Nim Game

Course on Game Theory and Greedy Algorithms

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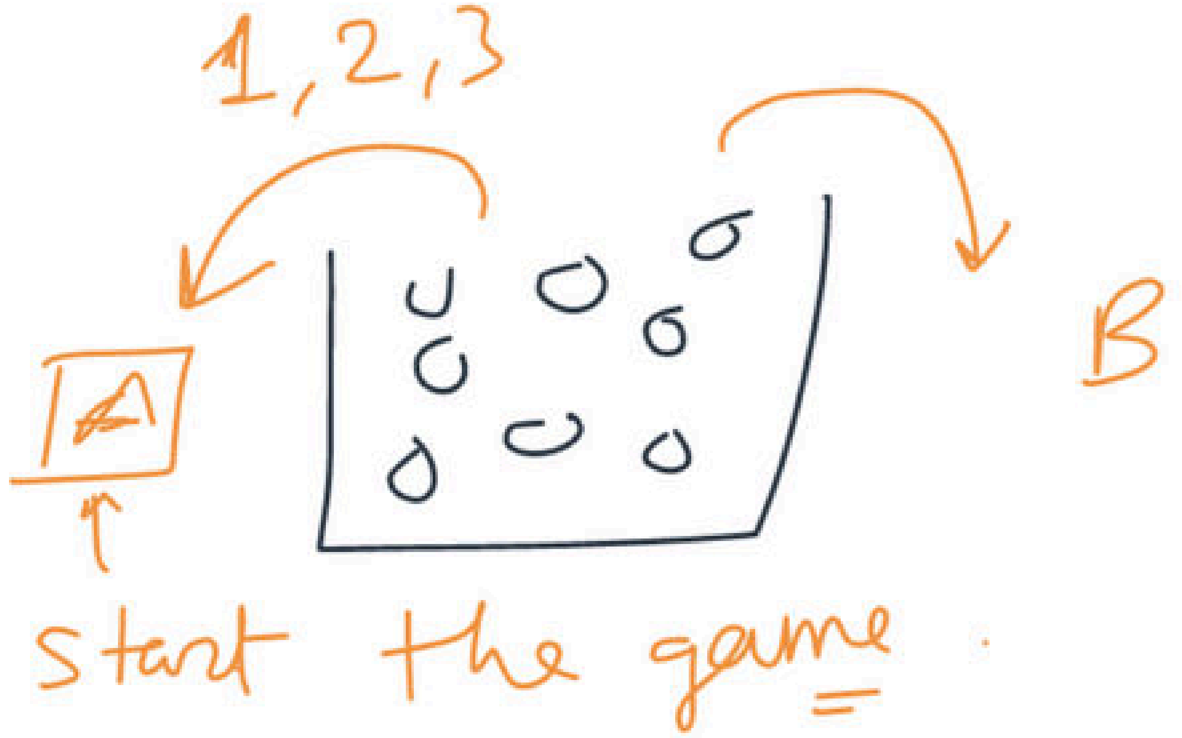
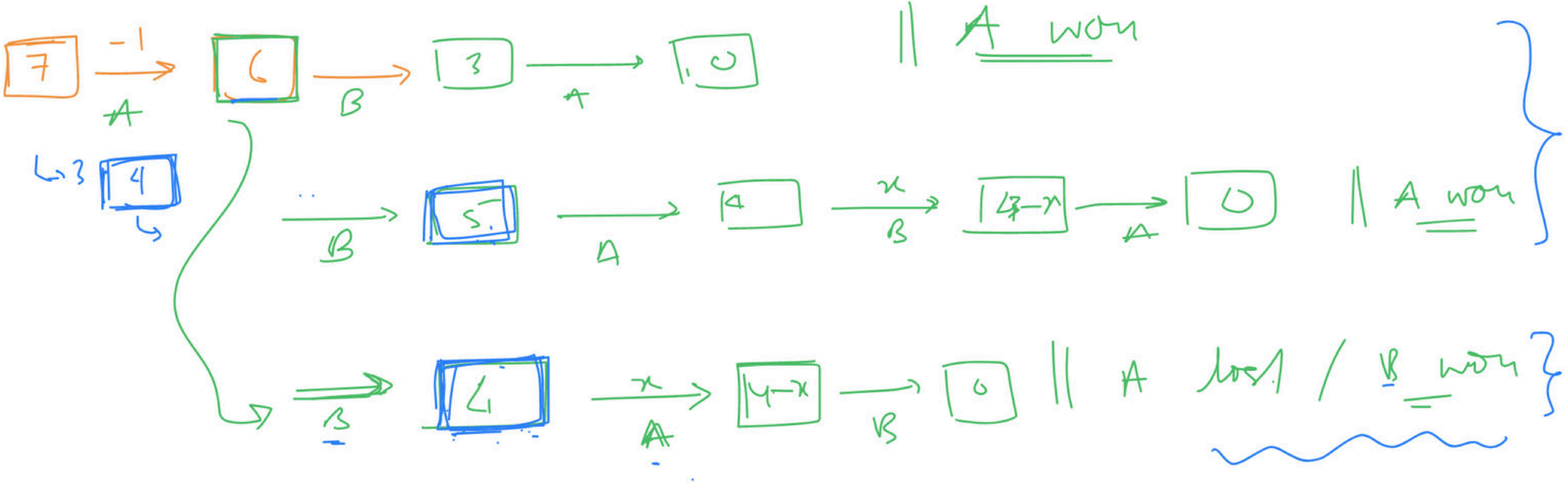
Two player games

↳ Opponent is very intelligent

↳ No randomness

__Example__: Given 'n' sticks, and a player is allowed to select either 1, 2 or 3 sticks from the pile. Whoever picks the last stick wins. Players take alternative turns.





Outcome of the game depends entirely on the intial given condition. When we assume 'A' and 'B' to be intelligents and having infinite processing power etc.

$\boxed{7} \xrightarrow[A]{} \boxed{4} \xrightarrow[B]{x} \boxed{4-x} \longrightarrow \boxed{0}$ || $\underline{A}$ won

$\boxed{7}$ stones $\Rightarrow$ 'A' is always going to win.

$\downarrow -3$

$\boxed{4}$ stones $\Rightarrow$ 'A' is always going to lose

4

first player

'x' : we try to figure out if it is a winning state or losing state.

Dynamic Prog Approach.

1	$\boxed{2}$	3	4	5	6	7	8	$\boxed{9}$	10	11	12	13
W	W	W	L	W	W	W	L	W	W	W	L	. . .
1	2	3	0	1	2	3	0	1	2	3	0	

g..n

<u>Conclusion</u>: If n is a multiple of "4" then you lose otherwise you win.

<u>State</u>: The circumstances on which the game depends.

<u>Ex</u>: number of sticks / stones.

W L

7	5	11	6
3	1	3	2

Suppose the game starts with stage "s" and according to the game rules you can reach to states $\{s_1\ s_2\ \dots\ s_k\}$ in one step. Then to determine whether 's' is winning or losing, we can do the following:

If all s_i $(1 \le i \le k)$ are winning $\Rightarrow$'s' is losing

o/w $\Rightarrow$'s' is winning

We noticed a pattern: $\left\{ \begin{array}{l} n \% 4 = 0 \\ n \% 4 \neq 0 \end{array} \right\}$ || losing
|| winning

we still need to prove it.

'W'

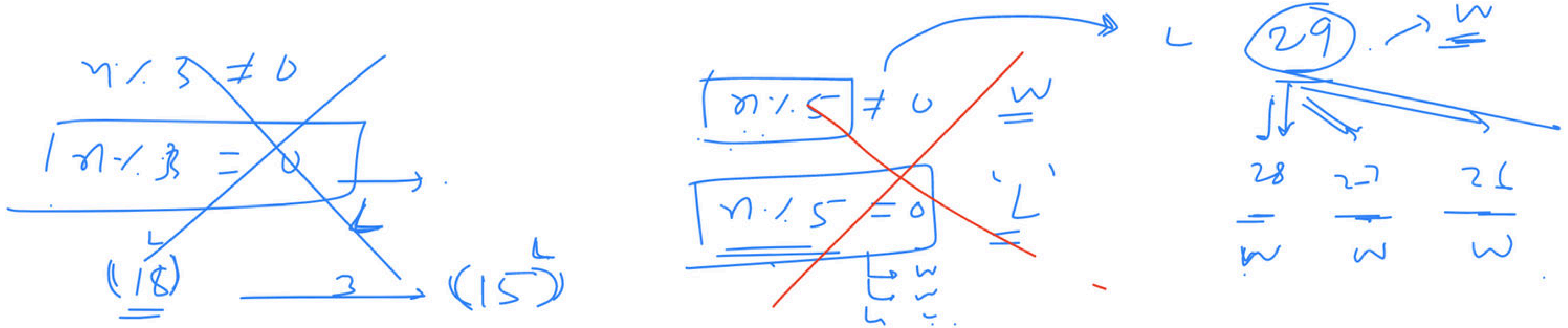
If $n \% 4 \neq 0 \longrightarrow$ You can always make sure that the next state is $m \% 4 = 0$.

'L'

If $n \% \boxed{4} = 0 \longrightarrow$ Anything you do, the next state is always going to satisfy $m \% 4 \neq 0$.

1, 2, 3

Base Case : $n = 0 \longrightarrow$ Losing Case || $n \% 4 = 0$



$N \xrightarrow{-d} (N-d) \longrightarrow \ldots$

'L' $\boxed{n \% 2 \neq 0}$ $\xrightarrow{d}$ W $\boxed{m \% 2 = 0}$

All steps leads to winning condition

$\downarrow$

d / n

$\boxed{d \% 2 \neq 0}$

d is a divisor of N and $d \neq N$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
L	W	L	W	L	W	L	W	L	W	L	W	L	W	L

$\boxed{n \% 2 \neq 0} \longrightarrow L$

$\boxed{n \% 2 = 0} \longrightarrow W$

} we need to prove it

$\boxed{n \% 2 = 0} \xrightarrow[\text{one step}]{\text{At-least}} m \% 2 \neq 0$

'1' is always a divisor

$m = \text{"}n-1\text{"} \% 2 \neq 0$

a_1 a_2 a_3 a_4 a_5 . . . a_n

You can remove any number of stones from any one pile/bucket.

$3 \quad 2 \quad 3 \Rightarrow W$

$\Downarrow A$

$3 \quad 2 \quad 1$

$\Downarrow B$

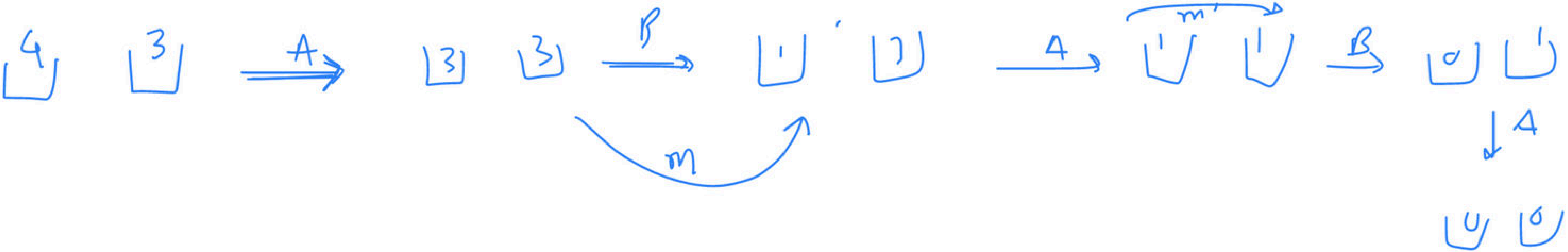
$0 \quad 2 \quad 1$

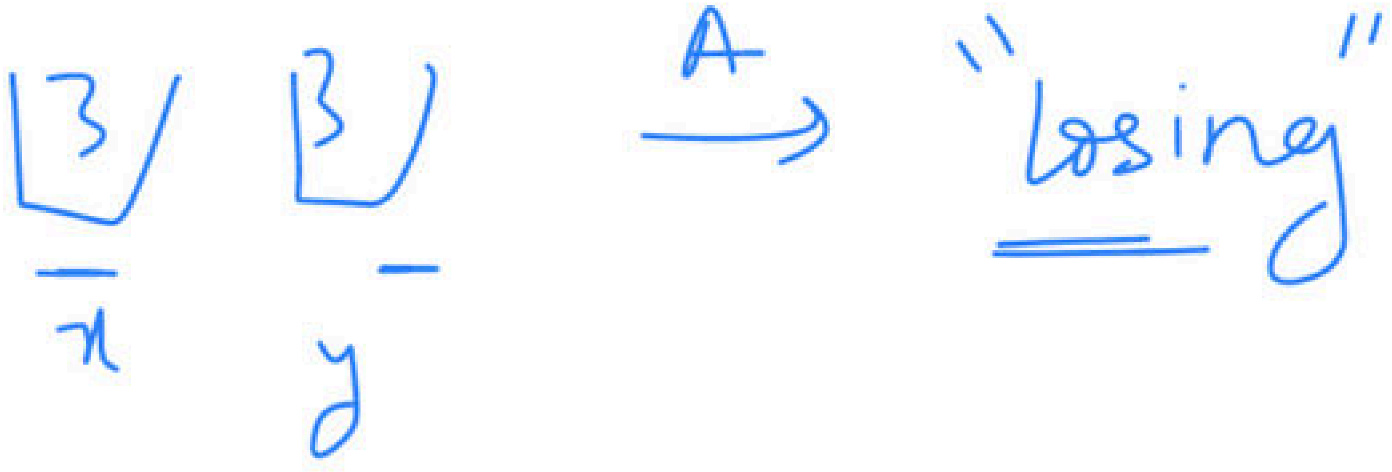
$\Downarrow A$

$0 \quad 1 \quad 1 \xrightarrow{B} 0 \quad 0 \quad 1 \xrightarrow{A} 0 \quad 0 \quad 0$

A wins

$\boxed{a_1 \times a_2 \times a_3 \times a_4 \cdots a_n} \rightarrow$ states





"You loose"

if $b[0] == b[1] \Longrightarrow$ Losing

o/w $\longrightarrow$ winning

$7 \; 4 \; 3 \xrightarrow{B} 5 \; 4 \; 3 \xrightarrow{A} 5 \; 4 \; 1 \xrightarrow{B} 3 \; 4 \; 1 \xrightarrow{A} 3 \; 2 \; 1$

$3 \; 2 \; 1 \xrightarrow{B} 0 \; 2 \; 1 \xrightarrow{A} 0 \; 1 \; 1$

$3 \; 2 \; 1 \xrightarrow{B} 2 \; 2 \; 1 \xrightarrow{A} 2 \; 2 \; 0 \xrightarrow{B}$

$a_1 \wedge a_2 \wedge a_3 = 0 \;\; \| \;\; \text{'L'}$

$\neq 0 \;\; \| \;\; \text{'W'}$

$$\begin{array}{c} 0111 \\ 0100 \\ 0011 \\ \hline \boxed{0000} \end{array} \xrightarrow{B} \begin{array}{c} 101 \\ 100 \\ 011 \\ \hline 010 \end{array} \xrightarrow{A} \begin{array}{c} 101 \\ \boxed{110} \\ 001 \\ \hline 000 \end{array} \xrightarrow{B} \begin{array}{cc} 011 & 3 \\ 010 & 2 \\ 001 & 1 \\ \hline 010 & \end{array}$$

$$\begin{array}{c} 11 \\ 10 \\ 01 \end{array}$$

Hypothesis: if $a_1 \wedge a_2 \wedge \dots a_n = 0 \Longrightarrow L$

$\neq 0 \longrightarrow W$

$W \xrightarrow{\text{atleast 1 way}} L$ } Identify which bits are non zero.

So find any number whose s^{th} bit is $\boxed{0\ 0\ \boxed{1}\ 0\ 1\ \dots}$ $\leftarrow$

s^{th} bit

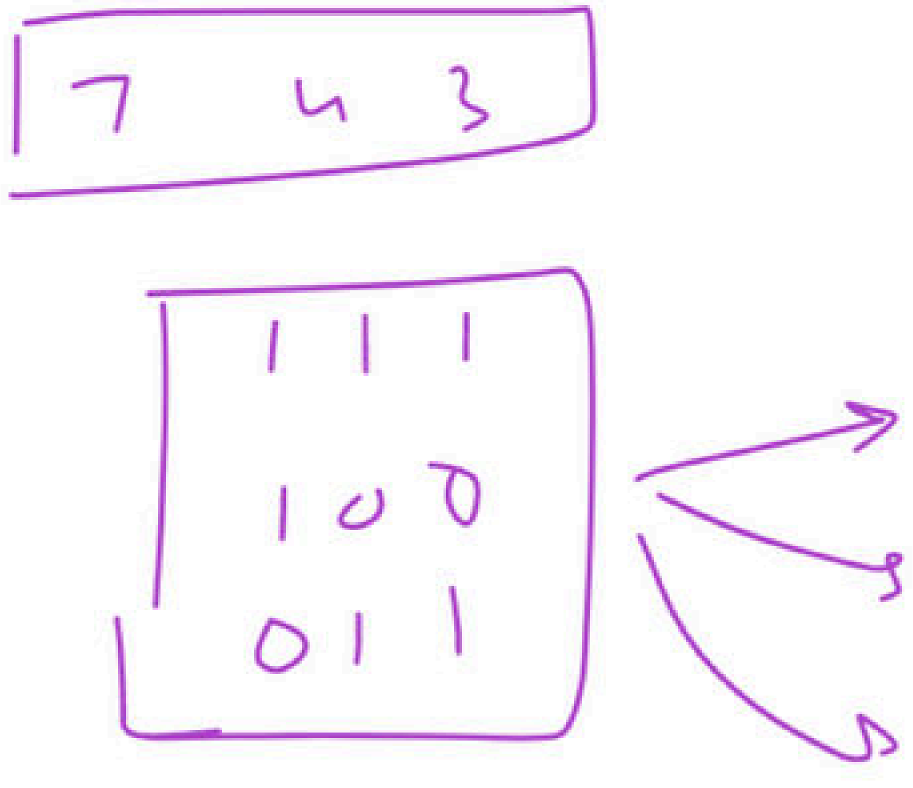
Set. "a_x" $\equiv$ $\boxed{x\ x\ x\ \underset{\uparrow\, s^{th}\text{ bit}}{1}\ x\ y\ y\ y} \longrightarrow \boxed{x\ x\ x\ 0\ x\ 1\ x\ x}$

Set the s^{th} bit to zero. And on the same number, change all other bits suitably so that the overall XOR is zero.

It is always true that the new number is less than the original number.

$$L \xrightarrow[\text{move}]{\text{any}} W$$

$$\boxed{a_1 \wedge a_2 \wedge a_3 \ldots \wedge a_x = 0} \xrightarrow[\text{number of stones from anywhere}]{\text{pick any}} (S_{new})_{\text{xor sum}} \neq 0.$$

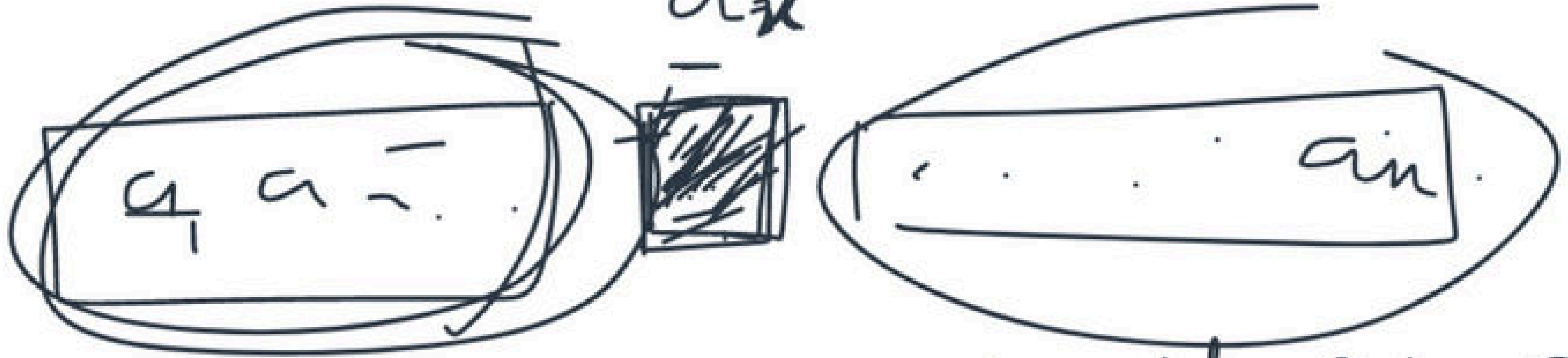


$$\boxed{a' \wedge x = 0} \Longrightarrow \boxed{x = a}$$

a_x'

find all possible 'x'.

a_x



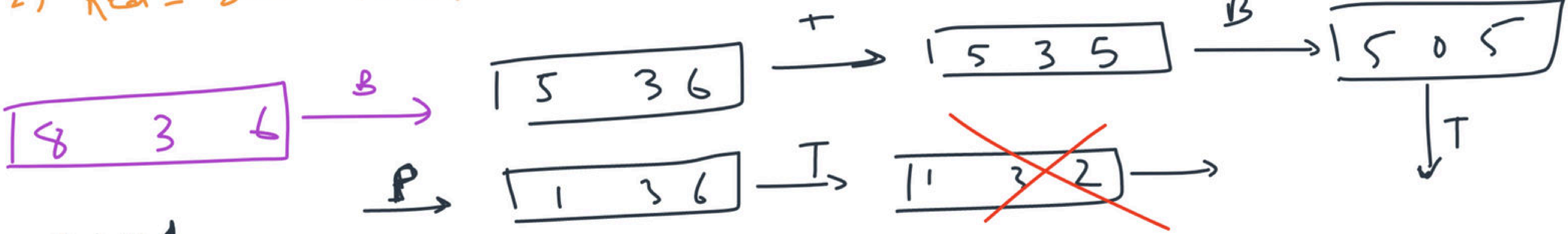
$$y \wedge a_x = 0$$

$$a_x \wedge a_x' \neq 0$$

xor sum of all a_i except $a_x = a_x$

1) NIM Game

2) Red - Blue Hackenbush



0 0 0 1
0 0 1 1
0 1 1 0
____
4 1 0 0

Modified NIM GAME

NIM $\rightarrow$ any number of stones

M-NM $\rightarrow \{x_1, x_2, x_3 \dots\}$ $\{1, 2, 3\}$

$\{2, 5\}$

Grundy numbers: Replace each pile by its grundy numbers. And then play the NIM Game on those.

$$\begin{matrix} a_1 & a_2 & \dots & a_n \\ \downarrow & \downarrow & & \downarrow \\ g_1 & g_2 & & g_n \end{matrix}$$

Game Theory and Greedy Algorithms for Interview Preparation

Game Theory

A winning state is a state from which current player can win the match if opponent makes some silly mistake in future

A. True
B. False

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A. True

B. False

Given a heap with N sticks in it. A player can pick only 1 or 2 sticks at a time. The last person to pick any stick will win.

What is the condition to check if we will win on starting with N sticks?

A. N should be divisible by 3
B. N should not be divisible by 3
C. N should be divisible by 2
D. N should not be divisible by 2

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A losing state is a state from which the current player will always lose no matter what move he makes, and opponent play optimally.

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A. True

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What should be the xor sum of a nim game to say current position is winning position?

A. Zero
B. One
C. Equal to size of heap
D. Non zero

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That’s all!