对于一个面：有三个顶点 $\vec{a}, \vec{b}, \vec{c}$

第4维面法向量 $\vec{d} = \dfrac{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})}{\| \text{分子} \|}$

在不考虑面位置情况下，$V_i = [\vec{b}-\vec{a}, \vec{c}-\vec{a}, \vec{d}] \in R^{3\times3}$

有 Target $\tilde{V}_i = \underbrace{T_i}_{\text{变化矩阵}} V_i \Rightarrow T_i = \tilde{V}_i V_i^{-1}$

n 代表 n 个顶点

m 代表 m 个面

$$\vec{x} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n+m})$$

求解目标 $= \min \sum_{j=1}^{M} \| S_j - T_j \|^2$

$S_j = \tilde{S}_j S_j^{-1}$ 已知.

$$T_j = \tilde{V}_j V_j^{-1}$$

$$= \begin{bmatrix} \vec{b}_j - \vec{a}_j \\ \vec{c}_j - \vec{a}_j \\ \vec{d}_j - \vec{a}_j \end{bmatrix} V_i^{-1}$$

$$= \left(\begin{bmatrix} \vec{b}_j \\ \vec{c}_i \\ \vec{d}_i \end{bmatrix} - \begin{bmatrix} \vec{a}_j \\ \vec{a}_j \\ \vec{a}_j \end{bmatrix} \right) V_i^{-1}$$

$$= [\vec{v}_1 \cdots \vec{v}_n \; \vec{d}_1 \; \vec{d}_2 \cdots \vec{d}_m] \begin{array}{c} a_j \text{行} \\ b_j \\ c_j \\ d_j \end{array} \begin{bmatrix} -1 & -1 & -1 \\ 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} V_i^{-1}$$

$$\min \sum \| S_j - T_j \|$$

$$= \min \sum \| T_j - \tilde{S}_j S_j^{-1} \| \qquad S_j \in R^{3\times 3}$$

$$= \min \sum \| x \hat{V}_i^{-1} - \underbrace{\tilde{S}_j S_j^{-1}}_{= C_j} \| \qquad \hat{V}_i^{-1} = \underbrace{\begin{pmatrix} -1 & -1 & -1 \\ 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}}_{(n+m)\times 3} \underbrace{V_i^{-1}}_{3\times 3}$$

$$= \min \sum \| \hat{V}_i^{-T} x^T - C_j^T \| \qquad x \in R^{3\times(n+m)}$$

$$= \min \left\| \begin{pmatrix} \hat{V}_1^{-T} \\ \hat{V}_2^{-T} \\ \vdots \\ \hat{V}_m^{-T} \end{pmatrix} x^T - \begin{pmatrix} C_1^T \\ C_2^T \\ \vdots \\ C_m^T \end{pmatrix} \right\|$$

设 $A = \begin{pmatrix} \hat{V}_1^{-T} \\ \hat{V}_2^{-T} \\ \vdots \\ \hat{V}_m^{-T} \end{pmatrix} \in R^{(3\times m)\times(n+m)}$

$\vec{b} = \begin{pmatrix} C_1^T \\ \vdots \\ C_m^T \end{pmatrix} \in R^{(3\times m)\times 3}$

即 $\min \| A x^T - b \|$， $A^T A x = A^T b$.