2 + 1 = 2 + 1 + 2 = 1 + 2 + 2$. Let $b(n)$ denote the number of ways of expressing n as an ordered sum of integers greater than 1, for example, $b(7) = 8$ because $7 = 3 + 2 + 2 = 2 + 3 + 2 = 2 + 2 + 3 = 3 + 4 = 4 + 3 = 2 + 5 = 5 + 2 = 7$. Prove that $a(n) = b(n + 2)$ for $n = 1, 2, \ldots$.

29. A pack of 13 distinct cards is shuffled in some particular manner and then repeatedly in exactly the same manner. What is the maximum number of shuffles required for the cards to return to their original positions?

30. Each of n boys attends a school-gathering with both his parents. In how many ways can the $3n$ people be divided into groups of three such that each group contains a boy, a male parent and a female parent, and no boy is with both his parents in his group?

31. A permutation $a_1, a_2, \ldots, a_n$ are $1, 2, 3, \ldots, n$ is said to be good if and only if $(a_j - j)$ is constant for all j, $1 \le j \le n$. Determine the number of good permutations for $n = 1999$, $n = 2000$.

32. An international society has its members from six different countries. The list of members contains 1978 names numbered 1, 2, 3, …, 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country. **[IMO, 1978]**

33. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that

$$a_{2n-1} = 0,\ a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1}),\ n = 1, 2, 3, \ldots, \text{ where } x = 2 + \sqrt{2}, y = 2 - \sqrt{2}.$$

Here a path of n jumps is a sequence of vertices $(P_0, \ldots, P_n)$ such that

(i) $P_0 = A, P_n = E$.

(ii) for every i, $0 \le i \le n - 1$, P_i is distinct from E.

(iii) for every i, $0 \le i \le n - 1$, P_i and $P_i + 1$ are adjacent. **[IMO, 1979]**

34. Let n and k be given relatively prime natural numbers $k < n$. Each number in the set $M = \{1, 2, \ldots, n - 1\}$ is coloured either blue or white. It is given that

(i) for each $i \in M$ both i and $(n - i)$ have the same colour;

(ii) for each $i \in M$, $i \neq k$, both i and $(f - k)$ have the same colour.

Prove that all numbers in M have the same colour. **[IMO, 1985]**

35. $2 \times 2 \times n$ hole in a wall is to be filled with $2n$, $1 \times 1 \times 2$ bricks. In how many different ways can this be done if the bricks are indistinguishable?

36. Let $P_1, P_2, \ldots, P_n$ be distinct two element subsets of the set of elements $\{a_1, a_2, \ldots, a_n\}$ such that if $P_i \cap P_j \neq \phi$, then (a_i, a_j) is one of the P's. Prove that each of the a_s appears in exactly two of the P's.

37. Ten airlines serve a total of 1983 cities. There is direct service without a stop over between any two cities and if an airline offers a direct flight from A to B, it also offers a direct flight from B to A. Prove that at least one of the airlines provides a round trip with an odd number of landings.

38. Five students A, B, C, D, E took part in a contest. One prediction was that the contestants could finish in the order A B C D E. This prediction was very poor. In fact, no contestant finished in the position predicted and no two contestants predicted to finish consecutively did so. A second prediction had the contestants finishing in the order D A E C B. This prediction was better. Exactly two of the contestants finished in the places predicted and two disjoint pairs of students predicted to finish consecutively actually did so. Determine the order in which the contestants finished.

39. Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections these perpendiculars can have.

40. In a plane, a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the segment we define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n.

41. In a mathematical contest, the three problems A, B and C were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem A, the number who solved problem B, was twice the number who solved C. The number of students who solved only problem A was one more than the number of students who solved A and at least one other problem. Of all students who solved just one problem, half did not solve problem A. How many students solved only problem B?

42. In a sports contest, there were m medals awarded on n successive days ($n > 1$), on the first day, one medal and $\frac{1}{7}$ of the remaining $(m - 1)$ medals were awarded on the second day, two medals and $\frac{2}{7}$ of the now remaining medals were awarded; and so on. On the nth and last day, the remaining n medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

43. Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of given points.

44. A certain organization has n members and it has $(n + 1)$ three member committees, no two of which have identical membership. Prove that there are two committees which share exactly one member. **[USA MO, 1979]**

45. In a party with 1982 persons, among any group of four there is at least one person who knows each of the other three. What is the minimum number of people in the party who know everyone else? **[USA MO, 1982]**

46. On an infinite chess board, a game is played as follows: At the start n^2 pieces are arranged on the chess board in $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square immediately beyond the piece who has been jumped over is then removed. Find those values of n for which the game will end with only one piece remaining on chess board. **[IMO, 1993]**

47. Find the number of ways in which one can place the numbers $1, 2, \ldots, n^2$ on square of $n \times n$ chess board, one on each such that the numbers in each row and each column are in AP (assume $n \geq 3$). **[INMO, 1992]**

48. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either coloured blue or red or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured, the set of coloured edges necessarily contains a triangle all of whose edges have the same colour. **[IMO, 1992]**

49. Nine mathematicians meet at an international conference and discover that among any three of them, at least two speak a common language. If each of the mathematicians can speak utmost three languages, prove that there are atleast three of the mathematicians who can speak the same language. **[USA MO, 1979]**

50. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an AP? Justify your answer. **[IMO, 1983]**