PROBABILITY AND STATISTICS (PMA303)

Lecture-[33]

(Chi-Square-test for one sample variance)

Para. & Non-para., Hypothesis Testing: (Unit VI-VII)

Dr. Rajanish Rai

Assistant Professor

Department of Mathematics

Thapar Institute of Engineering and Technology, Patiala

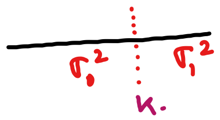
Let $X_1, X_2, \ldots X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$ μ is unknown

$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 = \sigma_1^2$ $\quad \sigma_1^2 > \sigma_0^2$, size of the test $= \alpha$

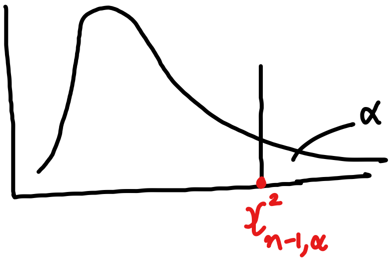
To test H_0 about σ^2 , the best thing we have at hand is S^2

We accept H_0 if S^2 is closer to σ_0^2



so , we $\begin{cases} \text{Reject } H_0 \text{ if } S^2 > k \\ \text{Retain } H_0 \text{ otherwise.} \end{cases}$

$$\alpha = P_{\theta_0}(\underline{X} \in \text{Reject}) = P_{\theta_0}(S^2 > k) \Rightarrow P\left(\frac{(n-1)S^2}{\sigma_0^2} > \frac{(n-1)k}{\sigma_0^2}\right)$$



$$\Rightarrow \alpha = P\left(chi^2 > \frac{(n-1)k}{\sigma_0^2}\right)$$

$$\Rightarrow \frac{(n-1)k}{\sigma_0^2} = \chi^2_{n-1,\alpha}$$

$$\Rightarrow k = \left(\frac{\sigma_0^2}{n-1}\right)\chi^2_{n-1,\alpha}$$

Alternative is to the right of Null

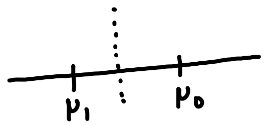
Right tailed χ^2 test. $\Rightarrow$ Critical Region $= \{(x_1, x_2 \ldots x_n) : s^2 > k\}$

$$\boxed{= \left\{(x_1 x_2 \ldots x_n) : s^2 > \frac{\sigma_0^2}{n-1} \cdot \chi^2_{n-1,\alpha}\right\}}$$

(ii) $X_1 X_2 \ldots X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$ μ unknown

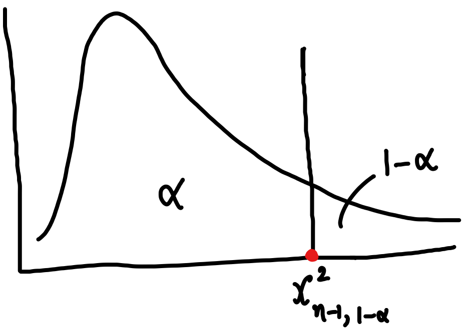
$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 = \sigma_1^2$ $\quad \sigma_1^2 < \sigma_0^2$ size of the test α.



Proceeding by previous argument

$\begin{cases} \text{Reject } H_0 \text{ if } S^2 < k \\ \text{Retain/Don't Reject otherwise} \end{cases}$



$$\alpha = P_{\theta_0}(\underline{X} \in \text{Reject})$$

$$= P_{\theta_0}(S^2 < k)$$

$$\alpha = P\left((n-1)\frac{S^2}{\sigma_0^2} < (n-1)\frac{k}{\sigma_0^2}\right)$$

$$\alpha = P\left(chi^2 < (n-1)\frac{k}{\sigma_0^2}\right)$$

$$\Rightarrow \frac{(n-1)k}{\sigma_0^2} = \chi^2_{n-1,1-\alpha} \Rightarrow k = \frac{\sigma_0^2}{n-1}\chi^2_{n-1,1-\alpha}.$$

Left Tail Chi-square test $\Rightarrow$ Critical Region $= \left\{(x_1 x_2 \ldots x_n) : s^2 < \frac{\sigma_0^2}{n-1}\chi^2_{n-1,1-\alpha}\right\}$

Suppose that the uniformity of the thickness of a part used in a semiconductor is critical and that measurements of the thickness of a random sample of 18 such parts have the variance $s^2 = 0.68$, where the measurements are in thousandths of an inch. The process is considered to be under control if the variation of the thicknesses is given by a variance not greater than 0.36. Assuming that the measurements constitute a random sample from a normal population, test the null hypothesis $\sigma^2 = 0.36$ against the alternative hypothesis $\sigma^2 > 0.36$ at the 0.05 level of significance. Assume Normality

$H_0: \sigma^2 = 0.36$ $\quad \alpha = 0.05 \quad s^2 = 0.68 \quad n = 18$

$H_1: \sigma^2 > 0.36$

$\Rightarrow$ Right tail Test : Reject if $s^2 > \frac{\sigma_0^2}{(n-1)} \chi^2_{n-1,\alpha}$

$0.68 \quad \frac{0.36}{17} \chi^2_{17, 0.05}$

$0.68 \quad \frac{(0.36)}{17} 27.587$

$0.68 > 0.58419$

Reject Null in favour of H_1

in this question if $\alpha = 0.01$ then we see

$0.68 \quad \frac{0.36}{17} \chi^2_{17, 0.01}$

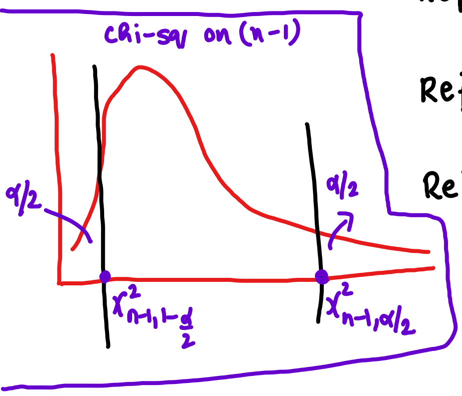
$0.68 \quad \frac{(0.36)}{17} 33.409$

$0.68 \not> 0.70748$

The null should be retained.

A manufacturer claims that the lifetime of a certain brand of batteries produced by his factory has a variance of 5000 (hours)^2. A sample of size 26 has a variance of 7200 (hours)^2. Assuming that it is reasonable to treat these data as a random sample from a normal population, let us test the manufacturer's claim at the $\alpha = 0.02$ level. Here $H_0: \sigma^2 = 5000$ is to be tested against $H_1: \sigma^2 \neq 5000$. We reject H_0 if either

Reject if $s^2 > \frac{\sigma_0^2}{n-1} \chi^2_{n-1,\alpha/2}$ or $s^2 < \frac{\sigma_0^2}{n-1} \chi^2_{n-1,1-\alpha/2}$



Reject if $7200 \; ? \; \frac{5000}{25} \chi^2_{25, 0.01}$ or $7200 \; ? \; \frac{5000}{25} \chi^2_{25, 0.99}$

Reject $7200 \; ? \; \left(\frac{5000}{25}\right) 44.314$ $\quad 7200 \quad \left(\frac{5000}{25}\right) 11.524$

$7200 \not> 8862.8$ $\quad 7200 \not< 2304.8$

Do not Reject H_0

In case two tailed Alternate $\sigma_1^2 \neq \sigma_0^2$: Reject if $s^2 > \frac{\sigma_0^2}{n-1} \chi^2_{n-1,\alpha/2}$ or $s^2 < \frac{\sigma_0^2}{n-1} \chi^2_{n-1,1-\alpha/2}$

Ratio of variances : $X_1 X_2 \ldots X_{n_1} \overset{\text{i.i.d}}{\sim} N(\mu_1, \sigma_1^2)$ μ_i unknown

$Y_1 Y_2 \ldots Y_{n_2} \overset{\text{i.i.d}}{\sim} N(\mu_2, \sigma_2^2)$

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 > \sigma_2^2$

$$\begin{cases} \text{Reject } H_0 \text{ if } \dfrac{S_1^2}{S_2^2} > k \\ \text{Retain } H_0 \text{ otherwise} \end{cases}$$

$$\alpha = P_{\theta_0}\left(\frac{S_1^2}{S_2^2} > k\right)$$

$$= P\left(\frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} > \frac{\sigma_2^2 k}{\sigma_1^2}\right)$$

$$\alpha = P(F > k)$$

$$\Rightarrow k = f_{n_1-1, n_2-1, \alpha}$$

$$\Rightarrow \boxed{\text{Reject } H_0 \text{ if } \frac{S_1^2}{S_2^2} > f_{n_1-1, n_2-1, \alpha}}$$

Right tailed

in case $H_1: \sigma_1^2 \neq \sigma_2^2$

we use two tailed test

Reject H_0 $\begin{cases} \text{if } \dfrac{S_1^2}{S_2^2} > f_{n_1-1, n_2-1, \alpha/2} \\ \text{or} \\ \dfrac{S_1^2}{S_2^2} < \dfrac{1}{f_{n_2-1, n_1-1, \alpha/2}} \end{cases}$

$$\boxed{F_{m,n} = \frac{1}{F_{n,m}}}$$

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 < \sigma_2^2$

Reject H_0 if $\dfrac{S_1^2}{S_2^2} < k$

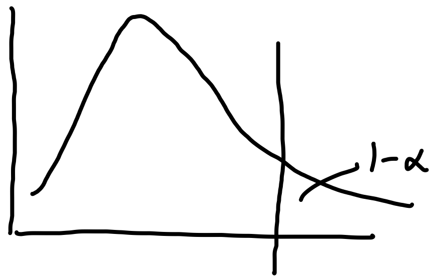
Retain H_0 otherwise

$$\alpha = P_{\theta_0}\left(\frac{S_1^2}{S_2^2} < k\right)$$

$$= P\left(\frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} < \frac{\sigma_2^2 k}{\sigma_1^2}\right)$$

$$\alpha = P(F < k)$$

$$\Rightarrow k = f_{n_1-1, n_2-1, 1-\alpha}$$



$f_{n_1-1, n_2-1, 1-\alpha}$

$$P\left(F_{n_1-1, n_2-1} < f_{n_1-1, n_2-1, 1-\alpha}\right) = \alpha$$

$$P\left(\frac{1}{F_{n_2-1, n_1-1}} < f_{n_1-1, n_2-1, 1-\alpha}\right) = \alpha$$

$$P\left(F_{n_2-1, n_1-1} > \frac{1}{f_{n_1-1, n_2-1, 1-\alpha}}\right) = \alpha$$

$$f_{n_2-1, n_1-1, \alpha} = \frac{1}{f_{n_1-1, n_2-1, 1-\alpha}}$$

$$\Rightarrow f_{n_1-1, n_2-1, 1-\alpha} = \frac{1}{f_{n_2-1, n_1-1, \alpha}} = k \text{ now}$$

$$\Rightarrow \text{Reject if } \boxed{\frac{S_1^2}{S_2^2} < \frac{1}{f_{n_2-1, n_1-1, \alpha}}}$$

Left Tailed

In comparing the variability of the tensile strength of two kinds of structural steel, an experiment yielded the following results: $n_1 = 13$, $s_1^2 = 19.2$, $n_2 = 16$, and $s_2^2 = 3.5$, where the units of measurement are 1,000 pounds per square inch. Assuming that the measurements constitute independent random samples from two normal populations, test the null hypothesis $\sigma_1^2 = \sigma_2^2$ against the alternative $\sigma_1^2 \neq \sigma_2^2$ at the 0.02 level of significance.

$n_1 = 13$, $n_2 = 16$ $\quad \alpha = 0.02$

$s_1^2 = 19.2$, $s_2^2 = 3.5$

Two sided:

Reject if $\frac{s_1^2}{s_2^2} > f_{12,15,0.01}$ or $\frac{s_1^2}{s_2^2} < \frac{1}{f_{15,12,0.01}}$

Reject if $\frac{s_1^2}{s_2^2} > 3.65$ or $\frac{s_1^2}{s_2^2} < \frac{1}{f_{15,12,0.01}}$

$\frac{19.2}{3.5} > 3.65$

$\boxed{5.48 > 3.65}$

Reject H_0 in favour of H_1