Answers to questions in
Lab 1: Filtering operations

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Instructions: Complete the lab according to the instructions in the notes and respond to the questions stated below. Keep the answers short and focus on what is essential. Illustrate with figures only when explicitly requested.

Good luck!

Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9), (17, 121), (5, 1) and (125, 1) respectively. What do you observe?

Answers:

The Fourier transform of the image shows the frequency components that have the most transcendence in the original image in spatial domain. It being one point (delta function = 1 in one coordinate) indicates that the original image will have the form of a sinusoidal wave with a specific frequency and orientation.
Mathematically the exact expression used by MATLAB is,

$$\mathbf{f(m,n)}= \frac{1}{128}\, e^{\frac{2\pi i((p-1)m+n(q-1))}{128}}$$

If there is a delta function at (p,q) in Fourier Domain.
But we will loosely follow the traditional formulae in the report.
After visualizing the results in the different points, some conclusions can be obtained:

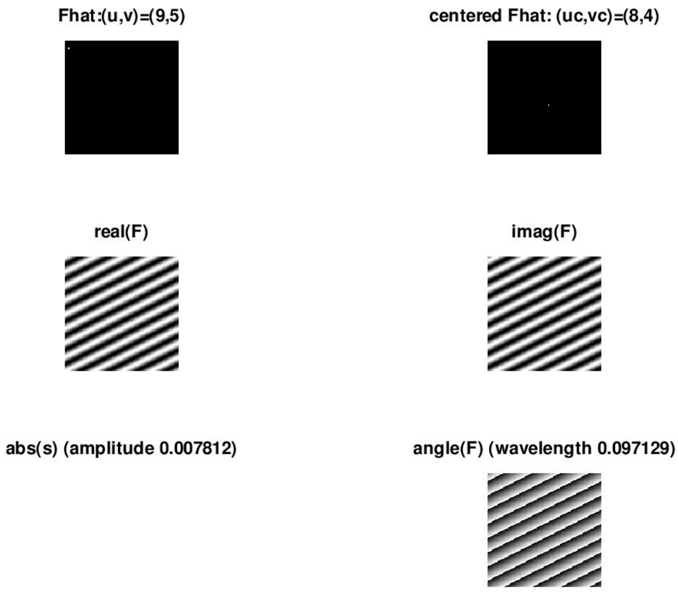
- Centering the fourier transform brings the zero frequency component at the center of the image.
- Magnitude of inverse Fourier transform will be 1 as the inverse fourier transform of Delta function is a complex exponential.
- The wavelength of

Question 2: Explain how a position (p, q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a Matlab figure.

Answers:

$$\mathrm{Re}(f(m,n))=\cos(2\pi((p-1)m+n(q-1))/128)$$
$$\mathrm{Img}(f(m,n))=\sin(2\pi((p-1)m+n(q-1))/128)$$

Taking a random point such as (9,5) in a 128x128 fourier domain image, we obtain the following images:



Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in the notes. Complement the code (variable amplitude) accordingly.

Answers:

The amplitude of the sine wave generated by this specific Fourier transform image (delta function) can be obtained by the general 2D inverse Fourier transform:

$$F(x,y)=\frac{1}{N}\cdot e^{\frac{i2\pi}{N}\cdot(px+qy)}=\frac{1}{N}\cdot\left(\cos\left(\frac{2\pi}{N}(px+qy)\right)+i\cdot\sin\left(\frac{2\pi}{N}(px+qy)\right)\right)$$

According to this solution, the amplitude of the wave generated is the module of the spatial image:

$$|F(x,y)|=\sqrt{Re^2(x,y)+Im^2(x,y)}$$

In this case, the amplitude will always result in being 1/N.

Question 4: How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

Answers:

Following the equation from the previous point:

$$F(x,y) = \frac{1}{N} \cdot e^{\frac{i2\pi}{N}(px+qy)} = \frac{1}{N} \cdot (\cos\left(\frac{2\pi}{N}(px+qy)\right) + i \cdot \sin\left(\frac{2\pi}{N}(px+qy)\right))$$

The direction of the sine wave is determined by $px + qy$.
The angle of the line can be found by $tan(\theta) = q/p$ where $tan(\theta)$ is the slope of the above mentioned line.

The length of the sinusoid is defined by the equation: $\lambda = \frac{1}{\sqrt{u^2+v^2}}$. This is logical as u and v are the frequencies in each axis of the image.

Question 5: What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically with Matlab!

Answers:

We know that

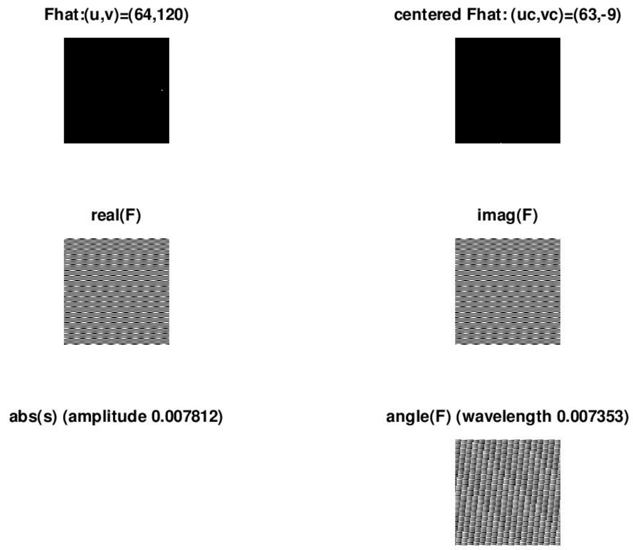
$$f(m,n) = \frac{1}{\sqrt{M}} \sum_{p=1}^{M} \left(\frac{1}{\sqrt{N}} \sum_{q=1}^{N} \widehat{f}(p,q)\ e^{\frac{2\pi inq}{N}}\ \right) e^{\frac{2\pi imp}{N}}$$

For a point which is passes through the center, for example (p,q)=(64,120)
The expression becomes

$$f(m,n) = \frac{1}{128}\ e^{\frac{2\pi inq}{N}}\ \ e^{\frac{2\pi im}{2}}$$

$$f(m,n) = \frac{1}{128}\ e^{\frac{2\pi inq}{N}}\ \ (-1)^m$$

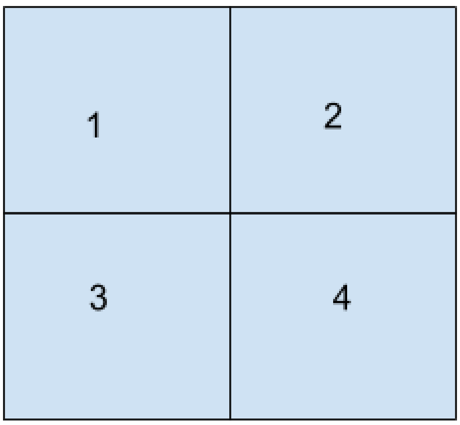
It can be easily inferred that Real and Imaginary part of the fourier transform fluctuate positive and negative. Hence the changes in the phase are also swift.



Question 6: What is the purpose of the instructions following the question *What is done by these instructions?* in the code?

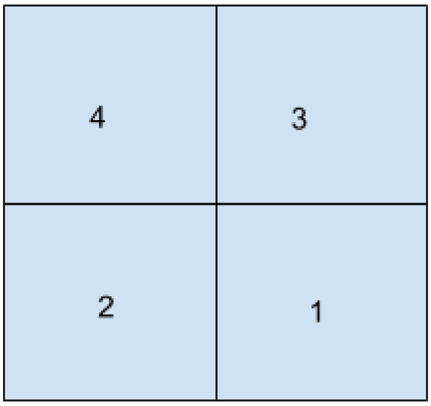
Answers:

Those instructions change the Fourier Transform such that **Zero frequency component is at center of the image/spectrum**.



Original Image In Frequency Domain

TO



Centered Image In Frequency Domain

Centering the image in fourier Image means multiplying the image by $(-1)^{m+n}$ in spatial domain. Centered Fourier Transform makes aides in interpretations also. Centering opertion is like changing domain from $(0, 2\pi)\ to\ (-\pi, \pi)$.

Question 7: Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Answers:

$$\widehat{f}\,(p+1,q+1)=\frac{1}{128}\sum_{m=1}^{128}\Big(\sum_{n=1}^{128} f(m,n)\, e^{\frac{-2\pi inq}{128}}\Big)\, e^{\frac{-2\pi imp}{128}}$$

Solving the above equation for Image F=[zeros(56, 128); ones(16, 128); zeros(56, 128)] gives:

$$\widehat{f}\,(p+1,q+1) = 0 \quad \text{if } q \neq 0$$
$$= \sum_{m} e^{\frac{-2\pi ipm}{128}} \text{ if } q=0$$

Hence it is clear from the above equation that fourier transform of F will have border in the first column.

Question 8: Why is the logarithm function applied?

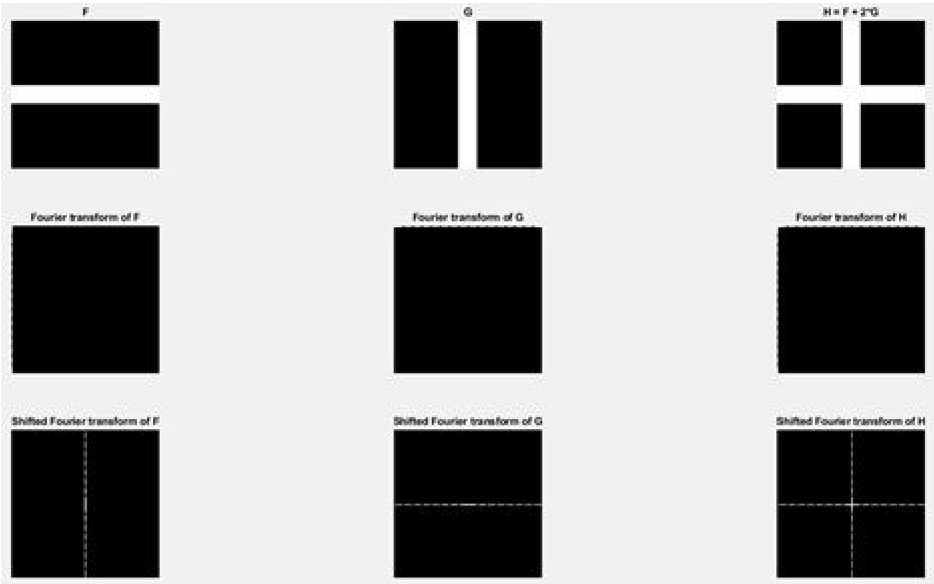
Answers:

The logarithm function is applied because we the raw magnitude data is not easily understandable, as the magnitude is much higher in low frequencies than higher ones. Therefore, the resulting image without applying the log function would be almost completely black.

The logarithm function enhances the low intensity pixels (corresponding to high frequency) and reduces the high intensity pixels (low frequency) so the transformed image can be more easily studied.

Question 9: What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answers:



As shown in the figure, linearity applies to the Fourier transform. The linearity property of the fourier transform allows to separate linear combinations of a spatial image accordingly in the fourier domain. Therefore, by applying the same linear operators separately, the same results would be obtained. The equation below sums up this property:

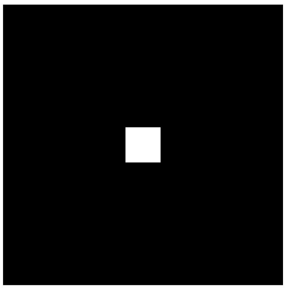
$$\mathcal{F}\left[a\, f_1(m,n) + b\, f_2(m,n))\right] = a\, \hat{f}_1(u,v) + b\, \hat{f}_2(u,v)$$

$$a\, f_1(m,n) + b\, f_2(m,n)) = \mathcal{F}^{-1}\left[a\, \hat{f}_1(u,v) + b\, \hat{f}_2(u,v)\right]$$

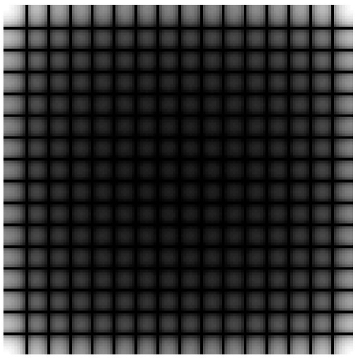
Question 10: Are there any other ways to compute the last image? Remember what multiplication in Fourier domain equals to in the spatial domain! Perform these alternative computations in practice.

Answers:

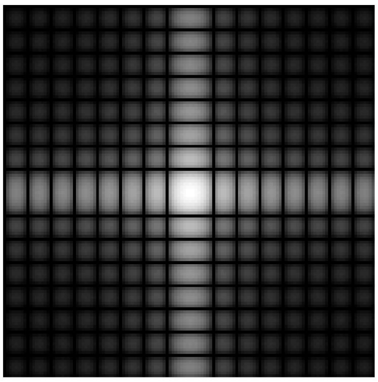
The image obtained is same is of 2D sinc function.



This is the image in Spatial Domain after multiplication



This is the image in Frequency Domain after fft



This is the centered Fourier Transform
(here 2D sinc function is clearly observed)

There are other ways to find the same image.

1.) Using the principle of Duality.
   a.) Rather than treating F and G in spatial domain, consider them to be in Fourier Domain.
   b.) Take Inverse Fourier Transform instead of Fourier Transform.
   c.) Multiply the Inverse Fourier Transform with $(128)^2$ due to technicalities in MATLAB.

2.) Multiplication in Fourier Domain is same as Convolution in Spatial Domain.
   a.) Take the fourier transform of F and G.
   b.) Recenter them using fftshift.
   c.) Perform 2D convolution on it.

Question 11: What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

Answers:

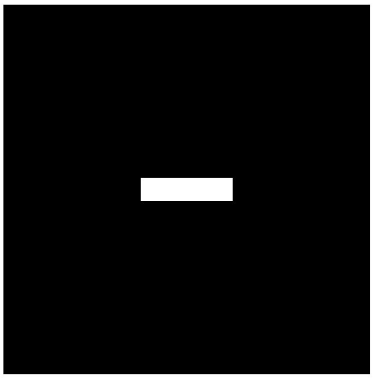
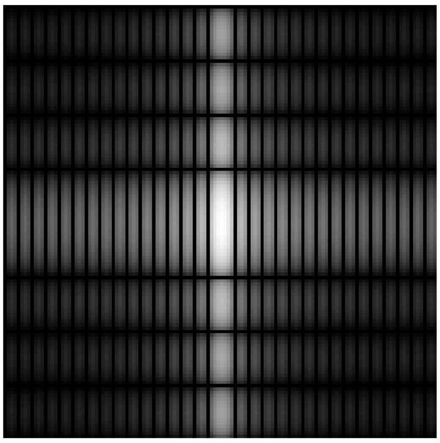
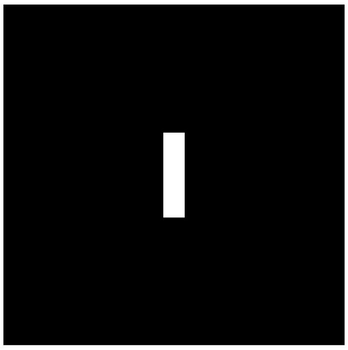


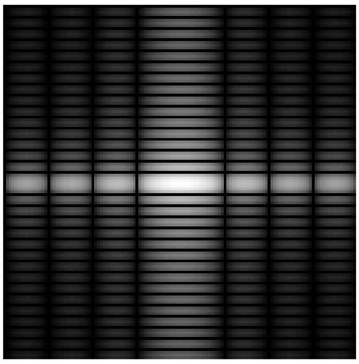
Image in Spatial Image



Corresponding Fourier Transform

1. We know that the fourier transform of a rectangular wave yields a sinc function in frequency domain.The width of the wave is inversely proportional to the zero magnitude points in sinc function. As we can see that the width of square wave is larger in horizontal direction than in vertical direction. Subsequently, the sinc function is wider in vertical direction than horizontal direction.
2. In the previous example, we have a square wave which implies that the sinc function in frequency domain will be symmetric which it actually is.

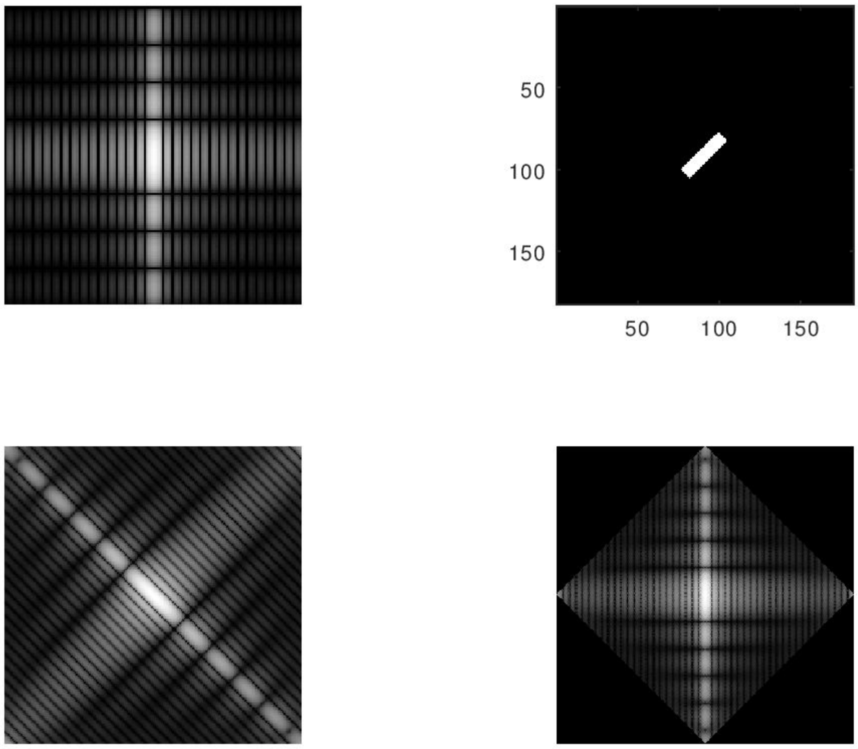




Expansion in Spatial Domain causes Compression in Fourier Domain and vice-versa.
The effect of scaling is clearly visible where the horizontal direction becomes wider than vertical in fourier domain as we rescaled the image in spatial domain.

Question 12: What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.

Answers:



Top left: Fourier Domain representation of F

Top Right: F when rotated 30^0 *anticlockwise*

Bottom Left: Fourier Transform of Top Right Image

Bottom Right: Bottom Left Image is rotated by 30^0 *in clockwise direction*

Similarities:

- Rotation of Original Image rotates the fourier transform with same angle.
- Bottom Left Image is same as Top Left Image rotated by 30^0 anticlockwise.

Differences:

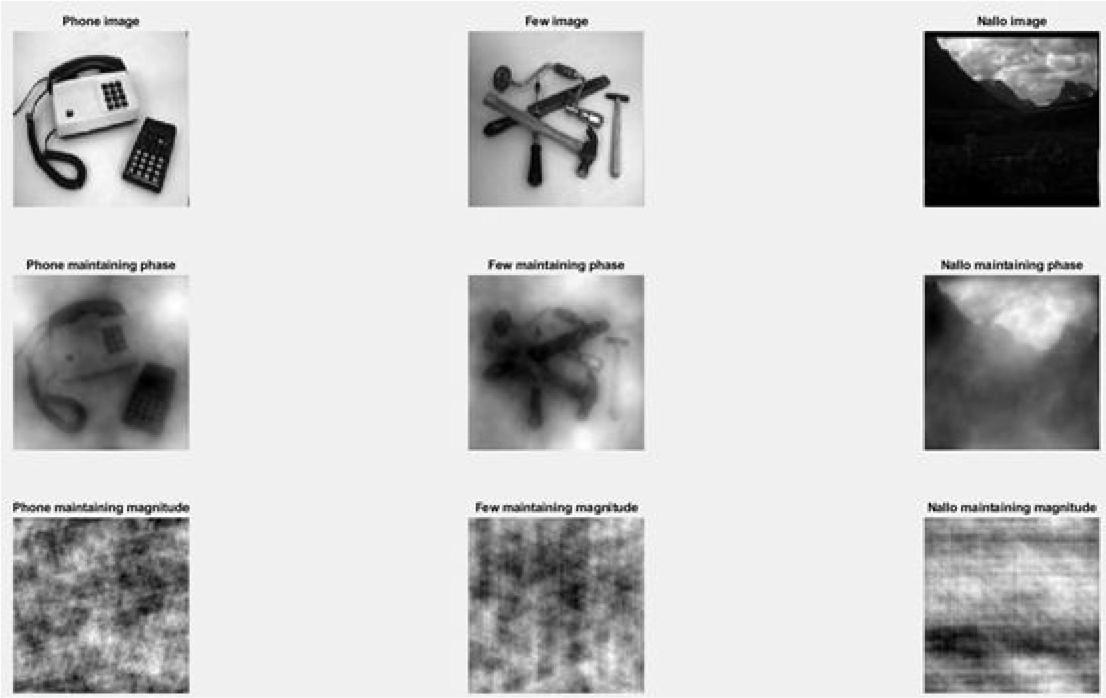
- Bottom Right Image is not exactly same as Image in Top Left.
- It is because Bottom Right Image is just a rotated version of Image Bottom Left not the fourier transform of original image.

Question 13: What information is contained in the phase and in the magnitude of the Fourier transform?

Answers:

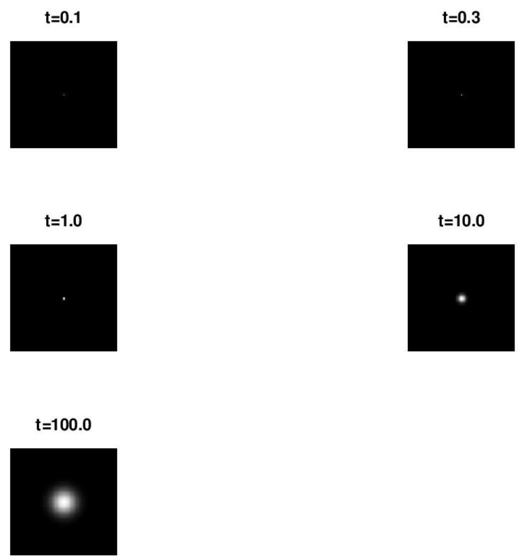
The phase contains the most important part of the information when transforming to the Fourier domain. It contains the information about the location in the spatial domain. Therefore, only retaining the magnitude will lead to an uninterpretable image.

On the other hand, the magnitude contains information about how much of each frequency there is in the image. If we alter the magnitude, the structure of the image will be maintained but the distribution of the color will be altered (e.g. if we convert the magnitude all equal to a number, the low frequency dominance of a typical image will no longer be true). The image below shows these facts.



Question 14: Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and
100.0?

Answers:



var_t_0.1 =

```
  1.3297e-02   1.9559e-14
  1.9559e-14   1.3297e-02

var_t_0.3 =

  2.8105e-01   1.7596e-15
  1.7596e-15   2.8105e-01

var_t_1.0 =

  1.0000e+00   2.1444e-15
  2.1444e-15   1.0000e+00

var_t_10.0 =

  1.0000e+01   7.7348e-16
  7.7348e-16   1.0000e+01

var_t_100.0 =

  1.0000e+02  -2.6881e-16
 -2.6881e-16   1.0000e+02
```

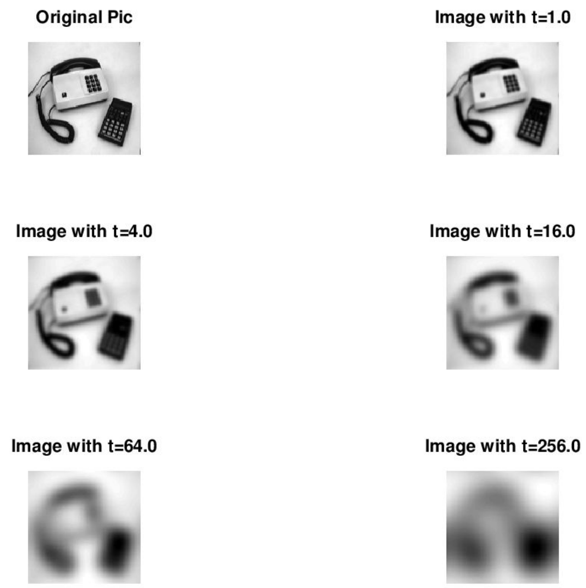
Question 15: Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.

Answers:

The results are slightly different from the estimated variance. The non-diagonal elements are non-zero. In the ideal continuous case, zero diagonal elements mean that coordinates in both directions of spatial domain are independent of each other. Discretization of Continuous Gaussian Filter leads to loss of the mentioned independence.

Question 16: Convolve a couple of images with Gaussian functions of different variances (like t = 1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

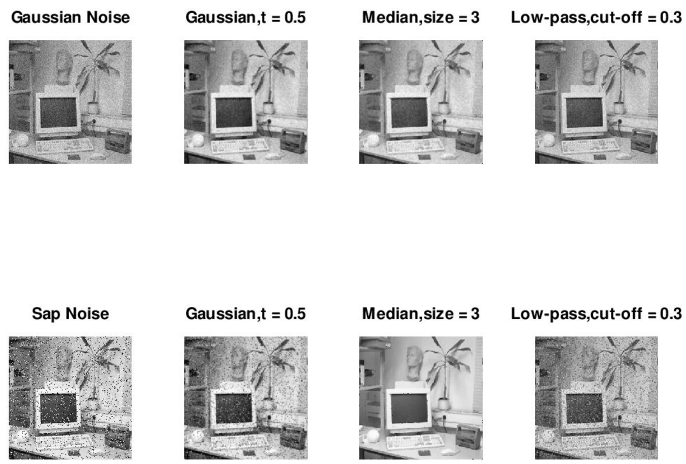
Answers:



The parameter “t” measures spread of Gaussian curve. Smaller the value, the larger the cutoff frequency and milder the filtering.The value of t=256 is so large that it also removes many low frequency values after filtering which causes loss of information. Gaussian Low Pass Filter makes the images blurry because the edges/sharp details are removed by it.

Question 17: What are the positive and negative effects for each type of filter? Describe what you observe and name the effects that you recognize. How do the results depend on the filter parameters? Illustrate with Matlab figure(s).

Answers:



Median Filtering

- works very well for Salt and Pepper Noise.
- Painting like images.

Low Pass Filtering

- Doesn't work well for Salt and Pepper Noise.
- Difficult to imagine in Spatial Domain.
- Removes high frequency noise very well.

Gaussian Filtering

- Doesn't work well for Salt and Pepper Noise
- It is a more robust low pass filter.
- Easy to imagine this filter in Spatial Domain.
- Removes high frequency noise.

The power of the filter certainly depends on their parameter.
Increasing Window Size removes the extreme values (Salt and Pepper) and adds more painting like effect.
The cut-off frequency and spread determines how mild will the filtering be.

Question 18: What conclusions can you draw from comparing the results of the respective methods?

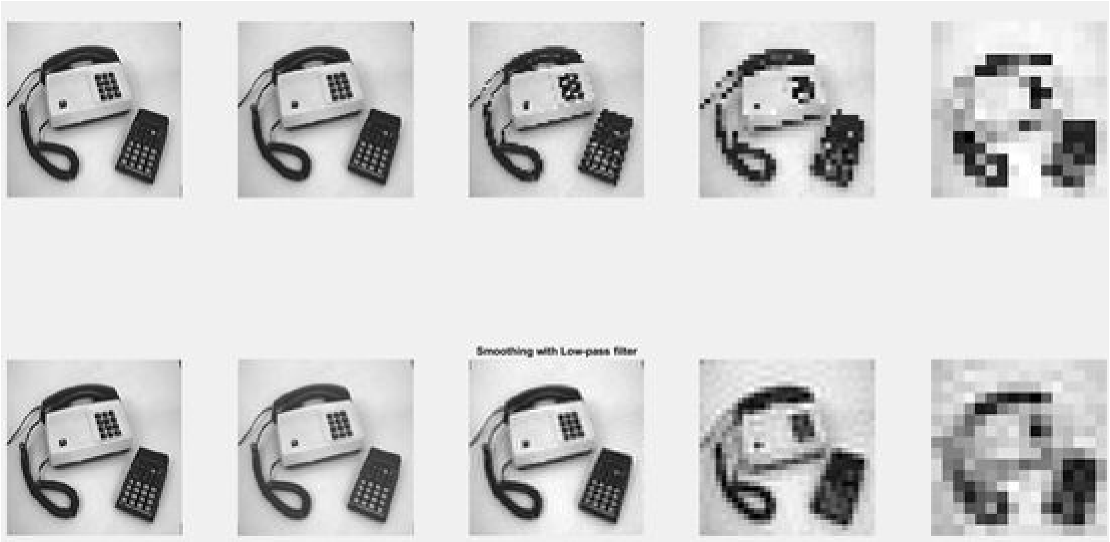
Answers:

The Gaussian noise can be correctly eliminated with either Gaussian filtering or low-pass filtering, having the down-size of blurring the image a bit. The median filter performs outstandingly against SAP noise and maintains the sharpness of edges in the image, but won't solve Gaussian noise as well.

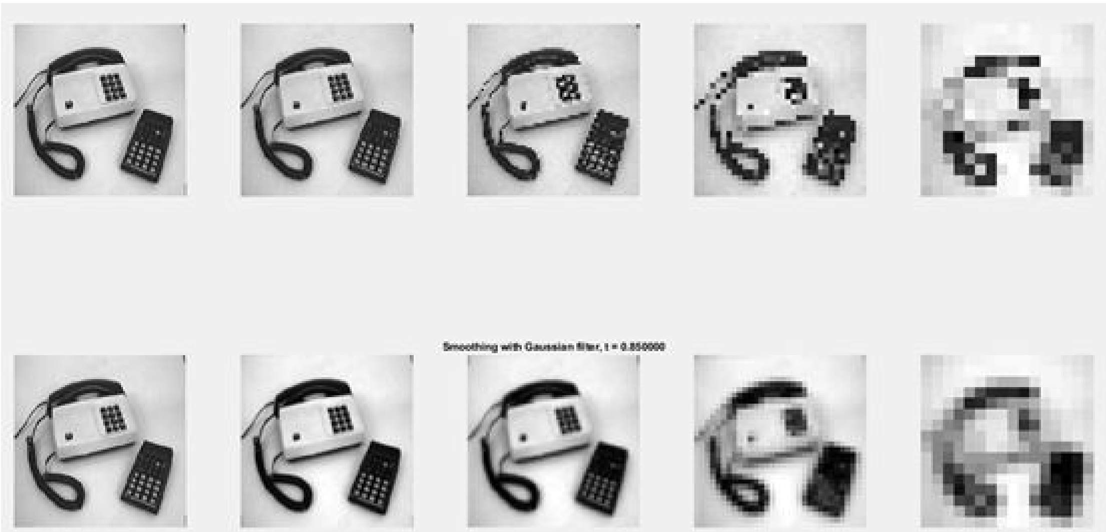
Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

Answers:

When blurring the image before subsampling, the resulting subsampled image seems more congruent and understandable.
First, by tuning the cut-off frequency for a low-pass filter (Optimal cut-off freq. = 0.26):



Secondly, using the Gaussian kernel, with t = 0.85, we obtain:



It is important to say that when the image has a lower resolution, the filter needs to blur the image more before subsampling.

Question 20: What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answers:

Blurring an image before subsampling is an effective method of preventing aliasing. This effect happens when subsampling with a frequency below the Nyquist Rate of the specific image (double of the maximum frequency of the image).
Thanks to the Gaussian or low-pass filter, the low frequencies of the image get prioritized or just cut off the image, so the Nyquist rate gets smaller, letting the subsampling rate be smaller as well. Therefore, if we blur the image before, the loss of information due to aliasing will get reduced or eliminated, with the drawback of having a blurred image.

When the subsample is with a lower frequency, the filter will have to reduce the frequency more as well.

References:

1) https://se.mathworks.com/help/matlab/ref/ifft2.html
2) https://se.mathworks.com/help/matlab/ref/fft2.html