Combinatorics

Motivation Problem :

only combinations

Q. Given infinite supply of N types of balls. find the no. of ways to choose K balls from the given set with repetition allowed.

(Don't consider permutations)

$\boxed{N=3 \\ K=3}$

Example

Type A, B, C

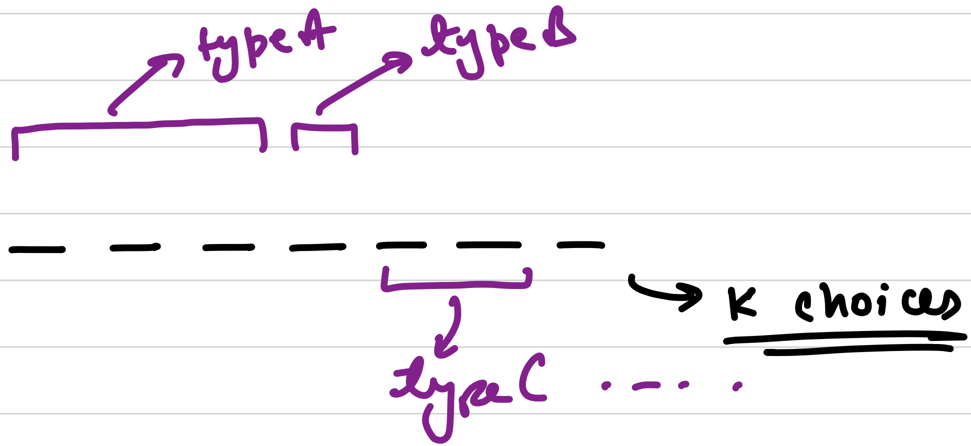
(A, B, C) (C C C) (A A C) (B B A)

(A A A) (A A B) (B C C)

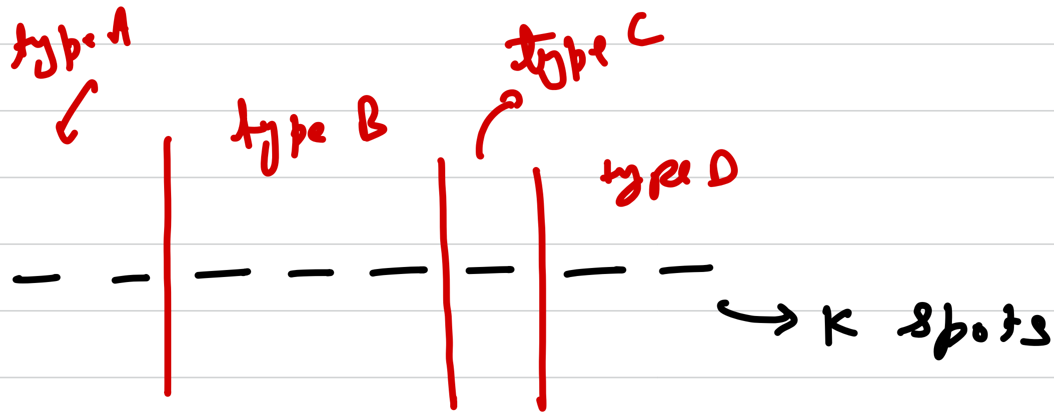
(B B B) (B B C) (C C A)

Because permutations doesn't matter, we can take type wise decision i.e.

How many balls to pick from any x type



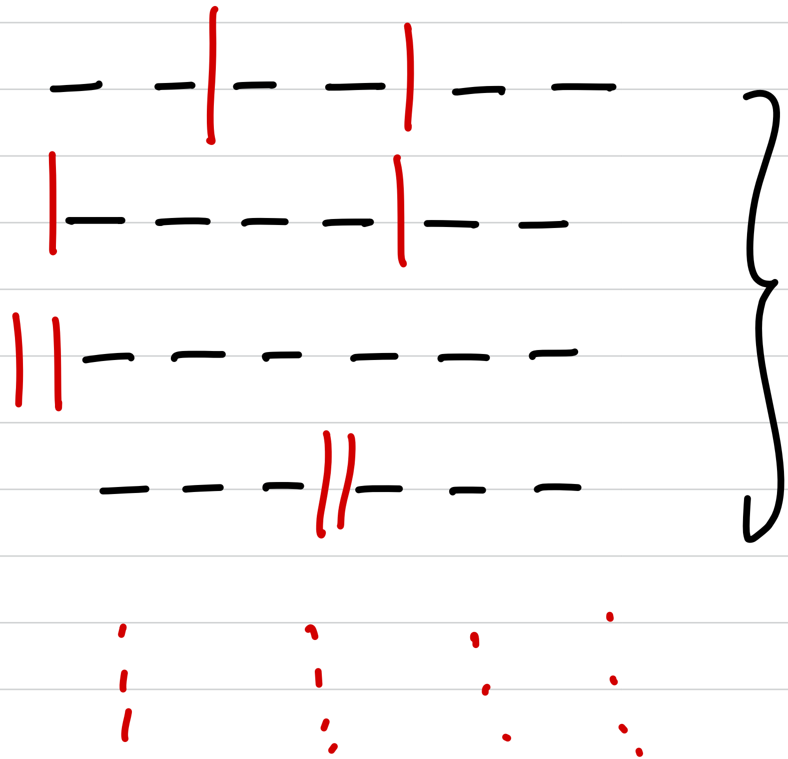
We can partition the selected choices.



for n types, we have to draw n-1 partition lines i.e, we need to count # of ways to put (n-1) lines here.

But how many positions are there ??

$\hookrightarrow \underline{\underline{(K+n-1)}}$



$k = 6$

$n = 3$

Choose $n-1$ spots from $n+k-1$ spots

$${}^{(n+k-1)}C_{n-1}$$

${}^{n}C_{r}$ → How to calc ??

${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$ → ${}^{n}C_{r} \% m$

fact[x] → precompute

→ array

→ Inv → Fermat theorem

$a^{-1} \% m$ → $a^{m-2} \% m$

Example Problem → You walk into a candy store, and got money to by 6 candies. The store has 3 diff candies C_1, C_2, C_3.
How many ways are there to select candies?

$n = 3$ $k = 6$

$^{6+3-1}C_{3-1} \Rightarrow {}^{8}C_{2} \Rightarrow 28$ ans

All Tracks > Math > Combinatorics > Basics of Combinatorics > Problem

Counting stars

971 36% 30 ★★★★★ 6 votes Approved, Combinatorics, Easy, Math, Ready, Recruit, approve

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Problem

There are N stars in the sky. A manual attempt at counting yielded K stars. It is possible that the same star may have been counted more than once. You need to determine the probability that any star may have been counted more than once. Probability can be represented as a rational number $\frac{P}{Q}$. If Q is not divisible by $10^9 + 7$ there is a unique integer $x \mid 0 \leq x < 10^9 + 7$ where $P \equiv Qx$ % $(10^9 + 7)$. Calculate value of this integer x.

Input Format:
First line of input consists of a single integer T denoting number of test cases.
Following T lines contain two space separated integers denoting N and K.

Output Format:
Print the answer to each test case in a new line.

Input Constraints:
$1 \leq T \leq 10$
$1 \leq N, K \leq 100000$

Sample Input	Sample Output
1 3 3	300000003

Time Limit: 1
Memory Limit: 256
Source Limit:

N type of stars

choose K stars

Total no. of ways $\rightarrow {}^{n+k-1}C_{n-1}$

# of way for only unique count $\rightarrow {}^{n}C_{k}$ $(k \leq n)$

# of ways for repeat only

${}^{n+k-1}C_{n-1} - {}^{n}C_{k}$

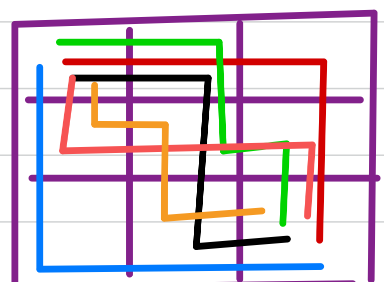
probability = $\dfrac{{}^{n+k-1}C_{n-1} - {}^{n}C_{k}}{{}^{n+k-1}C_{n-1}}$

→ **Fermat th^n**

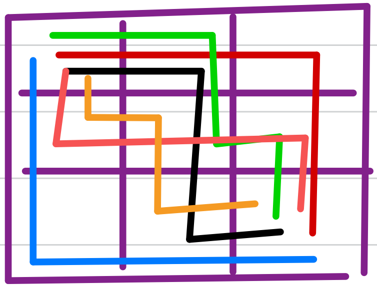
Q :- Given a grid of $m \times n$ dimensions. You are standing at the topleft of the grid. find the total no. of ways to reach bottom right if from any cell you can only go rightwards or downwards

$1 \le m, n \le 10^5$

$m = 3$, $n = 3$



ans → 6



m*n

R D D R
R R D D
D D R R
D R R D
R D R D
D R D R

6 ways

A A B B

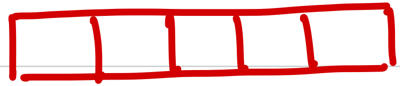
$$\frac{4!}{2!\,2!}$$

all of the 6 ways are permutations

R's → n − 1   # D's → m − 1   Total → m+n−2

total permutation → $\dfrac{(m+n-2)!}{(m-1)!\,(n-1)!}$

ans



$\underbrace{\qquad\qquad}^{m+n-2}$ spots

$m-1$ D's → identical

$n-1$ R's → identical

$${}^{m+n-2}C_{n-1} = {}^{m+n-2}C_{m-1} = \frac{(m+n-2)!}{(m-1)!\,(n-1)!}$$

$m+n-2-(m-1)$

$\cancel{m}+n-2-\cancel{m}+1 \to \underline{n-1}$

All Tracks > Math > Combinatorics > Basics of Combinatorics > Problem

K-Special Cells

4819 85% 20 ★★★★★ 5 votes Combinatorics, Easy, Math Share

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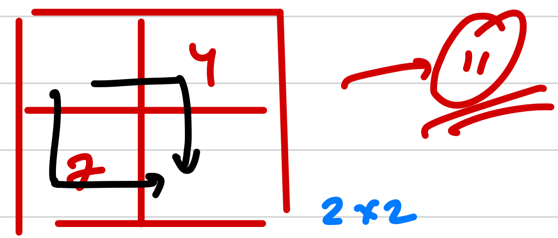
- You are given a $N \times M$ matrix which has K special cells in it. You have to reach (N, M) from $(1, 1)$. From any cell, you can only move **rightwards** or **downwards**.
- The $K - special$ cells are those cells in this grid which have special strength at them. i^{th} special cell has $P[i]$ units of strength and if you travel through this cell, you store the strength.
- Find the total strength you can store after travelling through all the possible paths in the grid to reach cell (N, M).

- **Note:**
  1. The strength of a path is the sum of strength $P[i]$ of all the special cells that are visited in this path.
  2. The cells that are not special have power quotient equals to zero.



Input format

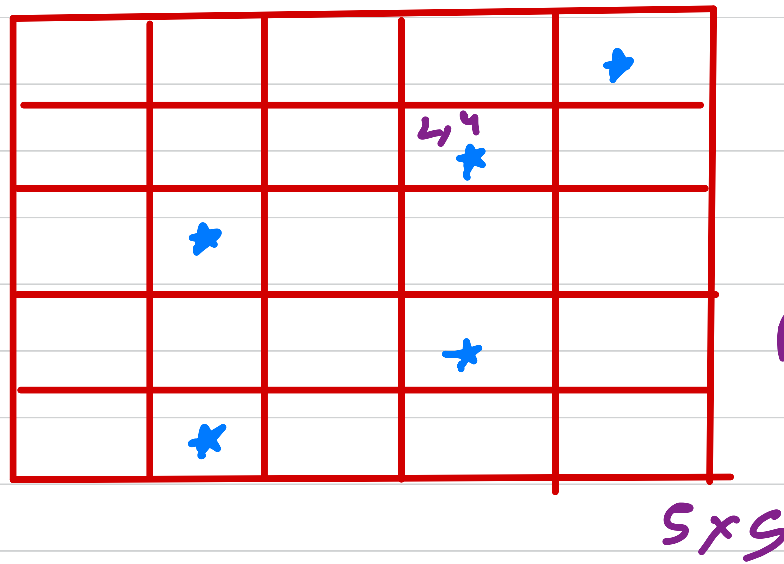
- The first line contains the total number of test cases denoted by **T**.
- The first line of each test case contains three space separated integers **N**, **M** and **K** where **N** x **M** is the size of grid and **K** is the total number of special cells in the grid.
- Each of next **K** lines contains **X[i]**, **Y[i]**, and **P[i]** where (X[i],Y[i]) is the location of special cell and P[i] is the cell strength.

$K \leq 10^6$

$\underline{M, N \leq 10^5}$ → We can't even make a $\underline{\text{grid}}$.

Output format

- For each test case, print in a new line a single integer representing the total strength that you can store, as the total strength can be too large, print it modulo $10^9 + 7$.



contri of 1 special cell = p[i][j] × (# of path it is a part of)

if we can count the contribution of each special cell into our ans, then we can solve it.

How to calc the no of path when i, j is also a part ??

Calc the no. of path from $0,0$ to i,j, say this is x.

Calc no. of path from i,j to $n-1, m-1$, say this is y.

Total paths $\Rightarrow$ $\underline{\underline{x * y}}$

$$x \rightarrow \frac{(i+j-2)!}{(i-1)!\,(j-1)!}$$

$$y = \frac{(n-i+m-j)!}{(n-i)!\,(m-j)!}$$

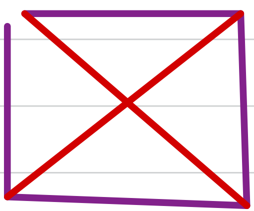
Q→ Given m, n that represents a $m \times n$ grid which as K blocked cells. find total no. of ways to reach from top left to bottom right.

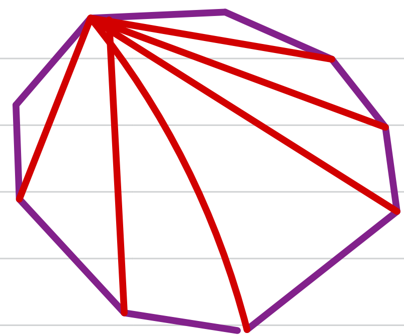
Total ways − Total ways when some blocked cell is a part.

almost similar to **prev**

Q:- Given a value n, which represents a n-sided polygon. Find the total no. of diagonals in the polygon.

$n = 4$ ans $\rightarrow 2$





$$\frac{n \times (n-3)}{2}$$ ans

$$\frac{n \times n-3}{2}$$

to avoid double count

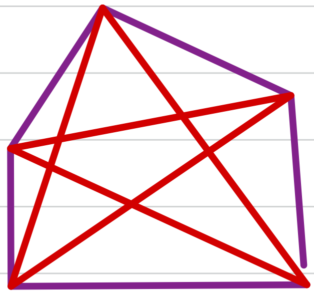
# of ways to choose 2nd vertex of the diagonal

# of ways to choose first vertex of the diagonal

why $\underline{\underline{n-3}}$??

if we now choose any i^{th} vertex for first vertex of diagonal, the 2 adjacent vertices, can't be the 2nd vertex

2^{nd} way



You've n vertices, for a diagonal you need any 2

# of ways to choose 2 vertices from a set of n is ${}^nC_2$

in ${}^nC_2$ we have also chosen the 2 vertices who will form sides. So

final ans $\rightarrow$ ${}^nC_2 - n$ $\hookrightarrow$ # of sides

$$\frac{n!}{2!\,(n-2)!} - n$$

$$\frac{n(n-1)}{2} - n$$

$$n\left(\frac{n-1}{2} - 1\right)$$

$$n\left(\frac{n-1-2}{2}\right)$$

$$\frac{n(n-3)}{2}$$

Binomial Coefficients.

$$(x+y)^n = {}^nC_0\, x^0 y^n + {}^nC_1\, x\, y^{n-1} \cdots\cdots {}^nC_n\, x^n y^0$$

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^r y^{n-r}$$

Example

$$(x+y)^3 = {}^3C_0\, x^0 y^3 + {}^3C_1\, x y^2 + {}^3C_2\, x^2 y + {}^3C_3\, x^3 y^0$$

$$= y^3 + 3xy^2 + 3x^2y + x^3$$

$(x+y)^n = {}^nC_0\, x^0 y^n + {}^nC_1\, x\, y^{n-1} \cdots\cdots {}^nC_n\, x^n y^0$

all x

all y

Total n objects choose

$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \cdots\ \_}_{r \to x}$

Let's say pick r x's

$\sum {}^nC_r\, x^r y^{n-r}$

What is $^nC_0 + {^nC_1} + {^nC_2} \cdots\cdots {^nC_n}$??

$$\downarrow$$

$$2^n$$

$$(x+y)^n = {^nC_0}\, x^0 y^n + {^nC_1}\, x y^{n-1} \cdots\cdots {^nC_n}\, x^n y^0$$

Put $x=1,\ y=1$

$$(1+1)^n = {^nC_0}\, 1^0 1^n + {^nC_1}\, 1^1 1^{n-1} \cdots\cdots {^nC_n}\, 1^n 1^0$$

$$2^n = {^nC_0} + {^nC_1} + {^nC_2} + {^nC_3} \cdots\cdots {^nC_n}$$

$n+1$

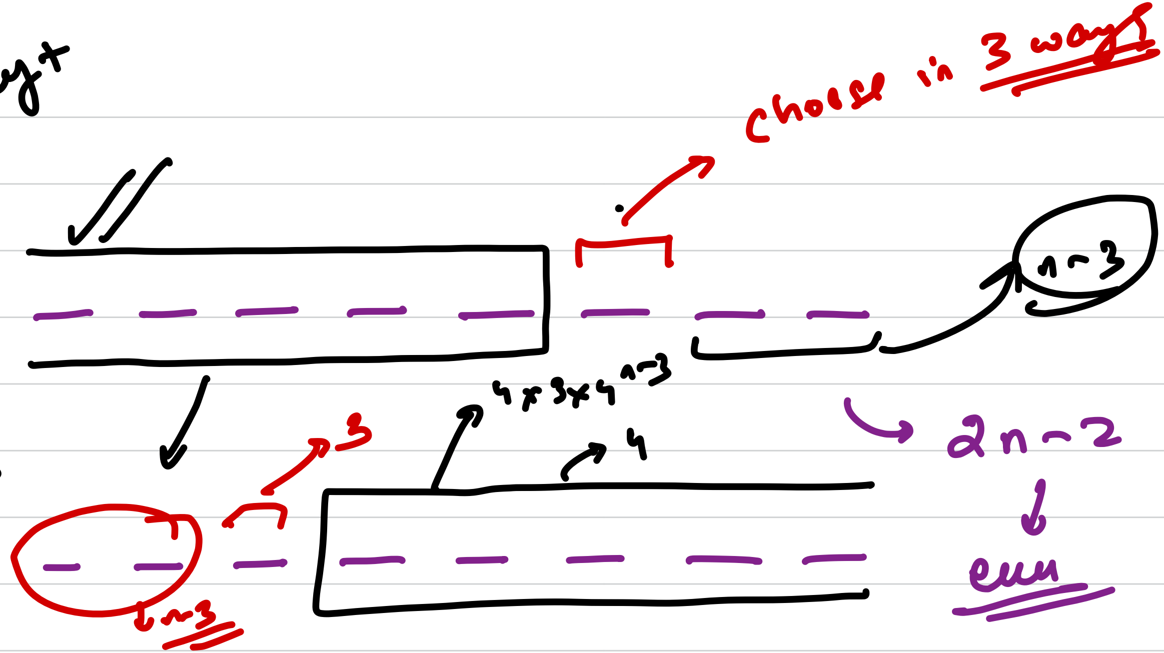
Q. There are some cars of 4 diff companies. There is a linear parking space of $2n-2$ capacity. Total cars of each type is more than $2n-2$. Calc the no. of ways in which exactly n consecutive cars of same type are parked there. $3 \le n \le 30$

$n = 3$ ans → 24

AAAB AAAC AAAD DBBB ....

$2n-2-n-1$

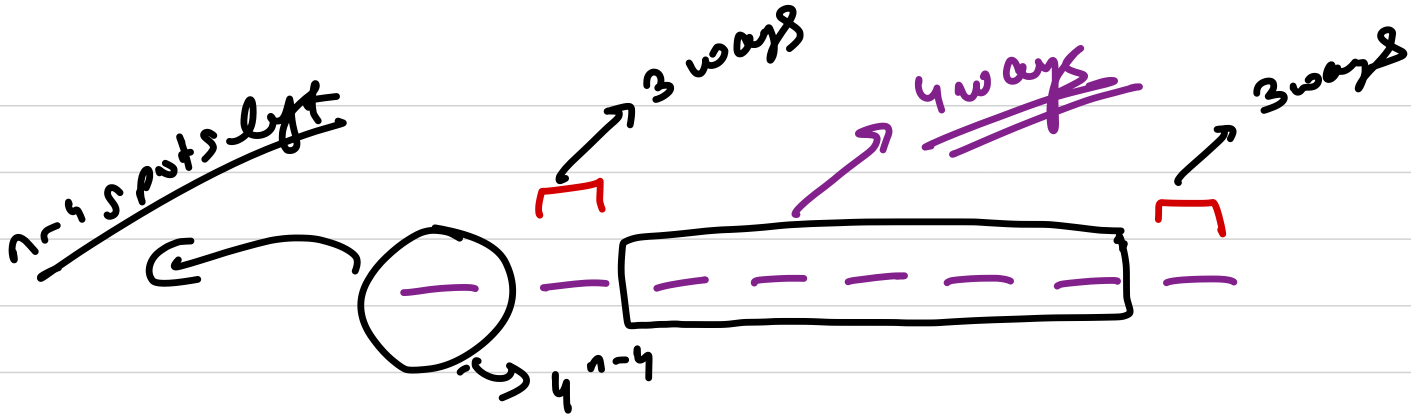
$n-3$ spots left



$4\times3\times4^{n-3}$

To arrange on initial n spots we have → 4 choices due to 4 type of car

Total ways needed → $2\times4\times3\times4^{n-3}$



$2n-2-n-2 \Rightarrow n-4$ spots left

Total ways for n consecutive car in between:

$(n-3)\times 4\times 3^2\times 4^{n-4} \rightarrow$ one window

ans $\rightarrow$ $2\times 4\times 3\times 4^{n-3} + (n-3)\,4\,3^2\,4^{n-4}$

I. Parking Lot

time limit per test: 0.5 seconds
memory limit per test: 64 megabytes
input: standard input
output: standard output

To quickly hire highly skilled specialists one of the new IT City companies made an unprecedented move. Every employee was granted a car, and an employee can choose one of four different car makes.

The parking lot before the office consists of one line of $(2n - 2)$ parking spaces. Unfortunately the total number of cars is greater than the parking lot capacity. Furthermore even amount of cars of each make is greater than the amount of parking spaces! That's why there are no free spaces on the parking lot ever.

Looking on the straight line of cars the company CEO thought that parking lot would be more beautiful if it contained exactly n successive cars of the same make. Help the CEO determine the number of ways to fill the parking lot this way.

Input

The only line of the input contains one integer n $(3 \le n \le 30)$ — the amount of successive cars of the same make.

Output

Output one integer — the number of ways to fill the parking lot by cars of four makes using the described way.

Examples

input

```
3
```

output

```
24
```

Note

Let's denote car makes in the following way: A — Aston Martin, B — Bentley, M — Mercedes-Maybach, Z — Zaporozhets. For $n = 3$ there are the following appropriate ways to fill the parking lot: AAAB AAAM AAAZ ABBB AMMM AZZZ BBBA BBBM BBBZ BAAA BMMM BZZZ MMMA MMMB MMMZ MAAA MBBB MZZZ ZZZA ZZZB ZZZM ZAAA ZBBB ZMMM

Originally it was planned to grant sport cars of Ferrari, Lamborghini, Maserati and Bugatti makes but this idea was renounced because it is impossible to drive these cars having small road clearance on the worn-down roads of IT City.

Q: Given a set of x letters (all uniq) — dict

made from given letters

count the no. of strings of length n, that doesn't contain any palindromic substring of length greater than 1.

Answer T such test cases.

$T \le 10^5$

$x, n \le 10^9$

Eg → $T = 1$

$x = 2 \quad n = 2$

ans → **2**

ab
ba

only $n-1$ chars

__ __ __ __ __ __ __ __ ... n length

all x choices available

only $x-2$ choices available

$$x \times (x-1)\underbrace{(x-2)^{n-2}}_{\text{binary expo}}$$

Anti-Palindromic Strings ★

You are given two integers, N and M. Count the number of strings of length N (under the alphabet set of size M) that doesn't contain any palindromic string of the length greater than 1 as a consecutive substring.

Input Format

Several test cases will be given to you in a single file. The first line of the input will contain a single integer, T, the number of test cases.

Then there will be T lines, each containing two space-separated integers, N and M, denoting a single test case. The meanings of N and M are described in the Problem Statement above.

Output Format

For each test case, output a single integer - the answer to the corresponding test case. This number can be huge, so output it modulo $10^9 + 7$.

Constraints

$1 \leq T \leq 10^5$
$1 \leq N, M \leq 10^9$

Sample Input

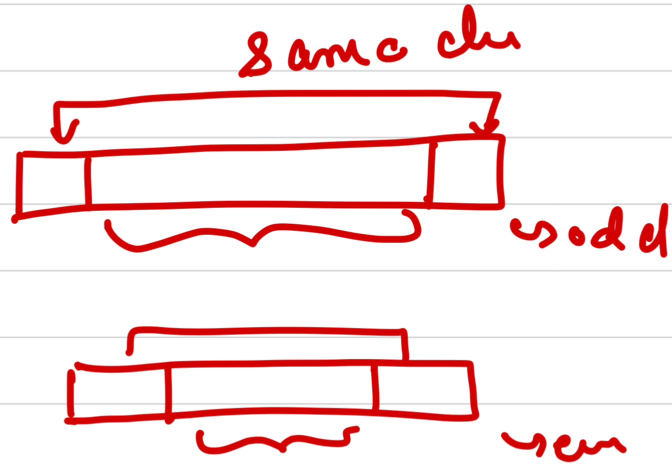
```
2
2 2
2 3
```

Sample Output

```
2
6
```

any palindrom → even

→ odd



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Pascal triangle

$$^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$$

dp[n][r] = dp[n-1][r-1] + dp[n-1][r]

& O(n) → space

Can we optimise space & make it less than O(n) ??

$$^nC_r = \frac{n!}{r!\,(n-r)!} \quad \text{------} \; 1$$

$$^nC_{r-1} = \frac{n!}{(r-1)!\,(n-r+1)!} \quad \text{------} \; 11$$

$1/11$

$$\frac{^nC_r}{^nC_{r-1}} = \frac{(r-1)!\,(n-r+1)!}{r!\,(n-r)!}$$

$$\frac{^nC_r}{^nC_{r-1}} = \frac{1}{r} \times (n-r+1)$$

$$\boxed{{}^{n}C_{r} = \frac{n-r+1}{r} \times {}^{n}C_{r-1}}$$

$\rightarrow O(1)$ space

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$$\cancel{4} \times \frac{4-4+1}{\cancel{4}}$$

$$^{n}C_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$$

physical intuition

among n objs choose r.

choose a subset of r elements

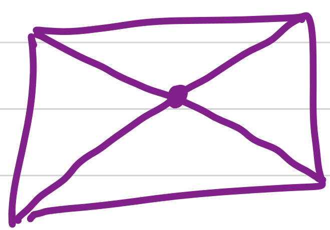
if we didn't pick n^{th} element then among n-1 elements we still need r elements

if we pick the n^{th} element then among n-1 elements pick only r-1 remaining elements

Qn Given a number n, find the no. of intersection points of the diagonals in a polygon of n vertices given that no 3 diagonals intersect in the polygon

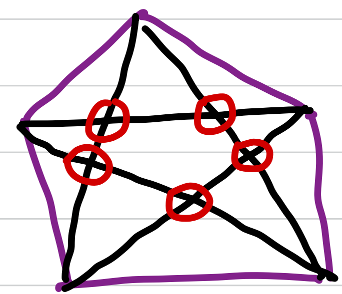
$n = 4$

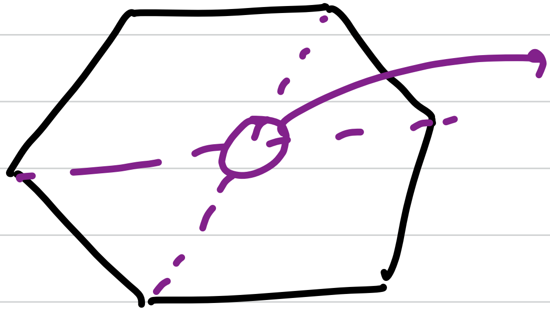
ans → 1



$n = 5$

ans → 5





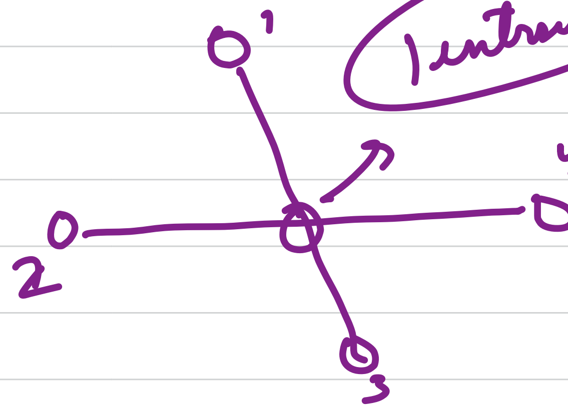
2 diagonals involved is 1 intersection

ans → $^nC_4$

$\frac{5!}{4!\,1!} \Rightarrow 5$

$\frac{7!}{4!\,3!} \Rightarrow \frac{7\times\cancel{6}\times5}{\cancel{6}}$

$\Rightarrow 35$



1 intersect

find the value of sum of odd terms of binomial ?

To find → ${}^nC_1 + {}^nC_3 + {}^nC_5$ - - - - - - -

Proof

$(x+y)^n = {}^nC_0 x^0 y^n + {}^nC_1 x y^{n-1} \cdot {}^nC_2 x^2 y^{n-2}$ - - - - -

Put $x = -1$ $y = 1$

$0 = {}^nC_0(-1)^0(+1)^n + {}^nC_1(-1)(+1)^{n-1} + {}^nC_2(-1)^2(+1)^{n-2}$ - - - -

$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3$ - - - - -

$$ {}^nC_0 + {}^nC_2 + {}^nC_4 \text{-------} = {}^nC_1 + {}^nC_3 + {}^nC_5 \text{------} $$

And we know

$$ {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 \text{---------} {}^nC_n = 2^n $$

$$ 2({}^nC_1 + {}^nC_3 + {}^nC_5 \text{------}) = 2^n $$

$$ {}^nC_1 + {}^nC_3 + {}^nC_5 \text{--------} = 2^{n-1} $$

ans

Q: In a tournament $2N$ teams are participating. Count the no. of pairs for the first round of tournament. Order of the rounds doesn't matter.

Ex $N=2$ T_1 T_2 T_3 T_4

$(T_1\ T_2)\ (T_3\ T_4)$

$(T_1\ T_3)\ (T_2\ T_4)$

$(T_1, T_4)\ (T_2\ T_3)$

ans → 3

1, 2, 3, 4 .... 2n-1, 2n

↳ Total 2n items

Total possibilities

$$^{2n}C_2 \times {}^{2n-2}C_2 \times {}^{2n-4}C_2 \cdots\cdots {}^{4}C_2 \times {}^{2}C_2$$

for 1st match

for 2nd match

for 3rd match

But we have repetition here.

1 2 3 4

$^4C_2 \times {}^2C_2$ → (1,2) (3,4)

→ $\frac{4!}{2!\,2!} \times 1$ → (2,3) (1,4)

⇒ 6 → (3,4) (1,2)

→ (1,3) (2,4)

→ (1,4) (2,3)

→ (2,4) (1,3)

Same

all possible order → n!

to make combinations agnostic of order divide by n!.

Total comb $\rightarrow$ $$\frac{{}^{2n}C_2 \times {}^{2n-2}C_2 \times {}^{2n-4}C_2 \text{ - - - - } {}^{4}C_2 \times {}^{2}C_2}{n!}$$

$$\rightarrow \frac{\frac{(2n)!}{2!\,\cancel{(2n-2)!}} \times \frac{\cancel{(2n-2)!}}{(2!)\,\cancel{(2n-4)!}} \times \frac{\cancel{(2n-4)!}}{(2!)\,\cancel{(2n-6)!}} \text{ - - - - - - } \frac{\cancel{4!}}{2!\,\cancel{2!}} \times \frac{\cancel{2!}}{2!\,0!}}{n!}$$

$$\rightarrow \frac{(2n)!}{2^n \times n!} \text{ ans}$$

ans $\rightarrow$ $$\frac{(2n)!}{2^n\, n!}$$

for $n=4$ $\rightarrow$ $$\frac{8!}{16 \times 4!} \Rightarrow \frac{\cancel{8} \times 2 \times \cancel{6}^{\,3} \times 5}{\cancel{16}\;\cancel{2}} \rightarrow 105$$

1, 2, 3, 4, 5, 6, 7, 8

1, 2, 3, 4, 5, 6

$$ans_{n=4} = 7 \times ans_{n=3} \rightarrow 105$$

$$ans_{n=3} = 5 \times ans_{n=2} \rightarrow 15$$

$$ans_{n=2} \Rightarrow 3$$

$$f(n) = (2n-1) \times f(n-1)$$

No. of ways to form pairs of 2n teams agnostic of order

DP

count

optim

Famous Qun → **Friends Pairing Problem**

↳ There are N friends who wants to go to party.

→ Either by making pairs

→ Or go alone

A B C → 4

→ (A) (B) (C)

(A B) (C)

(A C) (B)

(B C) (A)

A B C D

$$\boxed{f(n) = f(n-1) + (n-1)\,f(n-2)}$$

Q→ There are M children and N ~~ung~~ chocolate.
Count the no. of ways of distributing N chocolates to M children such that each children get at least x chocolate you can get

$x \in [1, 2, 3]$

T testcases

$N, M \le 10^3$

$T \le 10^2$

$N = 5$
$M = 2$
$x = 1$

ans → 25

"

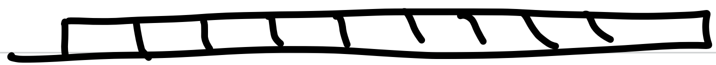
for $x = 1$

N choco

m children

choco

atleast 1



N

n^{th} choco $\longrightarrow$ m^{th} children

$(N-1) \longrightarrow (M-1)$ children distribute

$$f(n,m) = f(N-1, m) \quad f(n-1, m-1)$$

choco children

$$f(n, m) = {}^{m}C_1 \, f(n-1, m) + f(n-1, m-1)$$

no. of way to dist n choco to m children when everyone gets atleast 1

last one gets exactly 1 choco

last choco goes to someone with zero choco

atleast 1

$$f(n,m) = {}^nC_1\, f(n-1, m) + (n-1)\, f(n-2, m-1)$$

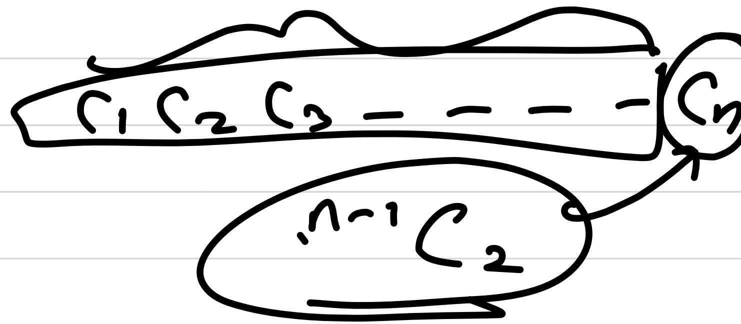
# of ways to dist
n choco to m
children such that
every one gets
atleast 2

${}^nC_2$ 2 ?

$C_1 \; C_2 \; C_3 \; C \text{-----} C_n \rightarrow 2$

$$f(n,m) = {}^{m}C_1 \, f(n-1, m) + {}^{n-1}C_2 \, f(n-3, m-1)$$

# of ways to dist n choco to m children such that every one gets atleast 3



atleast x → exact x

$> x$

atleast → 2

> 2

Qn Given n, m, count the no. of sol^n for the given inequality

$$1 \le x_1 < x_2 < x_3 < \ldots\ldots < x_n \le m$$

where x_i, n, m are integers

(1 —— m)

${}^mC_n$

↳ HW → Think for a dp solution

if the eq^n is of the form $\rightarrow$ $n=5$ $m=2$

$$1 \leq x_1 \leq x_2 \leq x_3 \cdots\cdots \leq x_n \leq m$$

There are m objects and because equality is there one object can be chosen more than once.

$$^{(n+m-1)}C_{n-1}$$