Quick Sort

Course on Sorting and Searching

Rohit Mazumder • Lesson 4 • Feb 22, 2021

Agenda:

1. Sorting Problem (Recap)
2. The partitioning problem
3. Quicksort
4. Critical Analysis of Quicksort
5. Merge Sort vs Quick Sort
6. QuickSort: Performance Summary
7. Assessment
8. HW

What is the sorting problem?

Arranging elements in a specific order.

Commonly used orders are:

- Ascending order
- Descending order

Example: Arranging the rank list in decreasing order of marks, Alphabetic ordering.

Target: Sort an array in ascending order

The Partitioning Problem

Given an array A of N elements, your task is to rearrange this array such that every element that is smaller than A[0] occurs to the left of it and every element larger than or equal to A[0] occurs to its right.

NOTE: There may be multiple possible solutions. Print any of them.

[20, 5, 27, 3, 45, 30, 19, 77, 1] => [1, 5, 3, 19, 20, 77, 45, 27, 30]

0 1 2 3 . . . .

Input

A[0] = 20

[19, 3, 5, 1, 20, 77, 27, 30, 55]

[1, 5, 3, 19, 20, 27, 30, 55, 77]

O(nlogn) ; O(log n)

Space → O(n) Time → O(n)

Solution 1: Naive Logic?

pivot

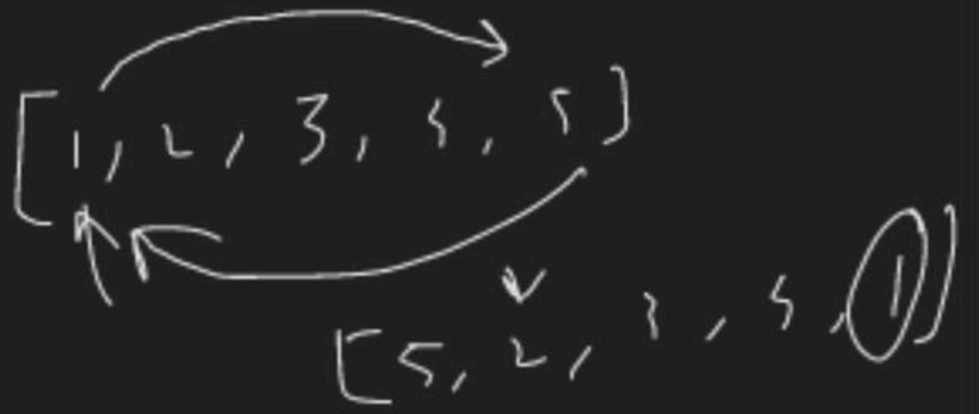
A = [20, 65, 27, 3, 45, 30, 19, 77, 1]

lesser = [3, 19, 1, _, _, _, _, _, _]

greater = [65, 27, 45, 30, 77, _, _, _, _]

A = [3, 19, 1, 20, 65, 27, 45, ...]

Partitioning: Lomuto Algorithm



Assume: Pivot element is at the end of the array

NOTE: You can always bring the pivot element at the end of the array, with an extra swap operation! :)

```
algorithm partition(A, lo, hi) is
    pivot := A[hi]
    i := lo
    for j := lo to hi do
        if A[j] < pivot then
            swap A[i] with A[j]
            i := i + 1
    swap A[i] with A[hi]
    return i
```

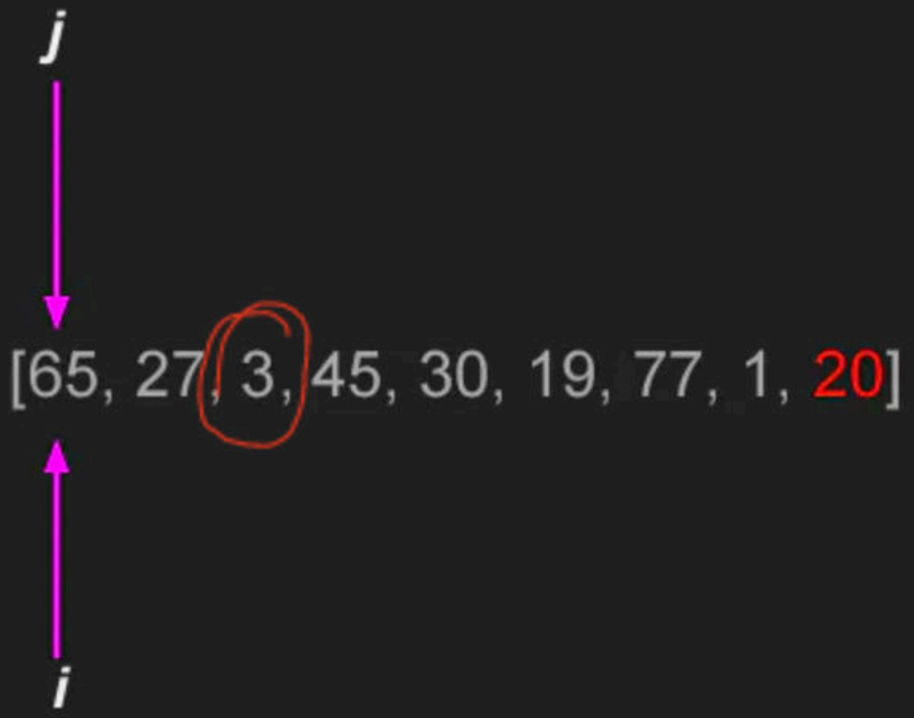
partition(A, 0, A.length-1)

i = 0 ← if lo = 0

Time: O(n)

Space O(1)

Core Idea: Search for A[j], such that A[j] < pivot and secure its position by placing it towards the beginning of the array



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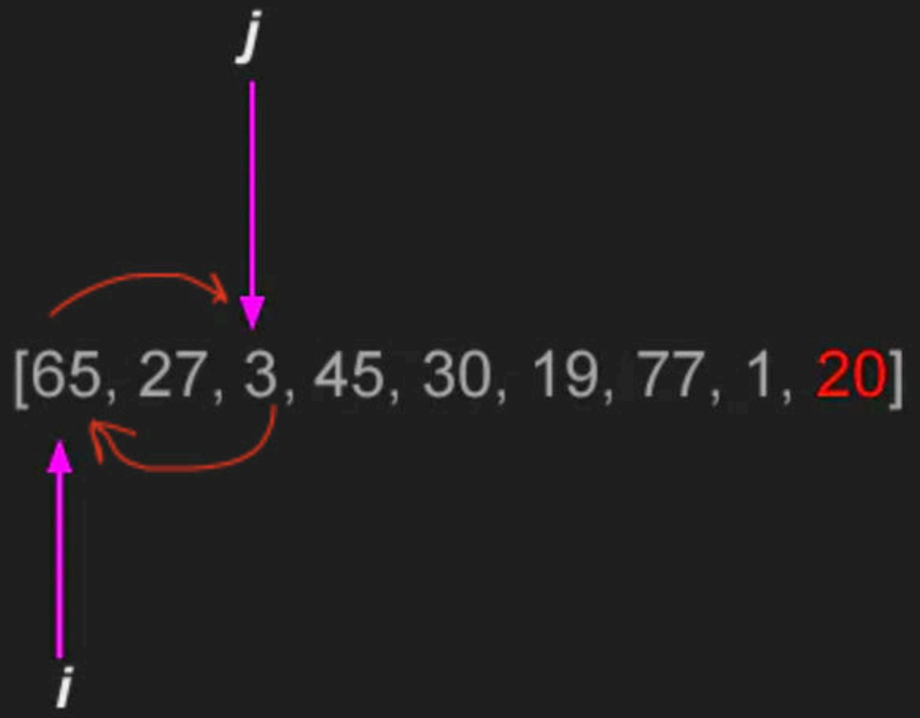

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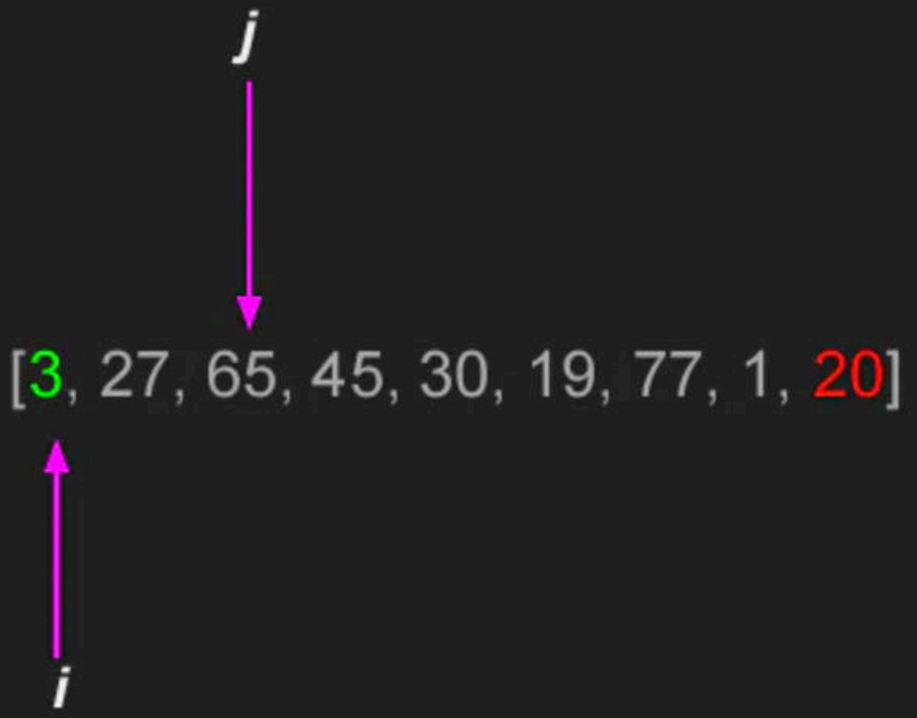

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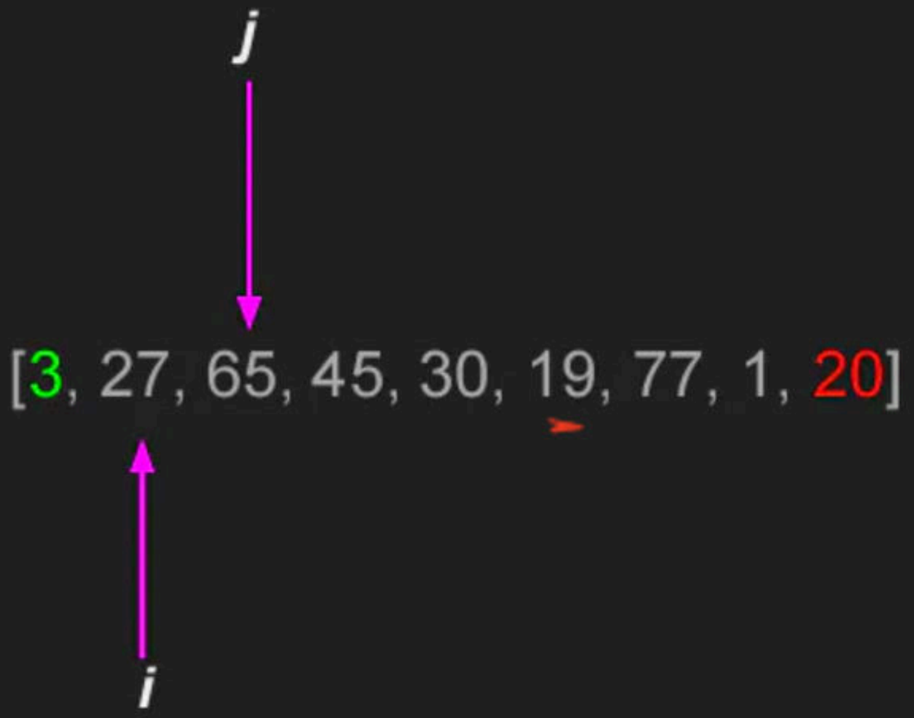

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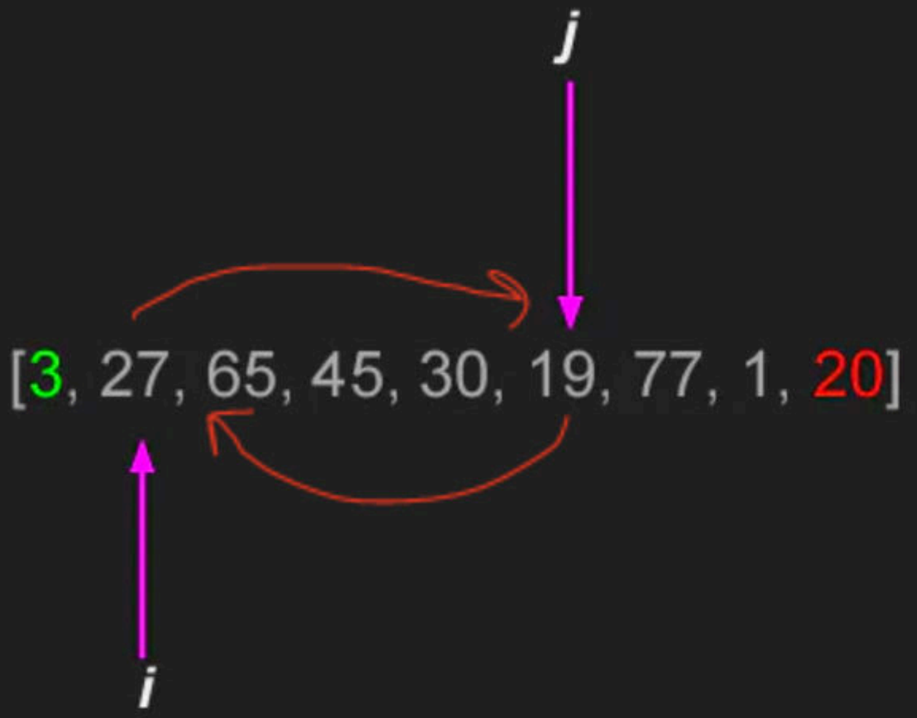

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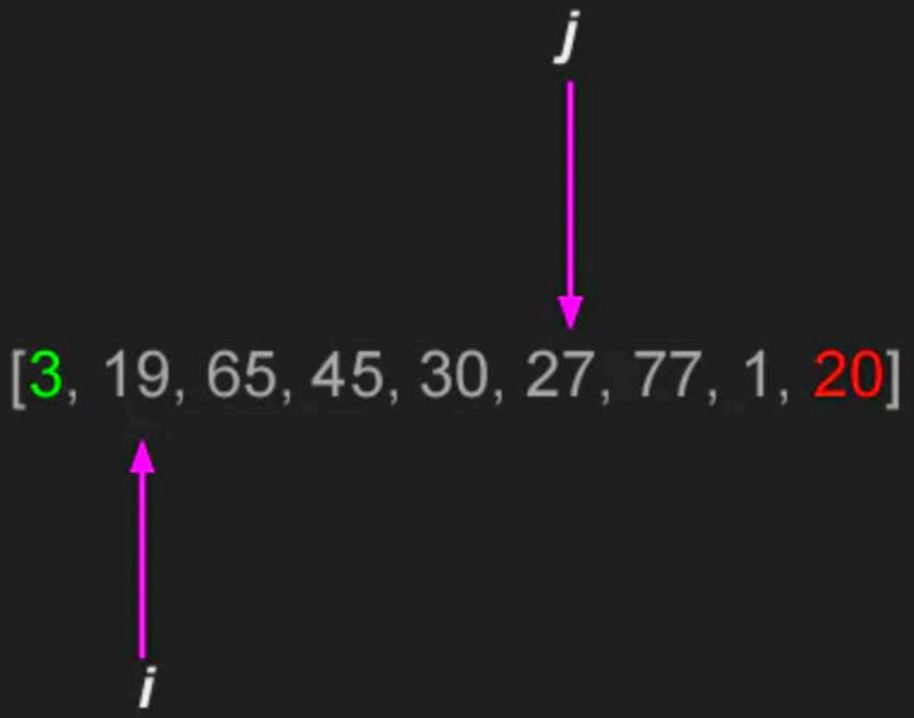

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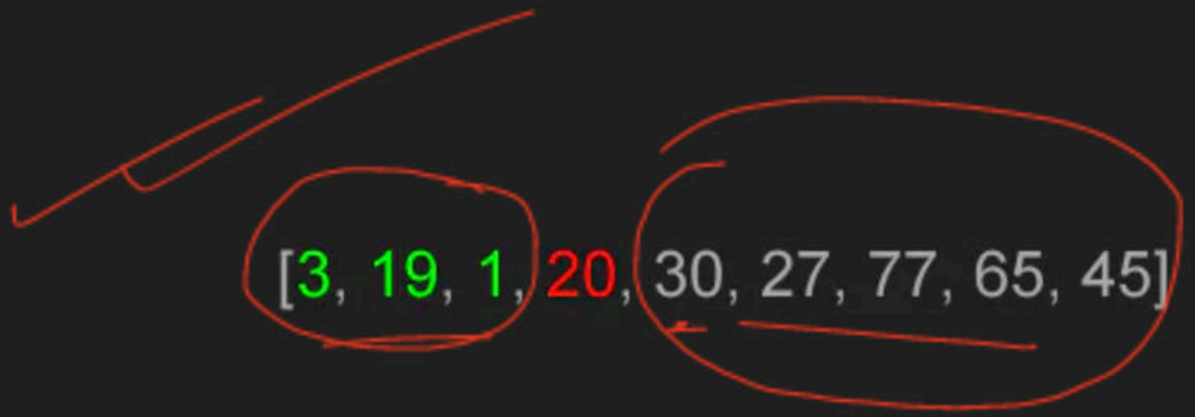


Exit condition encountered:
j has reached the end of the array!
swap(A[high], A[i]) and exit

Time, Space Complexity? Inplace? Stability?

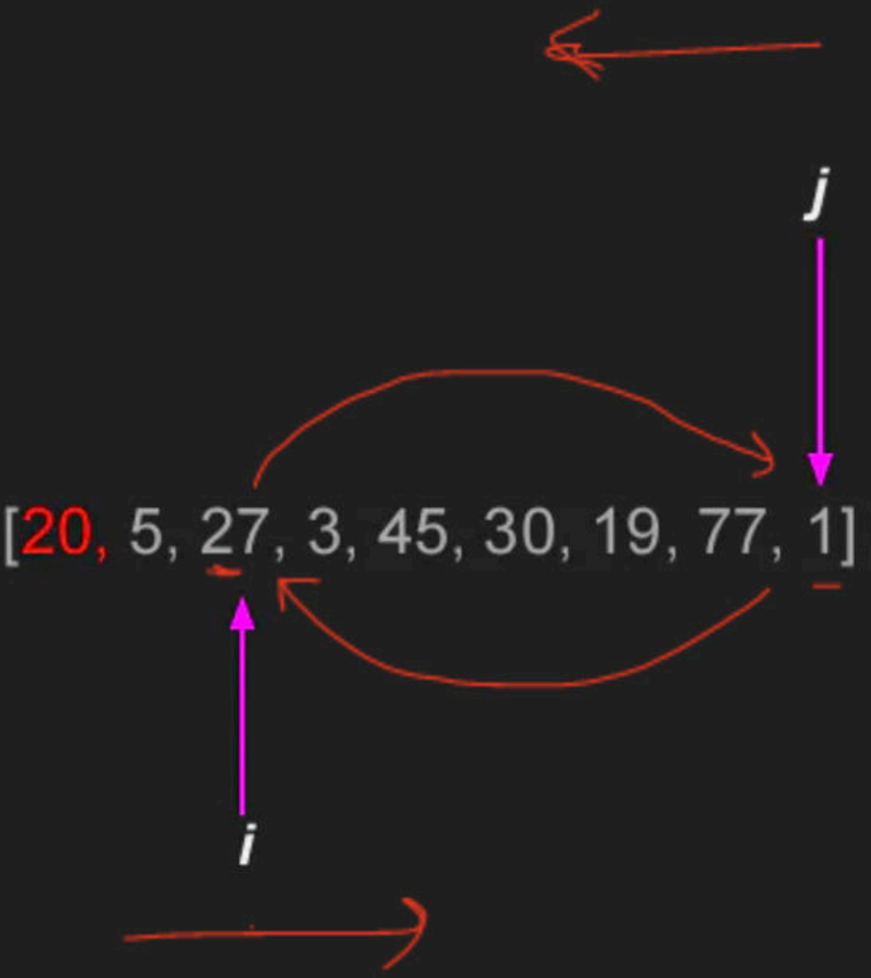
O(N) O(1) Inplace Not Stable

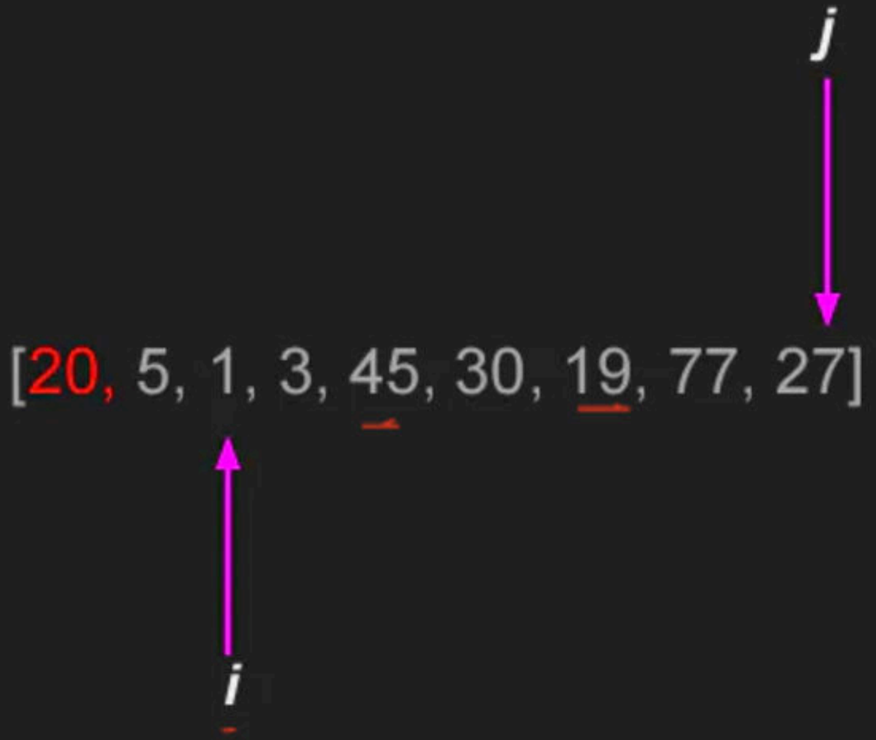
Solution-3: Hoare's Algorithm

1. Find i = index of first item from left larger than pivot
2. Find j = index of first item from right smaller than pivot
3. swap(A[i], A[j])
4. Repeat.

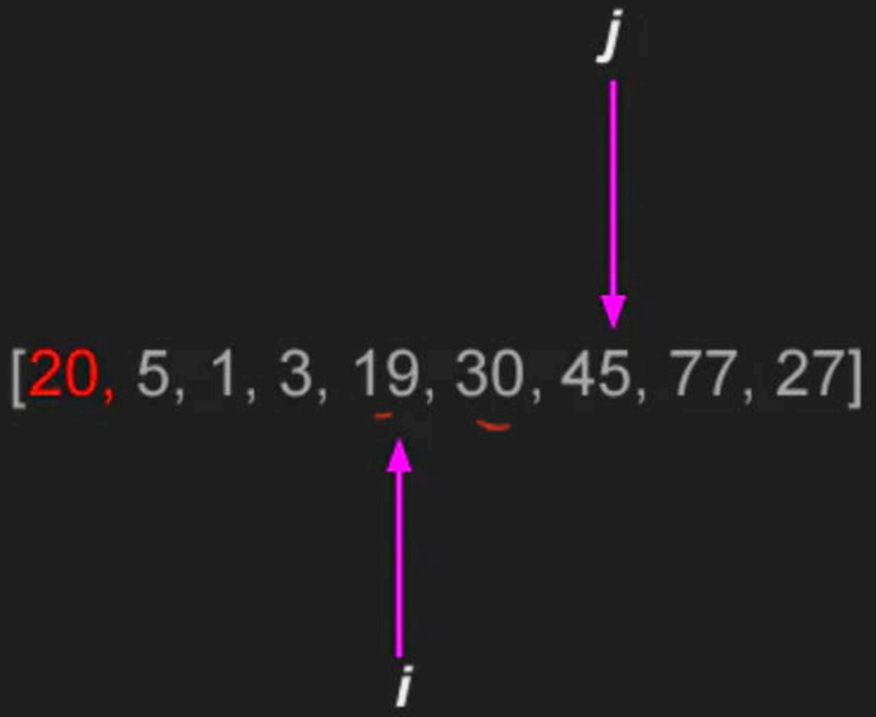
Exit condition: j <= i

After exiting swap(A[0], A[j])


j
[20, 5, 27, 3, 45, 30, 19, 77, 1]
i

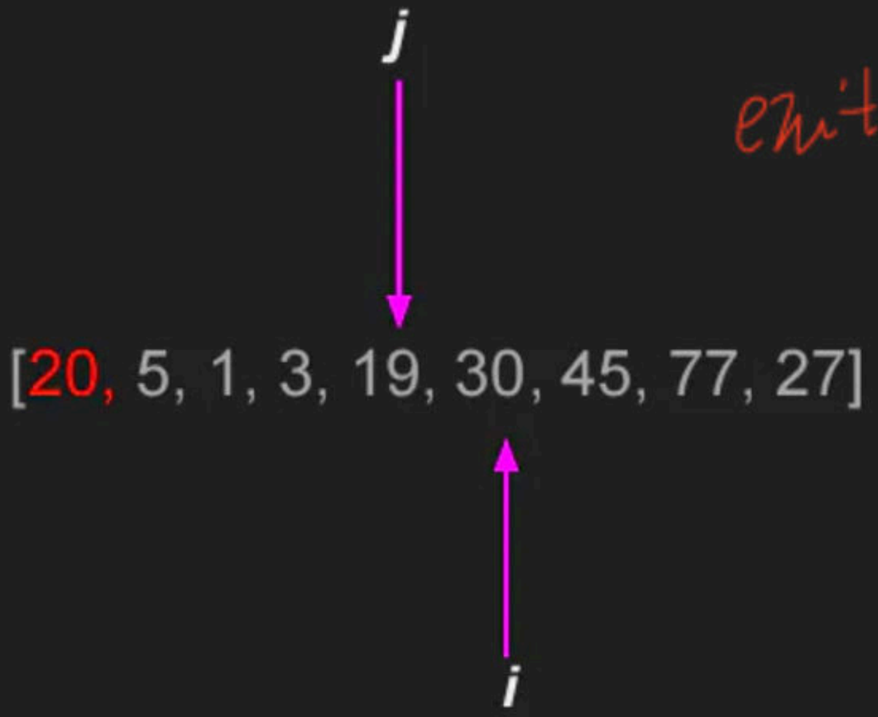

j
[20, 5, 1, 3, 45, 30, 19, 77, 27]
i


j
[20, 5, 1, 3, 45, 30, 19, 77, 27]
i


j
[20, 5, 1, 3, 19, 30, 45, 77, 27]
i

Exit Condition Encountered!
Swap(A[0], A[j]) and exit

exit: $j <= i$



Exit Condition Encountered!
Swap(A[0], A[j]) and exit

[19, 5, 1, 3, 20, 30, 45, 77, 27]

Time Complexity, Space Complexity, Inplace, Stability of Hoare's Scheme?

- Time Complexity = O(N)
- Space Complexity = O(1)

Inplace

Stable

<O(N), O(1)>

Comparison: Hoare's Scheme vs Lomuto's Scheme

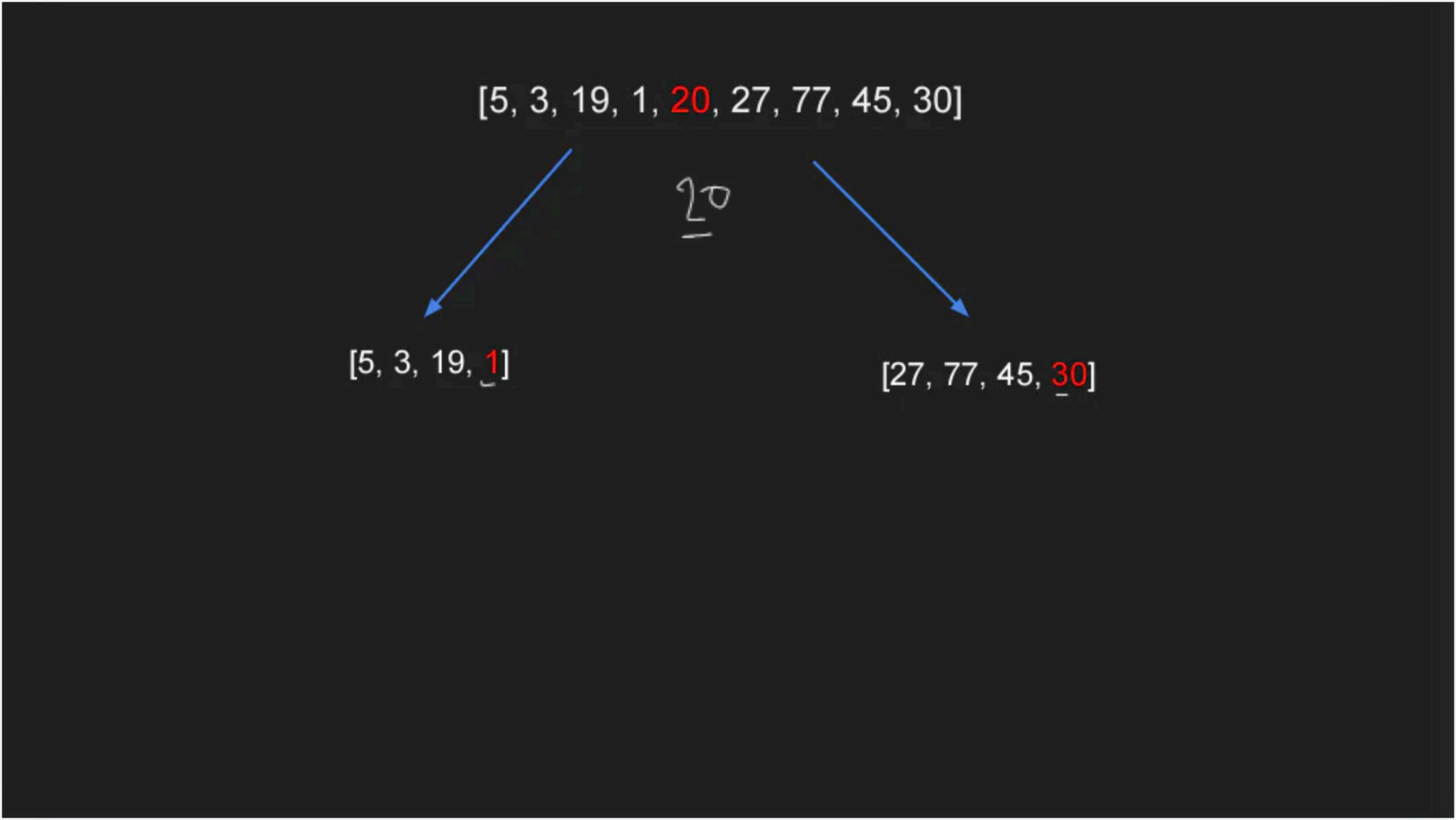
Interesting fact:

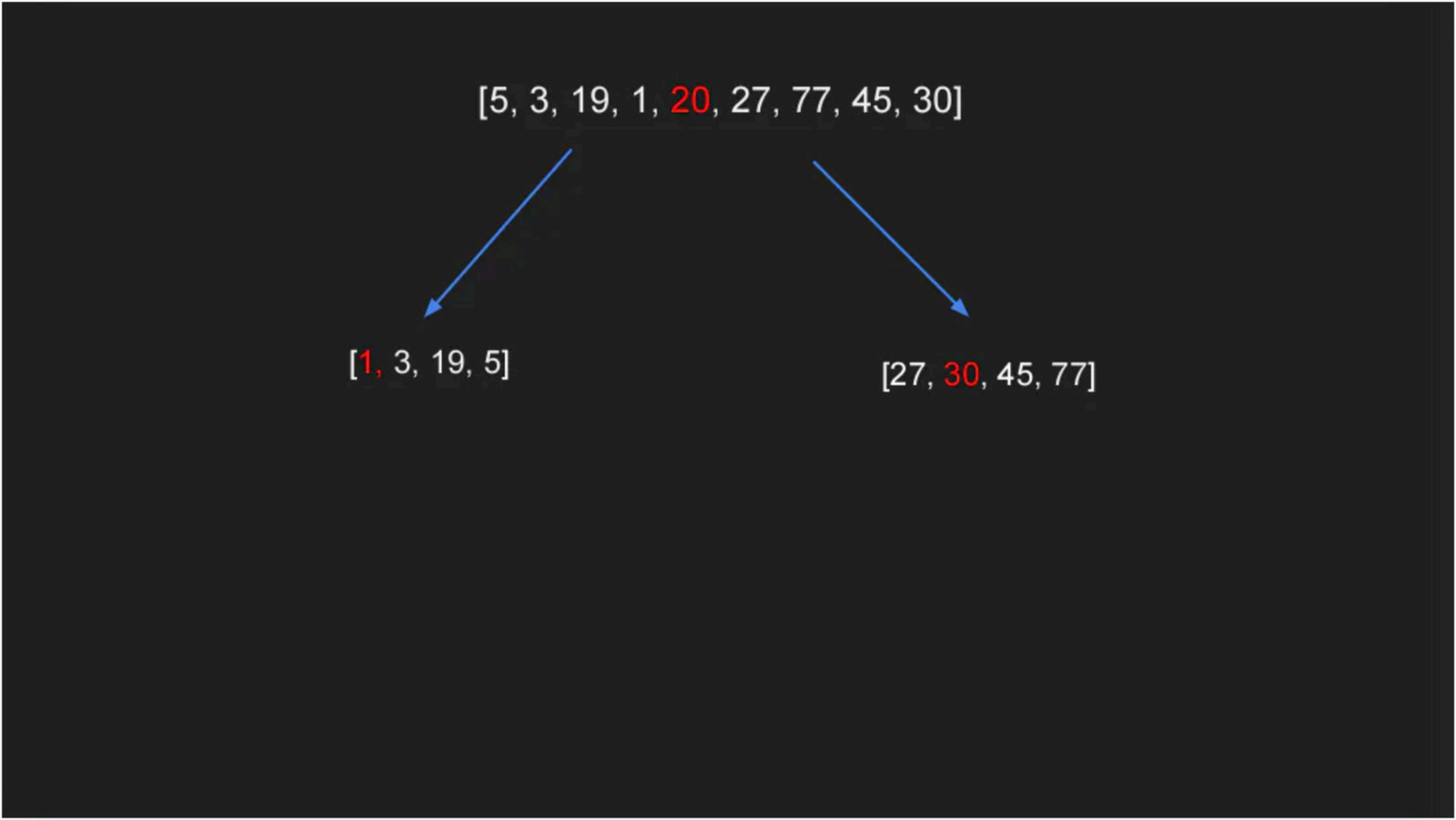
Upon partitioning an array, the position that the pivot ends up in, is same the position it would have taken on sorting the array.

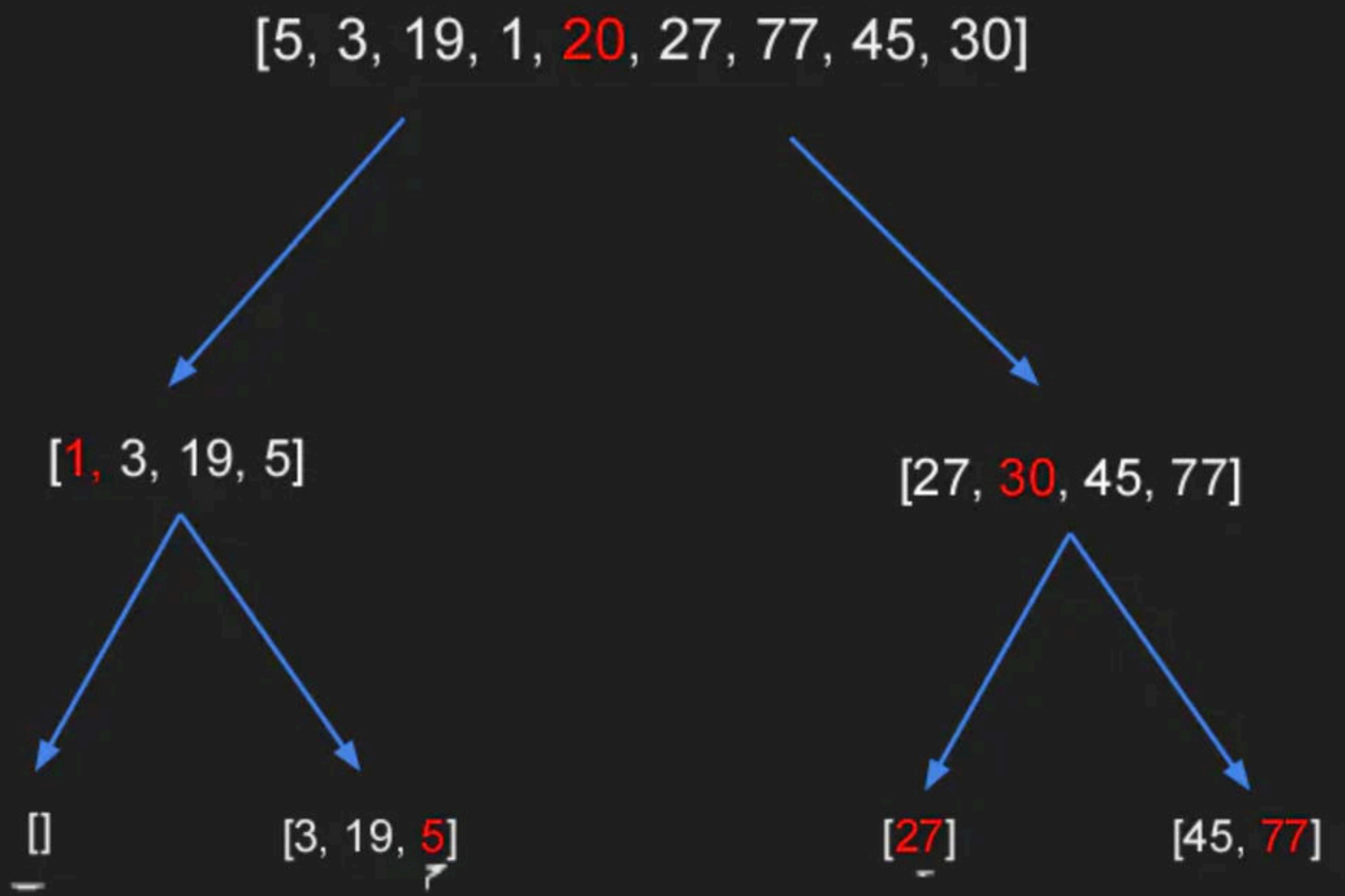
Quicksort: Core idea

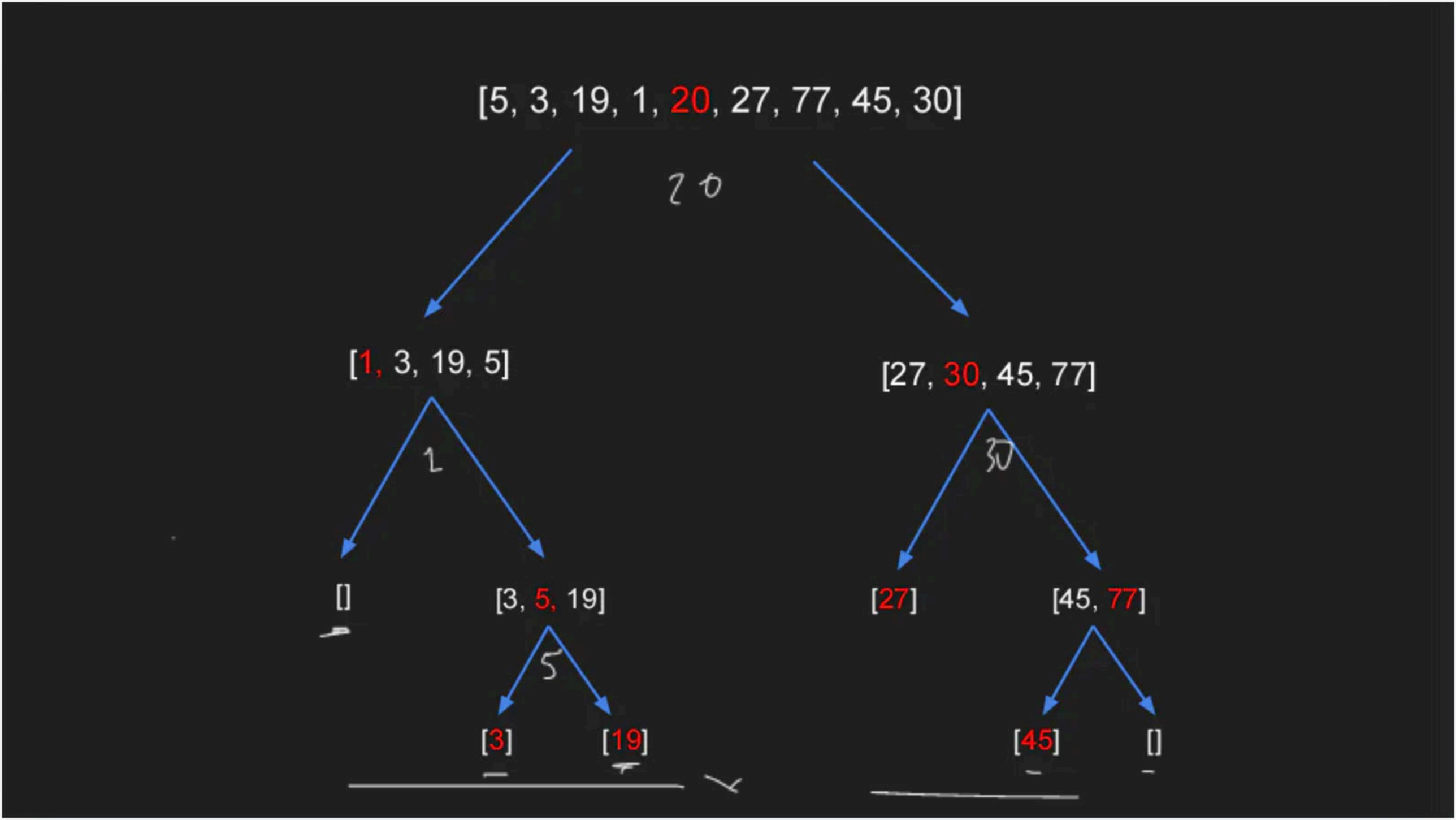
Input ⟶ [5, 27, 3, 45, 30, 19, 77, 1, 20]

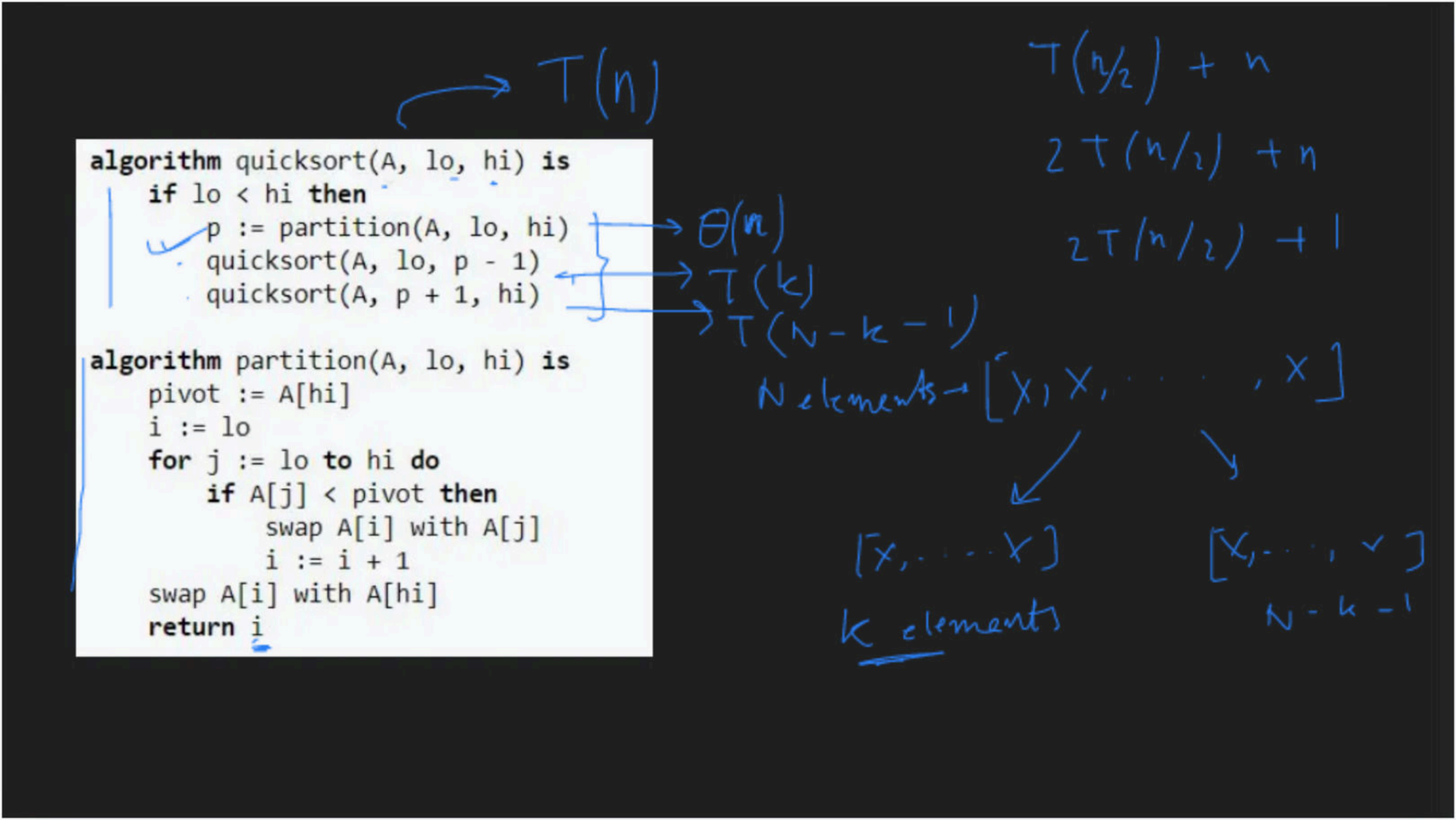

[5, 3, 19, 1, 20, 27, 77, 45, 30]


[5, 3, 19, 1, 20, 27, 77, 45, 30]
20
[5, 3, 19, 1]
[27, 77, 45, 30]


[5, 3, 19, 1, 20, 27, 77, 45, 30]
[1, 3, 19, 5]
[27, 30, 45, 77]


[5, 3, 19, 1, 20, 27, 77, 45, 30]
[1, 3, 19, 5]
[27, 30, 45, 77]
[]
[3, 19, 5]
[27]
[45, 77]


[5, 3, 19, 1, 20, 27, 77, 45, 30]
20
[1, 3, 19, 5]
[27, 30, 45, 77]
1
30
[]
[3, 5, 19]
[27]
[45, 77]
5
[3]
[19]
[45]
[]


T(n)
algorithm quicksort(A, lo, hi) is
if lo < hi then
p := partition(A, lo, hi)
quicksort(A, lo, p - 1)
quicksort(A, p + 1, hi)
algorithm partition(A, lo, hi) is
pivot := A[hi]
i := lo
for j := lo to hi do
if A[j] < pivot then
swap A[i] with A[j]
i := i + 1
swap A[i] with A[hi]
return i
Θ(n)
T(k)
T(N-k-1)
N elements → [x, x, ..., x]
[x, ..., x]
k elements
[x, ..., x]
N - k - 1
T(n/2) + n
2T(n/2) + n
2T(n/2) + 1

Recurrence Relation, Time Complexity & Space Complexity, Stability? Inplace ?

$$T(n) = T(k) + T(n-k-1) + \theta(n)$$

Best Scenario:

Best Case: When the pivot always comes to the center after partitioning

Recurrence Relation: (?)

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

$$T(n) = O(n \log n)$$

Worst Case: When the pivot always comes to one of the end when partitioning

Recurrence ?

$$T(n) = T(n-1) + T(0) + \theta(n)$$

$$T(n) = T(n-1) + T(0) + \Theta(n)$$

$$= \left[T(n-2) + T(0) + \Theta(n)\right] + T(0) + \Theta(n)$$

$$= \left[\left[T(n-3) + T(0) + \Theta(n)\right] + T(0) + \Theta(n)\right] + T(0) + \Theta(n)$$

$$\vdots$$

$$= \boxed{T(0)} + nT(0) + n\Theta(n)$$

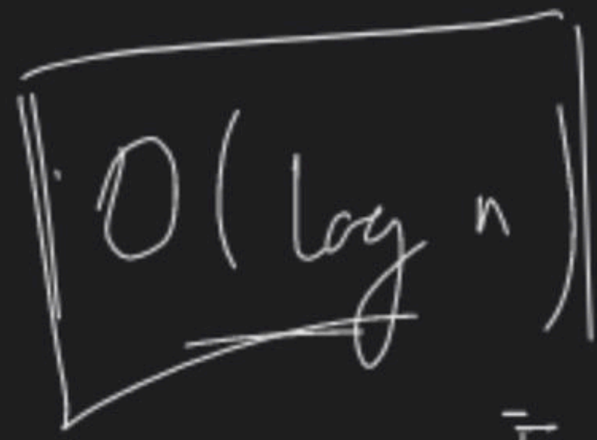
$$= k + kn + \Theta(n^2)$$

$$T(n) = k + kn + \theta(n^2)$$

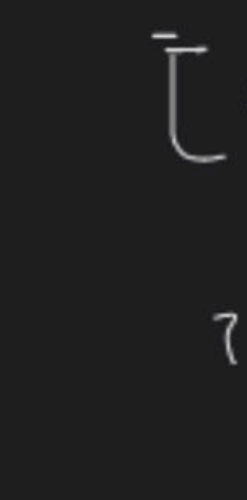
$$\boxed{T(n) = O(n^2)}$$

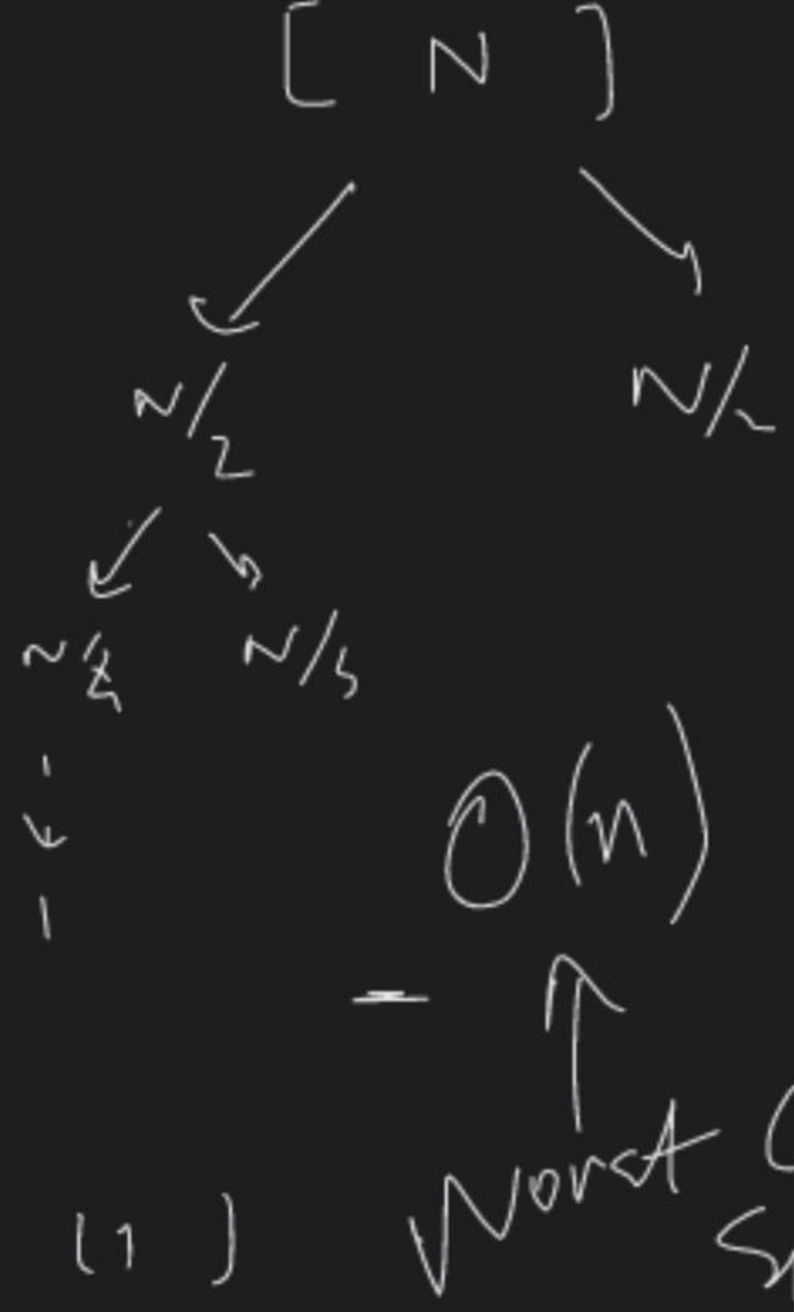
Average: $O(n \log n)$

- O(1) -
- log n -
- log n due to recursion

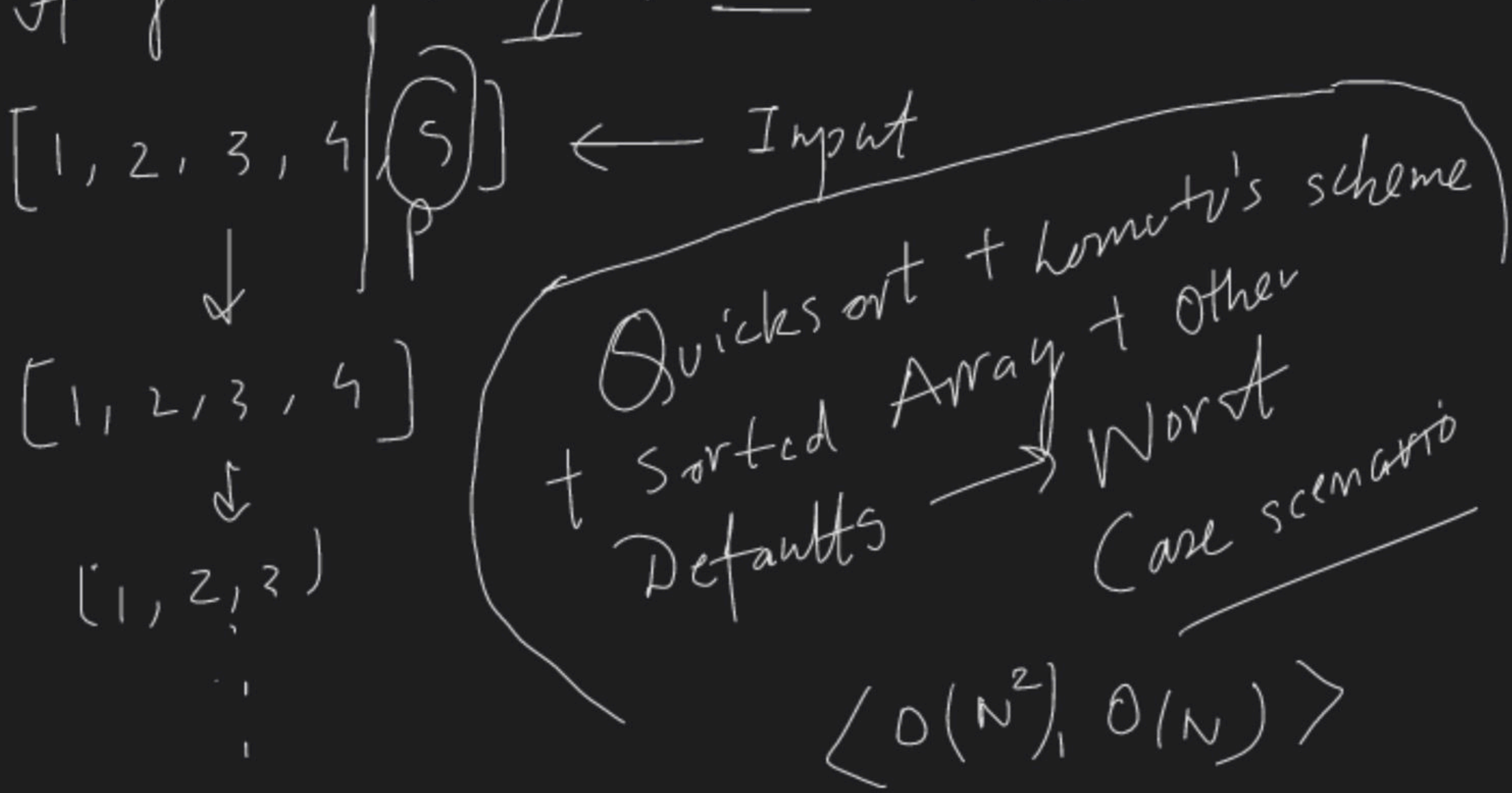


Space





- If you are using Lomuto's scheme



$$T(n) = T(n-1) + T(0) + \Theta(n)$$

Yes, inplace!

No, not stable!

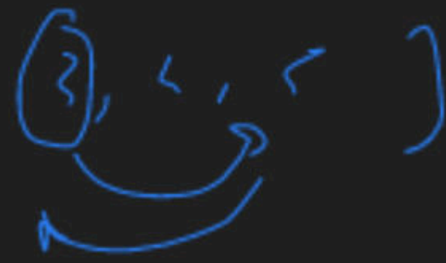
Time Complexity:

- Best → $O(n \log n)$
- Avg → $O(n \log n)$
- Worst → $O(n^2)$

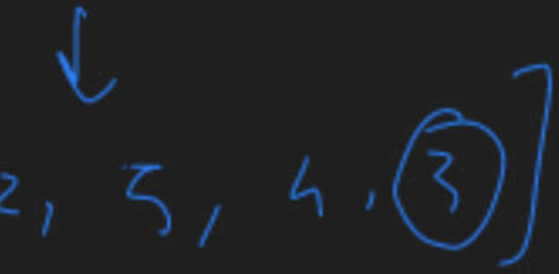
- Space → Best, Avg → $O(\log n)$
  Worst → $O(n)$
- Inplace
- Unstable
- Quicksort (By default) performs poorly on sorted Arrays

Selection of Pivot:

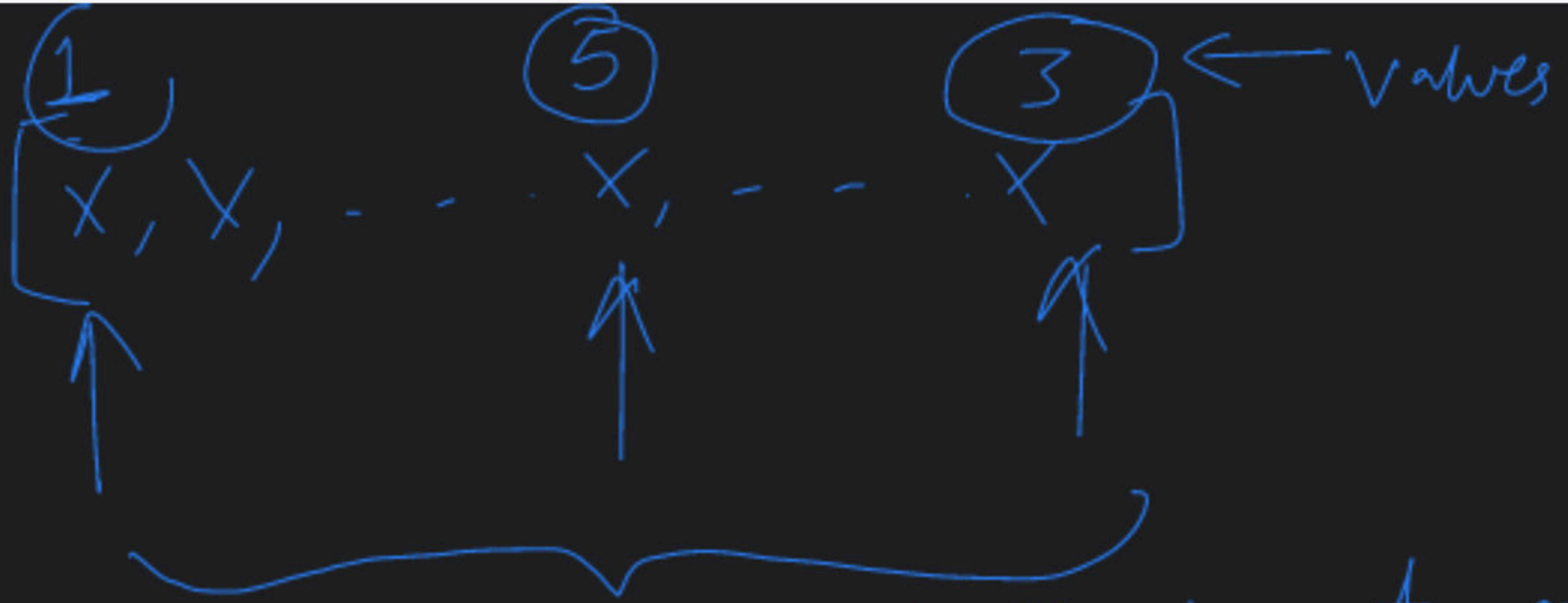
1. First or last element
2. Middle element
3. Random Pivot (MOST FAMOUS)
4. Median of three (BEST)



Math.random()



- random $\sim 1.36 n \log n$
- median $\sim 1.18 n \log n$



Median (1st element, middle, last)

median (1, 5, 3) = 3

pivot

Math.random() ⟶ [0, 1)

⌊Math.random() * N⌋ ⟶ [0, N)

↓

[0, N − 1]

Merge Sort vs QuickSort

→ $O(n \log n)$ $O(n \log n)$

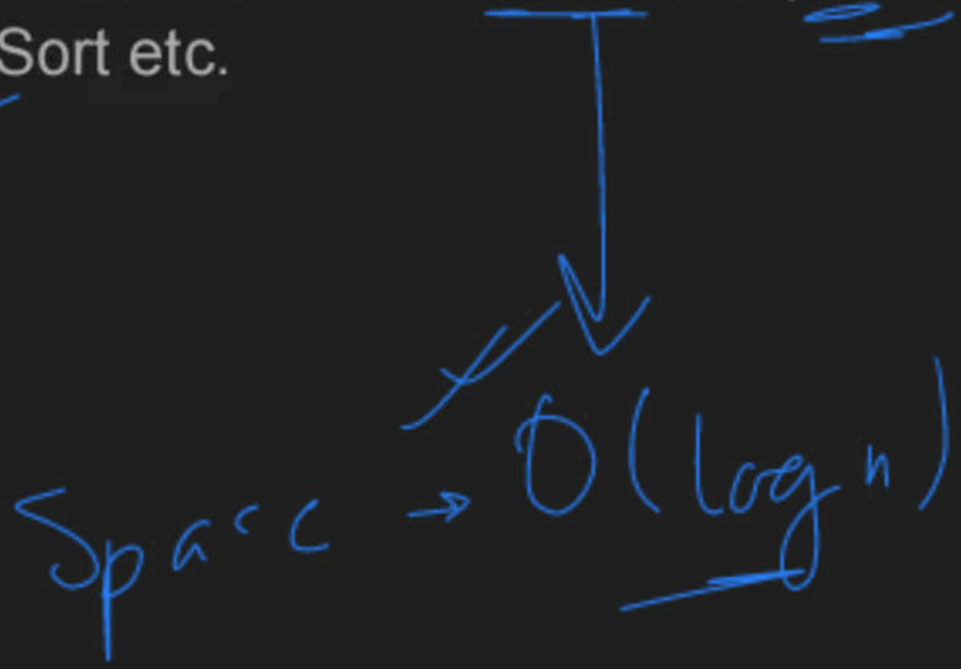
$T(n) = 6000 \, n \log n$ $T(n) = 3 n \log n$

→ Sorting LinkedList → MergeSort

QuickSort: Performance Summary

Performance is determined by:

1. Partition Schemes
2. Selection of Pivot
3. Other Optimizations like: Tail Recursion Optimization, Hybriding the algorithm with Insertion Sort etc.



1. What is the recurrence relation for worst case of quick sort?

A. $T(n) = T(n-1)+O(n)$
B. $T(n) = T(n-2) + O(n^2)$
C. $T(n) = 2*T(n/2) + O(1)$
D. $T(n) = 2*T(n/2) + O(n)$

A. $T(n) = T(n-1)+O(n)$
B. $T(n) = T(n-2) + O(n^2)$
C. $T(n) = 2*T(n/2) + O(1)$
D. $T(n) = 2*T(n/2) + O(n)$

Solution : In the worst case the pivot element can be greatest or smallest element.

2. Suppose we have a O(n) time algorithm that finds median of an unsorted array. Now consider a quicksort implementation where we first find the median using the above algorithm, then use median as pivot. What will be the worst case time complexity of this quick sort.?

A. $O(n^2)$
B. O(nlogn)
C. O(nloglogn)
D. O(n)

A. $O(n^2)$
B. O(nlogn)
C. O(nloglogn)
D. O(n)

Solution : After this the recurrence becomes same as merge sort : $T(n) = 2*T(n/2) + O(n)$ which is known to have O(nlogn)

3. Suppose we are sorting an array of eight integers using quicksort and we have just finished partitioning with the array looking like this : [1,5,1,7,9,12,11,10]

A. Pivot could be 7 or 9
B. Pivot could be 7 but not 9
C. Pivot is not 7 but could be 9
D. Neither 7 nor 9 is the pivot.

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B. Pivot could be 7 but not 9
C. Pivot is not 7 but could be 9
D. Neither 7 nor 9 is the pivot.

Solution : For every element check if all small numbers are on the left and all big numbers are on the right.

H.W.

1. Segregate positives and negatives: Revisited!