Recurrence Relation

A recurrence relation in mathematics is an eqⁿ that defines a sequence based on some rules where current term is dependent on prev terms.

Ex

$$f_n = f_{n-1} + f_{n-2}$$

$$\boxed{f_n = f_{n-1} + f_{n-2}}$$

$$f_n = f_{n/2} + \square$$

$$f_n = \max(f_{n-1},\ f_{n-2} + a_n)$$

MergeSort

$$f_n = f_{n/2} + f_{n/2} + g(n/2, n/2)$$

Multiple methods to solve recurrence

1) Recursion tree ✓

2) Guessing ✓

3) Master theorem } { ⟶ Bloomberg

4) Akra Bazzi formula

→ Master Theorem

↳ This theorem helps us to solve some DnC recurrences

DnC → Ex → binary search

merge sort

quicksort

BS $\rightarrow T(n) = T(n/2) + O(1)$

MS $\rightarrow T(n) = 2T(n/2) + O(n)$

QS $\rightarrow T(n) = 2T\left(n/2\right) + O(n)$

Best Case

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n)$$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$a \geq 1$

$b > 1$

func^n given you # of ops to apply on a problem of size n

a ? ⟷ const ⟷ b ?

no. of smaller subproblems we need to divide into

this denotes by how much fraction size of the problem will reduce.

$\frac{n}{b}$ → denotes size of the subproblems

f(n) → func^n applied on smaller subproblems to get the ans of bigger problem.

$$f(n) \rightarrow n^k \log^p n$$

p is a real no.

$k \geq 0$

Ex

$$T(n) = 3T\left(\frac{n}{2}\right) + n\log^2 n \checkmark$$

$$T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} \checkmark$$

master theory

$$T(n) = aT\left(\frac{n}{b}\right) + n^k \log^p n$$

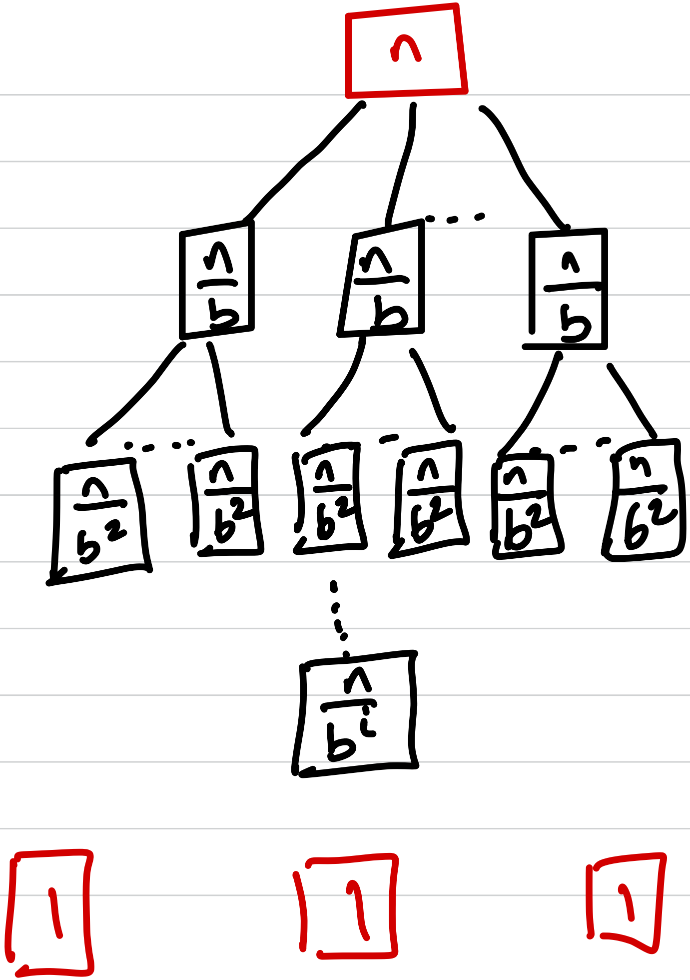
$$n \rightarrow \frac{n}{b} \rightarrow \frac{n}{b^2} \rightarrow \frac{n}{b^3} \cdots\cdots \frac{n}{b^x}$$

$$\frac{n}{b^k} = 1$$

$$k = \log_b n$$

$$a^{\log_b x} \longleftrightarrow x^{\log_b a}$$

$a, b \rightarrow$ const



0	1	$O(n^k)$
1	a	$a \times O\left(\frac{n}{b}\right)^k \rightarrow \frac{a}{b^k} O(n^k)$
2	a^2	$a^2 \times O\left(\frac{n}{b^2}\right)^k \rightarrow \left(\frac{a}{b^k}\right)^2 O(n^k)$
i	a^i	general term $\Rightarrow \left(\frac{a}{b^k}\right)^i O(n^k)$
$\log_b n$	$a^{\log_b n}$	$\underbrace{a^{\log_b n} \longleftrightarrow n^{\log_b a}}$

$$T \to \sum_{i=0}^{\log_b n} O(n^k) \left(\frac{a}{b^k}\right)^i \to \text{general sol}^n$$

$$gp$$

for $r \neq 1$

$$a + ar + ar^2 + ar^3 \text{ -------------- } ar^{n-1}$$

$$\frac{ar(1-r^n)}{1-r} \checkmark$$

asymptotic

$$\begin{cases} O(a) & r<1 \\ O(ar^{n-1}) & r>1 \\ O(na) & r=1 \end{cases}$$

$T \rightarrow \sum_{i=0}^{\log_b n} O(n^k) \left(\frac{a}{b^k}\right)^i$

$r \rightarrow \frac{a}{b^k}$

1 $r < 1 \rightarrow \frac{a}{b^k} < 1 \rightarrow a < b^k$

$O(n^k \log^p n)$

$p < 0$

ans $\rightarrow O(n^k)$

$p \geq 0$

ans $\rightarrow O(n^k \log^p n)$

(2) $r = 1 \rightarrow (\#\text{ of terms}) \times O(a)$

$\hookrightarrow \underline{(1 + \log_b n)}$

$\frac{a}{b^k} = 1 \rightarrow a = b^k$

$\rightarrow (1 + \log_b n)\, O(n^k \log^p n)$

$p > -1 \rightarrow O(n^k \log^{p+1} n)$

$p < -1 \rightarrow O(n^k)$

③ $r > 1 \Rightarrow \frac{a}{b^k} > 1$ $\quad a > b^k$

$n \to$ # of terms

$$O(n^k \log^p n)\left(\frac{a}{b^k}\right)^{\log_b n}$$

$$\downarrow$$

$$O(n^k \log^p n)\,\frac{a^{\log_b n}}{b^{k \log_b n}} \to O(n^k \log^p n)\,\frac{n^{\log_b a}}{n^k}$$

$$\underline{\underline{O(n^{\log_b a})}} \leftarrow \underline{\underline{O(n^{\log_b a} \times \log^p n)}}$$

Q 3

$$T(n) = 16\,T\left(\frac{n}{4}\right) + n^{1}$$

$$a > b^k$$

$$O\left(n^{\log_2 16}\right) \to O(n^2)$$

Q 4

$$T(n) = 2^n\,T\left(\frac{n}{2}\right) + n^n$$

Can't solve using master theorem

$$T(n) = 3T\left(\frac{n}{3}\right) + \sqrt{n}$$

$$n^{\log_b a} \rightarrow O\left(n^{\log_3 3}\right)$$

$$\underline{\underline{O(n)}}$$

$$T(n) = 16T\left(\frac{n}{4}\right) + \underline{\underline{n!}}$$

$$\begin{cases} T(n) = 3T(n-1) & \text{if } n > 0 \\ T(n) = 1 & \text{else} \end{cases}$$

$$T(n) = 3T(n-1) \rightarrow 3(3T(n-2)) \rightarrow 3^2 T(n-2)$$

$$3^2 [3T(n-3)] \rightarrow 3^3 T(n-3) \text{ - - - -}$$

$$3^n T(n-n) \rightarrow 3^n T(0)$$

$$\rightarrow 3^n$$

$$T(n) = 2T\left(\frac{n}{2}\right) + \underline{\underline{O(n)}}$$

recurrence tree

$$T(n) = \begin{cases} 2T(n-1) - 1 & \text{if } n > 0 \\ 1 & \text{else} \end{cases}$$

$$2T(n-1) - 1 \rightarrow 2(2T(n-2) - 1) - 1$$

$$\rightarrow 2^2 T(n-2) - 2 - 1$$

$$2^2(2T(n-3) - 1) - 2 - 1 \rightarrow 2^3 T(n-3) - 2^2 - 2 - 1$$

$$\vdots$$

$$2^n(\underbrace{T(n-n)}) - (\underbrace{2^{n-1} + 2^{n-2} + 2^{n-3} - - - - \cdots 2^2 + 2 + 1})$$

$$\Rightarrow 2^n - (1 \times 2^n - 1) \Rightarrow 2^n - 2^n + 1 \Rightarrow \underline{1}$$