PROBABILITY AND STATISTICS (PMA303)

Lecture-[26]

(Estimation and confidence interval with illustrations)

Para. & Non-para., Hypothesis Testing: (Unit VI-VII)

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Student's t distribution

(i) Let $X \sim N(0,1)$, $Y \sim \chi^2_n$ $\overset{\text{indep}}{\Longrightarrow}$ $\dfrac{X}{\sqrt{Y/n}} \sim T_n$

T_n : T on 'n' degrees of freedom

(ii) p.d.f. of T:

$$f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2^{\frac{n+1}{2}} \sqrt{\pi n}\, \Gamma\left(n/2\right) \left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \qquad -\infty < t < \infty$$

Observation : $f_T(t) = f_T(-t)$ $\Rightarrow$ symmetric about $t=0$

$\text{Med}(T) = 0$ and also $\mathbb{E}(T) = 0$

How to make T variable (Assuming Normal population to derive)

We need a standard Normal : $\dfrac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$

a chi-square : $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\overset{\text{ind}}{\Longrightarrow} \frac{\left(\dfrac{\bar{X}-\mu}{\sqrt{\sigma^2/n}}\right)}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2(n-1)}}} = \frac{\bar{X}-\mu}{\sqrt{S^2/n}}$$

Thus, $$\boxed{\frac{\bar{X}-\mu}{\sqrt{S^2/n}} \sim t_{n-1}}$$

F- distribution $F_{m,n}$

Let

$X \sim \chi^2_m$, $Y \sim \chi^2_n$ $\xrightarrow{\text{indep}}$ $\dfrac{X/m}{Y/n} \sim F_{m,n}$

How to make a F- variable

- $\dfrac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2_{n_1-1}$
- $\dfrac{(n_2-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n_2-1}$

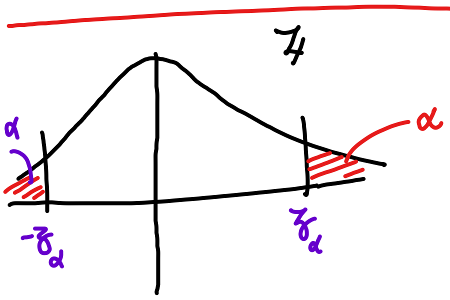
$$\overset{\text{indep}}{\Longrightarrow} \frac{\left(\dfrac{(n_1-1)S_1^2}{\sigma_1^2(n_1-1)}\right)}{\left(\dfrac{(n_2-1)S_2^2}{\sigma_2^2(n_2-1)}\right)} = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \sim F_{n_1-1,\, n_2-1}$$

$$\boxed{\frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \sim F_{n_1-1,\, n_2-1} \Rightarrow \frac{\sigma_1^2 S_1^2}{\sigma_2^2 S_2^2} \sim F_{n_2-1,\, n_1-1}}$$

$$\boxed{F_{n_1-1,\, n_2-1} = \frac{1}{F_{n_2-1,\, n_1-1}}}$$

Assuming Normal population, then a General Rule.

Symm				
✓	Z	(i)	$\dfrac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$	Used when some claim about μ is to be done when σ^2 is known. Based on sample
✗	χ^2	(ii)	$\dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$	Used when some claim about σ^2 is to be done when μ is unknown. Based on sample
✓	t	(iii)	$\dfrac{\bar{X}-\mu}{\sqrt{s^2/n}} \sim t_{n-1}$	used when some claim about μ is to be done when σ^2 is unknown. Based on sample
✗	F	(iv)	$\dfrac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2} \sim F_{n_1-1,n_2-1}$	used when some claim about the ratio of two variances of populations is to be done. Based on sample.



Observations heavily used while doing statistical inference, testing of Hypothesis.

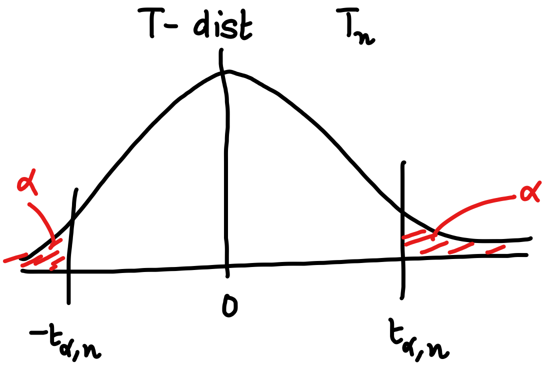
What is 'α' : $P(Z \geq z_\alpha) = \alpha$

and $P(Z \leq -z_\alpha) = \alpha$

$\Rightarrow P(Z \geq -z_\alpha) = 1-\alpha$

$P(Z \geq z_{1-\alpha}) = 1-\alpha$

$\Rightarrow \boxed{-z_\alpha = z_{1-\alpha}}$



$P(T \geq t_{\alpha,n}) = \alpha$

$P(T \leq -t_{\alpha,n}) = \alpha$

$\boxed{-t_{\alpha,n} = t_{1-\alpha,n}}$

using the same argument as above.

Z and t are symmetric about 0.

χ^2 and F are not symmetric distributions

Confidence interval

- Let $X_1, X_2, \dots X_n$ be a random sample from a distribution $f(x;\theta)$
- Let $0 < \alpha < 1$ be specified
- Let $L = L(X_1, X_2 \dots X_n)$ and $U = U(X_1, X_2 \dots X_n)$ be two STATISTICS.

Then we say that (L, U) is a $(1-\alpha)100\%$ CONFIDENCE INTERVAL for θ if

$$P_\theta\left(\theta \in (L, U)\right) = 1-\alpha$$

i.e we can assert with $(1-\alpha)100\%$ confidence that $\theta \in (L, U)$

this α is called the confidence Level

Q A potential question is can there be more than one confidence interval for the same Level of confidence α.

(A) Yes!! we can have more than interval

Q How do we choose b/w them.?

(A) the more efficient will be whose expected Length is SMALLER.

Q What are the procedures to find the this confidence interval

(A) Among several procedures, we will focus only on one of them.

- Procedure based on the concept of $\boxed{\text{PIVOT VARIABLE}}$
- which is usually a function of the estimator of θ
- whose distribution we know.

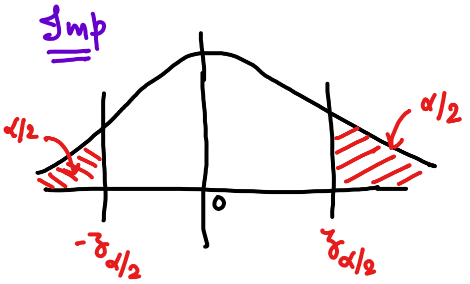
(i) Let $X_1, X_2 \dots X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$ $\boxed{\sigma^2 \text{ is known.}}$

Find a $(1-\alpha)100\%$ conifidence interval for μ.

$\because E(\bar{X}) = \mu$
$V(\bar{X}) = \frac{\sigma^2}{n}$

(Ans) We know that $Z = \dfrac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$

Imp



- $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1-\alpha$

- $\Rightarrow P\left(-z_{\alpha/2} < \dfrac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} < z_{\alpha/2}\right) = 1-\alpha$

$$\Rightarrow P\left(-\frac{\sigma}{\sqrt{n}} z_{\alpha/2} < \bar{X}-\mu < \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right) = 1-\alpha$$

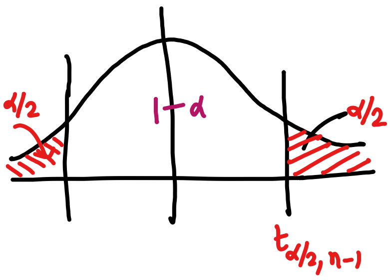
$$\Rightarrow P\left(-X - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} < -\mu < -\bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right) = 1-\alpha$$

Confidence interval for μ.

$$\Rightarrow P\left(\underbrace{\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}}_{L} < \mu < \underbrace{\bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}}_{U}\right) = 1-\alpha$$

(ii) What if X_i's $\sim N(\mu, \sigma^2)$, But now $\boxed{\sigma^2 \text{ is also unknown}}$

We know that $T = \dfrac{\bar{X} - \mu}{\sqrt{s^2/n}} \sim t_{n-1}$



Thus, $\mathbb{P}(-t_{\alpha/2, n-1} < T < t_{\alpha/2, n-1}) = 1-\alpha$

$$\Rightarrow \mathbb{P}\left(-t_{\alpha/2, n-1} < \frac{\bar{X} - \mu}{\sqrt{S^2/n}} < t_{\alpha/2, n-1}\right) = 1-\alpha$$

$$\Rightarrow \mathbb{P}\left(-\frac{S}{\sqrt{n}} t_{\alpha/2, n-1} < \bar{X} - \mu < \frac{S}{\sqrt{n}} t_{\alpha/2, n-1}\right) = 1-\alpha$$

$$\Rightarrow \mathbb{P}\Big(\underbrace{\bar{X} - \frac{S}{\sqrt{n}} t_{\alpha/2, n-1}}_{L} < \mu < \underbrace{\bar{X} + \frac{S}{\sqrt{n}} t_{\alpha/2, n-1}}_{U}\Big) = 1-\alpha$$

⊙ the distribution of X_i is also not known if $n \geq 30$ } Replace σ by s and use Z_p. However, we will get approximate interval in these cases.

(iii) $X_1, X_2 \dots X_{n_1} \overset{i.i.d.}{\sim} N(\mu_1, \sigma_1^2)$, $Y_1, Y_2 \dots Y_{n_2} \overset{i.i.d.}{\sim} N(\mu_2, \sigma_2^2)$ } i.i.d

find a confidence interval for $\mu_1 - \mu_2$

We know $\dfrac{\bar{X} - \mu_1}{\sqrt{\sigma_1^2/n_1}} \sim N(0,1)$ $\qquad \dfrac{\bar{Y} - \mu_2}{\sqrt{\sigma_2^2/n}} \sim N(0,1)$

$$\Rightarrow \bar{X} \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right), \quad \bar{Y} \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$$

$$\Rightarrow \bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

$$\Rightarrow \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

$$\Rightarrow \mathbb{P}(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1-\alpha$$

$$\Rightarrow \mathbb{P}\left(-z_{\alpha/2} < \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} < z_{\alpha/2}\right) = 1-\alpha$$

$$1-\alpha = \mathbb{P}\left((\bar{X} - \bar{Y}) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < \bar{X} - \bar{Y} + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$$

$n = 150$

$\sigma = 6.2$.

A team of efficiency experts intends to use the mean of a random sample of size $n = 150$ to estimate the average mechanical aptitude of assembly-line workers in a large industry (as measured by a certain standardized test). If, based on experience, the efficiency experts can assume that $\sigma = 6.2$ for such data, what can they assert with probability 0.99 about the maximum error of their estimate?

$(1-\alpha)100 = 99$.

$\Rightarrow 1 - 0.99 = \alpha$.

$\Rightarrow \alpha = 0.01$

$\Rightarrow \alpha/2 = 0.005$, we know that.

$$P\left(|\bar{X} - \mu| < z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right) = 0.99.$$

Error is $z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = z_{0.005} \frac{6.2}{\sqrt{150}} = (2.57583) \frac{6.2}{\sqrt{150}}$

$= \underline{\underline{1.30}}$.

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090

• $n = 20$.

$\sigma^2 = 225$

$\bar{x} = 64.3$

If a random sample of size $n = 20$ from a normal population with the variance $\sigma^2 = 225$ has the mean $\bar{x} = 64.3$, construct a 95% confidence interval for the population mean μ.

$(1-\alpha)100 = 95$

$\Rightarrow \alpha = 0.05$

$\Rightarrow \alpha/2 = 0.025$

$$64.3 - z_{0.025} \frac{15}{\sqrt{20}} < \mu < 64.3 + z_{0.025} \frac{15}{\sqrt{20}}$$

$$64.3 - (1.96) \frac{15}{\sqrt{20}} < \mu < 64.3 + (1.96) \frac{15}{\sqrt{20}}.$$

$$57.7 < \mu < 70.9.$$

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090

An industrial designer wants to determine the average amount of time it takes an adult to assemble an "easy-to-assemble" toy. Use the following data (in minutes), a random sample, to construct a 95% confidence interval for the mean of the population sampled:

17	13	18	19	17	21	29	22	16	28	21	15
26	23	24	20	8	17	17	21	32	18	25	22
16	10	20	22	19	14	30	22	12	24	28	11

$n = 36.$

$\bar{x} = 19.92, \; s = 5.73.$

Sample size is greater than 30 and population variance is not known= we will use confidence interval for Z

$(1-\alpha)100 = 95.$

$\alpha = 0.05.$

$\Rightarrow \alpha/2 = 0.025.$

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090

$$19.92 - (1.96)\frac{5.73}{\sqrt{36}} < \mu < 19.92 + (1.96)\frac{5.73}{\sqrt{36}}$$

$$18.05 < \mu < 21.79.$$

A paint manufacturer wants to determine the average drying time of a new interior wall paint. If for 12 test areas of equal size he obtained a mean drying time of 66.3 minutes and a standard deviation of 8.4 minutes, construct a 95% confidence interval for the true mean μ.

- $n = 12$

$\bar{x} = 66.3$

$s = 8.4$

sample is less than 30 and variance is not known use T

$(1-\alpha)100 = 95.$

$\alpha = 0.05$

$\Rightarrow \alpha/2 = 0.025.$

Assuming normal population.

$$66.3 - t_{\alpha/2, n-1}\frac{s}{\sqrt{n}} < \mu < 66.3 + t_{\alpha/2, n-1}\frac{s}{\sqrt{n}}.$$

Table IV: Values of $t_{\alpha,\nu}$ †

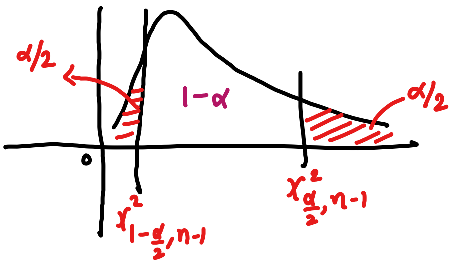
ν	$\alpha = .10$	$\alpha = .05$	$\alpha = .025$
1	3.078	6.314	12.706
2	1.886	2.920	4.303
3	1.638	2.353	3.182
4	1.533	2.132	2.776
5	1.476	2.015	2.571
6	1.440	1.943	2.447
7	1.415	1.895	2.365
8	1.397	1.860	2.306
9	1.383	1.833	2.262
10	1.372	1.812	2.228
11	1.363	1.796	2.201

$$- (2.201)\frac{8.4}{\sqrt{12}} < \mu < 66.3 + (2.201)\frac{8.4}{\sqrt{12}}$$

$$61 < \mu < 71.6.$$

(iv) $X_1, X_2 \dots X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$ (μ is unknown)

Find a confidence interval for σ^2.



We know $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\mathbb{P}\left(\chi^2_{1-\frac{\alpha}{2}, n-1} < \text{Pivot} < \chi^2_{\frac{\alpha}{2}, n-1}\right) = 1-\alpha$$

$$\Rightarrow \mathbb{P}\left(\chi^2_{1-\frac{\alpha}{2}, n-1} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\frac{\alpha}{2}, n-1}\right) = 1-\alpha$$

Confidence interval for σ^2

Rearranging terms

$$\Rightarrow \mathbb{P}\left(\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}, n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}}\right) = 1-\alpha$$

$$\underbrace{\quad\quad}_{L} < \sigma^2 < \underbrace{\quad\quad}_{U}$$

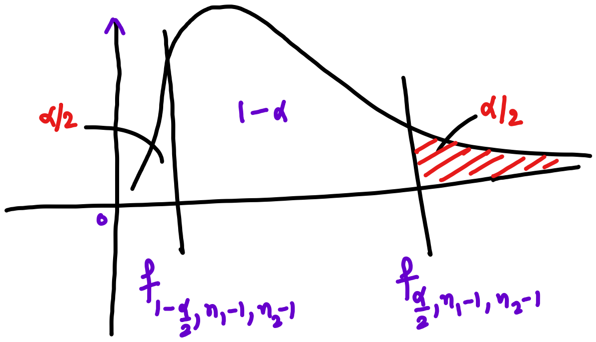
Confidence interval of σ

$$\Rightarrow \mathbb{P}\left(\underbrace{\sqrt{\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}, n-1}}}}_{L} < \sigma < \underbrace{\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}}}}_{U}\right) = 1-\alpha$$

(v) $X_1 X_2 \dots X_{n_1} \sim N(\mu_1, \sigma_1^2)$

$Y_1 Y_2 \dots Y_{n_2} \sim N(\mu_2, \sigma_2^2)$

Find a $(1-\alpha)100\%$ confidence interval for σ_1^2/σ_2^2



We know

$$\frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2} \sim F_{n_1-1, n_2-1}$$

$$\Rightarrow \mathbb{P}\left(f_{1-\frac{\alpha}{2}, n_1-1, n_2-1} < \text{Pivot} < f_{\frac{\alpha}{2}, n_1-1, n_2-1}\right) = 1-\alpha$$

$$\mathbb{P}\left(f_{1-\frac{\alpha}{2}, n_1-1, n_2-1} < \frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2} < f_{\frac{\alpha}{2}, n_1-1, n_2-1}\right) = 1-\alpha$$

$$\Rightarrow \mathbb{P}\left(\frac{1}{f_{\frac{\alpha}{2}, n_2-1, n_1-1}} < \frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2} < f_{\frac{\alpha}{2}, n_1-1, n_2-1}\right) = 1-\alpha$$

$$\mathbb{P}\left(F_{m,n} > f_{1-\frac{\alpha}{2}, m, n}\right) = 1-\frac{\alpha}{2}$$

$$\Rightarrow \mathbb{P}\left(F_{m,n} \le f_{1-\frac{\alpha}{2}, m, n}\right) = \frac{\alpha}{2}$$

$$\Rightarrow \mathbb{P}\left(\frac{1}{F_{n,m}} \le f_{1-\frac{\alpha}{2}, m, n}\right) = \frac{\alpha}{2}$$

$$\Rightarrow \mathbb{P}\left(F_{n,m} \geq \frac{1}{f_{1-\frac{\alpha}{2}, m, n}}\right) = \frac{\alpha}{2}$$

But $\mathbb{P}\left(F_{n,m} \geq f_{\alpha/2, n, m}\right) = \alpha/2$

$$\Rightarrow f_{\frac{\alpha}{2}, n, m} = \frac{1}{f_{1-\frac{\alpha}{2}, m, n}}$$

In 16 test runs the gasoline consumption of an experimental engine had a standard deviation of 2.2 gallons. Construct a 99% confidence interval for σ^2, which measures the true variability of the gasoline consumption of the engine.

$n = 16$

$s = 2.2$

$\alpha/2 = 0.005.$

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2, n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}}$$

$$\frac{(15)(2.2)^2}{\chi^2_{0.005, 15}} < \sigma^2 < \frac{(15)(2.2)^2}{\chi^2_{0.995, 15}}$$

$$\frac{(15)(2.2)^4}{32.801} < \sigma^2 < \frac{(15)(2.2)^2}{4.601}$$

$$2.21 < \sigma^2 < 15.78.$$

$$1.49. < \sigma < 3.97.$$

find 98% C.I for σ_1^2/σ_2^2

A study has been made to compare the nicotine contents of two brands of cigarettes. Ten cigarettes of Brand *A* had an average nicotine content of 3.1 milligrams with a standard deviation of 0.5 milligram, while eight cigarettes of Brand *B* had an average nicotine content of 2.7 milligrams with a standard deviation of 0.7 milligram. Assuming that the two sets of data are independent random samples from normal populations with equal variances, construct a 95% confidence interval for the difference between the mean nicotine contents of the two brands of cigarettes.

$n_1 = 10, \; s_1 = 0.5$

$n_2 = 8 \quad s_2 = 0.7$

$\alpha = 0.02.$

$\Rightarrow \alpha/2 = 0.01$

$$\frac{s_1^2}{s_2^2} \frac{1}{f_{\alpha/2, n_1-1, n_2-1}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2} f_{\alpha/2, n_2-1, n_1-1}$$

$$\frac{0.25}{0.49} \frac{1}{f_{0.01, 9, 7}} < \sigma_1^2/\sigma_2^2 < \frac{0.25}{0.49} f_{0.01, 7, 9}.$$

$$\frac{0.25}{0.49} \frac{1}{(6.72)} < \sigma_1^2/\sigma_2^2 < \frac{0.25}{0.49} (5.61)$$

$$\Rightarrow 0.076 < \sigma_1^2/\sigma_2^2 < 2.862$$

Let $X_1 X_2 \dots X_{n_1} \overset{i.i.d}{\sim} N(\mu_1, \sigma^2)$
$Y_1 Y_2 \dots Y_{n_2} \overset{i.i.d}{\sim} N(\mu_2, \sigma^2)$ $\Big\}$ $\underline{\underline{i.i.d}}$

find 'a' confidence interval for $(\mu_1 - \mu_2)$ when the variances are assumed to be equal but UNKNOWN.

$$\bar{X} \sim N\left(\mu_1, \frac{\sigma^2}{n_1}\right) \qquad \bar{Y} \sim N\left(\mu_2, \frac{\sigma^2}{n_2}\right)$$

i.i.d

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}\right)$$

$$Z = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}} \sim N(0,1).$$

Define.

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}.$$

$\downarrow$

pooled sample variance

We see that.

$$\frac{(n_1-1)S_1^2}{\sigma^2} \sim \chi^2_{n_1-1} \quad \text{and} \quad \frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2_{n_2-1}.$$

i.i.d

$$\tilde{Y} = \frac{(n_1-1)S_1^2}{\sigma^2} + \frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2_{n_1+n_2-2}.$$

$$\tilde{Y} = \frac{(n_1+n_2-2)S_p^2}{\sigma^2} \sim \chi^2_{n_1+n_2-2}.$$

$$T = \frac{Z}{\sqrt{\frac{\tilde{Y}}{d.f \text{ of } \tilde{Y}}}}$$

$$= \frac{\dfrac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}}}{\sqrt{\dfrac{(n_1+n_2-2)S_p^2}{\sigma^2 (n_1+n_2-2)}}}$$

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}.$$

$$1-\alpha = \mathbb{P}\left(-t_{\alpha/2, n_1+n_2-2} < T < t_{\alpha/2, n_1+n_2-2}\right)$$

$$= \mathbb{P}\left(-t_{\alpha/2, n_1+n_2-2} < \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} < t_{\alpha/2, n_1+n_2-2}\right)$$

$$\boxed{1-\alpha = \mathbb{P}\left(\bar{X} - \bar{Y} - t_{\alpha/2, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < (\mu_1 - \mu_2) < \bar{X} - \bar{Y} + t_{\alpha/2, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)}$$

so $(\bar{x} - \bar{y}) \pm t_{\alpha/2, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ is a $(1-\alpha)100\%$ confidence interval for $(\mu_1 - \mu_2)$

Construct a 94% confidence interval for the difference between the mean lifetimes of two kinds of light bulbs, given that a random sample of 40 light bulbs of the first kind lasted on the average 418 hours of continuous use and 50 light bulbs of the second kind lasted on the average 402 hours of continuous use. The population standard deviations are known to be $\sigma_1 = 26$ and $\sigma_2 = 22$.

$n_1 = 40$

$n_2 = 50$

$\sigma_1 = 26$, $\bar{x} = 418$.

$\sigma_2 = 22$. $\bar{y} = 402$.

$(1-\alpha)100 = 94$.

$\alpha = 0.06$.

$\Rightarrow \alpha/2 = 0.03$.

$$(418 - 402) \pm 1.88\sqrt{\frac{26^2}{40} + \frac{22^2}{50}}$$

$$6.3 < \mu_1 - \mu_2 < 25.7.$$

A study has been made to compare the nicotine contents of two brands of cigarettes. Ten cigarettes of Brand A had an average nicotine content of 3.1 milligrams with a standard deviation of 0.5 milligram, while eight cigarettes of Brand B had an average nicotine content of 2.7 milligrams with a standard deviation of 0.7 milligram. Assuming that the two sets of data are independent random samples from normal populations with equal variances, construct a 95% confidence interval for the difference between the mean nicotine contents of the two brands of cigarettes.

$n_1 = 10$

$n_2 = 8$

$s_1 = 0.5$, $\bar{x} = 3.1$

$s_2 = 0.7$, $\bar{y} = 2.7$.

$(1-\alpha)100 = 95$.

$\alpha = 0.05$

$\Rightarrow \alpha/2 = 0.025$.

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}.$$

$$s_p = \sqrt{\frac{9(0.5)^2 + (7)(0.7)^2}{10+8-2}}$$

$$= \underline{\underline{0.596}}.$$

$$(3.1 - 2.7) \pm t_{0.025,16}(0.596)\sqrt{\frac{1}{10} + \frac{1}{8}}.$$

$$= (3.1 - 2.7) \pm (2.120)(0.596)\sqrt{\frac{1}{10} + \frac{1}{8}}.$$

$$= -.20 < \mu_1 - \mu_2 < 1.00.$$

Table VI: (*continued*) Values of $f_{0.01,\nu_1,\nu_2}$

	ν_1 = Degrees of fre								
reedom for denominator	1	2	3	4	5	6	7	8	9
1	4,052	5,000	5,403	5,625	5,764	5,859	5,928	5,982	6,023
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35

Rule that might help

If the population in NON-NORMAL + $n > 30$ (large)
use the Z statistic.
if σ is given use σ
if σ is not given use s.

If the population is Non-normal + $n \leq 30$ use.
T statistic.

Table IV: Values of $t_{\alpha,\nu}$ †

ν	$\alpha = .10$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$	ν
1	3.078	6.314	12.706	31.821	63.657	1
2	1.886	2.920	4.303	6.965	9.925	2
3	1.638	2.353	3.182	4.541	5.841	3
4	1.533	2.132	2.776	3.747	4.604	4
5	1.476	2.015	2.571	3.365	4.032	5
6	1.440	1.943	2.447	3.143	3.707	6
7	1.415	1.895	2.365	2.998	3.499	7
8	1.397	1.860	2.306	2.896	3.355	8
9	1.383	1.833	2.262	2.821	3.250	9
10	1.372	1.812	2.228	2.764	3.169	10
11	1.363	1.796	2.201	2.718	3.106	11
12	1.356	1.782	2.179	2.681	3.055	12
13	1.350	1.771	2.160	2.650	3.012	13
14	1.345	1.761	2.145	2.624	2.977	14
15	1.341	1.753	2.131	2.602	2.947	15
16	1.337	1.746	2.120	2.583	2.921	16
17	1.333	1.740	2.110	2.567	2.898	17
18	1.330	1.734	2.101	2.552	2.878	18
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20
21	1.323	1.721	2.080	2.518	2.831	21
22	1.321	1.717	2.074	2.508	2.819	22
23	1.319	1.714	2.069	2.500	2.807	23
24	1.318	1.711	2.064	2.492	2.797	24
25	1.316	1.708	2.060	2.485	2.787	25
26	1.315	1.706	2.056	2.479	2.779	26
27	1.314	1.703	2.052	2.473	2.771	27
28	1.313	1.701	2.048	2.467	2.763	28
29	1.311	1.699	2.045	2.462	2.756	29
inf.	1.282	1.645	1.960	2.326	2.576	inf.

†Based on Richard A. Johnson and Dean W. Wichern, *Applied Multivariate Statistical Analysis*, 2nd ed., © 1988, Table 2, p. 592. By permission of Prentice Hall, Upper Saddle River, N.J.

Table V: Values of $\chi^2_{\alpha,\nu}$ †

ν	$\alpha = .995$	$\alpha = .99$	$\alpha = .975$	$\alpha = .95$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$	ν
1	.0000393	.000157	.000982	.00393	3.841	5.024	6.635	7.879	1
2	.0100	.0201	.0506	.103	5.991	7.378	9.210	10.597	2
3	.0717	.115	.216	.352	7.815	9.348	11.345	12.838	3
4	.207	.297	.484	.711	9.488	11.143	13.277	14.860	4
5	.412	.554	.831	1.145	11.070	12.832	15.086	16.750	5
6	.676	.872	1.237	1.635	12.592	14.449	16.812	18.548	6
7	.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278	7
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955	8
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589	9
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188	10
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757	11
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300	12
13	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.819	13
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319	14
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801	15
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267	16
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718	17
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156	18
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582	19
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997	20
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401	21
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796	22
23	9.260	10.196	11.689	13.091	35.172	38.076	41.638	44.181	23
24	9.886	10.856	12.401	13.848	36.415	39.364	42.980	45.558	24
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928	25
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290	26
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645	27
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993	28
29	13.121	14.256	16.047	17.708	42.557	45.722	49.588	52.336	29
30	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672	30

† Based on Table 8 of *Biometrika Tables for Statisticians*, Vol. 1, Cambridge University Press, 1954, by permission of the *Biometrika* trustees.

Table VI: Values of $f_{0.05,\nu_1,\nu_2}$ †

	ν_1 = Degrees of freedom for numerator																		
ν_2 = Degrees of freedom for denominator	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07

† Reproduced from M. Merrington and C. M. Thompson, "Tables of percentage points of the inverted beta (F) distribution," *Biometrika*, Vol. 33 (1943), by permission of the *Biometrika* trustees.

Table VI: (*continued*) Values of $f_{0.05,\nu_1,\nu_2}$

ν_2 = Degrees of freedom for denominator	ν_1 = Degrees of freedom for numerator																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table VI: (*continued*) Values of $f_{0.01,\nu_1,\nu_2}$

ν_2 = Degrees of freedom for denominator	ν_1 = Degrees of freedom for numerator																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	4,052	5,000	5,403	5,625	5,764	5,859	5,928	5,982	6,023	6,056	6,106	6,157	6,209	6,235	6,261	6,287	6,313	6,339	6,366
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9	26.7	26.6	26.5	26.4	26.3	26.2	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2	14.0	13.9	13.8	13.7	13.7	13.6	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	6.91	5.82	5.74	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87

Table VI: (*continued*) Values of $f_{0.01,\nu_1,\nu_2}$

ν_2 = Degrees of freedom for denominator	ν_1 = Degrees of freedom for numerator																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.53	2.45	2.36	2.27	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00