Design & Analysis of Algorithms

CS11313 - Fall 2020

Day # 2 and 3

The Big-*O* Notation and its Relatives

Today's Agenda

- Running Time Orders of Growth.
- A formal definition of Big-O
- Big-O Relatives

Orders of Growth (Review)

- Order of Growth of the running time: How quickly the running time of an algorithm grows as the input size grows.
  Examples: $\log n, \quad n, \quad n^2, \quad n^3, \quad 2^n,$ etc.

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! Assumption is good for theoretical study of algorithms, not fo practical comparison between algorithms (*stay tuned*).

Quiz # 1

Assume $T(n)$ is the order of growth of the running time of Bubble Sort as a function of the input size n. Which of the following is *true* about $T(n)$?

- **A.** $T(n) = O(n^2)$
- **B.** $T(n) = O(n^3)$
- **C.** $T(n) = O(2^n)$
- **D.** All of the above.
- **E.** None of the above.

What is

Big-*O* anyway?

Big-O

Definition. Let $f(n)$ and $g(n)$ be two functions that are always positive, $f(n)$ is said to be $O(g)$ if and only if :

> There are two constants c and n_o , such that
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Less formally: If multiplying $g(n)$ by a constant makes it an upper bound for $f(n)$ after some point, then f is $O(g)$.

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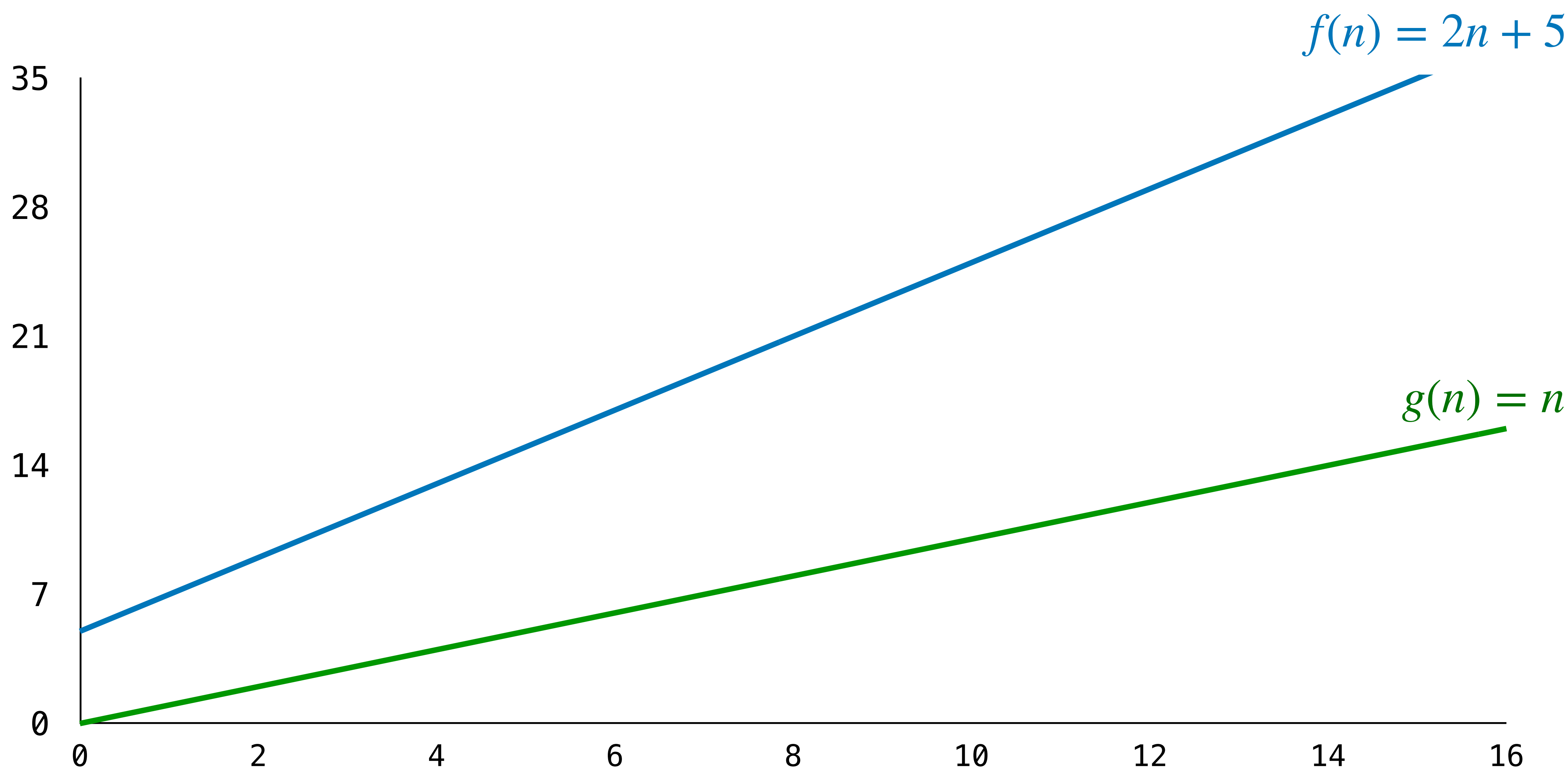
If $c = 3$, then $3 \bullet g(n) \geq f(n)$ for all $n \geq 5$



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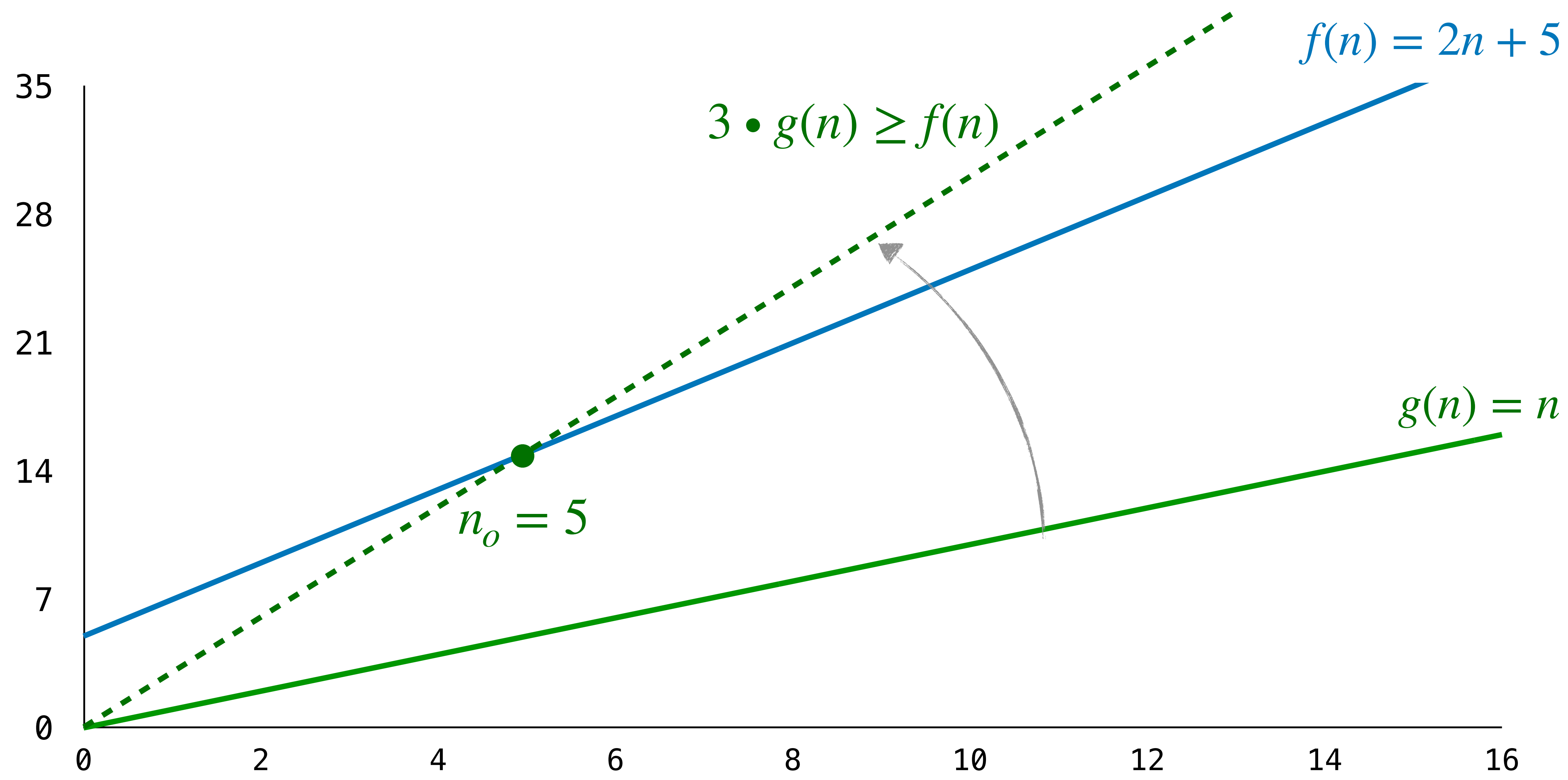

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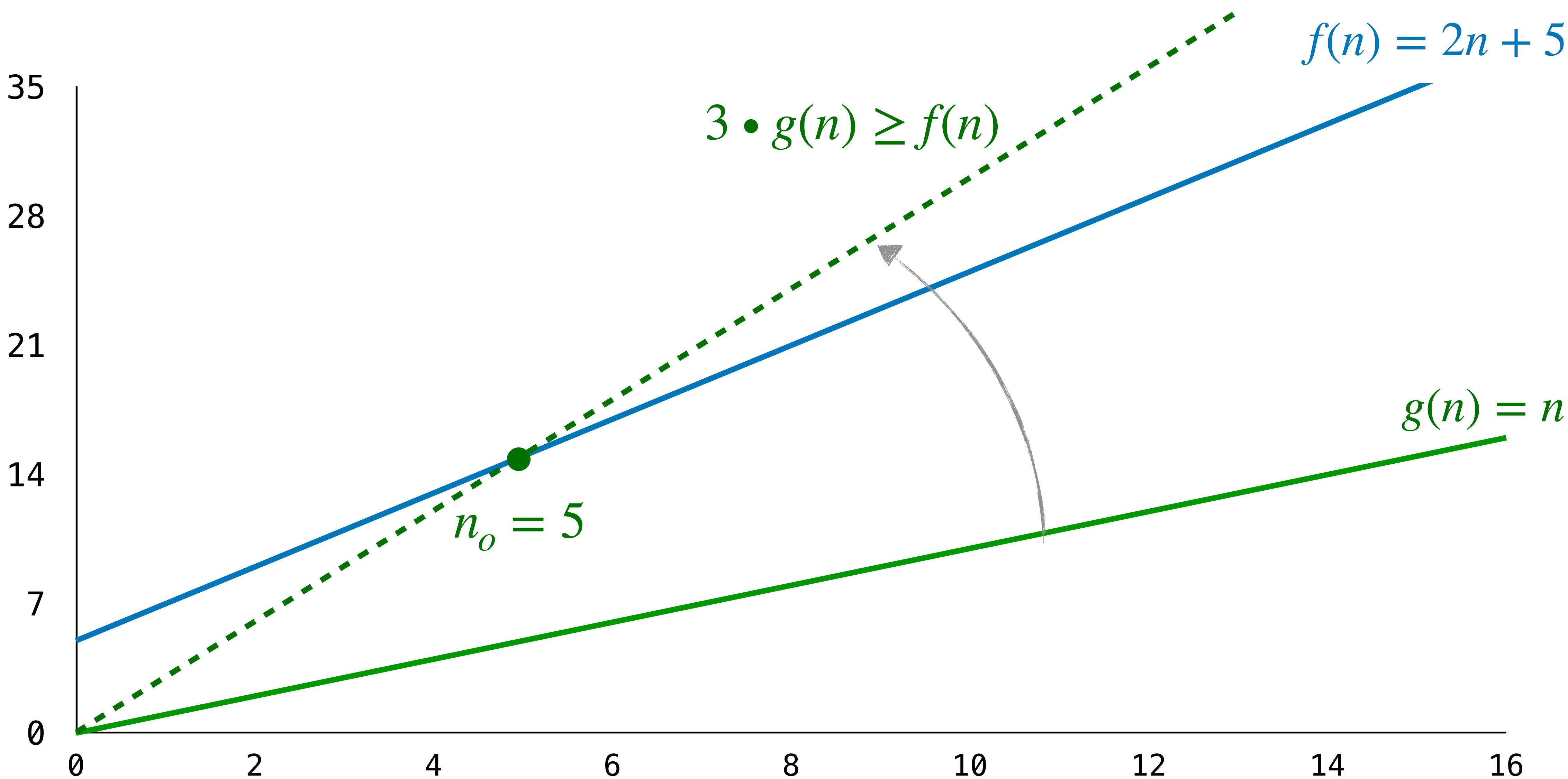
not the only possible c and n_o



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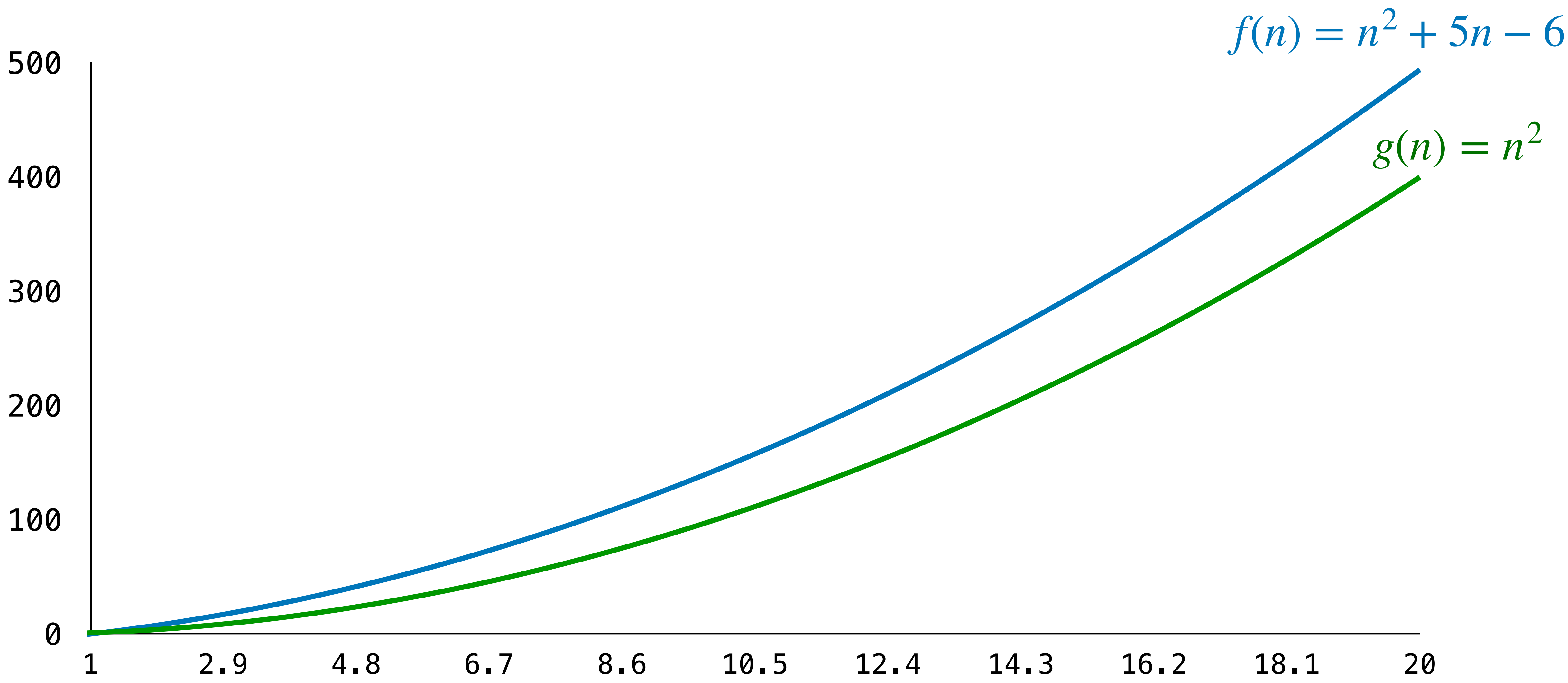

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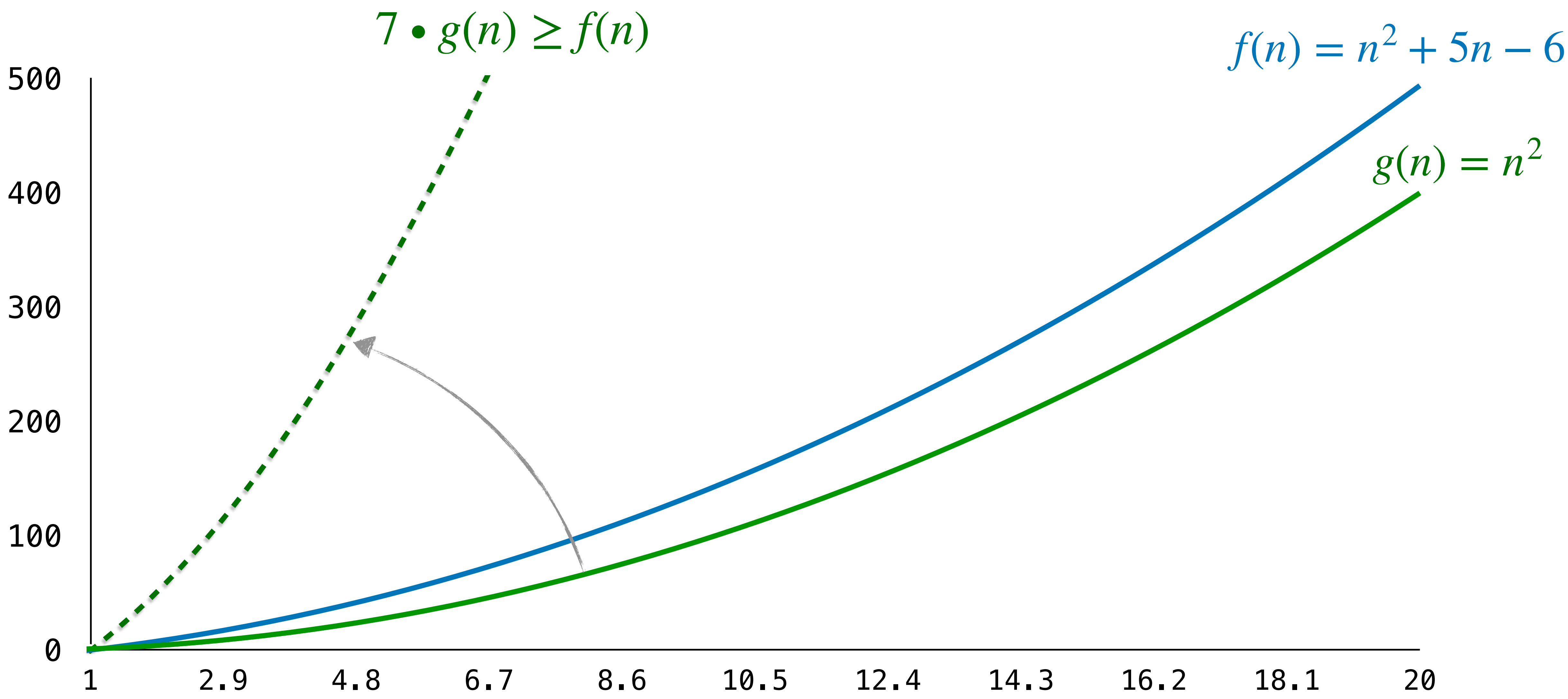

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$$\underbrace{n^2 + 5n - 6}_{f(n)} \leq n^2 + 5n \leq n^2 + 5n^2 \leq 6n^2 \leq \underbrace{7n^2}_{c \bullet f(n)} \quad \text{for all } n \geq 1.$$

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C. $f(n) = n^2 \quad \text{and} \quad g(n) = n^3$

Solution.

If we pick $c = 1$, It is clear that $c \bullet g(n) \geq f(n)$ for all $n \geq 1$.

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We need to show that: $2^n \leq c \cdot 3^n$ for all $n \geq n_o$.

Divide both sides by 2^n: $1 \leq c \cdot (\frac{3}{2})^n$

The left hand side is constant, whereas the right hand side is an increasing function. Therefore, there must be constants n_o and c such that $1 \leq c \cdot (\frac{3}{2})^n$ for all $n \geq n_o$.

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E. $f(n) = k^n$ and $g(n) = n!$ where k is a constant.

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$$n \log_2 k \leq \log_2 c + \underline{n \log_2 n - n \log_2 e + r \log_2 n}$$

Stirling's Approximation
(r is a positive constant)

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$$n \log_2 k \leq \log_2 c + n \log_2 n - n \log_2 e + r \log_2 n$$

Divide both sides by n:
$$\log_2 k \leq \frac{\log_2 c}{n} + \log_2 n - \log_2 e + \frac{r}{n} \log_2 n$$

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Divide both sides by n: $\log_2 k \leq \frac{\log_2 c}{n} + \log_2 n - \log_2 e + \frac{r}{n} \log_2 n$

The left hand side is constant, whereas the right hand side is an increasing function. Therefore, there must be some constants n_o, c and r such that the inequality is true.

Remember!

$$\boxed{\log(n!) = \Theta(n \log n)}$$

$$\begin{aligned}
\log(n!) &= \log(1 \times 2 \times 3 \times \ldots \times n-1 \times n) \\
&= \log(1) + \log(2) + \log(3) + \ldots + \log(n-1) + \log(n) \\
&\leqslant \log(n) + \log(n) + \log(n) + \ldots + \log(n) + \log(n) && n \text{ times} \\
&\leqslant n\log(n) \quad \text{for all} \quad n \geqslant 1 && \text{clearly } O(n \log n)
\end{aligned}$$

$$\begin{aligned}
\log(n!) &= \log(1 \times 2 \times \ldots \times \tfrac{n}{2} \times (\tfrac{n}{2}+1) \times \ldots \times n) \\
&= \log(1) + \log(2) + \ldots + \log(\tfrac{n}{2}) + (\log(\tfrac{n}{2}) + 1) + \ldots + \log(n) \\
&\geqslant \log(\tfrac{n}{2}) + (\log(\tfrac{n}{2}) + 1) + \ldots + \log(n) && \text{drop the first half of the sum} \\
&\geqslant \log(\tfrac{n}{2}) + \log(\tfrac{n}{2}) + \ldots + \log(\tfrac{n}{2}) && \tfrac{n}{2} \text{ times} \\
&\geqslant \tfrac{n}{2}\log(\tfrac{n}{2}) \quad \text{for all} \quad n \geqslant 1
\end{aligned}$$

a few more steps are needed to show that it is $\Omega(n \log n)$

Big-*O* Relatives

Big-Ω

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Less formally: If multiplying $g(n)$ by a constant makes it a lower bound for $f(n)$ after some point, then f is $\Omega(g)$.

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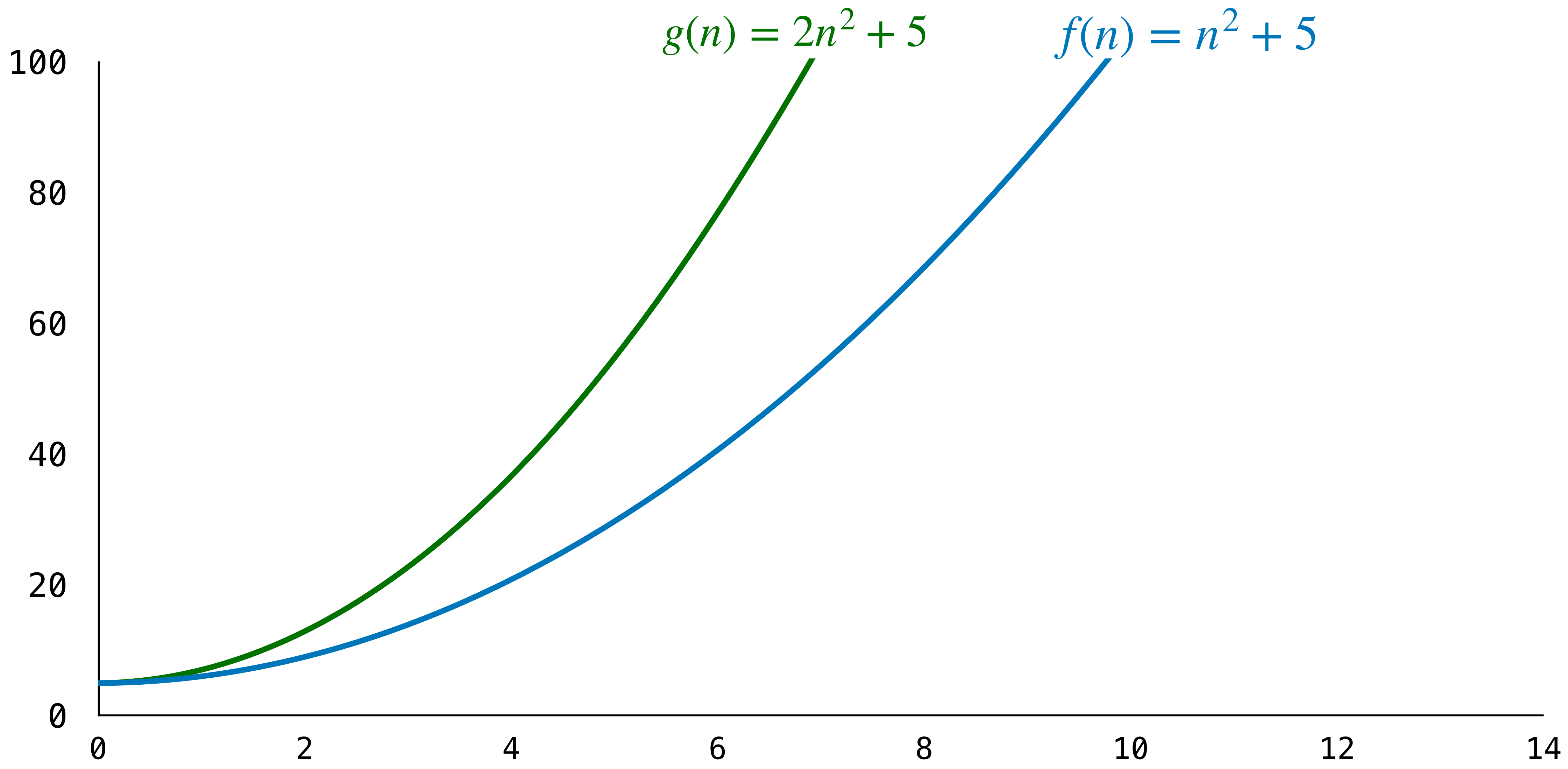

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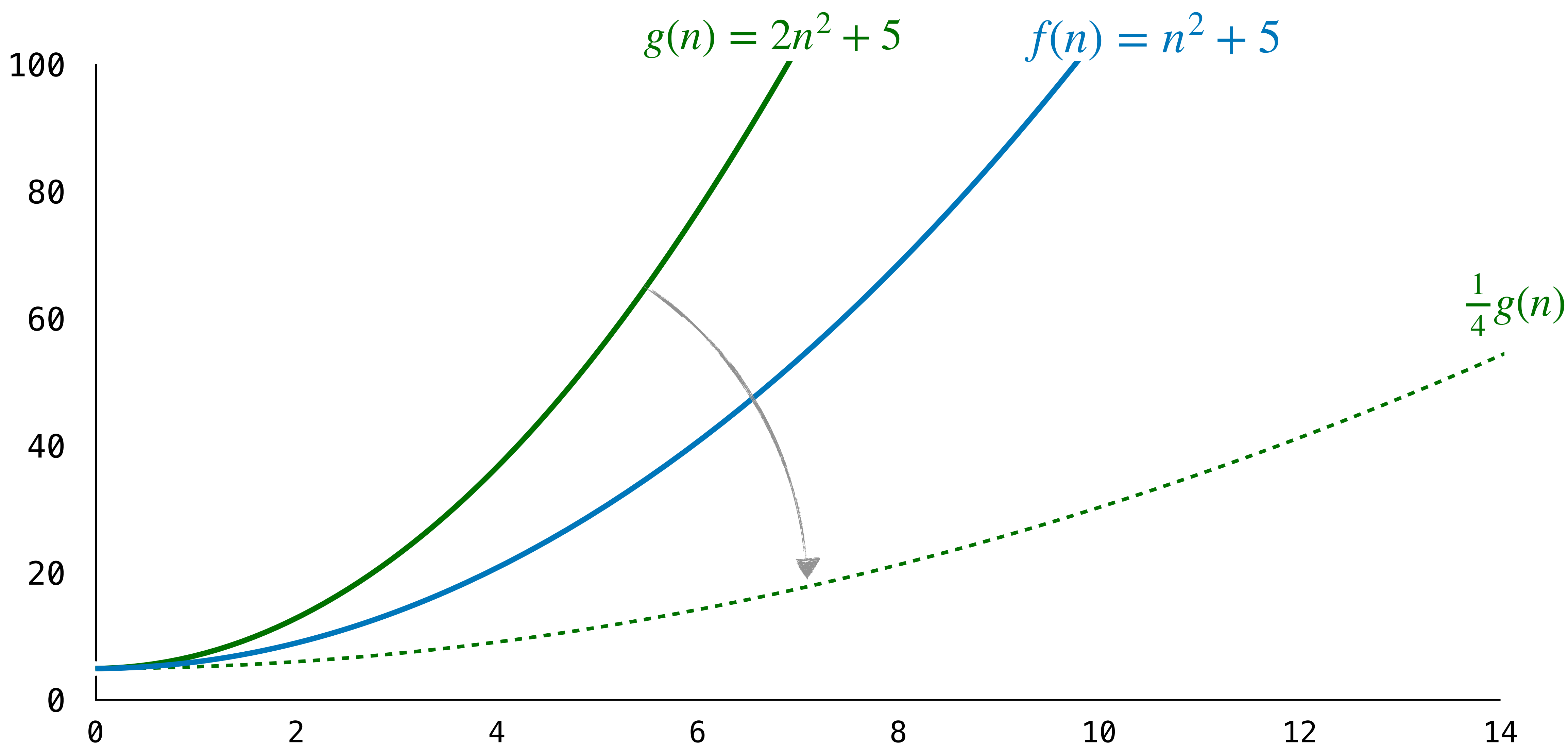

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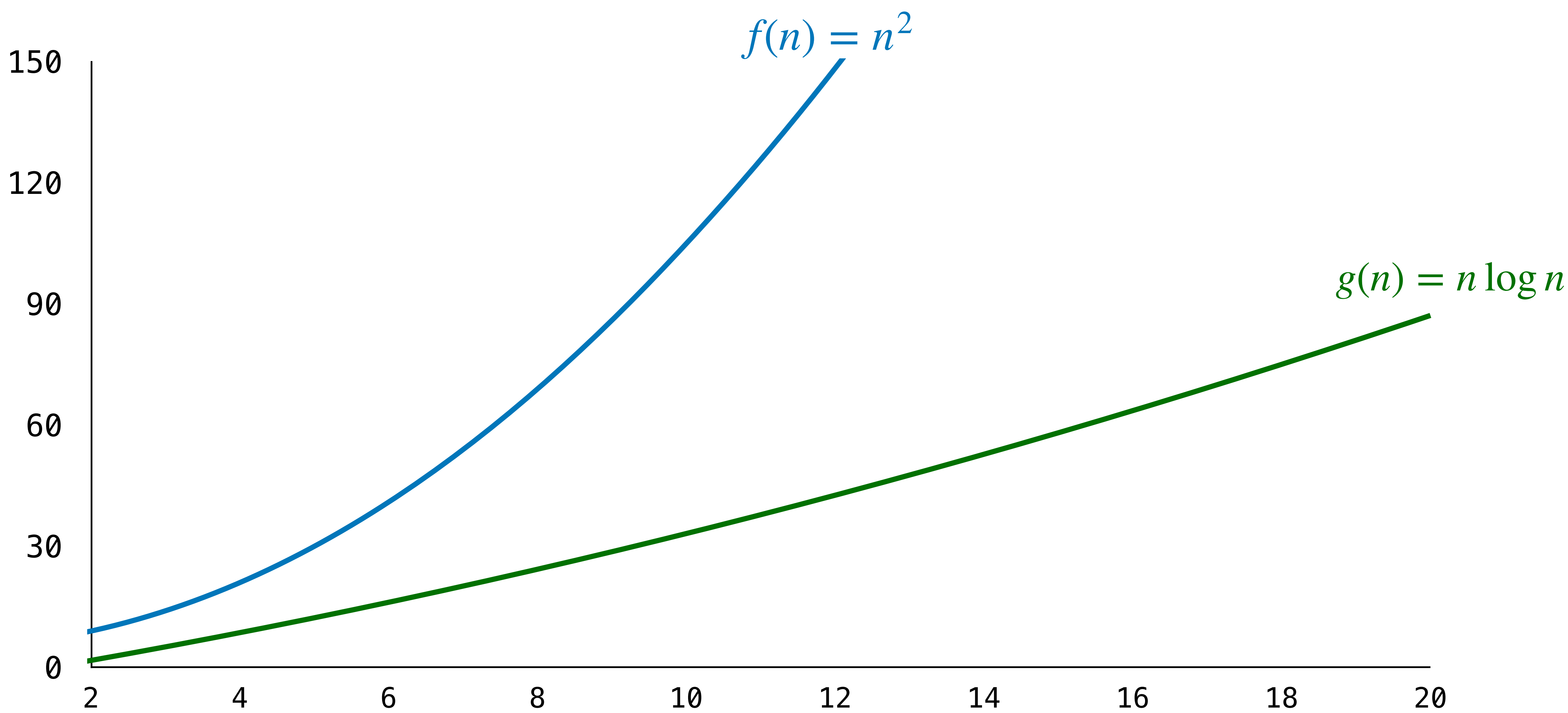
f is $\Omega(g)$ because there are c and n_o such that $0 \leq c \bullet g(n) \leq f(n)$ for all $n \geq n_o$:

If $c = 1$, then $g(n) \leq f(n)$ for all $n \geq 1$



Big-Θ

Definition. Let $f(n)$ and $g(n)$ be two functions that are always positive, $f(n)$ is said to be $\Theta(g)$ if and only if :

f is $O(g)$ and f is also $\Omega(g)$

Big-Θ

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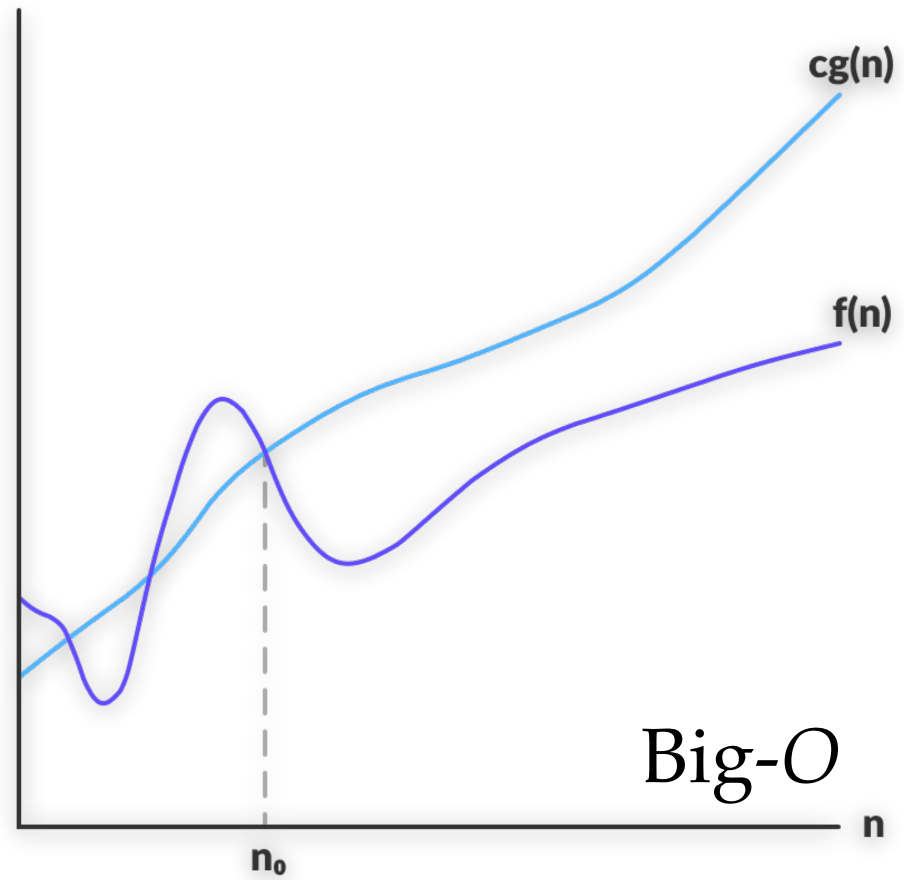
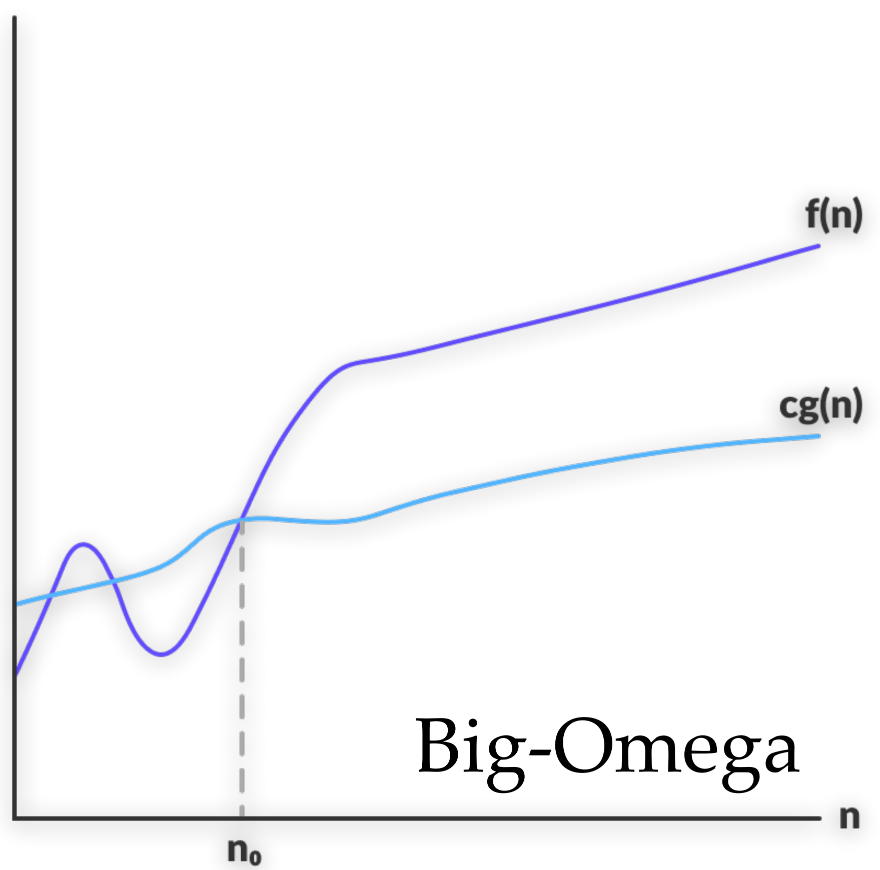
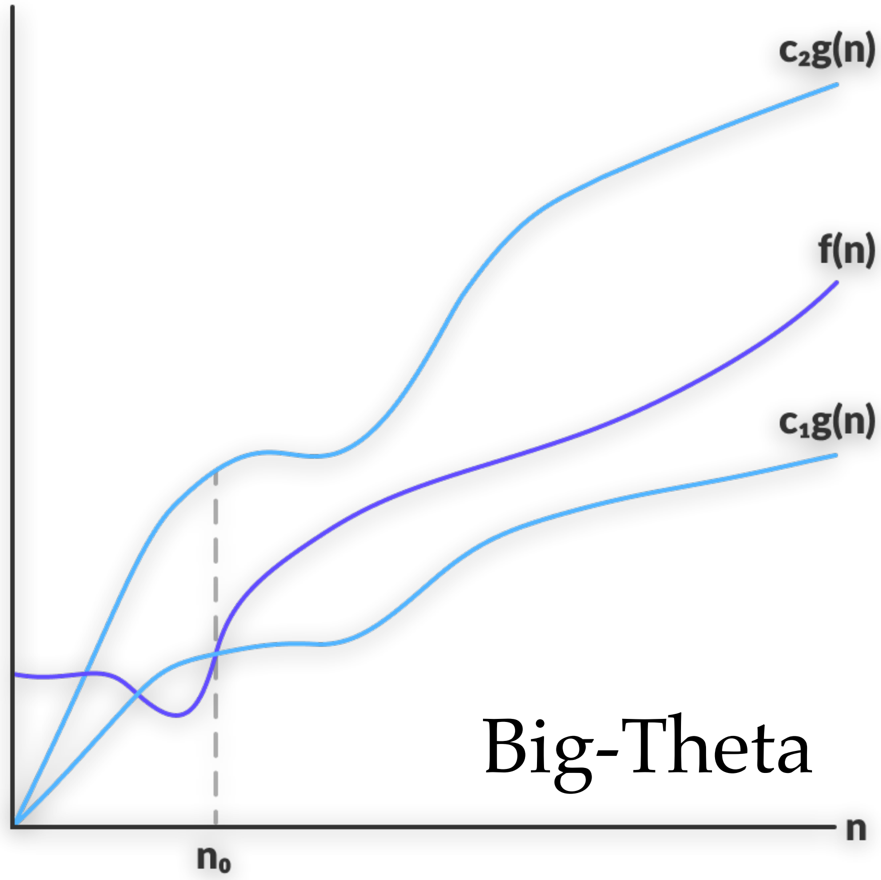
> f is $O(g)$ and f is also $\Omega(g)$

Less formally: If multiplying $g(n)$ by a constant makes it an upper bound for $f(n)$ after some point and also multiplying $g(n)$ by another constant makes it a lower bound for $f(n)$ after some point, then f is $\Theta(g)$.

Big-Θ







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For each of the following function, show that f is $\Theta(n)$.

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$$4n + 8 = O(n) \longrightarrow \text{pick } c = 20 \text{ and } n_o = 1$$

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We need to show that:

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More Relatives

Small-o and Small-ω

Informal Definition. f is said to be $o(g)$ if it grows strictly slower than g.

Informal Definition. f is said to be $\omega(g)$ if it grows strictly faster than g.

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$3n^2$ vs n^2

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$3n^2 = \Omega(n^2)$

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Quiz # 2

Which of the following is true about the running time of **insertion sort**?

A. The running time is $O(n^2)$

B. The running time is $\Omega(n)$

C. The best case is $\Theta(n)$.

D. The worst case is $\Theta(n^2)$.

E. All of the above.

Quiz # 3

Consider $f(n) = O(g(n))$.

Which of the following is true?

A. $g = \Omega(f)$

B. $0 \leqslant \lim_{n \to \infty} \dfrac{f(n)}{g(n)} < \infty$

C. All of the above.

D. None of the above.

Quiz # 3

Consider $f(n) = O(g(n))$.

Which of the following is true?

A. $g = \Omega(f)$ ←

$g = \Omega(f) \iff f = O(g)$
$g = \omega(f) \iff f = o(g)$
Sketch a graph to see that it's true!

B. $0 \leqslant \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$ ← See next slide!

C. All of the above.

D. None of the above.

Alternative Definitions

if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ *then*

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for all $n \geq n_o$

Properties

- Reflexivity. f is $\Theta(f)$

Properties

- **Reflexivity.** f is $\Theta(f)$ and $O(f)$ and $\Omega(f)$ but not $o(f)$ or $\omega(f)$

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  Example: $4n^2 + 5$ is $\Theta(n^2)$ and $4 \times (4n^2 + 5)$ is also $\Theta(n^2)$.

  Similarly: If f is $O(g)$ and $c > 0$, then $c \bullet f$ is $O(g)$.

  If f is $\Omega(g)$ and $c > 0$, then $c \bullet f$ is $\Omega(g)$.

  If f is $o(g)$ and $c > 0$, then $c \bullet f$ is $o(g)$.

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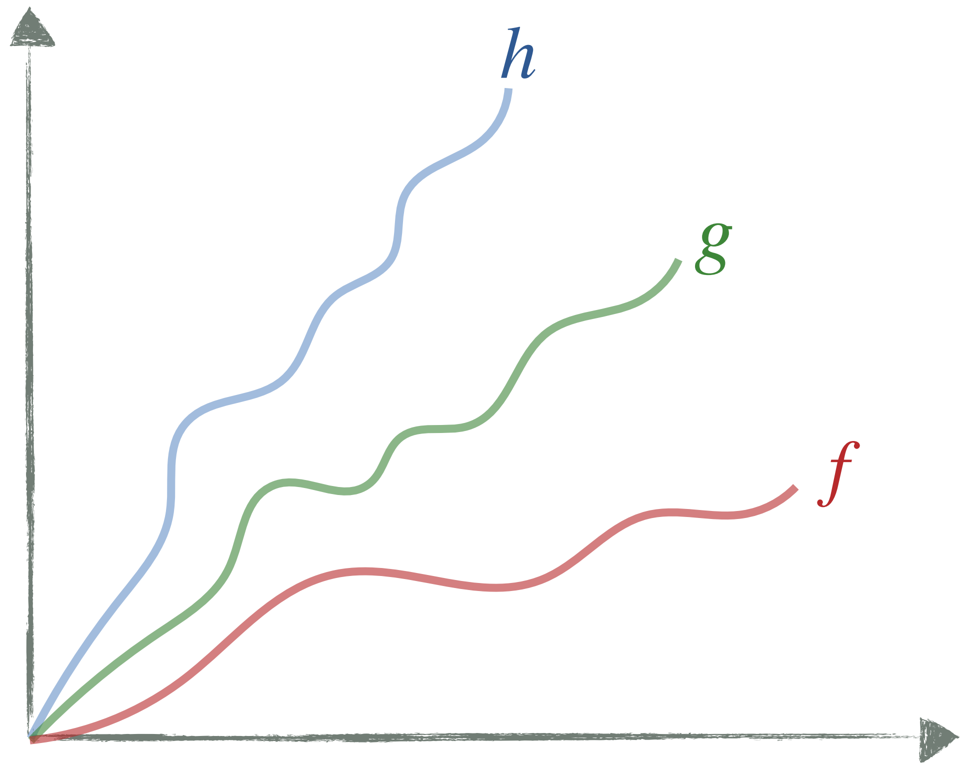

h is an upper bound for both g and f

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- Sums. If f_1 is $\Theta(g_1)$ and f_2 is $\Theta(g_2)$, then $f_1 + f_2$ is $\Theta(\max\{g_1, g_2\})$.

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! $O(g(n))$ is a set of functions, but computer scientists often *abuse* the notation by writing $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$.

Caution!

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  Although $\Theta(2^{\sqrt{n}})$ grows faster, it is performs less operations for $n \lessapprox 112000$.

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! To compare the **actual running time** of algorithms, other factors need to be taken into account (e.g. typical input sizes, likelihood of worst case, constant factors, lower order terms).