Assignment-7

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***Abstract*—This document contains solution of Problem Ramsey**(4.1.4)

Download latex-tikz codes from

https://github.com/Hrithikraj2/MatrixTheory_EE5609/blob/master/Assignment_7/A7.tex

1 Problem

1)Find QR decomposition of matrix

$$\mathbf{V} = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} \tag{1.0.1}$$

2)Find the vertex $\mathbf{c}$ of the parabola using SVD for

$$16x^2 + 24xy + 9y^2 - 5x - 10y + 1 = 0 \tag{1.0.2}$$

by changing η to $\eta/2$ also verify the result using least squares.

2 Solution

2.1 Part 1:QR Decomposition of V

Let $\mathbf{x}$ and $\mathbf{y}$ be the column vectors of the given matrix.

$$\mathbf{x} = \begin{pmatrix} 16 \\ 12 \end{pmatrix} \tag{2.1.1}$$

$$\mathbf{y} = \begin{pmatrix} 12 \\ 9 \end{pmatrix} \tag{2.1.2}$$

The column vectors can be expressed as ,

$$\mathbf{x} = k_1\mathbf{u}_1 \tag{2.1.3}$$

$$\mathbf{y} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.1.4}$$

$$k_1 = \|\mathbf{x}\| \tag{2.1.5}$$

$$\mathbf{u}_1 = \frac{\mathbf{x}}{k_1} \tag{2.1.6}$$

$$r_1 = \frac{\mathbf{u}_1^T\mathbf{y}}{\|\mathbf{u}_1\|^2} \tag{2.1.7}$$

$$\mathbf{u}_2 = \frac{\mathbf{y} - r_1\mathbf{u}_1}{\|\mathbf{y} - r_1\mathbf{u}_1\|} \tag{2.1.8}$$

$$k_2 = \mathbf{u}_2^T\mathbf{y} \tag{2.1.9}$$

The (2.1.3) and (2.1.4) can be written as,

$$\begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \tag{2.1.10}$$

$$\begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} = \mathbf{QR} \tag{2.1.11}$$

Now, $\mathbf{R}$ is an upper triangular matrix and also,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.1.12}$$

Now using equations (2.1.5) to (2.1.9) we get,

$$k_1 = \sqrt{16^2 + 12^2} = 20 \tag{2.1.13}$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \tag{2.1.14}$$

$$r_1 = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 12 \\ 9 \end{pmatrix} = 15 \tag{2.1.15}$$

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.16}$$

$$k_2 = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 12 \\ 9 \end{pmatrix} = 0 \tag{2.1.17}$$

Thus putting the values from (2.1.13) to (2.1.17) in (2.1.10) we obtain QR decomposition,

$$\begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & 0 \\ \frac{3}{5} & 0 \end{pmatrix} \begin{pmatrix} 20 & 15 \\ 0 & 0 \end{pmatrix} \tag{2.1.18}$$

2.2 Part 2:Finding Vertex using SVD

$$\mathbf{V} = \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} \tag{2.2.1}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{5}{2} \\ -5 \end{pmatrix} \tag{2.2.2}$$

$$f = 1 \tag{2.2.3}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \tag{2.2.4}$$

$$\eta = \mathbf{u}^T\mathbf{p_1} = -\frac{5}{2} \tag{2.2.5}$$

So the equation of perpendicular line passing through focus and intersecting parabola at vertex c is given as

$$\begin{pmatrix}\mathbf{u^T}+\frac{\eta}{2}\mathbf{p_1^T}\\ \mathbf{V}\end{pmatrix}\mathbf{c}=\begin{pmatrix}-f\\ \frac{\eta}{2}\mathbf{p_1}-\mathbf{u}\end{pmatrix} \tag{2.2.6}$$

using (2.2.1),(2.2.2) ,(2.2.3) and (2.2.4)

$$\begin{pmatrix}-\frac{7}{4} & -6\\ 16 & 12\\ 12 & 9\end{pmatrix}\mathbf{c}=\begin{pmatrix}-1\\ \frac{13}{4}\\ 4\end{pmatrix} \tag{2.2.7}$$

$$\mathbf{Mc}=\mathbf{b} \tag{2.2.8}$$

where

$$\mathbf{M}=\begin{pmatrix}-\frac{7}{4} & -6\\ 16 & 12\\ 12 & 9\end{pmatrix}, b=\begin{pmatrix}-1\\ \frac{13}{4}\\ 4\end{pmatrix} \tag{2.2.9}$$

To solve (2.2.8), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M}=\mathbf{USV^T} \tag{2.2.10}$$

Substituting the value of $\mathbf{M}$ from (2.2.10) in (2.2.8), we get

$$\mathbf{USV^Tc}=\mathbf{b} \tag{2.2.11}$$

$$\implies \mathbf{c}=\mathbf{VS_+U^Tb} \tag{2.2.12}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{MM^T}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M^TM}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M^TM}$.

$$\mathbf{M}^T\mathbf{M}=\begin{pmatrix}\frac{6449}{16} & \frac{621}{2}\\ \frac{621}{2} & 261\end{pmatrix} \tag{2.2.13}$$

$$\mathbf{MM}^T=\begin{pmatrix}\frac{625}{16} & -100 & -75\\ -100 & 400 & 300\\ -75 & -300 & 225\end{pmatrix} \tag{2.2.14}$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left|\mathbf{M^TM}-\lambda\mathbf{I}\right|=0 \tag{2.2.15}$$

$$\begin{pmatrix}\frac{6449}{16}-\lambda & \frac{621}{2}\\ \frac{621}{2} & 261-\lambda\end{pmatrix}=0 \tag{2.2.16}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1=13.51 \tag{2.2.17}$$

$$\lambda_2=650.6 \tag{2.2.18}$$

$$\tag{2.2.19}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1=\begin{pmatrix}-0.7971\\ 1\end{pmatrix}, \mathbf{v}_2=\begin{pmatrix}1.255\\ 1\end{pmatrix} \tag{2.2.20}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1=\begin{pmatrix}-0.6233\\ 0.782\end{pmatrix}\mathbf{v}_2=\begin{pmatrix}0.7816\\ 0.6228\end{pmatrix} \tag{2.2.21}$$

Hence we obtain $\mathbf{V}$ of (2.2.10) as follows,

$$\mathbf{V}=\begin{pmatrix}-0.6233 & 0.7816\\ 0.782 & 0.6228\end{pmatrix} \tag{2.2.22}$$

Similarly,eigen values of $\mathbf{MM}^T$ are,

$$\left|\mathbf{MM^T}-\lambda\mathbf{I}\right|=0 \tag{2.2.23}$$

$$\begin{pmatrix}\frac{625}{16}-\lambda & -100 & -75\\ -100 & 400-\lambda & 300\\ -75 & 300 & 225-\lambda\end{pmatrix}=0 \tag{2.2.24}$$

$$\lambda_3=13.51 \tag{2.2.25}$$

$$\lambda_4=650.6 \tag{2.2.26}$$

$$\lambda_5=0 \tag{2.2.27}$$

Hence the corresponding eigen vectors of $\mathbf{MM}^T$ are,

$$\mathbf{u}_1=\begin{pmatrix}8.153\\ 1.333\\ 1\end{pmatrix}, \mathbf{u}_2=\begin{pmatrix}-0.3407\\ 1.333\\ 1\end{pmatrix}, \mathbf{u}_3=\begin{pmatrix}0\\ -0.75\\ 1\end{pmatrix} \tag{2.2.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1=\begin{pmatrix}0.9798\\ 0.1601\\ 0.1201\end{pmatrix}, \mathbf{u}_2=\begin{pmatrix}-0.2\\ 0.7841\\ 0.5882\end{pmatrix}, \mathbf{u}_3=\begin{pmatrix}0\\ -0.6\\ 0.8\end{pmatrix} \tag{2.2.29}$$

Hence we obtain $\mathbf{U}$ of (2.2.10) as follows,

$$\mathbf{U}=\begin{pmatrix}0.9798 & -0.2 & 0\\ 0.1601 & 0.7841 & -0.6\\ 0.1201 & 0.5882 & 0.8\end{pmatrix} \tag{2.2.30}$$

After computing the singular values from eigen values $\lambda_3, \lambda_4, \lambda_5$ we get $\mathbf{S}$ of (2.2.10) as follows,

$$\mathbf{S}=\begin{pmatrix}3.675 & 0\\ 0 & 25.5\\ 0 & 0\end{pmatrix} \tag{2.2.31}$$

From (2.2.10) we get the Singular Value Decomposition of $\mathbf{M}$,

$$\mathbf{M} = \begin{pmatrix} 0.9798 & -0.2 & 0 \\ 0.1601 & 0.7841 & -0.6 \\ 0.1201 & 0.5882 & 0.8 \end{pmatrix} \begin{pmatrix} 3.675 & 0 \\ 0 & 25.5 \\ 0 & 0 \end{pmatrix} \tag{2.2.32}$$

$$\begin{pmatrix} -0.6233 & 0.7816 \\ 0.782 & 0.6228 \end{pmatrix}^T \tag{2.2.33}$$

$$= \begin{pmatrix} -1.75 & -6 \\ 16 & 12 \\ 12 & 9 \end{pmatrix} \tag{2.2.34}$$

Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_{+} = \begin{pmatrix} 0.272 & 0 & 0 \\ 0 & 0.0391 & 0 \end{pmatrix} \tag{2.2.35}$$

From (2.2.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0.02093 \\ 5.101 \\ 1.25 \end{pmatrix} \tag{2.2.36}$$

$$\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0.005692 \\ 0.1995 \end{pmatrix} \tag{2.2.37}$$

$$\mathbf{c} = \mathbf{V}\mathbf{S}_{+}\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0.16 \\ 0.12 \end{pmatrix} \tag{2.2.38}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0.16 \\ 0.12 \end{pmatrix} \tag{2.2.39}$$

Verifying the solution of (2.2.39) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{c} = \mathbf{M}^T\mathbf{b} \tag{2.2.40}$$

Evaluating the R.H.S in (2.2.40) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{c} = \begin{pmatrix} \frac{1018}{10} \\ 81 \end{pmatrix} \tag{2.2.41}$$

$$\implies \begin{pmatrix} \frac{6449}{16} & \frac{621}{2} \\ \frac{621}{2} & 261 \end{pmatrix} \mathbf{c} = \begin{pmatrix} \frac{1018}{10} \\ 81 \end{pmatrix} \tag{2.2.42}$$

Solving the augmented matrix of (2.2.42) we get,

$$\begin{pmatrix} \frac{6449}{16} & \frac{621}{2} & \frac{1018}{10} \\ \frac{621}{2} & 261 & 81 \end{pmatrix} \xleftrightarrow{R_1 = \frac{16}{6449}R_1} \begin{pmatrix} 1 & \frac{4968}{6449} & \frac{8144}{32245} \\ \frac{621}{2} & 261 & 81 \end{pmatrix} \tag{2.2.43}$$

$$\xleftrightarrow{R_2 = R_2 - \frac{621}{2}R_1} \begin{pmatrix} 1 & \frac{4968}{6449} & \frac{8144}{32245} \\ 0 & \frac{140625}{6449} & \frac{83133}{32245} \end{pmatrix} \tag{2.2.44}$$

$$\xleftrightarrow{R_2 = \frac{6449}{140625}R_2} \begin{pmatrix} 1 & \frac{4968}{6449} & \frac{8144}{32245} \\ 0 & 1 & \frac{3}{25} \end{pmatrix} \tag{2.2.45}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{4968}{6449}R_2} \begin{pmatrix} 1 & 0 & \frac{197}{1250} \\ 0 & 1 & \frac{3}{25} \end{pmatrix} \tag{2.2.46}$$

From equation (2.2.46), solution is given by,

$$\mathbf{c} = \begin{pmatrix} \frac{197}{1250} \\ \frac{3}{25} \end{pmatrix} \tag{2.2.47}$$

$$\mathbf{c} = \begin{pmatrix} 0.16 \\ 0.12 \end{pmatrix} \tag{2.2.48}$$

Comparing results of $\mathbf{c}$ from (2.2.39) and (2.2.48), we can say that the solution is verified.