Connectivity
Discrete Mathematics

Path

Definition

Let n be a non negative integer and G an undirected graph. A **path of length** n from vertices u to v in G is a sequence of n edges e_1, e_2, ..., e_n of G such that e_1 is associated with $\{x_0, x_1\}$, e_2 is associated with $\{x_1, x_2\}$, and so on, with e_n associated with $\{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$.

When the graph is simple, we denote this path by the vertex sequence $x_0, x_1, \ldots, x_n$ because listing these vertices uniquely determines the path.

Circuit

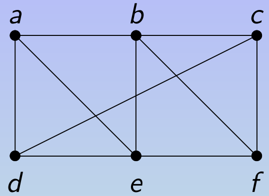
Definition

In an undirected graph, a path is a **circuit** if it begins and ends at the same vertex, that is, if $x_0 = u = v = x_n$. and has length greater than zero.

The path or circuit is said to **pass through** the vertices x_1, x_2, ..., x_{n-1} or **traverse** the edges e_1, e_2, ..., e_n.

A path or circuit is **simple** if it does not contain the same edge more than once.

Example



a, d, c, f, e is a simple path of length 4.

d, e, c, a is not a path because $\{e, c\}$ and $\{c, a\}$ are not an edge.

b, c, f, e, b is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$ and $\{e, b\}$ are edges, and this path begins and ends at b.

a, b, e, d, a, b, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice.

Path in Directed Graph

Note: Take care, here edges are directed, as (x_i, x_{i+1}), not undirected, as $\{x_i, x_{i+1}\}$.

Definition

Let n be a non negative integer and G an directed graph. A **path of length** n from vertices u to v in G is a sequence of n edges e_1, e_2, ..., e_n of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$.

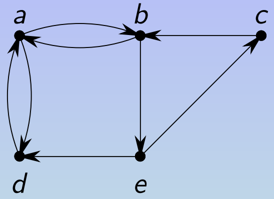
When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence x_0, x_1,x_2, ..., x_n.

Circuit

Definition

A path of length greater than zero that begins and ends at the same vertex is called a **circuit** or **cycle**. A path, circuit or cycle, is called **simple** if it does not contain the same edge more than once.

Examples of Path in Directed Graph (ex 2.)



a, b, e, c, b is a simple path of length 4.

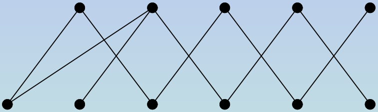
a, d, a, d, a is a circuit of length 4. However, it is not a simple circuit.

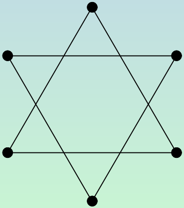
a, d, b, e, a is not a path because (d, b) is not an edge.

a, b, e, c, b, a is a simple circuit or cycle.

Definition

An undirected graph is called **connected** is there is a path between every pair of distinct vertices of the graph.

Connected graph: 

Unconnected graph.: 

Simple Path in a Connected Graph

Theorem

There is a simple path between every pair of distinct vertices of a connected undirected graph.

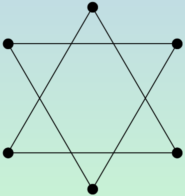
Proof.

Let u and v be two distinct vertices of the connected undirected graph $G = (V, E)$. Because G is connected, there is at least one path between u and v. Let x_0, x_1, ..., x_n, where $x_0 = u$ and $x_n = v$ be the vertex sequence of a path of least length. This path of least length is simple. To see this, suppose it is not simple. Then $x_i = x_j$ for some i and j with $0 \leq i < j \leq n$. This means that there is path from u to v of shorter length with vertex sequence x_0, x_1, ..., x_{i-1}, x_j, ..., x_n obtained by deleting the edges corresponding to the vertex sequence x_i, ..., x_{j-1}. $\square$

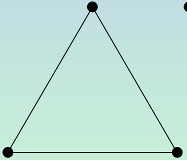
Connected Components

Definition

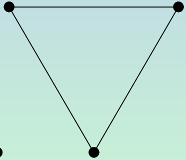
A **connected component** of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.



G



G_1



G_2

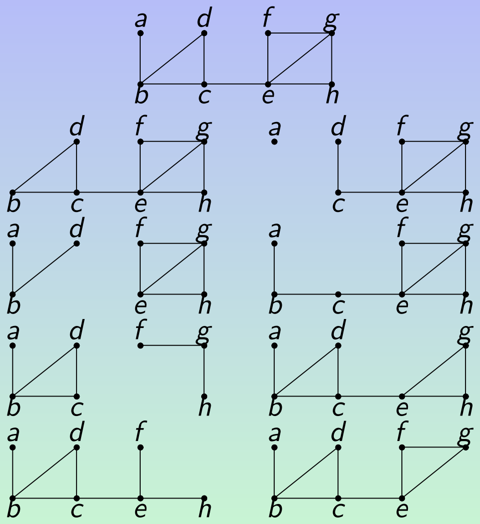
Articulation Points

Definition

Sometimes the removal of a vertex and all edges incident with it produces a subgraph with more connected components than in the original graph. Such vertices are called **cut vertices** (or **articulation points**). The removal of a cut vertex from a connected graph produces a subgraph that is not connected.

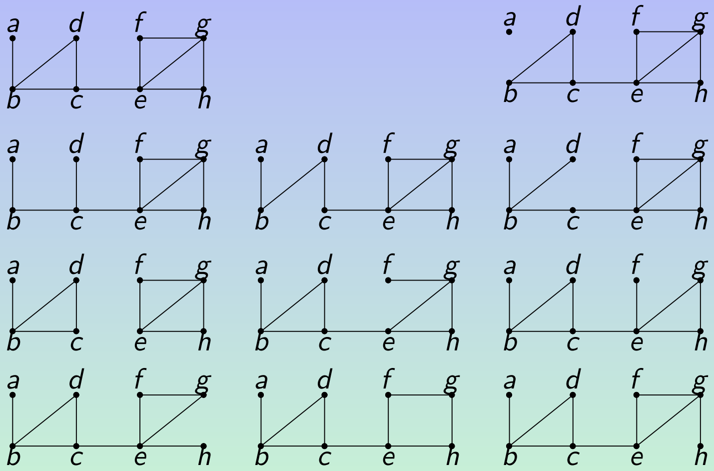
Analogously, an edge whose removal produces a graph with more connected components than in the original graph is called a **cut edge** or **bridge**.

Examples of Articulation Points



The vertices b, c and e are articulation points.

Examples of Cut Edges



The edges $\{a, b\}$ and $\{c, e\}$ are cut edges.

Connectedness in Directed Graphs

Definition

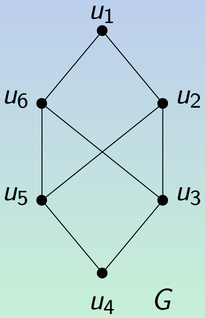
A directed graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

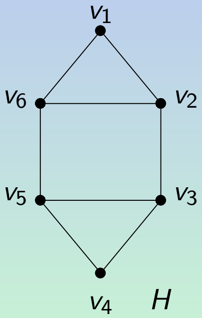
Definition

A directed graph is **weakly connected** if there is a path between every two vertices in the underlying undirected graph.

A strongly connected graph is also a weakly connected graph, but not the converse.

A useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer greater than 2. This is useful to show that two graphs are not isomorphic.

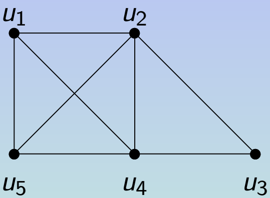




Theorem

Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i,j)th entry of $\mathbf{A}^r$.

Example of Path Counts

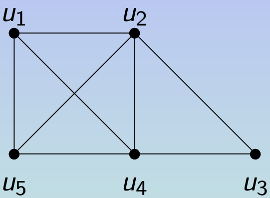


The vertices are u_1, u_2, u_3, u_4, u_5.

$$\mathbf{A}^1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Number of paths of length 1.

Example of Path Counts (cont.)

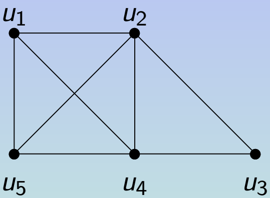


The vertices are u_1, u_2, u_3, u_4, u_5.

$$\mathbf{A}^2 = \begin{pmatrix} 3 & 2 & 2 & 2 & 2 \\ 2 & 4 & 1 & 3 & 2 \\ 2 & 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 4 & 2 \\ 2 & 2 & 2 & 2 & 3 \end{pmatrix}$$

Number of paths of length 2.

Example of Path Counts (cont.)



The vertices are u_1, u_2, u_3, u_4, u_5.

$$\mathbf{A}^3 = \begin{pmatrix} 6 & 9 & 4 & 9 & 7 \\ 9 & 8 & 7 & 9 & 9 \\ 4 & 7 & 2 & 7 & 4 \\ 9 & 9 & 7 & 8 & 9 \\ 7 & 9 & 4 & 9 & 6 \end{pmatrix}$$

Number of paths of length 3.