1. $y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2$

by Euler's $\Rightarrow y(t_{i+1}) = y(t_i) + h \cdot y'(t_i, y(t_i))$

by Taylor's of order 2

$y(t_{i+1}) = y(t_i) + h T^{(2)}(t_i + y(t_i))$

$T^{(2)} = y' + \frac{h}{2}\left[\frac{\partial y'}{\partial t} + y'\frac{\partial y'}{\partial y}\right]$

t	Euler	Taylor	Exact
1.0	0.000000	0.000000	0.000000
1.1	0.100000	0.105000	0.105160
1.2	0.209917	0.220919	0.221243
1.3	0.330471	0.348612	0.349121
1.4	0.462354	0.488954	0.489682
1.5	0.606285	0.642883	0.643875
1.6	0.763041	0.811438	0.812753
1.7	0.933475	0.995787	0.997494
1.8	1.118537	1.197252	1.199439
1.9	1.319293	1.417344	1.420116
2.0	1.536943	1.657795	1.661282

2.

$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t \quad , u_1(0) = \frac{4}{3}$

$u_2' = -24u_1 - 52u_2 - 9\cos t + \frac{1}{3}\sin t \quad , u_2(0) = \frac{2}{3}$

$$u_1 = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t$$

$$u_2 = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t$$

by Runge-Kutta method

$h = 0.05$

$h = 0.1$

$$\begin{cases} w_i = \begin{bmatrix} u_1(i) \\ u_2(i) \end{bmatrix} \\ k_1 = hu'(t_i, w_i) \\ k_2 = hu'(t_i + \frac{h}{2}, w_i + k_1) \\ k_3 = hu'(t_i + \frac{h}{2}, w_i + k_2) \\ k_4 = hu'(t_{i+1}, w_i + k_3) \\ w_{i+1} = w_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

$h = 0.05$

=== Error Table for h = 0.05 ===

t	RK4 u1	Exact u1	Error in u1	RK4 u2	Exact u2	Error in u2
0.00	1.333333	1.333333	0.000000	0.666667	0.666667	0.000000
0.05	1.721880	1.912059	0.190178	-0.499599	-0.909077	0.409477
0.10	1.726915	1.793063	0.066148	-0.832598	-1.032002	0.199405
0.15	1.617161	1.601967	0.015194	-0.890373	-0.961459	0.071086
0.20	1.481687	1.423902	0.057785	-0.861042	-0.874681	0.013639
0.25	1.348945	1.267646	0.081299	-0.807505	-0.795221	0.012284
0.30	1.227063	1.131577	0.095487	-0.750341	-0.724999	0.025342
0.35	1.117478	1.012999	0.104480	-0.695886	-0.663060	0.032826
0.40	1.019525	0.909409	0.110117	-0.645732	-0.608214	0.037518
0.45	0.931977	0.818630	0.113347	-0.599934	-0.559389	0.040545
0.50	0.853541	0.738788	0.114753	-0.558092	-0.515658	0.042435
0.55	0.783017	0.668275	0.114743	-0.519706	-0.476225	0.043482
0.60	0.719337	0.605710	0.113627	-0.484290	-0.440411	0.043880
0.65	0.661560	0.549909	0.111651	-0.451407	-0.407635	0.043772
0.70	0.608868	0.499860	0.109007	-0.420673	-0.377404	0.043269
0.75	0.560547	0.454695	0.105852	-0.391754	-0.349296	0.042459
0.80	0.515980	0.413671	0.102309	-0.364365	-0.322954	0.041411
0.85	0.474633	0.376158	0.098475	-0.338259	-0.298076	0.040183
0.90	0.436043	0.341614	0.094428	-0.313226	-0.274409	0.038817
0.95	0.399812	0.309583	0.090229	-0.289089	-0.251739	0.037351
1.00	0.365600	0.279675	0.085925	-0.265698	-0.229888	0.035810

$h = 0.1$

=== Error Table for h = 0.1 ===

t	RK4 u1	Exact u1	Error in u1	RK4 u2	Exact u2	Error in u2
0.00	1.333333e+00	1.333333e+00	0.000000e+00	6.666667e-01	6.666667e-01	1.110223e-16
0.10	-3.052437e+00	1.793063e+00	4.845500e+00	8.989305e+00	-1.032002e+00	1.002131e+01
0.20	-2.384779e+01	1.423902e+00	2.527170e+01	5.119270e+01	-8.746810e-01	5.206739e+01
0.30	-1.301652e+02	1.131577e+00	1.312968e+02	2.692692e+02	-7.249986e-01	2.699942e+02
0.40	-6.802315e+02	9.094086e-01	6.811409e+02	1.399369e+03	-6.082142e-01	1.399977e+03
0.50	-3.531300e+03	7.387878e-01	3.532038e+03	7.258242e+03	-5.156577e-01	7.258757e+03
0.60	-1.831280e+04	6.057096e-01	1.831340e+04	3.763496e+04	-4.404108e-01	3.763540e+04
0.70	-9.495133e+04	4.998603e-01	9.495183e+04	1.951319e+05	-3.774038e-01	1.951322e+05
0.80	-4.923065e+05	4.136715e-01	4.923069e+05	1.011722e+06	-3.229535e-01	1.011722e+06
0.90	-2.552514e+06	3.416143e-01	2.552514e+06	5.245579e+06	-2.744088e-01	5.245579e+06
1.00	-1.323428e+07	2.796749e-01	1.323428e+07	2.719729e+07	-2.298878e-01	2.719729e+07