

Basic Algebra

Functions, polynomials,
coordinate systems, complex numbers

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#MathForDevs

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Polynomials

Definition, storing, basic operations

Polynomials

- We already looked at linear and quadratic polynomials
- Term (monomial): $2x^2$
 - Coefficient (number), variable, power (number ≥ 0)
- Polynomial: sum of monomials
 - $2x^4 + 3x^2 - 0,5x + 2,72$
 - Degree: the highest degree of the variable (with coefficient $\neq 0$)
- Operations
 - Defined the same way as with numbers
 - Addition and subtraction
 - $(2x^2 + 5x - 8) + (3x^4 - 2) = 3x^4 + 2x^2 + 5x - 10$
 - Multiplication and division
 - $(2x^2 + 5x - 8)(3x^4 - 2) = 6x^6 + 15x^5 - 24x^4 - 4x^2 - 10x + 16$

Polynomials in Python

- numpy has a module for working with polynomials
 - Includes the "general" polynomials, as well as a few special cases
 - Chebyshev, Legendre, Hermit
- Storing polynomials
 - As arrays (index \Rightarrow power, value \Rightarrow coefficient)
 - Keep in mind this will look "reversed" relative to the way we write

```
import numpy.polynomial.polynomial as p
p.polyadd([-8, 5, 2], [-2, 0, 0, 0, 3])
# array([-10., 5., 2., 0., 3.])
p.polymul([-8, 5, 2], [-2, 0, 0, 0, 3])
# array([ 16., -10., -4., 0., -24., 15., 6.])
```

Polynomials in Python (2)

- Pretty printing
 - Use sympy to print the polynomial
 - If it's a list, use it directly
 - If it's a Polynomial object, call the coef property
 - Reverse the order of coefficients (sympy expects them from highest to lowest)

```
import sympy
from sympy.abc import x
polynomial = p.Polynomial([-2, 0, 0, 0, 3])
sympy.init_printing()
print(sympy.Poly(reversed(polynomial.coef), x).as_expr())
# Output: 3.0*x**4 - 2.0
```



Sets

Set notation and basic operations

Set

- An **unordered collection** of things
 - Usually numbers
 - No repetitions
- Set notation: $\{x \in \mathbb{R} \mid x \geq 0\}$
 - "The set of numbers x , which are a subset of the real numbers, which are greater than or equal to zero"
 - Left: example **element**
 - Right: **conditions** to satisfy
- Python set comprehensions
 - Very similar to what we already wrote
 - Also very similar to list comprehensions (but with curly braces)

```
positive_x = {x for x in range(-5, 5) if x >= 0}
# {0, 1, 2, 3, 4}
```

Set Operations

- Cardinality: number of elements
- Checking whether an element is in the set: $x \in S$
- Checking whether a set is subset of another set: $S_1 \subseteq S_2$
- Union $S_1 \cup S_2$, intersection $S_1 \cap S_2$, difference $S_1 \setminus S_2$

```
set1 = { 1, 2, 3, 4 }
set2 = {3, 4, 5, 10, 3, 5, 10, 3, 3}
print(len(set2)) # 4
print(1 in set1) # True
print(10 not in set1) # True
print({1, 2}.issubset(set1)) # True
print(set1.union(set2)) # {1, 2, 3, 4, 5, 10}
print(set1.difference(set2)) # {1, 2}
print(set2.difference(set1)) # {10, 5}
print(set1.symmetric_difference(set2)) # {1, 2, 5, 10}
```



Functions

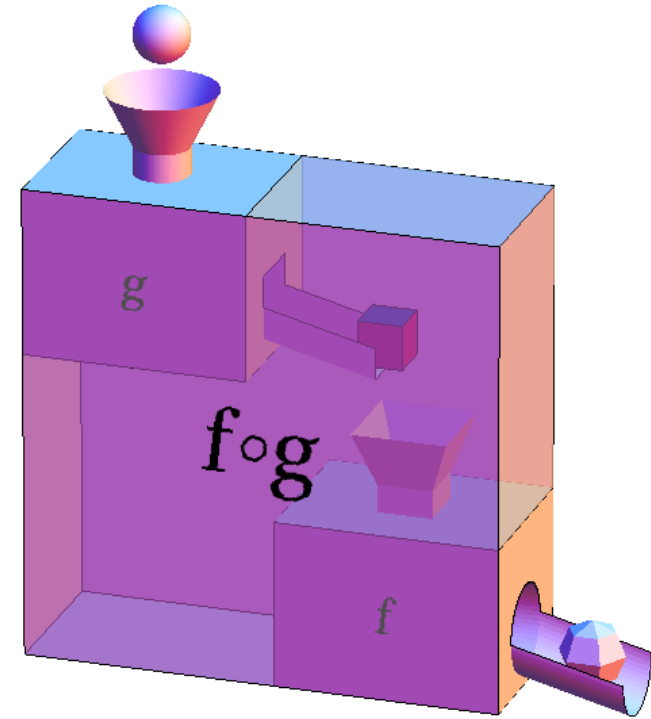
Mappings from one thing to another

Function

- A **relation** between
 - A set of inputs X (**domain**)
 - ... and a set of outputs Y (**codomain**)
 - **One input produces exactly one output**
 - The inputs don't need to be numbers
 - Functions don't know how to compute the output, they're just mappings
 - In programming, we're writing procedures
- Math notation: $f : X \rightarrow Y$
 - Commonly abbreviated as $y = f(x)$
- Some more definitions
 - **Injective** (one-to-one): unique inputs \Rightarrow unique outputs
 - **Surjective** (onto): every element in the codomain is mapped
 - **Bijjective** (one-to-one correspondence): injective and surjective
 - Here is [a graphical view](#)

Function Composition

- Also called **pipelining** in most languages
- Takes two functions and applies them in order
 - **Innermost to outermost**
 - Math notation: $f \circ g = f(g(x))$
 - Can be generalized to more functions
- Note that the order matters
$$f(x) = 2x + 3, \quad g(x) = x^2$$
$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 3$$
$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2$$
- This kind of notation can be confusing sometimes
 - x is only a placeholder for the input
 - We've used the same letter x for different inputs
 - Tip: When working with complicated functions, be very careful what the inputs and outputs are, and how variables depend on other variables
- Functions and composition are the basis of functional programming



Function Graphs

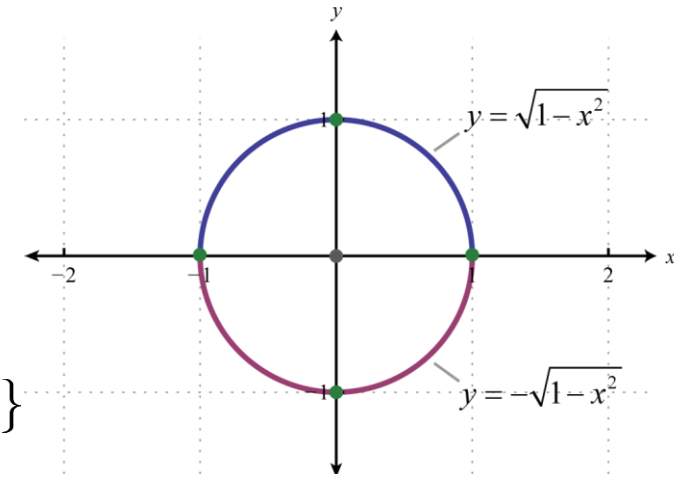
- One very intuitive way to get to know functions is to plot them
 - We already did that in the last exercise
 - Generate values in the domain (independent variable)
 - For each value compute the output (dependent variable)
 - Create a graph; plot all computed points and connect them with tiny straight lines
- **lambda** in Python is a short syntax for a function
 - We can define it outside as well (it's just shorter and simpler to use it inline)

```
import numpy as np
import matplotlib.pyplot as plt
def plot_function(f, x_min = -10, x_max = 10, n_values = 2000):
    x = np.linspace(x_min, x_max, n_values)
    y = f(x)
    plt.plot(x, y)
    plt.show()

plot_function(lambda x: np.sin(x))
```

Graphing a Circle

- Let's try to graph the unit circle
 - Equation: $x^2 + y^2 = 1$
- This cannot be represented as one function
 - We have multiple values of y , e. g. $x = 0 \rightarrow y = \{-1, 1\}$
- We can try two functions (see graph)
 - But we want to represent the circle as one object



```
def plot_function(f, x_min = -10, x_max = 10, n_values = 2000):
```

```
    plt.gca().set_aspect("equal")
```

```
    x = np.linspace(x_min, x_max, n_values)
```

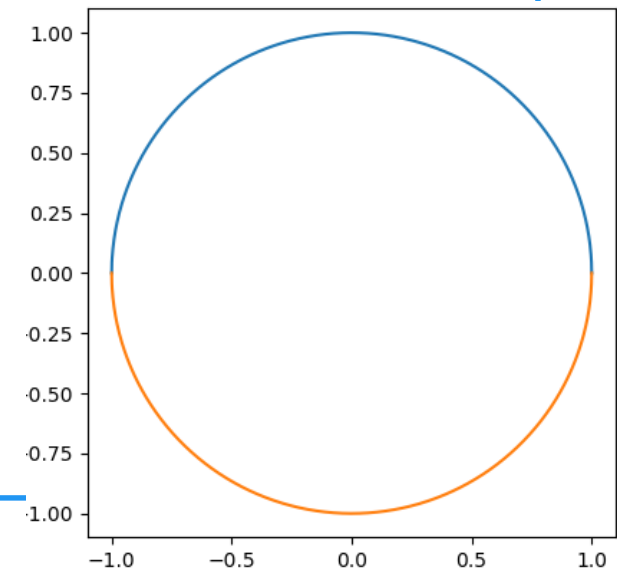
```
    y = f(x)
```

```
    plt.plot(x, y)
```

```
plot_function(lambda x: np.sqrt(1 - x**2), -1, 1)
```

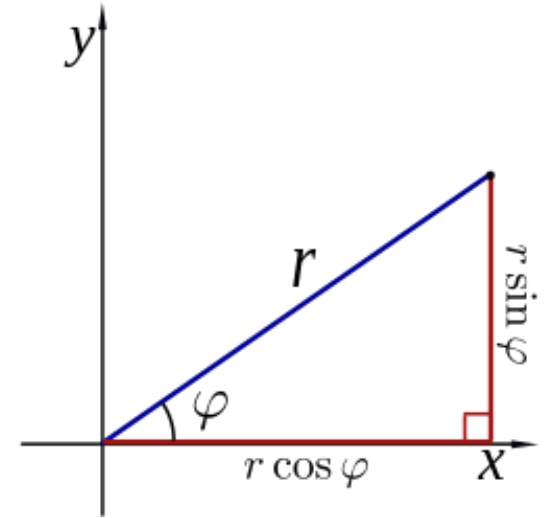
```
plot_function(lambda x: -np.sqrt(1 - x**2), -1, 1)
```

```
plt.show()
```



Graphing a Circle (2)

- In math and science, many problems can be solved by changing our viewpoint
- We can use another type of reference system
 - One which incorporates angles naturally
 - Polar coordinate system (r, φ) :
 - (r : distance from origin ($r \geq 0$); φ : angle to x-axis)
 - We can easily convert Cartesian to polar coordinates
$$x^2 + y^2 = 1$$
$$(r \cos \varphi)^2 + (r \sin \varphi)^2 = 1$$
$$r^2 \cos^2 \varphi + r^2 \sin^2 \varphi = 1$$
$$r^2 (\cos^2 \varphi + \sin^2 \varphi) = 1$$
$$r^2 = 1, \quad r \geq 0 \Rightarrow r = 1$$
- Now we can see the equation is very, very simple
 - Doesn't even depend on φ
 - This is why we needed the change of viewpoint (coordinates)



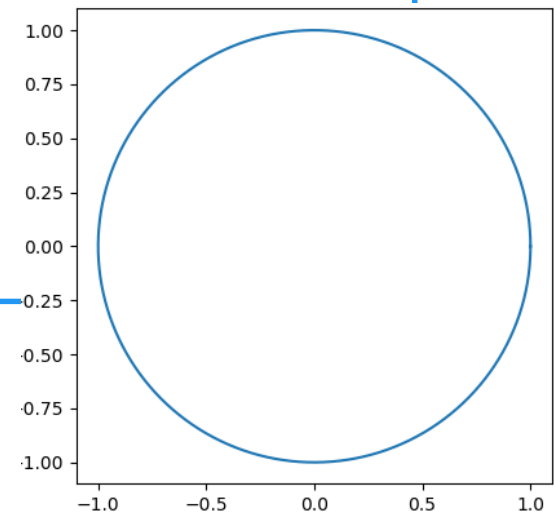
Graphing a Circle (3)

- Graphing a function in polar coordinates
 - This applies to any function, circles in particular
 - Generate initial values of r and φ
 - Convert them to rectangular coordinates
 - Plot the rectangular coordinates

```
import numpy as np
import matplotlib.pyplot as plt
r = 1 # Radius
phi = np.linspace(0, 2 * np.pi, 1000) # Full circle
x = r * np.cos(phi)
y = r * np.sin(phi)
plt.plot(x, y)
plt.gca().set_aspect("equal")
plt.show()
```

- For most other applications we can do this directly

```
plt.polar(phi, r)
```





Complex Numbers

Not so complex as they seem

Number Fields

- Field
 - A collection of values with operations "plus" and "times"
 - Algebra is so abstract we can redefine these operations (stay tuned)
- History of number fields
 - Natural (counting) numbers $\mathbb{N} = \{0, 1, 2, \dots\}$
 - Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
 - Subtraction
 - Rational numbers \mathbb{Q} : ratio of two integers
 - Division
 - This is **the smallest field**
 - Real numbers $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$
 - Most roots (e.g. $\sqrt{2}$)
 - **Complex numbers** \mathbb{C}
 - All roots (including square roots of negative numbers)
 - "Imaginary unit": i is the positive solution of $x^2 = -1$

Complex Numbers

- Pairs of real numbers: $(a; b) : a, b \in \mathbb{R}$
 - Commonly written as $a + bi$
 - Real part: $\text{Re}(a + bi) = a$, imaginary part: $\text{Im}(a + bi) = b$
- In Python, we use **j** instead of **i**
- We can get the real and imaginary parts
- Adding and multiplying complex numbers

3j

1j

3 + 2j

- Note that we write **1j** to prevent confusion with the variable **j**
- For the same reason, we don't write **2 * j** if **j** is the imaginary unit

```
z = 3 + 2j
print(z.real) # 3
print(z.imag) # 2
```

```
print((3 + 2j) + (8 - 3j)) # (11-1j)
print((3 + 2j) * (8 - 3j)) # (30+7j)
```

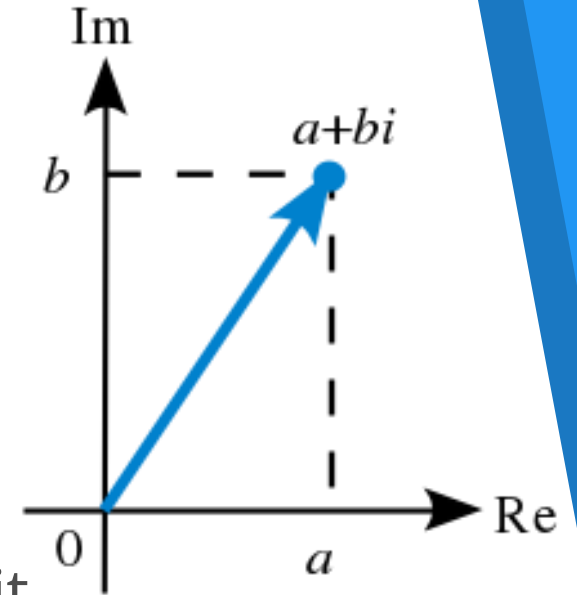
Geometric Interpretation

- Intuition

- We can plot the coordinate pairs on the plane
- Each point in the 2D space represents one complex number

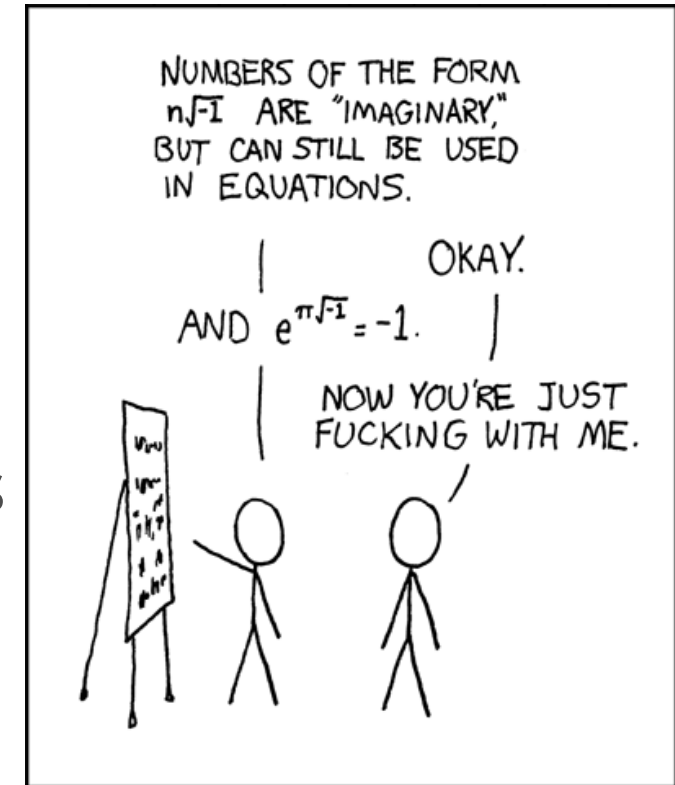
- But...

- We saw that we can change our perspective a little bit
- Polar coordinates: we can use the same transformation
 - $\rho = |z|$ – **module** of the complex number
 - $\varphi = \arg(z)$ – **argument** of the complex number
 - $a = \rho \cos(\varphi)$, $b = \rho \sin(\varphi)$
- Why do we do this?
 - Some operations (e.g. multiplication and division) are easier in polar coordinates
 - Powers of complex numbers become extremely easy
- Polar form
 - $z = a + bi = \rho(\cos(\varphi) + i \sin(\varphi))$



Euler's Formula

- Leonhard Euler proved that $e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$
 - Here's a [summary of the proof](#) if you're interested
 - It involves series which we haven't covered yet
 - A very beautiful consequence: $e^{i\pi} + 1 = 0$
- Now we can write our complex number as $z = |z|e^{i\varphi}$
- Why and how does multiplication work?
 - Multiplication by a real number
 - Scales the original vector
 - Multiplication by an imaginary number
 - Rotates the original vector
 - You can see a thorough explanation [here](#)
- **Main point:** Multiplication of complex numbers is the same as scaling and rotating 2D vectors
 - Algebra is abstract and we love it :)





Fundamental Theorem of Algebra

Roots, roots, and more roots

Fundamental Theorem of Algebra

- "Every non-zero, single-variable, degree- n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots"
 - More simply said, **every algebraic equation has as many roots as its power**
- Back to quadratic equations
 - How do we get all roots?
 - Simply use the complex math Python module: **cmath**

```
import cmath
def solve_quadratic_equation(a, b, c):
    discriminant = cmath.sqrt(b * b - 4 * a * c)
    return [
        (-b + discriminant) / (2 * a),
        (-b - discriminant) / (2 * a)]

print(solve_quadratic_equation(1, -3, -4)) # [(4+0j), (-1+0j)]
print(solve_quadratic_equation(1, 0, -4)) # [(2+0j), (-2+0j)]
print(solve_quadratic_equation(1, 2, 1)) # [(-1+0j), (-1+0j)]
print(solve_quadratic_equation(1, 4, 5)) # [(-2+1j), (-2-1j)]
```




Some More Notes

Taking abstraction to the max

Galois Field

- In everyday algebra, we usually think about fields as those we already know
 - E.g. the field of real numbers
- But since algebra is abstract, we can define our own fields
- Galois field: $GF(2)$
 - Elements $\{0, 1\}$
 - Addition: equivalent to XOR
 - Multiplication: as usual
- Usage: in cryptography
 - If you're interested, you can have a look at [this](#) paper

+	0	1
0	0	1
1	1	0

*	0	1
0	0	0
1	0	1

A Note about Vectors

- One more application of abstractions
- **Vector**
 - A line segment with a direction
- We saw that 2D vectors and 2D points have a one-to-one correspondence
 - A point can be represented as its **radius-vector**
- A vector is also an ordered tuple of coordinates
 - That's why we were able to take out thinking of points and apply it to complex numbers
- We usually represent vectors as Python lists: `[2, 3, -5]`
- **Idea**
 - Can we think of the list as a mapping: $0 \Rightarrow 2, 1 \Rightarrow -3, 2 \Rightarrow -5$?
 - What does this mean?
 - ... we'll find out more next time
- What does this imply about fields?

Summary

- Polynomials
- Sets
- Functions
- Coordinates
- Complex numbers
- Abstraction

The image features a white background with two blue decorative bars. The top bar is a solid blue strip. The bottom bar is a gradient blue strip that transitions from a lighter blue on the left to a darker blue on the right. The word "Questions?" is centered in a blue, sans-serif font.

Questions?