Matrix Theory Assignment 3

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Abstract—**This document contains the solution to problem No.66 from Lines and Planes**

1 Problem Statement

In ΔABC, AD is the perpendicular bisector of BC. Show that ΔABC is an isosceles triangle in which AB = AC.

2 Solution

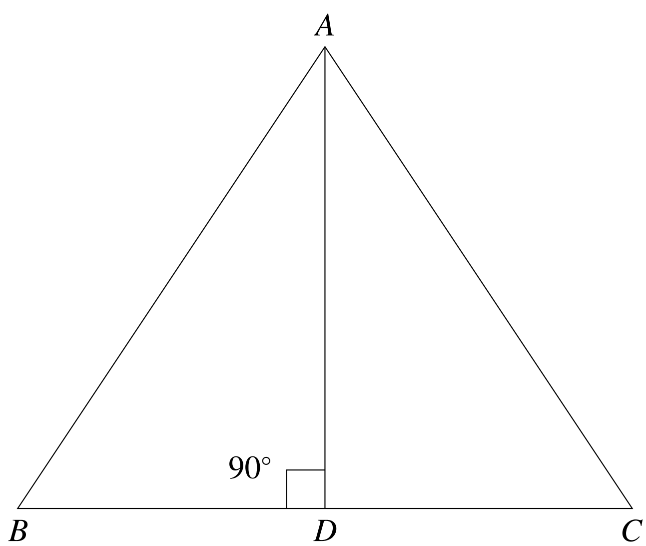

Fig. 0

Consider the above ΔABC. Given that AD is the perpendicular bisector of BC. So, BD = DC and ∠ADB = ∠ADC = 90.Since D is the midpoint of BC

$$\mathbf{D} = (\mathbf{B}+\mathbf{C})/2$$
$$2\mathbf{D} = (\mathbf{B}+\mathbf{C})$$
$$(\mathbf{B}-\mathbf{D}) = (\mathbf{D}-\mathbf{C})$$
$$\|\mathbf{B}-\mathbf{D}\| = \|\mathbf{D}-\mathbf{C}\| \quad (2.0.1)$$

Since, AD is the perpendicular bisector of BC

$$(\mathbf{A}-\mathbf{D})^T(\mathbf{B}-\mathbf{C}) = (\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = 0$$
$$(\mathbf{A}-\mathbf{D})^T(\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{C}) = 0$$
$$(\mathbf{A}-\mathbf{D})^T(\mathbf{B}-\mathbf{D}) + (\mathbf{A}-\mathbf{D})^T(\mathbf{D}-\mathbf{C}) = 0$$
$$\implies (\mathbf{A}-\mathbf{D})^T(\mathbf{B}-\mathbf{D}) = (\mathbf{A}-\mathbf{D})^T(\mathbf{D}-\mathbf{C}) = 0 \quad (2.0.2)$$

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = 0$$
$$(\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = 0$$
$$[(\mathbf{B}-\mathbf{D})^T + (\mathbf{D}-\mathbf{C})^T](\mathbf{A}-\mathbf{D}) = 0$$
$$(\mathbf{B}-\mathbf{D})^T(\mathbf{A}-\mathbf{D}) + (\mathbf{D}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = 0$$
$$\implies (\mathbf{B}-\mathbf{D})^T(\mathbf{A}-\mathbf{D}) = (\mathbf{D}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = 0 \quad (2.0.3)$$

Find the length of AB,

$$\begin{aligned}\|\mathbf{A}-\mathbf{B}\|^2 &= (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B})\\ &= (\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{B})\\ &= [(\mathbf{A}-\mathbf{D})^T + (\mathbf{D}-\mathbf{B})^T][(\mathbf{A}-\mathbf{D}) + (\mathbf{D}-\mathbf{B})]\\ &= (\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{D}) + (\mathbf{A}-\mathbf{D})^T(\mathbf{D}-\mathbf{B}) +\\ &\quad (\mathbf{D}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) + (\mathbf{D}-\mathbf{B})^T(\mathbf{D}-\mathbf{B}) \quad (2.0.4)\end{aligned}$$

From Eq (2.0.2) and Eq (2.0.3),

$$\begin{aligned}&(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B})\\ &= (\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{D}) + (\mathbf{D}-\mathbf{B})^T(\mathbf{D}-\mathbf{B})\\ &\implies \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{D}\|^2 + \|\mathbf{D}-\mathbf{B}\|^2 \quad (2.0.5)\end{aligned}$$

Similarily, find the length of AC

$$\begin{aligned}\|\mathbf{A}-\mathbf{C}\|^2 &= (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{C})\\ &= (\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{C})\\ &= [(\mathbf{A}-\mathbf{D})^T + (\mathbf{D}-\mathbf{C})^T][(\mathbf{A}-\mathbf{D}) + (\mathbf{D}-\mathbf{C})]\\ &= (\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{D}) + (\mathbf{A}-\mathbf{D})^T(\mathbf{D}-\mathbf{C}) +\\ &\quad (\mathbf{D}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) + (\mathbf{D}-\mathbf{C})^T(\mathbf{D}-\mathbf{C}) \quad (2.0.6)\end{aligned}$$

From Eq (2.0.2) and Eq (2.0.3)

$$
\begin{aligned}
&(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{C}) \\
&\quad = (\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{D}) + (\mathbf{D}-\mathbf{C})^T(\mathbf{D}-\mathbf{C}) \\
&\implies \|\mathbf{A}-\mathbf{C}\|^2 = \|\mathbf{C}-\mathbf{D}\|^2 + \|\mathbf{D}-\mathbf{A}\|^2 \quad (2.0.7)
\end{aligned}
$$

Eq (2.0.1), Eq (2.0.5) and Eq (2.0.7)

$$\|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2 \quad (2.0.8)$$

Since the lengths AB and AC are equal, We can conclude that the ΔABC is an isosceles triangle with AB = AC.

Latex codes:

https://github.com/Hrithikraj2/MatrixTheory_EE5609/tree/master/Assignment_3/latex