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Sampling distribution (χ^2, t & F distribution)

Content Covered :

- χ^2 – distribution
- t – distribution
- F – distribution.

Note that :

* Let $x_1, x_2, x_3, \ldots x_n$ be a random sample from a population with mean μ and Variance σ^2, M.g.f. $M_X(t)$. Then

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{as } n \to \infty .$$

* If population is itself Normal then

- $$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$ irrespective of large or small n.

$$\Rightarrow \bar{X}-\mu \sim N\left(0, \sigma^2/n\right)$$

$$\Rightarrow \bar{X} \sim N\left(\mu, \sigma^2/n\right).$$

* Population $A \sim N(\mu_1, \sigma_1^2)$ Population $B \sim N(\mu_2, \sigma_2^2)$

$$\Rightarrow \quad \overline{X_1} \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right) \qquad \overline{X_2} \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$$

$$\overline{X_1} - \overline{X_2} \sim N\left(\mu_1 - \mu_2, \underbrace{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}_{\text{Variance always gets added.}}\right)$$

$$\frac{(\overline{X_1} - \overline{X_2}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1)$$

Note that : "If populations are not Normal then this is valid for large n."

i.e., $n > 30$.

Chi-square distribution :

A Continuous random variable X is $\chi^2_{(n)}$ if its p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{2^{n/2}\,\Gamma(n/2)} e^{-x/2} x^{n/2-1} & x>0 \\ & n>0 \\ 0 & o/w \end{cases}$$

$$\int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{2^{n/2}\Gamma(n/2)} e^{-x/2} x^{n/2-1}\,dx$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty e^{-x/2} x^{n/2-1}\,dx$$

$$\because \int_0^\infty e^{-ax} x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$$

$$= \frac{1}{\cancel{2^{n/2}}\,\cancel{\Gamma(n/2)}} \frac{\cancel{\Gamma(n/2)}}{\cancel{(1/2)^{n/2}}} = 1$$

So X is basically Gamma$\left(\frac{n}{2}, \frac{1}{2}\right)$

$X \sim G(\alpha, \beta)$

$$f(x) = \frac{\beta^\alpha e^{-\beta x} x^{\alpha-1}}{\Gamma(\alpha)} \quad x>0.$$

$E(X) = \alpha/\beta \quad Var(X) = \frac{\alpha}{\beta^2} \quad M_X(t) = \left(1 - t/\beta\right)^{-\alpha};\ |t|<\beta$

$$\Rightarrow E(X) = \frac{(n/2)}{1/2} = n$$

$$\Rightarrow Var(X) = \frac{(n/2)}{(1/4)} = 2n$$

$$\Rightarrow M_X(t) = (1-2t)^{-n/2} \quad |t| < \frac{1}{2}.$$

i.e., If $X \sim \chi^2_{(n)}$ then

$$E(X) = n, \quad Var(X) = 2n$$

& $M_X(t) = (1-2t)^{-n/2}$; $|t| < \frac{1}{2}$.

Results: If $Y \sim N(0,1)$ then $X = Y^2 \sim \chi^2_{(1)}$	If $X_i \sim \chi^2_{(1)}$, all X_i are independent $\Rightarrow \sum_{i=1}^{n} X_i \sim \chi^2_{(n)}$.

$\Downarrow$

- $P(X \leq x) = F_X(x)$

$$F_X(x) = P(Y^2 \leq x)$$

$$= P(|Y| \leq \sqrt{x})$$

$$= P(-\sqrt{x} \leq Y \leq \sqrt{x})$$

$\because \quad Y \sim N(0,1)$

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= P(-\sqrt{x} \leq Y \leq \sqrt{x})$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} f(y)\, dy$$

$$= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

$$\Rightarrow \quad F_X(x) = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

we know that

$$\frac{d}{dx}\left(F_X(x)\right) = f(x)$$

we know by Leibnitz integral rule

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\, dt\right) = f(x, b(x))\frac{d}{dx}(b(x))$$

$$- f(x, a(x))\frac{d}{dx}(a(x))$$

$$+ \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt.$$

In particular

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\frac{d}{dx}(b(x)) - f(a(x))\frac{d}{dx}(a(x)).$$

Therefore,

$$\frac{d}{dx}F_X(x) = f(x) = \frac{d}{dx}\left(\int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy\right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x/2}\frac{d}{dx}(\sqrt{x}) - \frac{1}{\sqrt{2\pi}} e^{-x/2}\frac{d}{dx}(-\sqrt{x})$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x/2}\left(\frac{1}{2\sqrt{x}}\right) + \frac{1}{\sqrt{2\pi}} e^{-x/2}\left(\frac{1}{2\sqrt{x}}\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}\sqrt{x}} e^{-x/2}$$

$$f(x) = \frac{1}{2^{1/2}\,\Gamma\frac{1}{2}} e^{-x/2} x^{\frac{1}{2}-1} \quad ; \quad 0 < x < \infty$$

$$X \sim \chi^2_{(1)} \qquad \because \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$$

$\because$ Square of a standard Normal $\sim \chi^2_{(1)}$

$$\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$$

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$

and Gamma $\left(\frac{1}{2}, \frac{1}{2}\right) = \chi^2_{(1)}$.

Results: If

$X_i \sim N(0,1)$ be independent

random variables

then $Y_n = \sum_{i=1}^{n} X_i^2 \sim \chi^2_{(n)}$.

Result: If $\bar{X}$ and s^2 are the mean and variance of a random sample of size n from a normal population with mean μ and standard deviation σ, then $\bar{X}$ and s^2 are independent and $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$