PROBABILITY AND STATISTICS (PMA303)

Lecture-[27]

(Type I & Type-II Error of Hypothesis with illustrations)

Para. & Non-para., Hypothesis Testing: (Unit VI-VII)

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Confidence interval for Proportion

Let $X_1, X_2 \dots X_n \overset{i.i.d}{\sim} Bin(1, p)$ find a $(1-\alpha)100\%$ confidence interval of p

Consider $\hat{p} = \frac{\sum X_i}{n}$ (known as $\bar{x}$ in previous notes

$$\mathbb{E}(\hat{p}) = \boxed{p}$$

$$V(\hat{p}) = V\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2}[\overbrace{pq + pq + \dots pq}^{n \text{ times}}] = \boxed{\frac{pq}{n}} = \boxed{\frac{p(1-p)}{n}}$$

$$\Rightarrow \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

$$\Rightarrow \mathbb{P}\left(-z_{\alpha/2} < \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} < z_{\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow \mathbb{P}\left(-\hat{p} - z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} < -p < -\hat{p} + z_{\alpha/2}\sqrt{\frac{(1-p)p}{n}}\right) = 1-\alpha$$

using the fact $\hat{p} \approx p$

$p(1-p) \approx \hat{p}(1-\hat{p})$

$\because \mathbb{E}(S^2) = \sigma^2$

we get, after multiplying by (-1) and reversing inequ

Confidence Interval for p = sample proportion

$$\mathbb{P}\left(\underbrace{\hat{p} - z_{\alpha/2}\sqrt{\frac{(1-\hat{p})\hat{p}}{n}}}_{L} < p < \underbrace{\hat{p} + z_{\alpha/2}\sqrt{\frac{(1-\hat{p})\hat{p}}{n}}}_{U}\right) = 1-\alpha$$

Hypothesis Testing (ii)

Based on the $\boxed{\text{sample}}$ drawn say $\underline{X}$
we either accept H_0 or reject it in favour of H_1
Since the decision is based on the sample/'evidence' at hand
it is not the ultimate truth, it is only inferential.
$\Rightarrow$ There is bound to be some error in the decision making
and we wish to minimize this error of wrong deductions.

The Errors are of Two types.

- Type-I : Reject H_0 when it was actually TRUE
- Type-II Accept H_0 when it was actually False.

- Which error is more fatal ??

(A) There is no rule, but usually Type-I error is considered more serious.

- How to frame the Null hypothesis.

(A) There is no rule, but generally the scenario which one wants to avoid is framed as the null H_0.

- The probability of Type-I error is denoted by α

i.e $\alpha = \mathbb{P}(\text{Type-I error}) = \mathbb{P}(\text{Reject when } H_0 \text{ is true})$

$$\Rightarrow \boxed{\alpha = \mathbb{P}_{\theta_0}(\underline{X} \in \text{Reject})}$$

- The probability of Type-II error is denoted by β

i.e $\beta = \mathbb{P}(\text{Type-II error})$
$= \mathbb{P}(\text{Accept when } H_0 \text{ is false})$
$\beta = \mathbb{P}_{\theta_1}(\underline{X} \in \text{Accept})$

Since α and β both are errors
our goal should be to minimize both. $\Rightarrow$ $\boxed{\text{Ideally } \alpha = \beta = 0}$

However can we minimize both $\boxed{\text{simultaneously}}$

Ex: for any H_0 set Rejection Region $= \phi$ = Empty set $\Rightarrow$ You never reject
$\Rightarrow \mathbb{P}_{\theta_0}(\underline{X} \in \text{Reject}) = \mathbb{P}_{\theta_0}(\underline{X} \in \phi) = 0$

if Rejection Region is $\phi \Rightarrow$ Acceptance Region is $\phi^c = \mathbb{R}$
$\Rightarrow \beta = \mathbb{P}_{\theta_1}(\underline{X} \in \text{Accept}) = \mathbb{P}_{\theta_1}(\underline{X} \in \mathbb{R}) = 1$

$\Rightarrow \alpha = 0, \beta = 1$ $\Rightarrow \alpha$ gets minimized but β gets maximized.

Thus, There is always a trade-off.

Q What strategy should be deployed.

(A) In general, we specify a Critical Region that bounds α. Among $\boxed{\text{all such CR,}}$ choose that CR which minimizes β

Mathematically, we say a Critical Region C is of size α if

$$\alpha = \max_{\theta_0 \in \Omega_0} \left\{ \mathbb{P}_{\theta_0}(\underline{X} \in C) \right\} \quad \text{———} \quad (1)$$

" α may be thought of as the maximum Type-I error that is allowed".

Since there can be more than one critical regions such that (1) holds we choose that which minimizes β.

$$\begin{aligned}
\text{so} \quad & \text{minimize } \beta \\
= \quad & \text{maximize } (1-\beta) \\
= \quad & \max_{\theta_1 \in \Omega_1} \left\{ 1 - \mathbb{P}_{\theta_1}(\underline{X} \in \text{Accept}) \right\} \\
= \quad & \max \left\{ \mathbb{P}_{\theta_1}(\underline{X} \in \text{Reject}) \right\}
\end{aligned}$$

$$\boxed{1-\beta = \mathbb{P}_{\theta_1}(\underline{X} \in \text{Reject})}$$

if Ω_1 is the just a single point

This $(1-\beta)$ is referred to as the Power of the test-

$\Rightarrow$ You reject H_0 when it should actually be rejected.

Ex: $X \sim \text{Bin}(5, p)$

$H_0 : p = \frac{1}{2}$ Given $\overbrace{\alpha}^{\text{Max Type-I error}} = \frac{1}{32}$ find critical Region(CR) for this α.

$H_1 : p = 3/4$

$\because \underbrace{\Omega_0 = \{1/2\}}_{\text{single point}}$ $\underbrace{\Omega_1 = \{3/4\}}_{\text{single point}}$, Assume CR = $\{0\}$

$\Rightarrow$

$$\alpha = \mathbb{P}_{\theta_0}(\underline{X} \in \text{Reject}) = \mathbb{P}_{p=\frac{1}{2}}(X \in \{0\})$$

$$\Rightarrow \mathbb{P}_{\frac{1}{2}}(X=0) = {}^5C_0\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^{5-0} = \boxed{\frac{1}{32}}$$

$\Rightarrow$ if $\alpha = \frac{1}{32}$, $C_1 = \{0\}$ is a candidate critical region

also observe that if CR = $\{5\}$ we see

$$\mathbb{P}_{\frac{1}{2}}(X \in \{5\}) = \mathbb{P}_{\frac{1}{2}}(X=5) = {}^5C_5\left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^{5-5} = \boxed{\frac{1}{32}}$$

$\Rightarrow$ if $\alpha = \frac{1}{32}$, $C_2 = \{5\}$ is also a candidate critical Region

But do we choose C_1 or C_2 ??

We would choose C_1 or C_2 based on which CR minimizes β or which maximizes the power $1-\beta$

So let

$C_1 = \{0\} = CR \Rightarrow Acc = \{1,2,3,4,5\}$

$\beta = \mathbb{P}(\text{Type-II error})$

$= \mathbb{P}_{\theta_1}(\underline{X} \in Accept)$

$= \mathbb{P}_{3/4}(X \in \{1,2,3,4,5\})$

$= 1 - \mathbb{P}_{3/4}(X \in \{0\})$

$\beta = 1 - {}^5C_0 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^5$

Power $= 1-\beta = \left(\frac{1}{4}\right)^5$

$C_1 = \{5\} = C\cdot R \Rightarrow Accep = \{0,1,2,3,4\}$

$\beta = \mathbb{P}(\text{Type-II error})$

$= \mathbb{P}_{\theta_1}(\underline{X} \in Accept)$

$= \mathbb{P}_{3/4}(X \in \{0,1,2,3,4\})$

$= 1 - \mathbb{P}(X \in \{5\})$

$\beta = 1 - {}^5C_5 \left(\frac{3}{4}\right)^5 (1/4)^0$

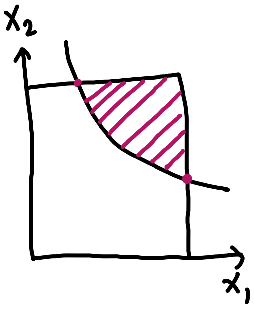
$1-\beta = \left(\frac{3}{4}\right)^5 =$ Power.

$\Rightarrow \because \left(\frac{3}{4}\right)^5 > \left(\frac{1}{4}\right)^5 \Rightarrow$ Power is greater when CR $= \{5\}$

$\Rightarrow \{5\}$ is a critical Region of size $\alpha = \frac{1}{32}$

Ex: **4.5.3.** Let X have a pdf of the form $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, zero elsewhere, where $\theta \in \{\theta : \theta = 1, 2\}$. To test the simple hypothesis $H_0 : \theta = 1$ against the alternative simple hypothesis $H_1 : \theta = 2$, use a random sample X_1, X_2 of size $n = 2$ and define the critical region to be $C = \{(x_1, x_2) : \frac{3}{4} \le x_1x_2\}$.

Find Probability of Type-1 error



$\alpha = \mathbb{P}_{\theta_0}(\underline{X} \in Reject)$

$\alpha = \mathbb{P}_1(X_1X_2 \geq 3/4)$

$= \int_{\frac{3}{4}}^{1}\int_{\frac{3}{4x_1}}^{1} 1 \cdot dx_2 dx_1$

when $\theta = \theta_0 = 1$ the joint p.d.f is

$f(x_1,x_2) = \theta_0 x_1^{\theta_0 - 1}\, \theta_0 x_2^{\theta_0 - 1}$

$f(x_1,x_2) = \begin{cases} 1 & 0 < x_1, x_2 < 1 \\ 0 & \text{else.} \end{cases}$

$\alpha = \int_{3/4}^{1}\left(1 - \frac{3}{4x_1}\right)dx_1 = x_1 - \frac{3}{4}\log x_1 \Big|_{\frac{3}{4}}^{1} = \frac{1}{4} - \frac{3}{4}\log\frac{3}{4} = 0.4657$

⊙ Let $X_1, X_2 \ldots X_n \overset{i.i.d}{\sim} N(\mu, 1)$ μ is unknown, But it is known that $\mu \in \{\mu_0, \mu_1\}$ with $\mu_0 < \mu_1$

size of test $= \alpha$

$H_0 : X_i \sim N(\mu_0, 1)$

$H_1 : X_i \sim N(\mu_1, 1)$

$\Rightarrow$ if an assertion has to be made about μ

$\Rightarrow$ The current best thing we have $\overline{X}$ from the sample

so if we see $\overline{X}$ closer to $\mu_0 \Rightarrow$ accept H_0

and if $\overline{X}$ closer to $\mu_1 \Rightarrow$ reject H_0

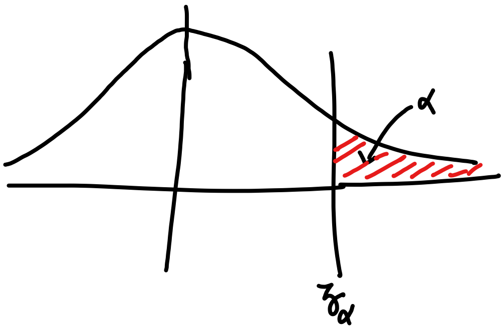
Thus Reject H_0 if $\overline{X} > k$

Retain/Accept H_0 otherwise

So we have to find $\underline{\underline{k}}$.

This k should be such that for a given α

$$\alpha = \mathbb{P}_{\mu_0}(\text{Type-I error}) = \mathbb{P}_{\mu_0}(\underline{x} \in \text{Reject})$$



$$= \mathbb{P}_{\mu_0}(\bar{X} > k) \qquad \because \bar{X} \sim N(\mu_0, \tfrac{1}{n})$$

$$\alpha = \mathbb{P}\left(\frac{\bar{X} - \mu_0}{\sqrt{1/n}} > \frac{k - \mu_0}{\sqrt{1/n}} \right)$$

$$\alpha = \mathbb{P}\left(Z_0 > \frac{k - \mu_0}{\sqrt{1/n}} \right)$$

$$\Rightarrow \frac{k - \mu_0}{\sqrt{1/n}} = z_\alpha \Rightarrow k = \mu_0 + z_\alpha \sqrt{1/n}$$

Thus, CRITICAL REGION is

$$C = \left\{ (x_1, x_2 \ldots x_n) : \bar{x} > \mu_0 + z_\alpha \sqrt{1/n} \right\}$$

$$\Rightarrow \boxed{\text{Reject } H_0 \text{ if } \bar{x} > \mu_0 + z_\alpha \sqrt{1/n}}$$

Right tail Z test

$$\boxed{\text{In general Reject } H_0 \text{ if } \bar{x} > \mu_0 + z_\alpha \sqrt{\sigma^2/n}}$$

Q $X_1, X_2 \ldots X_{16} \sim N(\mu, 4)$

$H_0: \mu_0 = 10$

$H_1: \mu_1 = 11$

size of test $= \alpha = 0.05$

find the critical region for this α.

$$C = \left\{ (x_1, x_2 \ldots x_n) : \bar{x} > \mu_0 + z_\alpha \sqrt{\sigma^2/n} \right\}$$

$$\Rightarrow C = \left\{ (x_1, x_2 \ldots x_{16}) : \bar{x} > 10 + z_{0.05} \sqrt{\tfrac{4}{16}} \right\}$$

$$\Rightarrow C = \left\{ (x_1 \ldots x_{16}) : \bar{x} > 10 + 1.645 \left(\tfrac{1}{2}\right) \right\}$$

$$\Rightarrow C = \left\{ (x_1 \ldots x_{16}) : \bar{x} > 10.8225 \right\}$$

$\Rightarrow$ when $\boxed{\bar{x} > 10.8225 \text{ we reject the null Hypothesis}}$

ex: if we see $\bar{x} = 12$ we reject $H_0 \Rightarrow$ Accept H_1

$\Rightarrow X_i \sim N(11, 4)$

if we see $\bar{x} = 10.7$ we retain H_0.

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090
$z_{\alpha/2}$	0.842	1.036	1.282	1.645	1.960	2.241	2.326	2.576	2.807	3.291

For the corresponding α and Critical Region How to find β and Power of the test

$$\beta = \mathbb{P}(\text{Type-II error})$$

$$= \mathbb{P}_{\mu_1}(\underline{X} \in \text{Accept}) = 1 - \mathbb{P}_{\mu_1}(\underline{X} \in \text{Reject})$$

under H_1
$\bar{X} \sim N(\mu_1, \frac{\sigma^2}{n})$

$$\beta = 1 - \mathbb{P}_{\mu_1}(\bar{X} > k)$$

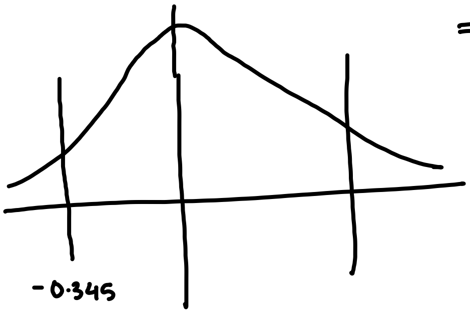
$$1-\beta = \mathbb{P}_{\mu_1}\left(\bar{X} > \mu_0 + z_\alpha \sqrt{\sigma^2/n}\right)$$

$$= \mathbb{P}\left(\frac{\bar{X}-\mu_1}{\sqrt{\sigma^2/n}} > \frac{\mu_0 + z_\alpha\sqrt{\sigma^2/n} - \mu_1}{\sqrt{\sigma^2/n}}\right)$$

$$= \mathbb{P}\left(Z > \frac{\mu_0-\mu_1}{\sqrt{\sigma^2/n}} + z_\alpha\right)$$

$$\underbrace{1-\beta}_{\text{Power}} = \mathbb{P}\left(Z > z_\alpha - \frac{\sqrt{n}(\mu_1-\mu_0)}{\sigma}\right)$$

for the above Q: $n = 16$, $\alpha = 0.05$
$\sigma^2 = 4$ $\quad \mu_0 = 10, \mu_1 = 11$



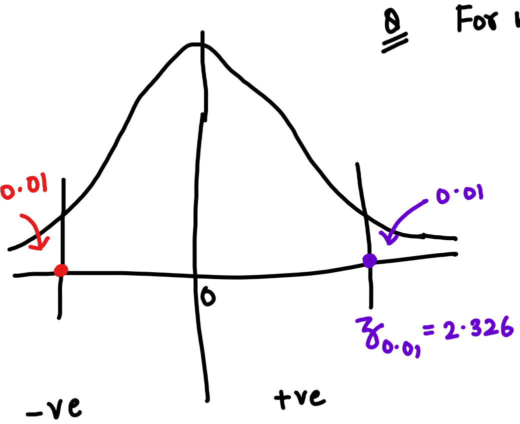
$$\Rightarrow \text{Power} = 1-\beta = \mathbb{P}\left(Z > z_{0.05} - \frac{\sqrt{16}\,(11-10)}{2}\right)$$

$$= \mathbb{P}(Z > 1.645 - 2)$$

$$= \mathbb{P}(Z > -0.345) = \mathbb{P}(Z \leq 0.345)$$ (see table)

$$= \boxed{0.63495}$$

Q For what least value of n will the power of the test be atleast 0.99



$$\Rightarrow \mathbb{P}\left(Z > z_{0.05} - \frac{\sqrt{n}}{2}(11-10)\right) \geq 0.99$$

$$\mathbb{P}\left(Z \geq 1.645 - \frac{\sqrt{n}}{2}\right) \geq 0.99$$

$$\Rightarrow 1.645 - \frac{\sqrt{n}}{2} = -z_{0.01}$$

$$1.645 - \frac{\sqrt{n}}{2} = -2.326$$

$$\Rightarrow n = 63.07$$

So let $\boxed{n = 64}$

n is the number of samples
$\Rightarrow n$ is an integer

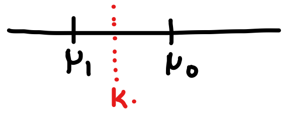
```
>> samples = sampsizepwr('z',[10 2],11,.99,[],'Tail','right')

samples =

    64
```

Test $\frac{\bar{X}-\mu_0}{\sqrt{\sigma^2/n}}$ $\quad$ μ_0 H_0 $\quad$ σ $\quad$ μ_1 H_1 $\quad$ Power $\quad$ Samples $\quad$ which test ?? Right tail

(ii) $X_i \overset{i.i.d}{\sim} N(\mu, \sigma^2)$



$H_0 : \mu_0$ size of the test α

$H_1 : \mu_1$ and $\boxed{\mu_1 < \mu_0}$, σ^2 is Already known

find the critical Region for the size α

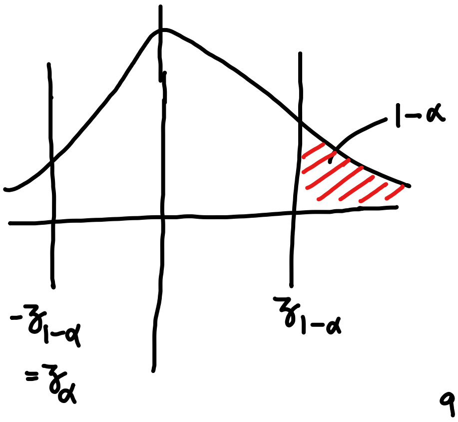
Proceeding as before Rejecting H_0 if $\boxed{\bar{X} < k.}$

Retaining otherwise

the critical region becomes

$$C = \left\{ (x_1 x_2 \cdots x_n) : \bar{x} < \overbrace{\mu_0 - z_\alpha \sqrt{\sigma^2/n}}^{k.} \right\}$$

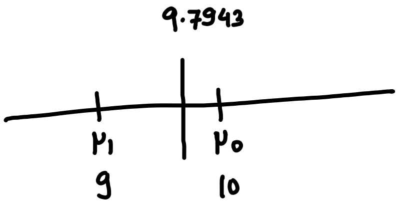
Left Tail z test



Ex : if the observed sample of size 64 has an average of 9, with $\sigma^2 = 1$. Test the hypothesis

$H_0 : \mu_0 = 10$ $\alpha = 0.05$

$H_1 : \mu_1 = 9$



$$\mu_0 - z_\alpha \sqrt{\frac{\sigma^2}{n}} = 10 - z_{0.05} \sqrt{\frac{1}{64}}$$

Left Tail Test

$$= 10 - \frac{1.645}{8}$$

$$= 9.794375$$

$\because \bar{x} = 9$ and $9 < 9.794375$

$\Rightarrow$ Null gets rejected in favour of H_1

$\Rightarrow$ we say $X \sim N(9,1)$