Lecture 2:
Asymptotics & Sorting

June 23, 2020

WHAT WE'LL COVER TODAY

- More Asymptotic Analysis: Big-O and friends!
  - Mathematical definitions & proving Big-O bounds
- Insertion Sort
  - Proof of Correctness (using induction) & Runtime Analysis
- MergeSort
  - Proof of Correctness (using induction) & Runtime Analysis

ASYMPTOTIC ANALYSIS

Big-O Notation & its relatives (Big-Ω and Big-Θ)

FROM LECTURE 1

THE POINT OF ASYMPTOTIC NOTATION

suppress constant factors and lower-order terms

too system dependent

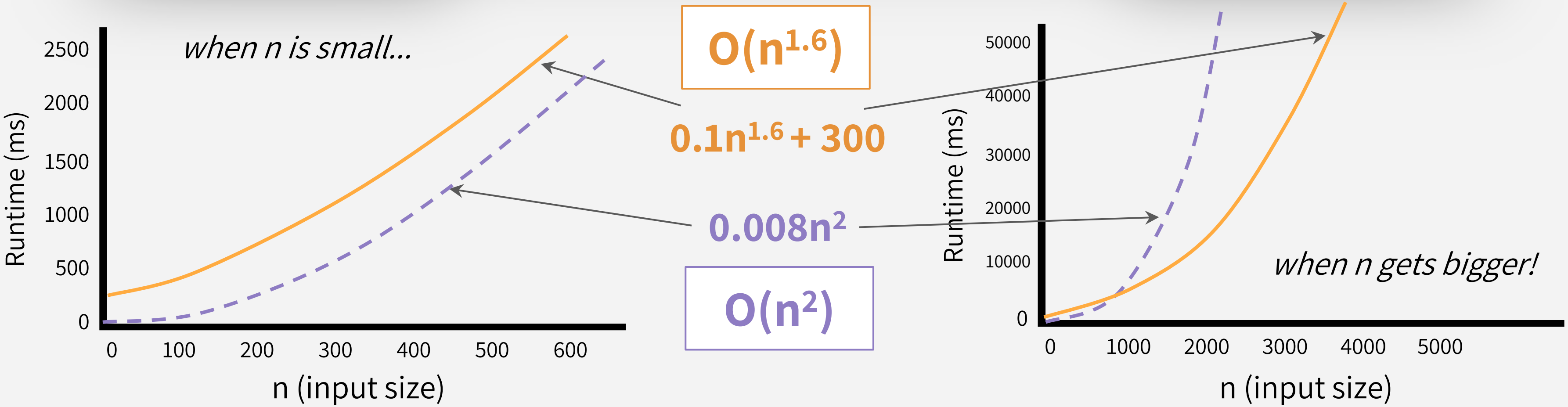
irrelevant for large inputs

- **Some guiding principles:** we care about how the running time/number of operations *scales* with the size of the input (i.e. the runtime's *rate of growth*), and we want some measure of runtime that's independent of hardware, programming language, memory layout, etc.
  - We want to reason about high-level algorithmic approaches rather than lower-level details

FROM LECTURE 1

THE POINT OF ASYMPTOTIC NOTATION

$$\text{suppress } \underbrace{\text{constant factors}}_{\textit{too system dependent}} \text{ and } \underbrace{\text{lower-order terms}}_{\textit{irrelevant for large inputs}}$$



A NOTE ON RUNTIME ANALYSIS

There are a few different ways to analyze the runtime of an algorithm:

Worst-case analysis:
What is the runtime of the algorithm on the *worst* possible input?

Best-case analysis:
What is the runtime of the algorithm on the *best* possible input?

Average-case analysis:
What is the runtime of the algorithm on the *average* input?

A NOTE ON RUNTIME ANALYSIS

There are a few different ways to analyze the runtime of an algorithm:

We'll mainly focus on worst case analysis since it tells us how fast the algorithm is on *any* kind of input

Worst-case analysis:
What is the runtime of the algorithm on the *worst* possible input?

Best-case analysis:
What is the runtime of the algorithm on the *best* possible input?

Average-case analysis:
What is the runtime of the algorithm on the *average* input?

We'll work with this more when we cover Randomized Algorithms!

BIG-O NOTATION

Let T(n) & f(n) be functions defined on the positive integers.

(In this class, we'll typically write T(n) to denote the worst case runtime of an algorithm)

What do we mean when we say "T(n) is O(f(n))"?

English Definition

Pictorial Definition

Mathematical Definition

BIG-O NOTATION

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What do we mean when we say "T(n) is O(f(n))"?

In English

T(n) = O(f(n)) if and only if T(n) is *eventually* **upper bounded** by a constant multiple of f(n)

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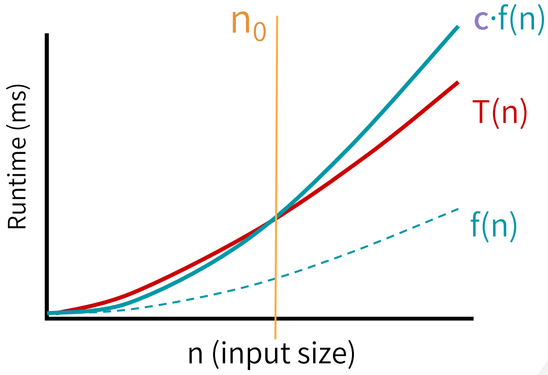

Mathematical Definition

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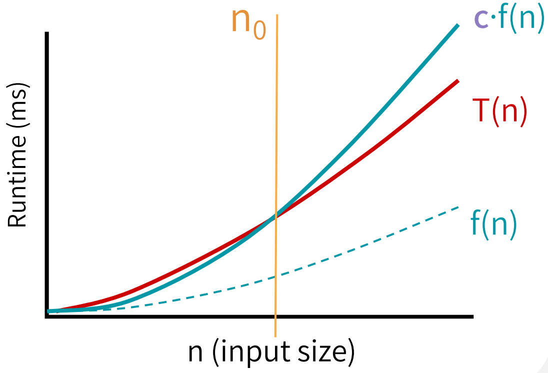

In Math

T(n) = O(f(n)) if and only if there exists positive **constants** **c** and $\mathbf{n_0}$ such that *for all* $n \geq n_0$

$$T(n) \leq c \cdot f(n)$$

BIG-O NOTATION

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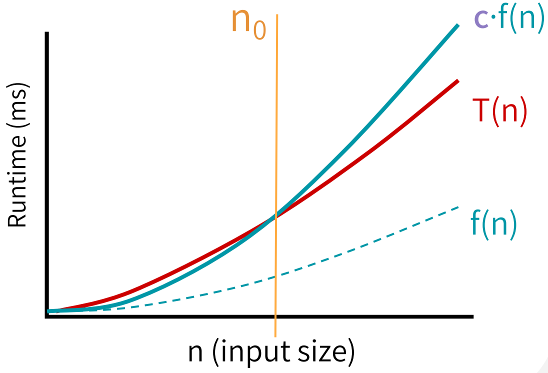

In *Math*

$$T(n) = O(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$\mathbf{T(n) \leq c \cdot f(n)}$$

BIG-O NOTATION

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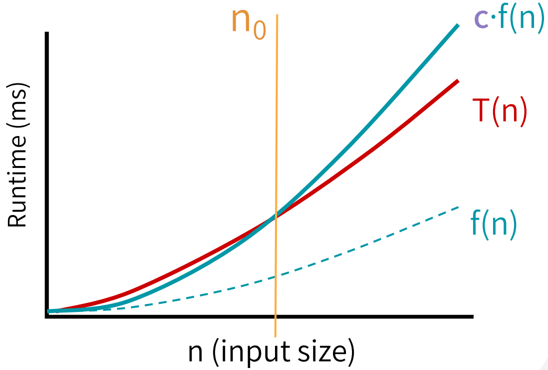

In *Math*

$$T(n) = O(f(n))$$

"if and only if" → $\Leftrightarrow$

"for all"

$$\exists\ c\ ,\ n_0 > 0\ \text{ s.t. } \forall\ n \geq n_0\ ,$$

$$T(n) \leq c \cdot f(n)$$

"such that"

"there exists"

PROVING BIG-O BOUNDS

If you’re ever asked to formally prove that T(n) is O(f(n)), use the *MATH* definition:

$$T(n) = O(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$\mathbf{T(n) \leq c \cdot f(n)}$$

must be constants! i.e. c & n_0 cannot depend on n!

- To **prove** T(n) = O(f(n)), you need to announce your c & n_0 up front!
  - Play around with the expressions to find appropriate choices of c & n_0 (positive constants)
  - Then you can write the proof! Here how to structure the start of the proof:

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- To **prove** T(n) = O(f(n)), you need to announce your c & n_0 up front!
  - Play around with the expressions to find appropriate choices of c & n_0 (positive constants)
  - Then you can write the proof! Here how to structure the start of the proof:

"Let c = __ and n_0 = __. We will show that T(n) ≤ c · f(n) for all n ≥ n_0."

PROVING BIG-O BOUNDS: EXAMPLE

$$T(n) = O(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c\ ,\ n_0 > 0\ \text{ s.t. } \forall\ n \geq n_0\ ,$$

$$T(n) \leq c \cdot f(n)$$

Prove that $3n^2 + 5n = O(n^2)$.

My thinking: I want to find a c & n_0 such that for all $n \geq n_0$:

$$3n^2 + 5n \leq c \bullet n^2$$

I can rearrange this inequality just to see things a bit more clearly:

$$5n \leq (c - 3) \bullet n^2$$

Now let's cancel out the n:

$$5 \leq (c - 3)\ n$$

PROVING BIG-O BOUNDS: EXAMPLE

$$T(n) = O(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c\ ,\ n_0 > 0\ \text{ s.t. } \forall\ n \geq n_0\ ,$$

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Now let's cancel out the n:

$$5 \leq (c - 3)\ n$$

Let's choose:

c = 4

n_0= 5

(other choices work too!
e.g. c= 10, n_0 = 10)

PROVING BIG-O BOUNDS: EXAMPLE

$$T(n) = O(f(n))$$
$$\Leftrightarrow$$
$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$
$$T(n) \leq c \cdot f(n)$$

Prove that $3n^2 + 5n = O(n^2)$.

Let $c = 4$ and $n_0 = 5$. We will now show that $3n^2 + 5n \leq c \cdot n^2$ for all $n \geq n_0$. We know that for any $n \geq n_0$, we have:

$$5 \leq n$$
$$5n \leq n^2$$
$$3n^2 + 5n \leq 4n^2$$

Using our choice of c and n_0, we have successfully shown that $3n^2 + 5n \leq c \cdot n^2$ for all $n \geq n_0$. From the definition of Big-O, this proves that $3n^2 + 5n = O(n^2)$. ■

DISPROVING BIG-O BOUNDS

If you're ever asked to formally disprove that T(n) is O(f(n)), use **proof by contradiction!**

DISPROVING BIG-O BOUNDS

If you're ever asked to formally disprove that T(n) is O(f(n)), use **proof by contradiction!**

This means you need to show that NO POSSIBLE CHOICE of c & n_0 exists such that the Big-O definition holds

DISPROVING BIG-O BOUNDS

If you're ever asked to formally disprove that T(n) is O(f(n)), use **proof by contradiction!**

For sake of contradiction, assume that T(n) is O(f(n)). In other words, assume there does indeed exist a choice of c & n_0 s.t. ∀ $n \geq n_0$, **$T(n) \leq c \cdot f(n)$**

pretend you have a friend that comes up and says "I have a c & n_0 that will prove T(n) = O(f(n))!!!",
and you say "ok fine, let's assume your c & n_0 does prove T(n) = O(f(n))"

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↓

Treating c & n_0 as "variables", derive a contradiction!

although you are skeptical, you'll entertain your friend by saying: "let's see what happens. [some math work... and then...] AHA! regardless of what your constants c & n_0, trusting you has led me to something *impossible!!!"*

DISPROVING BIG-O BOUNDS

If you're ever asked to formally disprove that T(n) is O(f(n)), use **proof by contradiction!**

For sake of contradiction, assume that T(n) is O(f(n)). In other words, assume there does indeed exist a choice of c & n_0 s.t. $\forall$ n ≥ n_0 , **T(n) ≤ c · f(n)**

pretend you have a friend that comes up and says "I have a c & n_0 that will prove T(n) = O(f(n))!!!", and you say "ok fine, let's assume your c & n_0 does prove T(n) = O(f(n))"

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↓

Conclude that the original assumption must be false, so **T(n) is *not* O(f(n))**.

you have triumphantly proven your silly (or lying) friend wrong.

DISPROVING BIG-O: EXAMPLE

$T(n) = O(f(n))$

$\Leftrightarrow$

$\exists\ c, n_0 > 0$ s.t. $\forall\ n \geq n_0$,

$T(n) \leq c \cdot f(n)$

Prove that $3n^2 + 5n$ is *not* $O(n)$.

For sake of contradiction, assume that $3n^2 + 5n$ is $O(n)$. This means that there exists positive constants c & n_0 such that $3n^2 + 5n \leq c \cdot n$ for all $n \geq n_0$. Then, we would have the following:

$$3n^2 + 5n \leq c \cdot n$$

$$3n + 5 \leq c$$

$$n \leq (c - 5)/3$$

However, since $(c - 5)/3$ is a constant, we've arrived at a contradiction since n cannot be bounded above by a constant for all $n \geq n_0$. For instance, consider $n = n_0 + c$: we see that $n \geq n_0$, but $n > (c - 5)/3$. Thus, our original assumption was incorrect, which means that $3n^2 + 5n$ is not $O(n)$. ■

BIG-O EXAMPLES

$$\log_2 n + 15 = O(\log_2 n)$$

$$3^n = O(4^n)$$

Polynomials

Say $p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$ is a polynomial of degree $k \geq 1$.

Then:

i. $p(n) = O(n^k)$

ii. $p(n)$ is **not** $O(n^{k-1})$

$$6n^3 + n \log_2 n = O(n^3)$$

$$25 = O(1)$$

$$[\text{any constant}] = O(1)$$

BIG-O EXAMPLES

lower order terms don’t matter!

$$\log_2 n + 15 = O(\log_2 n)$$

remember, big-O is upper bound!

$$3^n = O(4^n)$$

Polynomials

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constant multipliers & lower order terms don’t matter

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$$25 = O(1)$$

[any constant] = O(1)

BIG-Ω NOTATION

Let T(n) & f(n) be functions defined on the positive integers.

(In this class, we'll typically write T(n) to denote the worst case runtime of an algorithm)

What do we mean when we say "T(n) is Ω(f(n))"?

English Definition

Pictorial Definition

Mathematical Definition

BIG-Ω NOTATION

Let T(n) & f(n) be functions defined on the positive integers.

(In this class, we'll typically write T(n) to denote the worst case runtime of an algorithm)

What do we mean when we say "T(n) is Ω(f(n))"?

In English

T(n) = Ω(f(n)) if and only if T(n) is eventually **lower bounded** by a constant multiple of f(n)

Pictorial Definition

Mathematical Definition

BIG-Ω NOTATION

Let $T(n)$ & $f(n)$ be functions defined on the positive integers.

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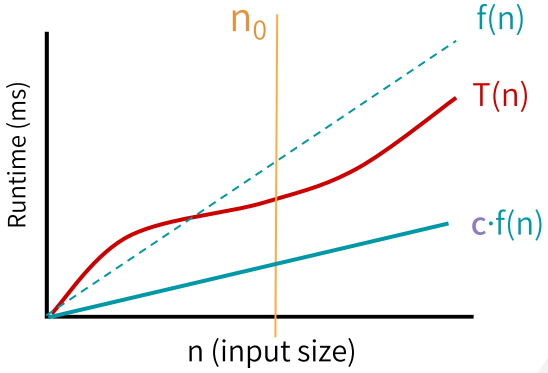

Mathematical Definition

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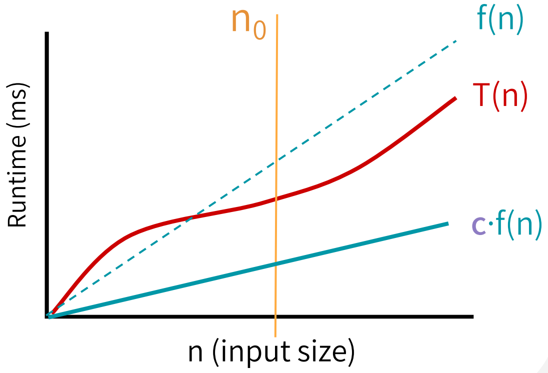


In *Math*

$$T(n) = \Omega(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$T(n) \geq c \cdot f(n)$$

inequality switched directions!

BIG-Θ NOTATION

We say **“T(n) is Θ(f(n))”** if and only if both

$$\mathbf{T(n) = O(f(n))}$$

and

$$\mathbf{T(n) = \Omega(f(n))}$$

$$T(n) = \Theta(f(n))$$

$$\Leftrightarrow$$

$$\exists\ c_1, c_2, n_0 > 0 \text{ s.t. } \forall\ n \geq n_0,$$

$$\mathbf{c_1 \cdot f(n) \leq T(n) \leq c_2 \cdot f(n)}$$

ASYMPTOTIC NOTATION CHEAT SHEET

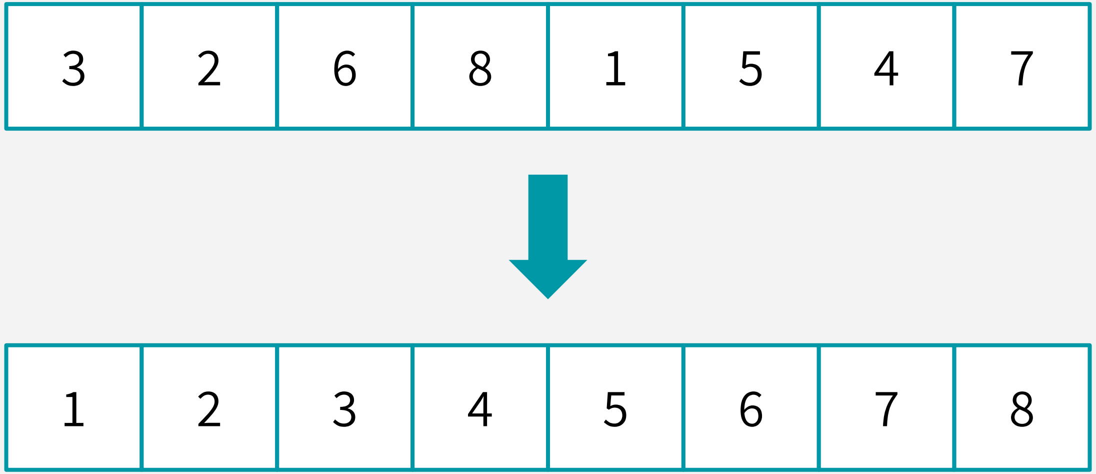
BOUND	DEFINITION (HOW TO PROVE)	WHAT IT REPRESENTS
$T(n) = O(f(n))$	$\exists\ c > 0,\ \exists\ n_0 > 0$ s.t. $\forall\ n \geq n_0,$ $T(n) \leq c \cdot f(n)$	upper bound
$T(n) = \Omega(f(n))$	$\exists\ c > 0,\ \exists\ n_0 > 0$ s.t. $\forall\ n \geq n_0,$ $T(n) \geq c \cdot f(n)$	lower bound
$T(n) = \Theta(f(n))$	$T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$	tight bound

INSERTION SORT

Algorithm, Proof of Correctness, Runtime

THE SORTING TASK

INPUT: a list of n elements (for today, we’ll assume all elements are distinct)



3	2	6	8	1	5	4	7

1	2	3	4	5	6	7	8

OUTPUT: a list with those n elements in sorted order!

INSERTION SORT: PSEUDOCODE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

```
InsertionSort(A):
  for i in range(1, len(A)):
        cur_value = A[i]
        j = i - 1
        while j >= 0 and A[j] >
cur_value:
      A[j+1] = A[j]
          j -= 1
        A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

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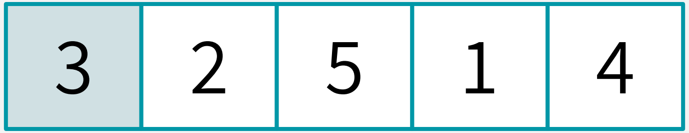

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INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

3	2	5	1	4

At the start, our growing sorted list only has one element (the first element):

3 is in its “correct” place within the growing list (shaded)

```
InsertionSort(A):
  for i in range(1, len(A)):
           cur_value = A[i]
           j = i - 1
           while j >= 0 and A[j] >
cur_value:
        A[j+1] = A[j]
             j -= 1
           A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

3	2	5	1	4

Now we look at A[1] = 2. We’ll move 2 into its “correct” place in the growing sorted list.

```
InsertionSort(A):
 for i in range(1, len(A)):
      cur_value = A[i]
      j = i - 1
      while j >= 0 and A[j] >
cur_value:
    A[j+1] = A[j]
       j -= 1
      A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

2	3	5	1	4

Now we look at A[1] = 2. We’ll move 2 into its “correct” place in the growing sorted list.

In other words, move 2 towards the start of the list until it hits something smaller (or if it can’t go any further).

```
InsertionSort(A):
 for i in range(1, len(A)):
      cur_value = A[i]
      j = i - 1
      while j >= 0 and A[j] >
cur_value:
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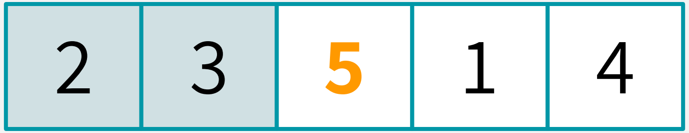

2	3	5	1	4

Now we look at A[2] = 5. We’ll move 5 into its “correct” place in the growing sorted list.

```
InsertionSort(A):
  for i in range(1, len(A)):
           cur_value = A[i]
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INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

2	3	5	1	4

Now we look at A[2] = 5. We’ll move 5 into its “correct” place in the growing sorted list.

It’s already where it should be in the growing sorted list, so we don’t need to move it anywhere. Moving on!

```
InsertionSort(A):
  for i in range(1, len(A)):
           cur_value = A[i]
           j = i - 1
           while j >= 0 and A[j] >
cur_value:
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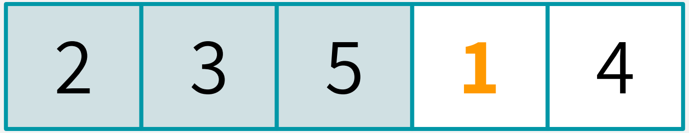

2	3	5	1	4

Now we look at A[3] = 1. We’ll move 1 into its “correct” place in the growing sorted list.

```
InsertionSort(A):
  for i in range(1, len(A)):
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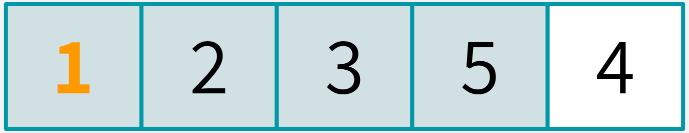

1	2	3	5	4

Now we look at A[3] = 1. We’ll move 1 into its “correct” place in the growing sorted list.
We move it all the way to the front, since that’s its “correct” position in this growing sorted list.

```
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           A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

1	2	3	5	4

Finally, we look at A[4] = 4. We’ll move 4 into its “correct” place in the growing sorted list.

```
InsertionSort(A):
 for i in range(1, len(A)):
     cur_value = A[i]
     j = i - 1
     while j >= 0 and A[j] >
cur_value:
    A[j+1] = A[j]
      j -= 1
     A[j+1] = cur_value
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INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

1	2	3	4	5

Finally, we look at A[4] = 4. We’ll move 4 into its “correct” place in the growing sorted list.

It just needs to squeeze in right before 5.

```
InsertionSort(A):
  for i in range(1, len(A)):
           cur_value = A[i]
           j = i - 1
           while j >= 0 and A[j] >
cur_value:
       A[j+1] = A[j]
             j -= 1
           A[j+1] = cur_value
```

INSERTION SORT: EXAMPLE

Intuition: Maintain a growing sorted list. For each element, put it into its “correct” place in this growing list.

1	2	3	4	5

And, that’s it! We’ve finished performing Insertion Sort on this example array of five elements.

Now we ask… does it work?

```
InsertionSort(A):
  for i in range(1, len(A)):
           cur_value = A[i]
           j = i - 1
           while j >= 0 and A[j] >
cur_value:
       A[j+1] = A[j]
             j -= 1
           A[j+1] = cur_value
```

INSERTION SORT: DOES IT WORK?

Since the algorithm isn't too complex, it might feel pretty obvious… but it won't be so obvious later, so let's take some time now to see how to prove the correctness of this algorithm rigorously..

3	2	5	1	4

We verified Insertion Sort worked for this particular input list.
However, we need to prove that the algorithm works for *all* possible input lists.

INSERTION SORT: DOES IT WORK?

Since the algorithm isn't too complex, it might feel pretty obvious… but it won't be so obvious later, so let's take some time now to see how to prove the correctness of this algorithm rigorously..

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We verified Insertion Sort worked for this particular input list.
However, we need to prove that the algorithm works for *all* possible input lists.

HERE'S WHAT WE FOCUS ON:

Insertion Sort is an *iterative* algorithm - **what does each iteration promise**?

INSERTION SORT: DOES IT WORK?

HERE’S WHAT WE FOCUS ON:

Each iteration of the algorithm promises to add one more element to the sorted region.

*In other words: by the end of iteration i, we’re guaranteed that the first **i+1** elements in the array are sorted.*

INSERTION SORT: DOES IT WORK?

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Each iteration of the algorithm promises to add one more element to the sorted region.

In other words: by the end of iteration i, we’re guaranteed that the first ***i+1*** *elements in the array are sorted.*

THIS IS A JOB FOR: ***PROOF BY INDUCTION!***

BUILDING AN INFINITE LADDER

You’re writing foolproof IKEA instructions
to build an infinite ladder

1. First, give instructions on how to build the first step of the ladder
2. Then, assuming we’ve built some step k, give instructions on how to build the next step (k+1)! This step may need to rely on the fact that step k is already built
3. Then, you can celebrate, knowing that your ladder can theoretically be built so that *for any (positive) value i, the i-th step exists!*

4 INGREDIENTS OF INDUCTION

INDUCTIVE HYPOTHESIS (IH)

This is a statement that's basically what you're trying to prove, except it's written in terms of some variable (e.g. **i**). We need to set up the inductive hypothesis clearly, and our goal in the next three steps is to prove that the IH holds for a whole *range* of values for **i**.

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BASE CASE

First establish that the inductive hypothesis holds for some base case value(s) of **i**.

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INDUCTIVE STEP *(weak induction version)*

Next, assume that the inductive hypothesis holds when **i** takes on some value **k**.
Now prove that the IH holds as well when **i** takes on the value **k+1**.

4 INGREDIENTS OF INDUCTION

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CONCLUSION

By induction, conclude that the IH holds across the range of **i** you're dealing with.

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BASE CASE

First establish that the inductive hypothesis holds for some base case value(s) of **i**.

INDUCTIVE STEP *(strong/complete induction version)*

Next, assume that the IH holds when **i** takes on any value *between [base case value(s)] and some number* **k**. Now prove that the IH holds as well when **i** takes on the value **k**.

CONCLUSION

By induction, conclude that the IH holds across the range of **i** you're dealing with.

INSERTION SORT: INDUCTION PROOF

INDUCTIVE HYPOTHESIS (IH)

After iteration i of the outer for-loop, A[:i+1] is sorted.

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INDUCTIVE STEP *(weak induction)*

Let k be an integer, where $0 < k < n$. Assume that the IH holds for $i = k-1$, so A[:k] is sorted after the (k-1)th iteration. We want to show that the IH holds for $i = k$, i.e. that A[:k+1] is sorted after the kth iteration.

Let j* be the largest position in {0, …, k-1} such that A[j*] < A[k]. Then, the effect of the inner while-loop is to turn:

[A[0], A[1], …, A[j*], …, A[k-1], **A[k]**] into [A[0], A[1], …, A[j*], **A[k]**, A[j*+1] …, A[k-1]]

We claim that the second list on the right is sorted. This is because A[k] > A[j*], and by the inductive hypothesis, we have A[j*] ≥ A[j] for all j ≤ j*, so A[k] is larger than everything positioned before it. Similarly, we also know that A[k] ≤ A[j*+1] ≤ A[j] for all j ≥ j*+1, so A[k] is also smaller than everything that comes after it. Thus, A[k] is in the right place, and all the other elements in A[:k+1] were already in the right place.

Thus, after the kth iteration completes, A[:k+1] is sorted, and this establishes the IH for k.

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[A[0], A[1],

We claim t s, we
have A[j*] ≥ [k] ≤
A[j*+1] ≤ A[lace, and
all the othe

TLDR, this inductive step is saying "if we assume the growing list on the left of A is properly sorted by iteration k-1, then when we're on iteration k, the algorithm correctly moves A[k] into the right place, and the growing list on the left of A is still going to be properly sorted."

Thus, after the kth iteration completes, A[:k+1] is sorted, and this establishes the IH for k.

CONCLUSION

By induction, we conclude that the IH holds for all 0 ≤ i ≤ n-1. In particular, after the algorithm ends, A[:n] is sorted.

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By induction, we conclude that the IH holds for all 0 ≤ i ≤ n-1. In particular, after the algorithm ends, A[:n] is sorted.

A NOTE ABOUT INDUCTION

We're going to be seeing/doing/skipping over a lot of induction proofs this quarter. I'm technically supposed to assume you're comfortable with them from CS 103 (one of the prereqs), but if any of this was too fast or confusing, ***come to section*** & ***OH***!

INSERTION SORT: DOES IT WORK?

We just used induction to prove that the Insertion Sort algorithm correctly produces a sorted array given *any input array of length n*.

(This is also what we mean by worst case analysis - even if a “bad guy” comes up with a worst-case input for our algorithm, we’ve proven that our algorithm will work).

INSERTION SORT: IS IT FAST?

FROM MONDAY!

THE POINT OF ASYMPTOTIC NOTATION

$$\textbf{suppress } \underbrace{\textbf{constant factors}}_{\textit{too system dependent}} \textbf{ and } \underbrace{\textbf{lower-order terms}}_{\textit{irrelevant for large inputs}}$$

- **Some guiding principles:** we care about how the running time/number of operations *scales* with the size of the input (i.e. the runtime's *rate of growth*), and we want some measure of runtime that's independent of hardware, programming language, memory layout, etc.
  - We want to reason about high-level algorithmic approaches rather than lower-level details

INSERTION SORT: IS IT FAST?

Instead of counting every little operation, we can think about:

How many iterations take place

How much work happens within each iteration

```
InsertionSort(A):
   for i in range(1, len(A)):
          cur_value = A[i]
          j = i - 1
          while j >= 0 and A[j] >
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         A[j+1] = A[j]
             j -= 1
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At most n
outer for-loop
iterations

At most n
inner while-loop
iterations

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          A[j+1] = cur_value
```

At most n
outer for-loop
iterations

At most n
inner while-loop
iterations

We have ~n for-loop iterations. Each iteration does O(n) work.

(Each for-loop iteration performs an inner-while loop which iterates up to n times and does O(1) work in each iteration).

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Instead of counting every little operation, we can think about:

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How much work happens within each iteration

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At most n
outer for-loop
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At most n
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OVERALL RUNTIME OF INSERTION SORT: $O(n^2)$

INSERTION SORT: IS IT FAST?

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How m **place**

How much **h iteration**

```
Insertio
    for i

cur_value
        A[
        j
    A[j+1] = cur_value
```

At most n
outer for-loop
iterations

At most n
inner while-loop
iterations

THE QUESTION IS...

CAN WE DO BETTE R?

OVERALL RUNTIME OF INSERTION SORT: $O(n^2)$

5-MINUTE BREAK

Stay hydrated, stretch, ask questions, etc.

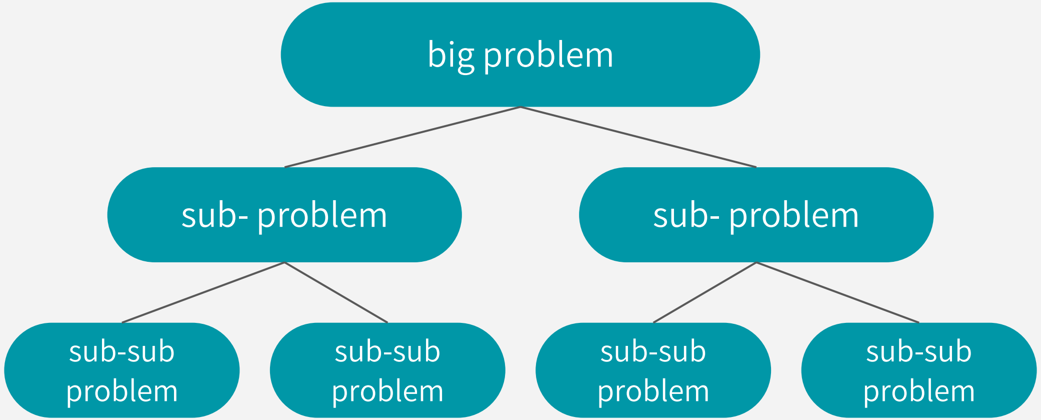
MERGESORT

Algorithm, Proof of Correctness, Runtime

MERGESORT

FROM MONDAY!

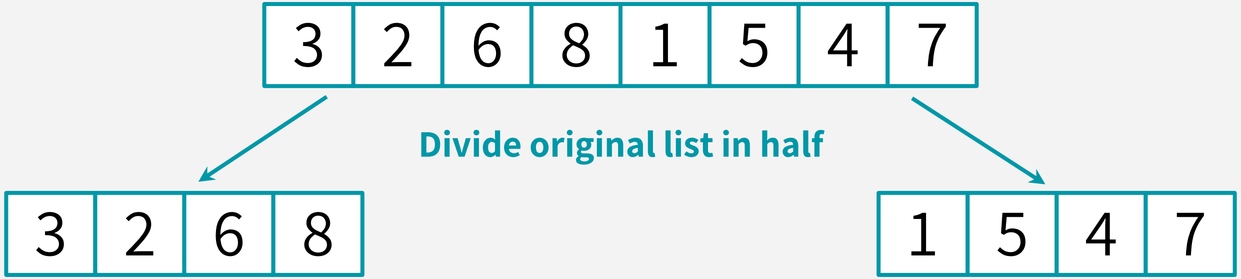
- **DIVIDE-AND-CONQUER: an algorithm design paradigm**
  1. break up a problem into smaller subproblems
  2. solve those subproblems *recursively*
  3. combine the results of those subproblems to get the overall answer



MERGESORT

3	2	6	8	1	5	4	7

MERGESORT



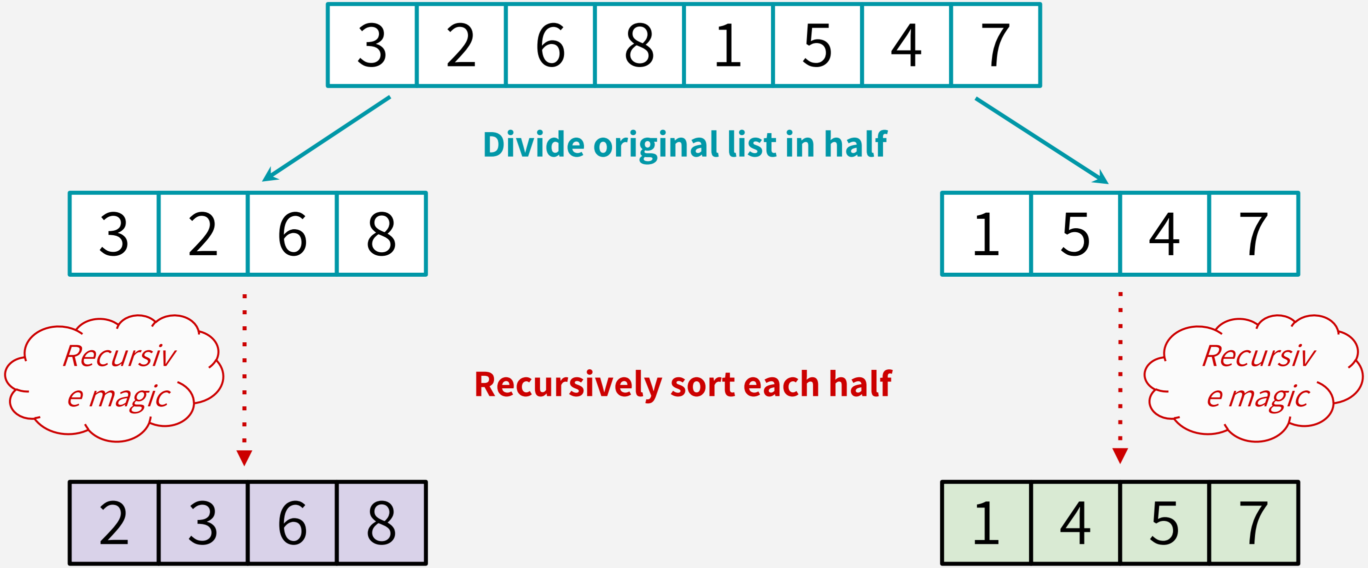
3	2	6	8	1	5	4	7

Divide original list in half

3	2	6	8

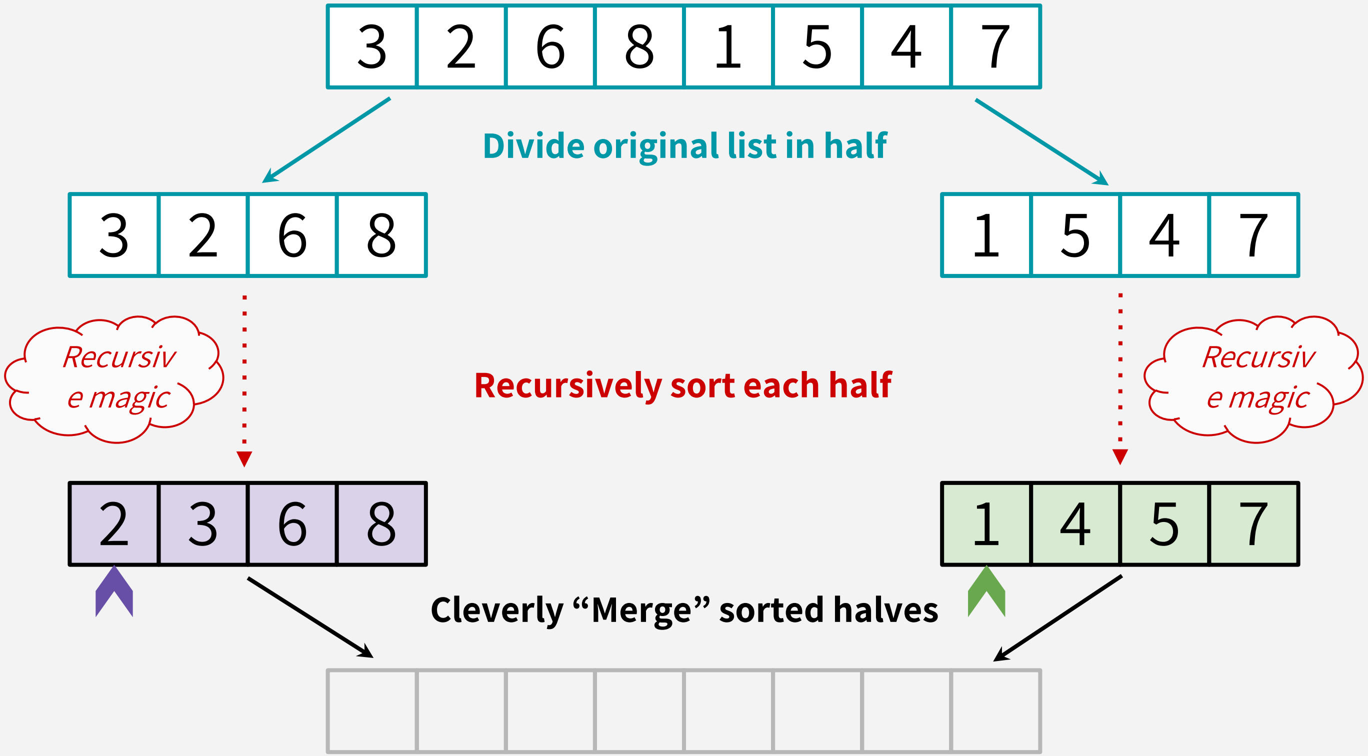
1	5	4	7

MERGESORT



3 2 6 8 1 5 4 7
Divide original list in half
3 2 6 8
1 5 4 7
Recursiv
e magic
Recursively sort each half
Recursiv
e magic
2 3 6 8
1 4 5 7

MERGESORT



3	2	6	8	1	5	4	7

Divide original list in half

3	2	6	8

1	5	4	7

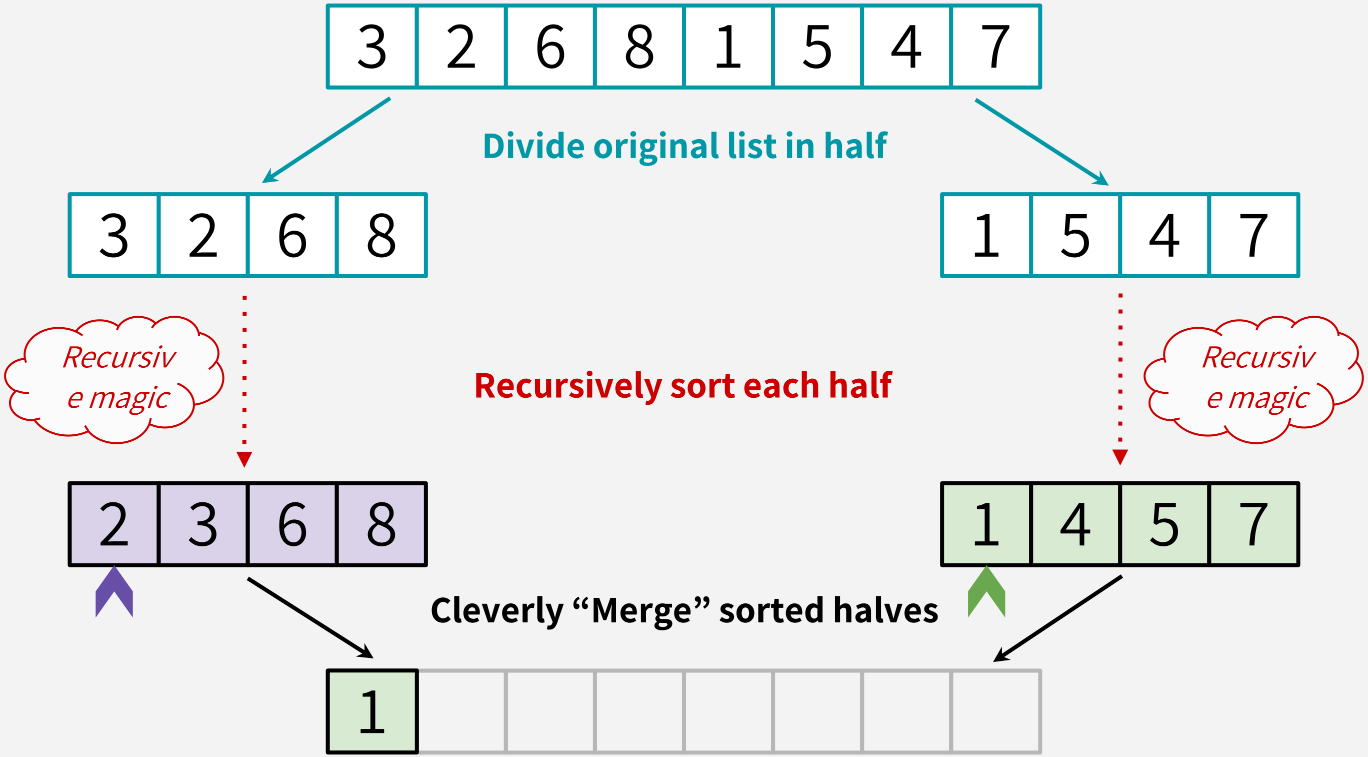
Recursively sort each half

2	3	6	8

1	4	5	7

Cleverly “Merge” sorted halves

MERGESORT



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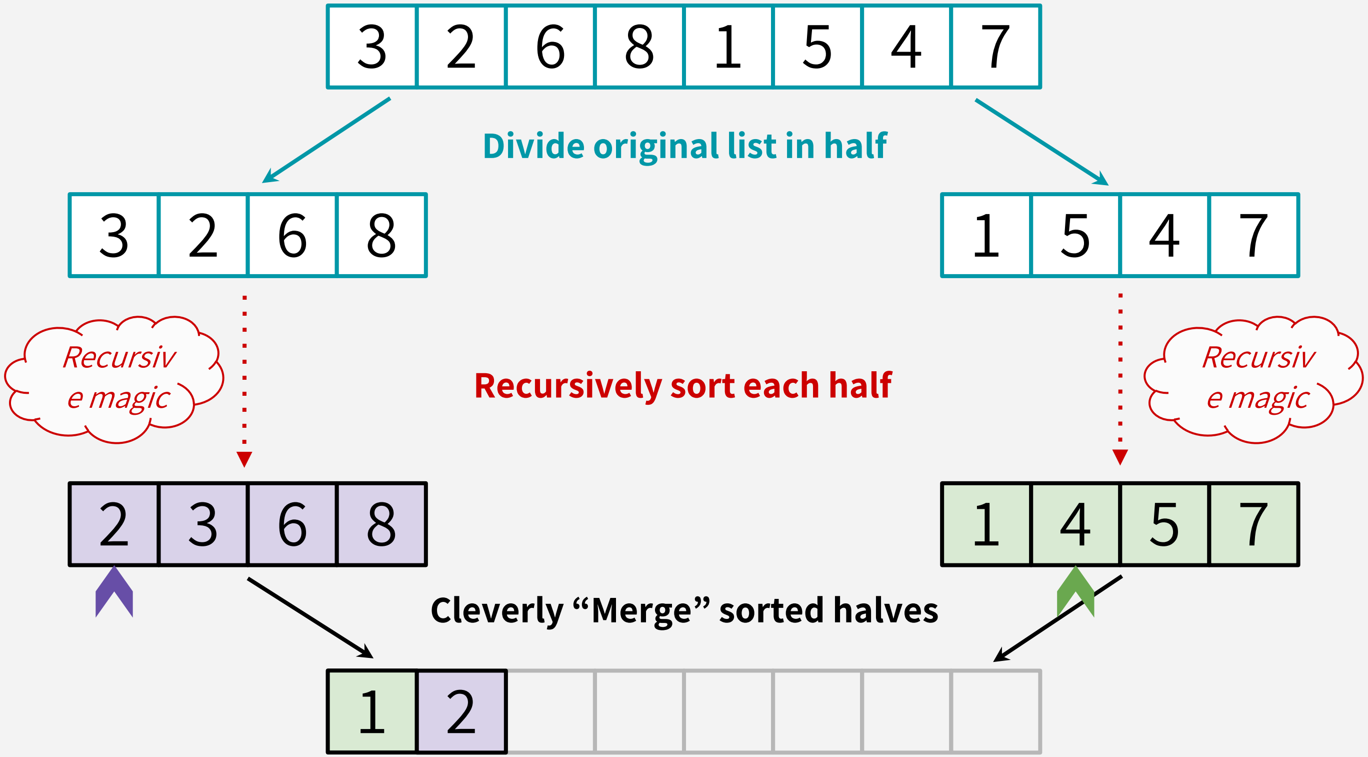
Recursively sort each half

Cleverly “Merge” sorted halves

MERGESORT



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Divide original list in half

3 2 6 8

1 5 4 7

Recursively sort each half

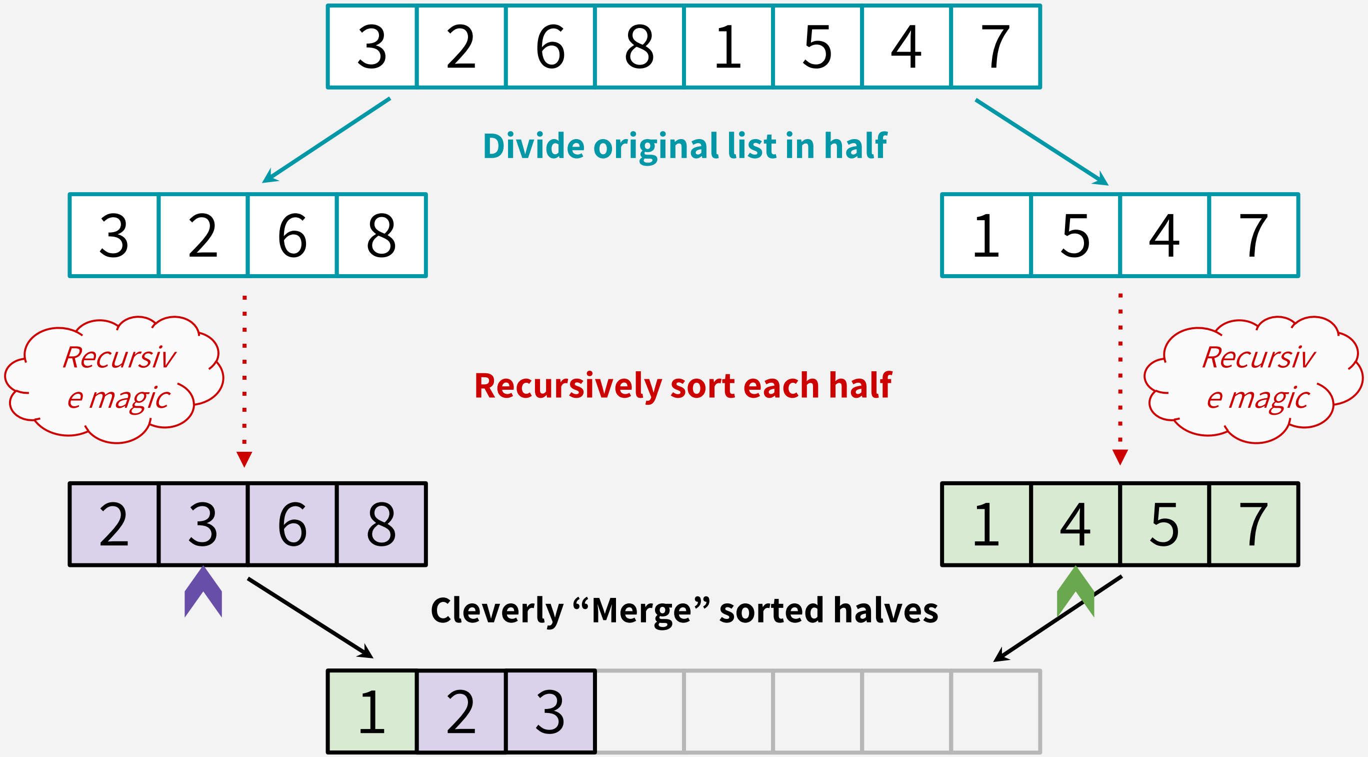
2 3 6 8

1 4 5 7

Cleverly “Merge” sorted halves

1 2

MERGESORT

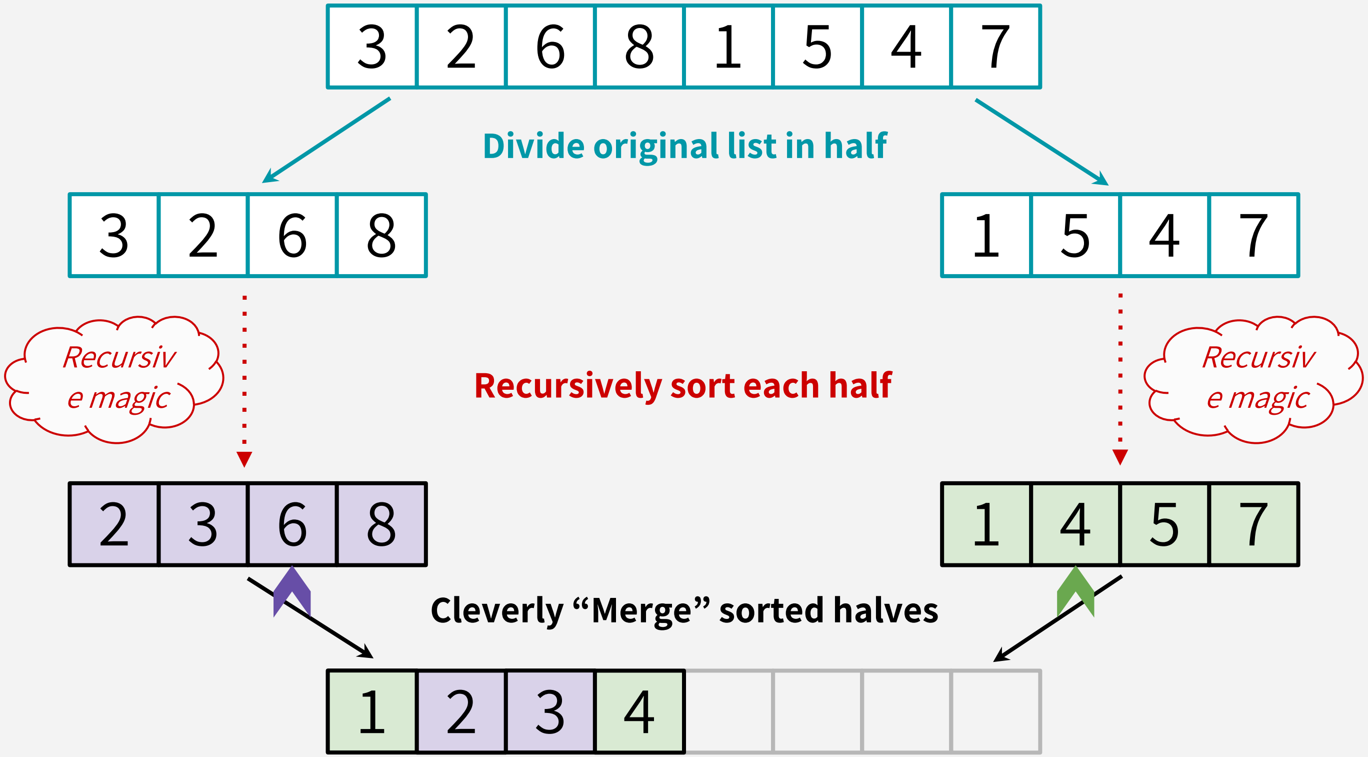


Divide original list in half

Recursively sort each half

Cleverly “Merge” sorted halves

MERGESORT



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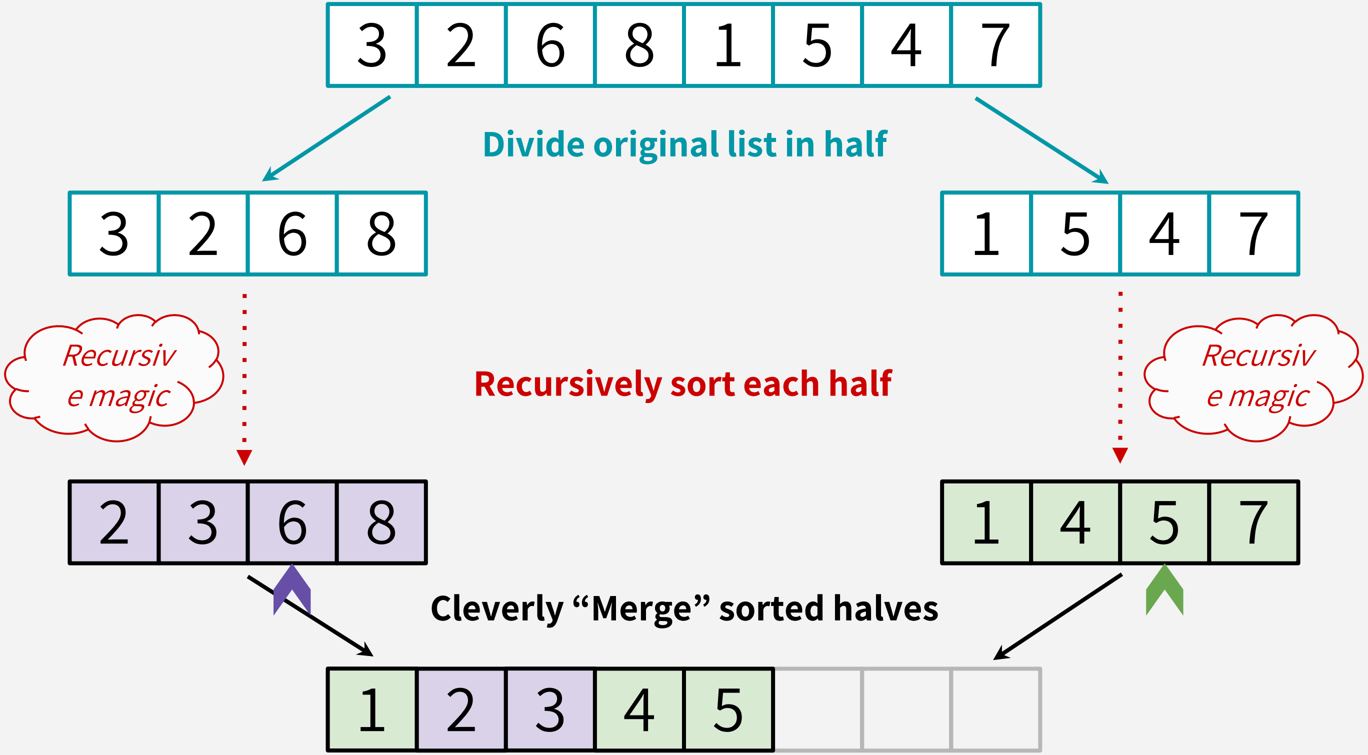
MERGESORT



3	2	6	8	1	5	4	7

Divide original list in half

3	2	6	8

1	5	4	7

Recursively sort each half

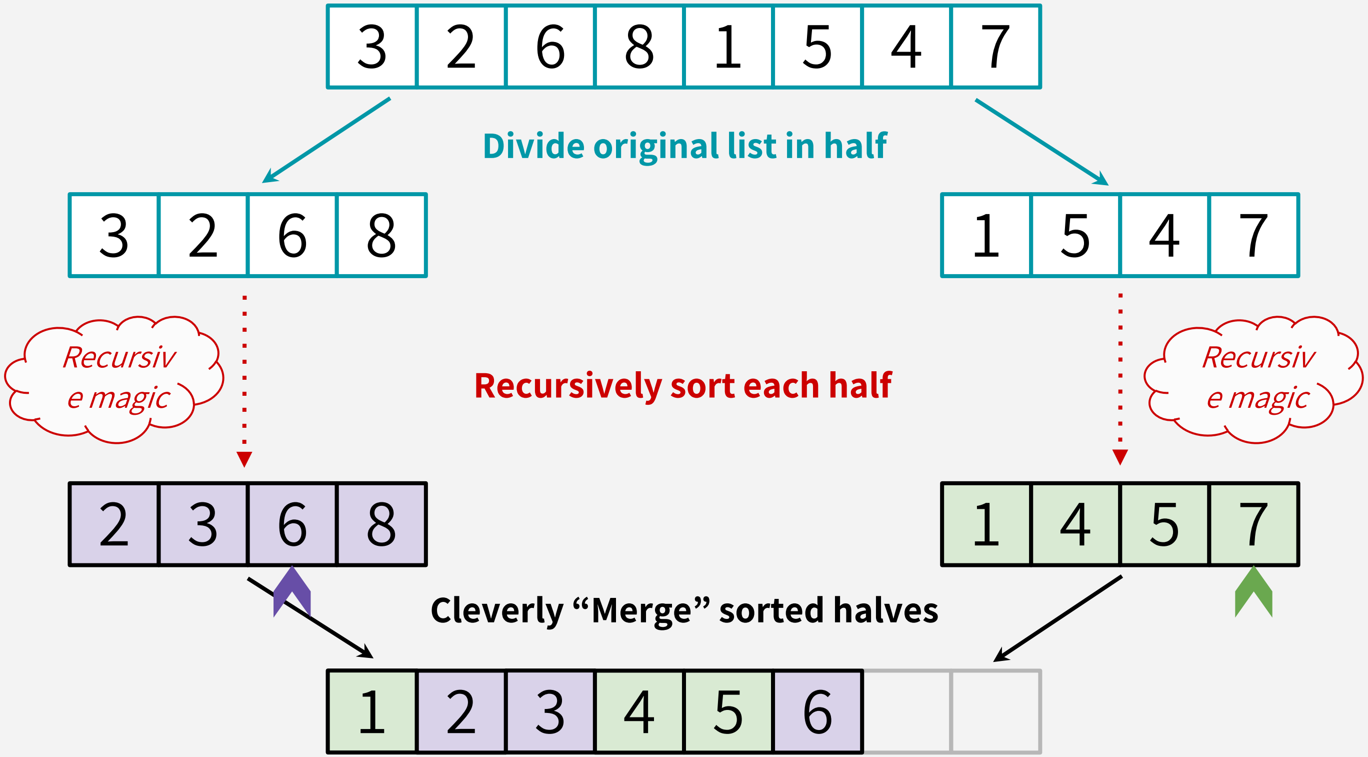
2	3	6	8

1	4	5	7

Cleverly "Merge" sorted halves

1	2	3	4	5			

MERGESORT

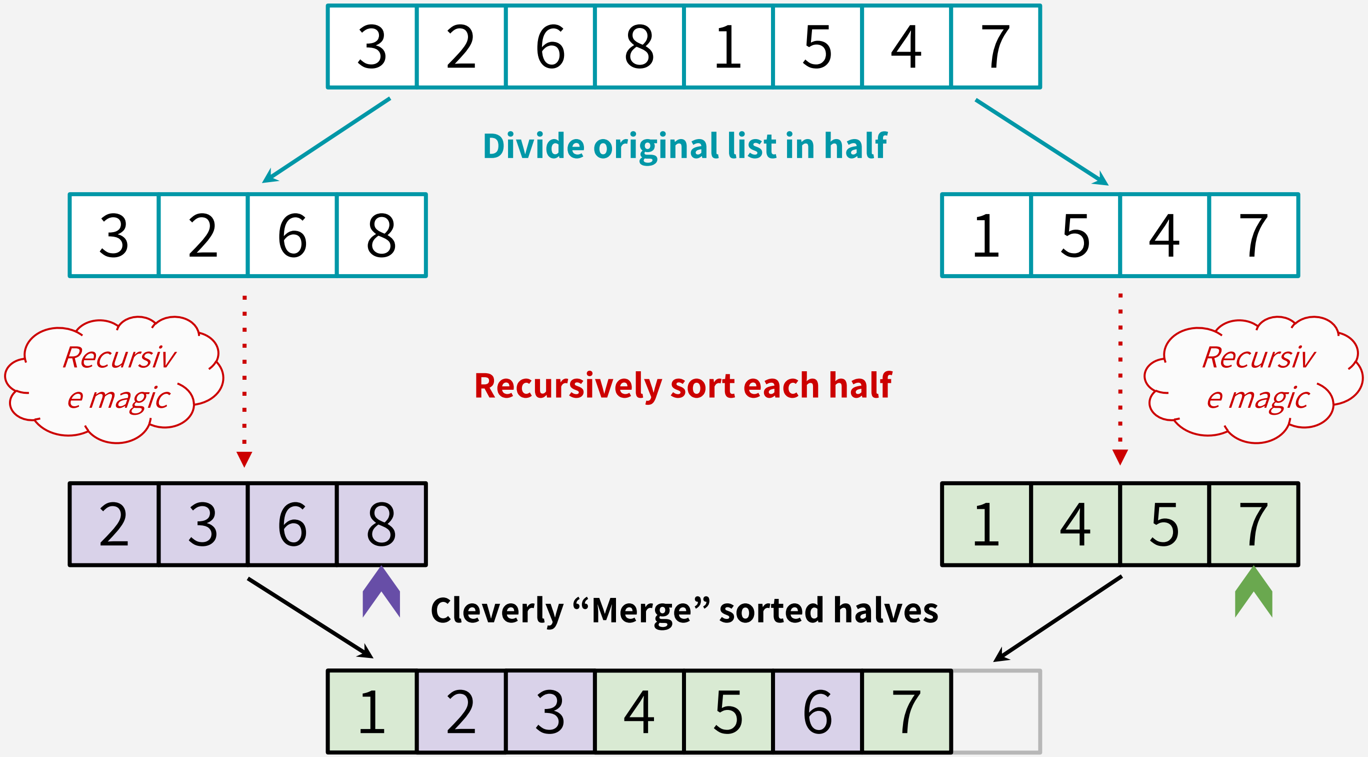


Divide original list in half

Recursively sort each half

Cleverly "Merge" sorted halves

MERGESORT



3	2	6	8	1	5	4	7

Divide original list in half

3	2	6	8

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Recursively sort each half

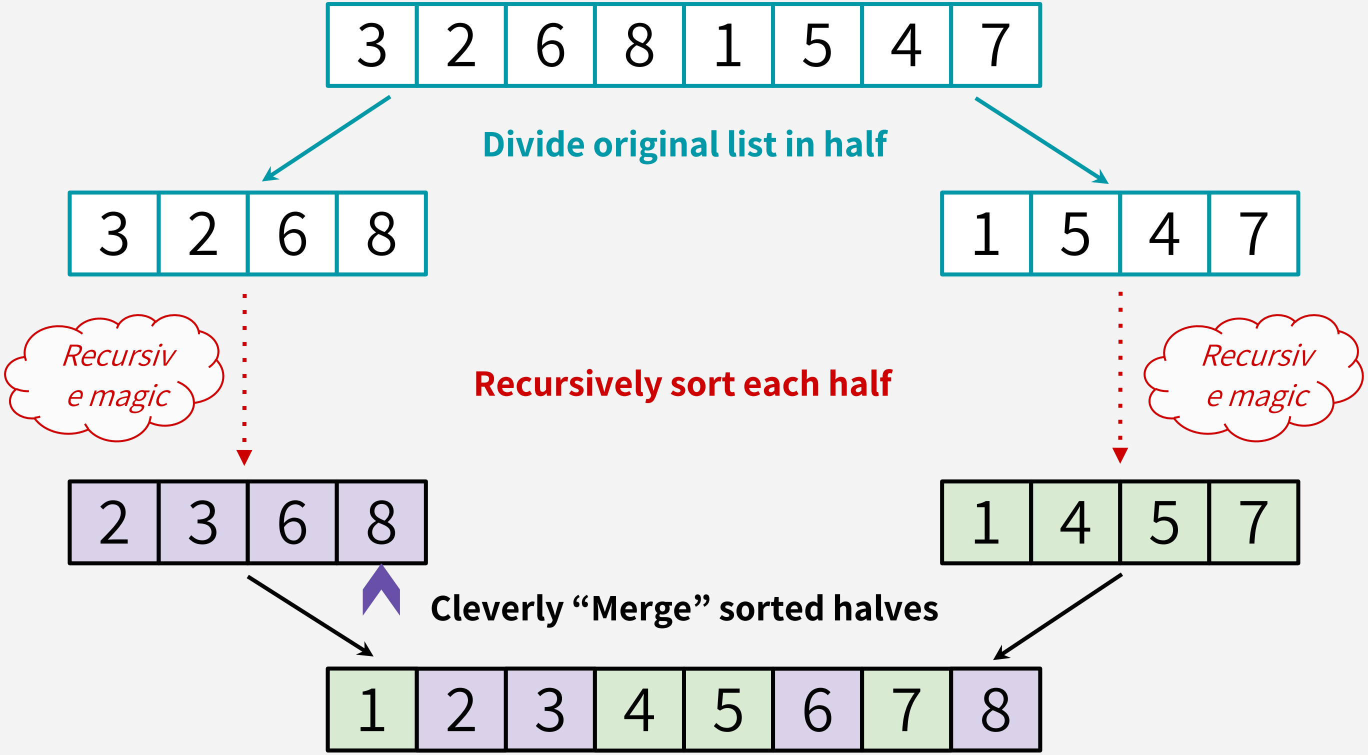
2	3	6	8

1	4	5	7

Cleverly “Merge” sorted halves

1	2	3	4	5	6	7	

MERGESORT



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Divide original list in half
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Recursiv e magic
Recursively sort each half
Recursiv e magic
2 3 6 8
1 4 5 7
Cleverly "Merge" sorted halves
1 2 3 4 5 6 7 8

MERGESORT: PSEUDOCODE

Intuition: Divide and Conquer. If you sort your left and right halves, it’s easier to “Merge” them into a sorted list.

```
MERGESORT(A):
```

MERGESORT: PSEUDOCODE

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```
MERGESORT(A):
   n = len(A)
   if n <= 1:
      return A
```

MERGESORT: PSEUDOCODE

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```
MERGESORT(A):
    n = len(A)
    if n <= 1:
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    L = MERGESORT(A[0:n/2])
```

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```

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    R = MERGESORT(A[n/2:n])
    return MERGE(L,R)
```

For today, let's assume that n is a power of 2.

MERGESORT: PSEUDOCODE

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    if n <= 1:
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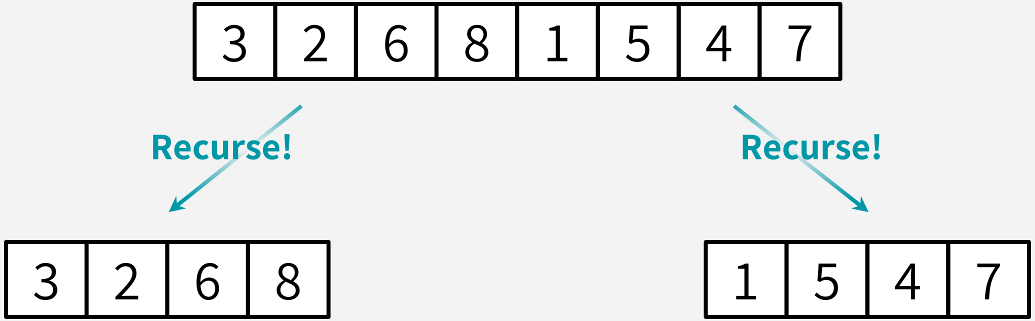
    R = MERGESORT(A[n/2:n])
    return MERGE(L,R)
```

```
MERGE(L,R):
    result = length n array
    i = 0, j = 0
    for k in [0,...,n-1]:
        if L[i] < R[j]:
            result[k] = L[i]
            i += 1
        else:
            result[k] = R[j]
            j += 1
    return result
```

MERGESORT: RECURSIVE CALLS

3	2	6	8	1	5	4	7

MERGESORT: RECURSIVE CALLS



3	2	6	8	1	5	4	7

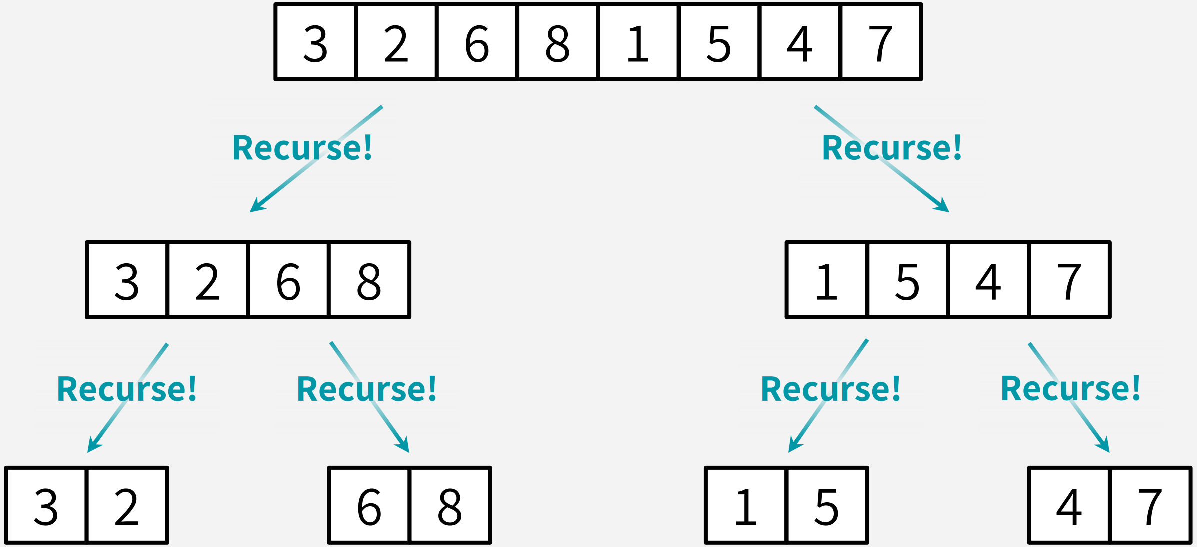
Recurse!

Recurse!

3	2	6	8

1	5	4	7

MERGESORT: RECURSIVE CALLS



3 2 6 8 1 5 4 7

Recurse!

Recurse!

3 2 6 8

1 5 4 7

Recurse!

Recurse!

Recurse!

Recurse!

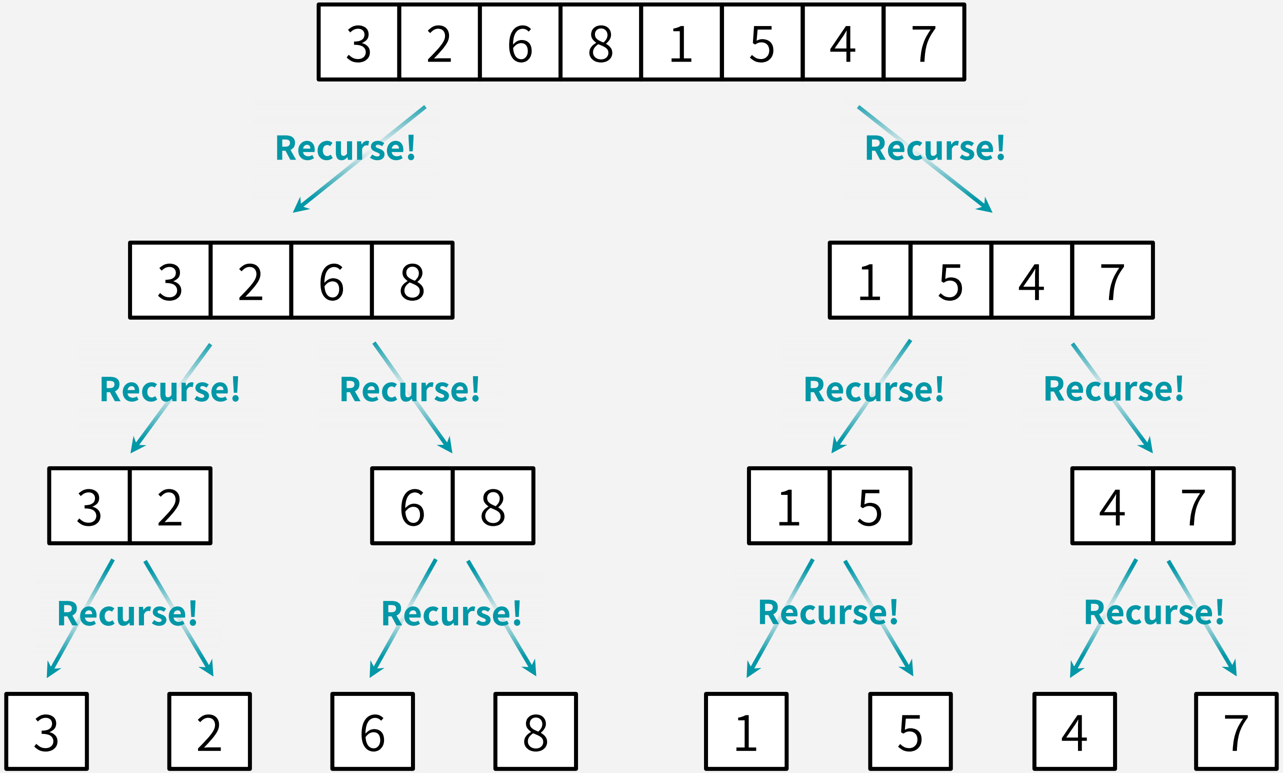
3 2

6 8

1 5

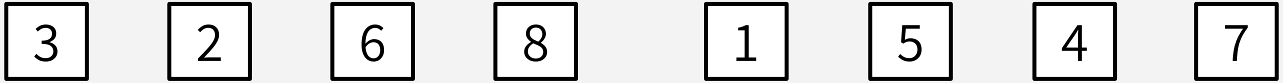
4 7

MERGESORT: RECURSIVE CALLS



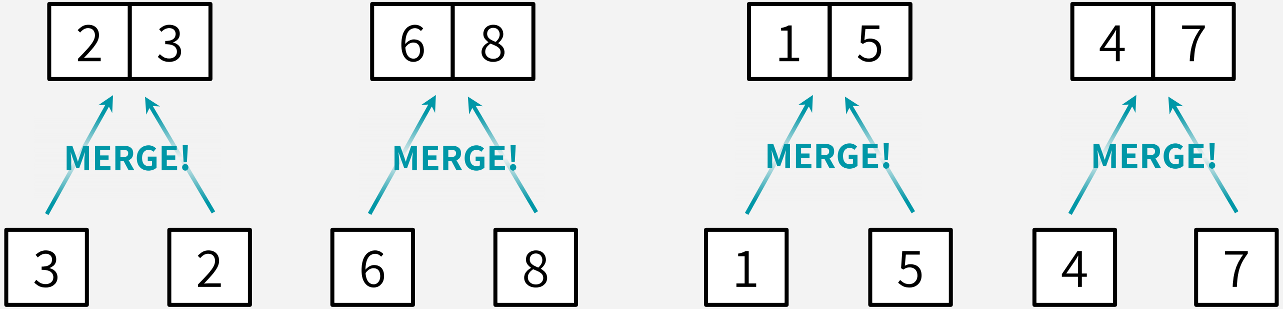
This is where we hit our base case!

MERGESORT: MERGE STEPS

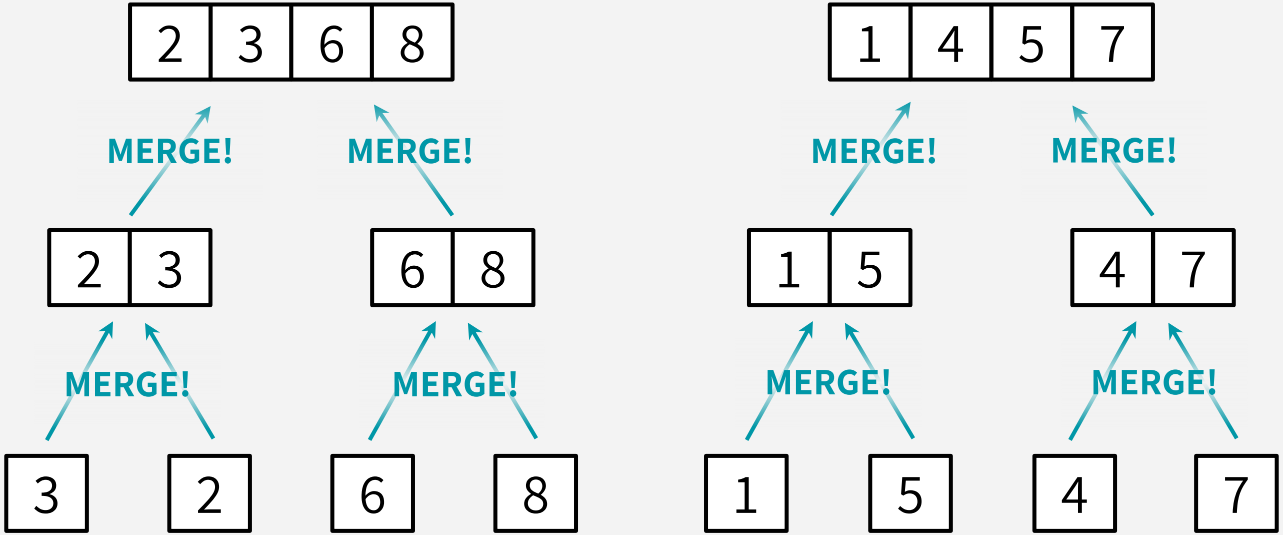


3 2 6 8 1 5 4 7

MERGESORT: MERGE STEPS



MERGESORT: MERGE STEPS



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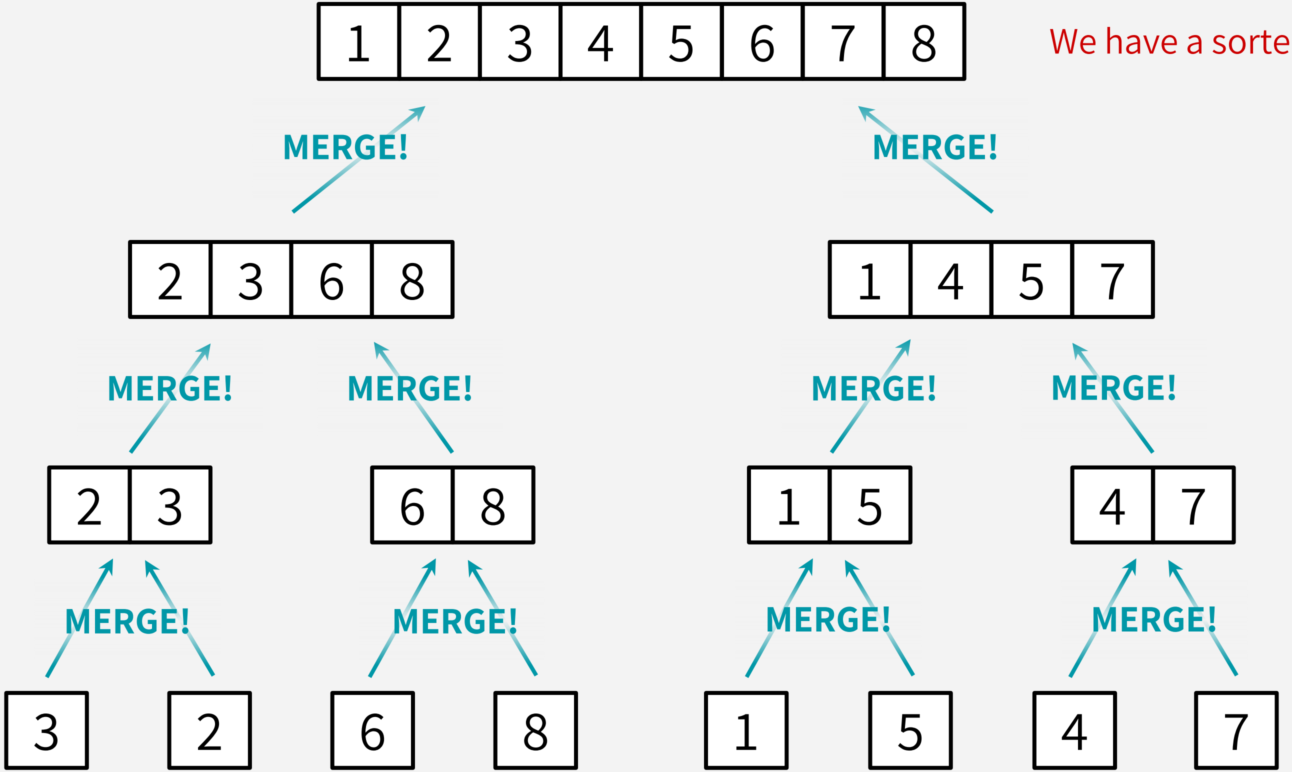

We have a sorted sequence!

MERGESORT: DOES IT WORK?

HERE'S WHAT WE FOCUS ON:

Whenever we make two "child" recursive calls, as long as those calls successfully sort our left and right halves, we'll safely merge them to create a fully sorted array.

In other words: as long as our recursive calls work on arrays of <u>smaller</u> lengths, then our algorithm will correctly return a sorted array.

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THIS IS A JOB FOR: ***PROOF BY INDUCTION!***

(This time, we perform induction on the *length of input list*, rather than # of iterations)

MERGESORT: INDUCTION PROOF

INDUCTIVE HYPOTHESIS (IH)

In every recursive call on an array of length *at most* **i**, MERGESORT returns a sorted array.

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INDUCTIVE STEP *(strong/complete induction)*

Let k be an integer, where $1 < k \leq n$. Assume that the IH holds for $i < k$, so MERGESORT correctly returns a sorted array when called on arrays of length less than k. We want to show that the IH holds for $i = k$, i.e. that MERGESORT returns a sorted array when called on an array of length k.

[INSERT INDUCTION PROOF TO PROVE THE MERGE SUBROUTINE IS CORRECT WHEN GIVEN TWO SORTED ARRAYS]

Try out this inner proof on your own!

Since the two "child" recursive calls are executed on arrays of length k/2 (which is strictly less than k), then our inductive hypothesis tells us that MERGESORT will correctly sort the left and right halves of our length-k array. Then, since the MERGE subroutine is correct when given two sorted arrays, we know that MERGESORT will ultimately return a fully sorted array of length k.

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Try out this inner proof on your own!

CONCLUSION

By induction, we conclude that the IH holds for all 1 ≤ i ≤ n. In particular, it holds for i = n, so in the top recursive call, MERGESORT returns a sorted array.

PROVE CORRECTNESS w/ INDUCTION

ITERATIVE ALGORITHMS

RECURSIVE ALGORITHMS

PROVE CORRECTNESS w/ INDUCTION

ITERATIVE ALGORITHMS

1. **Inductive hypothesis**: some state/condition will always hold throughout your algorithm by any iteration **i**
2. **Base case**: show IH holds for iteration 0 (i.e. start of algorithm)
3. **Inductive step**: Assume IH holds for k ⇒ prove k+1
4. **Conclusion**: IH holds for i = # total iterations ⇒ yay!

RECURSIVE ALGORITHMS

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RECURSIVE ALGORITHMS

1. **Inductive hypothesis**: your algorithm is correct for sizes *up to* **i**
2. **Base case**: IH holds for i < small const.
3. **Inductive step**:
   - ○ assume IH holds for k ⇒ prove k+1, *OR*
   - ○ assume IH holds for {1,2,...,k-1} ⇒ prove k (*it's not important that I chose k instead of k+1, using k is can just be syntactically cleaner!)
4. **Conclusion**: IH holds for i = n ⇒ yay!

MERGESORT: IS IT FAST?

```
MERGESORT(A):
    n = len(A)
    if n <= 1:
        return A
    L = MERGESORT(A[0:n/2])
    R = MERGESORT(A[n/2:n])
    return MERGE(L,R)
```

CLAIM: MergeSort runs in time **O(n log n)**

AN ASIDE: $O(n \log n)$ vs. $O(n^2)$?

log(n) grows very slowly! (Much more slowly than n)

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ALL LOGARITHMS IN THIS COURSE ARE BASE 2

log(2) = 1

log(4) = 2

…

log(64) = 6

log (128) = 7

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log(4096) = 12

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log(**# particles in the universe**) < 280

AN ASIDE: O(n log n) vs. O(n^2)?

log(n) grows very slowly! (Much more slowly than n)

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Logs are slow!

In fact,

log n = O(n^d)

for any d > 0

n log n grows much more slowly than n^2

Punchline: A running time of O(n log n) is a LOT better than O(n^2)

MERGESORT: O(n log n) PROOF

Instead of counting every little operation and tracing all recursive calls, we can think about:
THE RECURSION TREE!
(and we'll add up all the work done across levels to compute the Big-O runtime)

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```

```
MERGE(L,R):
   result = length n array
   i = 0, j = 0
   for k in [0,...,n-1]:
            if L[i] <
R[j]:
                result[k]
= L[i]
                i += 1
            else:
           result[k] =
R[j]
           j += 1
   return result
```

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        else:
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    return result
```

n iterations, O(1) work per iteration

We can see that MERGE is **O(n)**

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    result = length n array
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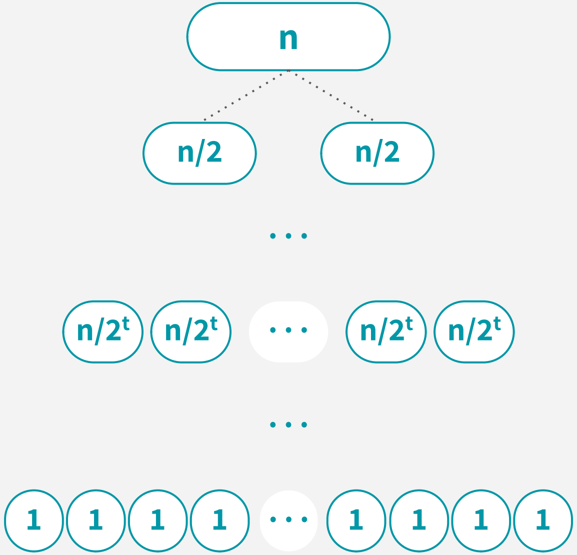
R[j]
                    j += 1
    return result
```

n i
C
i

This means that within one recursive call that processes an array/subarray of length ***n***, the work done in that subproblem (creating subproblems & "merging" those results) is **O(n)**.

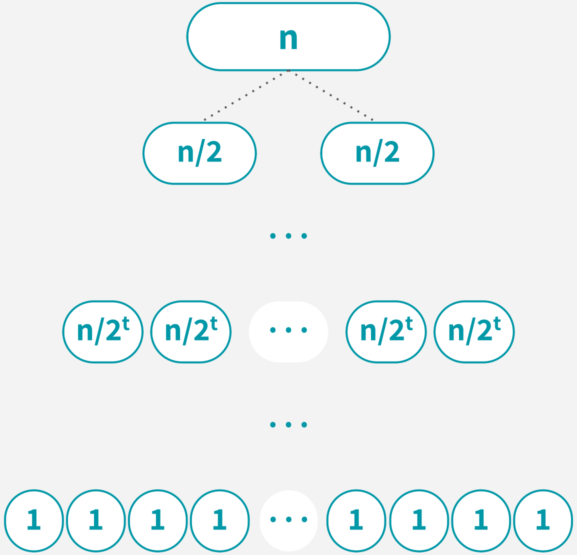
We can see that MERGE is **O(n)**

MERGESORT RECURSION TREE



Level	# of Problems	Size of each Problem	Work done per Problem ≤	Total work on this level
0				
1				
...				
t				
...				
$\log_2 n$				

MERGESORT RECURSION TREE

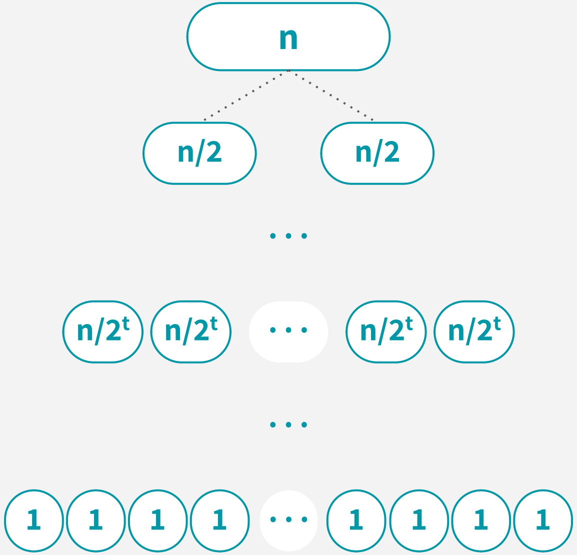


Level	# of Problems	Size of each Problem	Work done per Problem ≤	Total work on this level
0	1	n		
1	2^1	n/2		
...				
t	2^t	$n/2^t$		
...				
$\log_2 n$	$2^{\log_2 n} = n$	1		

MERGESORT RECURSION TREE

If a subproblem is of size **n**, then the work done in that subproblem is **O(n)**.

⇒ **Work ≤ c · n** (c is a constant)



Level	# of Problems	Size of each Problem	Work done per Problem ≤	Total work on this level
0	1	n	c · n	
1	2^1	n/2	c · (n/2)	
…				
t	2^t	$n/2^t$	$c \cdot (n/2^t)$	
…				
$\log_2 n$	$2^{\log_2 n} = n$	1	c · (1)	

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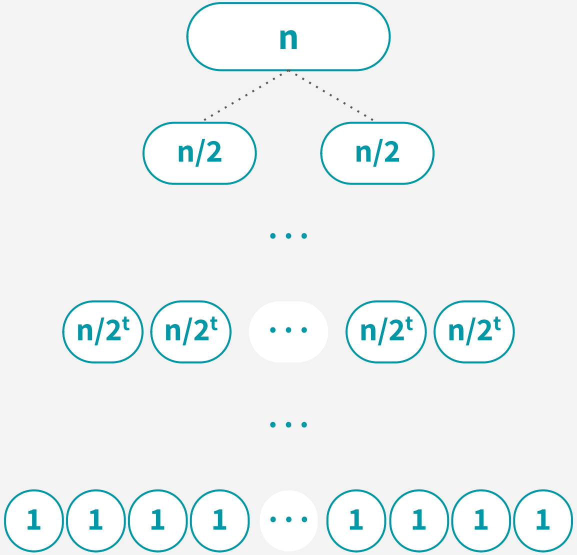

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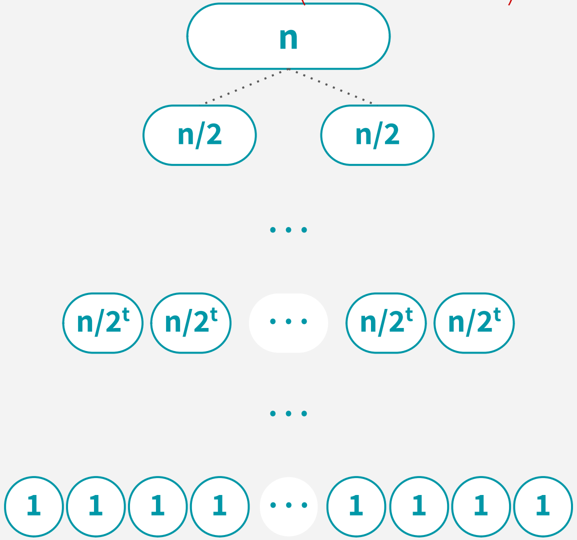

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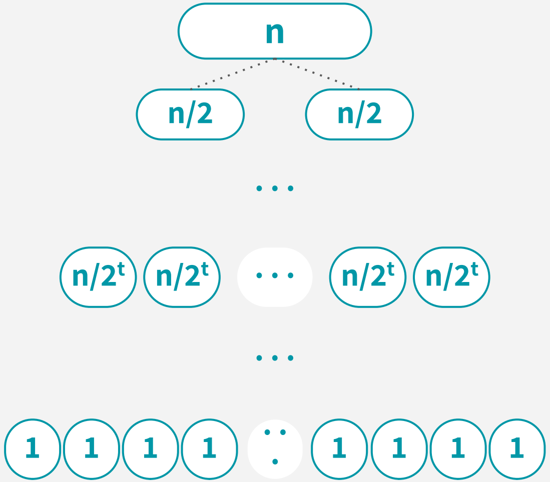
$\log_2 n$	$2^{\log_2 n} = n$	1	c · (1)	n · c · (1) = **O(n)**

We have ($\log_2 n$ + 1) levels, each level has O(n) work total ⇒ **O(n log n)** work overall!

MERGESORT: O(n log n) RUNTIME

Using the "Recursion Tree Method" (i.e. drawing the tree & filling out the table), we showed that the runtime of MergeSort is **O(n log n)**



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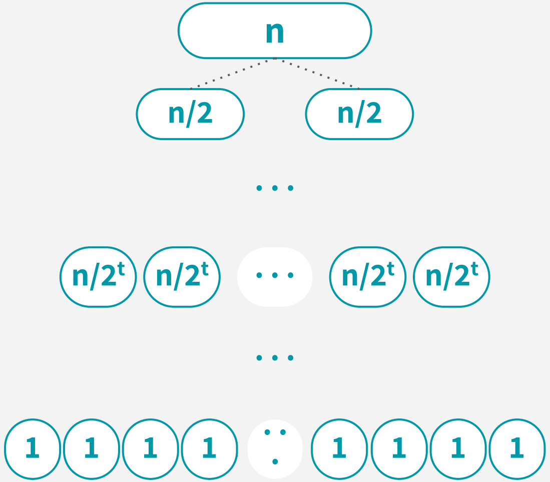

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RECAP

- Concept Check 1 is due Friday, & OH/Sections are happening today-Friday (see Calendar & course website)
- We learned about **Insertion Sort** (an iterative O(n^2) sorting algorithm)
- We learned about **MergeSort** (a divide & conquer O(n log n) sorting algorithm)
  - More practice with recursion trees!
- We proved the correctness of both Insertion Sort & MergeSort using induction!

NEXT TIME

- Recurrence relations:
    - The ~Master Theorem~ and the Substitution method!

Drop by Nooks today & tomorrow!

Don’t forget to join
Ed, Nooks, and Gradescope,
& read Homework Policies!