Matrix Theory Assignment 2

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***Abstract*—This document contains the solution to problem No.66 from Lines and Planes**

1 Problem Statement

Matrices A and B will be inverse of each other only if

(A)AB=BA (B)AB=BA=0

(C)AB=0,BA=I (D)AB=BA=I

2 Solution

Consider a matrix A. We define matrix A as follows

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.0.1}$$

The inverse of A is A^{-1}. Let $B = A^{-1}$. Evaluate the inverse of A.

$$B = A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$AB = AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{ad-bc}(ad-bc) & 0 \\ 0 & \frac{1}{ad-bc}(-bc+ad) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\implies AB = AA^{-1} = I \tag{2.0.2}$$

Now, We define B as follows

$$B = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{2.0.3}$$

$$A = B^{-1} = \frac{1}{ps-qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$$

$$BA = BB^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\frac{1}{ps-qr}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$$

$$= \frac{1}{ps-qr}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$$

$$= \frac{1}{ps-qr}\begin{pmatrix} ps-qr & sr-sr \\ -pq+pq & -qr+ps \end{pmatrix}$$

$$= \frac{1}{ps-qr}\begin{pmatrix} ps-qr & 0 \\ 0 & ps-qr \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{ps-qr}(ps-qr) & 0 \\ 0 & \frac{1}{ps-qr}(ps-qr) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\implies BA = BB^{-1} = I \tag{2.0.4}$$

From Eq (2.0.2) and Eq (2.0.4),

$AB = I, BA = I$.

We can conclude that option D is correct

Statement - If AB = BA , then both A and B are invertible.

Proof:

Case 1 : We define A and B as follows

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{2.0.5}$$

$$B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \tag{2.0.6}$$

$$AB = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$$

$$BA = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ca & 0 \\ 0 & bd \end{pmatrix}$$

$$= \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix} \tag{2.0.7}$$

In the above example, AB =BA and both A and B are non - singular(invertible) matrices.

Case 2 : We define A and B as follows

$$A = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \tag{2.0.8}$$

$$B = \begin{pmatrix} b & b \\ b & b \end{pmatrix} \tag{2.0.9}$$

$$\begin{aligned} AB &= \begin{pmatrix} a & a \\ a & a \end{pmatrix} \begin{pmatrix} b & b \\ b & b \end{pmatrix} \\ &= \begin{pmatrix} ab+ab & ab+ab \\ ab+ab & ab+ab \end{pmatrix} \\ &= \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix} \\ BA &= \begin{pmatrix} b & b \\ b & b \end{pmatrix} \begin{pmatrix} a & a \\ a & a \end{pmatrix} \\ &= \begin{pmatrix} ab+ab & ab+ab \\ ab+ab & ab+ab \end{pmatrix} \\ &= \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix} \end{aligned} \tag{2.0.10}$$

In the above example, AB = BA and both A and B are non - invertible(singular) matrices.

Python Code:

https://github.com/Hrithikraj2/MatrixTheory_EE5609/blob/master/Assignment_2/codes/A2_code1.py

https://github.com/Hrithikraj2/MatrixTheory_EE5609/blob/master/Assignment_2/codes/A2_code2.py

Latex codes:

https://github.com/Hrithikraj2/MatrixTheory_EE5609/blob/master/Assignment_2/latex/A2.tex