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Notice that we can prove easily, using Axioms 2.1, 2.2, and induction (Axiom 2.5), that the sum of two natural numbers is again a natural number (why?).

> **Axiom 2.1** 0 is a natural number.
> **Axiom 2.2** If n is a natural number, then $n{++}$ is also a natural number.
> **Axiom 2.5** (Principle of mathematical induction). Let $P(n)$ be any property per taining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{++})$ is also true. Then $P(n)$ is true for every natural number n.
> **Definition 2.2.1** (Addition of natural numbers). Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n{++}$ to m by defining $(n{++}) + m := (n + m){++}$.

Proof. Fix an any natural number m, we want to show that for any natural number n, $n + m$ is a natural number.

Base case: We have $0 + m = m$, which is a natural number by Axiom 2.1.

Inductive step: Assume that for some natural number n, $n + m$ is a natural number.

By Definition 2.2.1, we have $(n{++}) + m = (n + m){++}$. Since $n + m$ is a natural number, by Axiom 2.2, $(n + m){++}$ is also a natural number. Thus, $(n{++}) + m$ is a natural number.

Therefore, by the principle of mathematical induction (Axiom 2.5), we conclude that for any natural number n, $n + m$ is again a natural number.

Since the arbitrary choice of m was made, we can conclude that the sum of two natural numbers is again a natural number.

□

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As a particular corollary of Lemma 2.2.2 and Lemma 2.2.3 we see that $n{++} = n + 1$(why?).

> **Lemma 2.2.2** For any natural number n, $n + 0 = n$.
> **Lemma 2.2.3** For any natural numbers n and m, $n + (m{++}) = (n + m){++}$.

Proof. In Lemma 2.2.3, let $m = 0$. Then we have

$$n + (0{++}) = (n + 0){++}$$

By Lemma 2.2.2, we have $n + 0 = n$. Thus, the left hand side becomes $n + 1$. The right hand side becomes $(n + 0){++} = n{++}$. □

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When $a = 0$ we have $0 \leq b$ for all b (why?)

If $a > b$, then $a{++} > b$ (why?).

If $a = b$, then $a{++} > b$ (why?).

Proof. See solution to Exercise 2.2.4 □

4 Another handy lemma in textbook Chapter 2.2

Lemma. For any natural number a and b, $a < b{+}{+}$ iff. $a \leq b$.

Proof. Included in solution ot Exercise 2.2.5 $\square$

5 Exercise 2.2.7

Solution found from `https://math.stackexchange.com/a/4730660`

Exercise 2.2.7 Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m)$ is true, $P(m{+}{+})$ is true. Show that if $P(n)$ is true, then $P(m)$ is true for all $m \geq n$. (This principle is sometimes referred to as the principle of induction starting from the base case n.)

> **Definition 2.2.11** (Ordering of the natural numbers) Let n and m be natural numbers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$,iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$,iff $n \geq m$ and $n = m$.

Proof. Define $Q(m) := P(n + m)$. Note that, in particular, $Q(0) = P(n)$.

We are looking to prove the statement "Whenever $P(m)$ is true, $P(m{+}{+})$ is true. If $P(n)$ is true, then $P(m)$ is true for $m \geq n$ ", and we will do so via induction on Q.

For the base case, we are already told we can assume $P(n)$ is true, so $Q(0)$ is true.

Then for the inductive step, we have $Q(m) = P(n + m) \implies P((n + m){+}{+}) = P(n + (m{+}{+})) = Q(m{+}{+})$ (by Lemma2.2.3, already quoted in above section not far away). But since $Q(m) \implies Q(m{+}{+})$ and $Q(0)$, then $Q(m)$ is true for every natural m (by Axiom 2.5 Principle of mathematical induction).

Now, consider $P(m)$ for $m \geq n$. By Definition 2.2.11 (Ordering of the natural numbers), this means that $m = n + a$ for some natural number a. That means we can write $P(m) = P(n + a) = Q(a)$, but by our inductive proof $Q(a)$ is true for any natural number a, therefore $P(m)$ is also true, and that completes the proof. $\square$

6 Exercise 3.2.1

Exercise 3.2.1 Show that the universal specification axiom, Axiom 3.9, if assumed to be true, would imply Axioms 3.3, 3.4, 3.5, 3.6,and 3.7. (If we assume that all natural numbers are objects, we also obtain Axiom 3.8.) Thus, this axiom, if permitted, would simplify the foundations of set theory tremendously (and can be viewed as one basis for an intuitive model of set theory known as "naive set theory"). Unfortunately, as we have seen, Axiom 3.9 is "too good to be true" !

> **Axiom 3.3** (Empty set).There exists a set $\emptyset$, known as the empty set,which contains no elements, i.e., for every object x we have $x \notin \emptyset$.
>
> **Axiom 3.4** (Singleton sets ~~and pair sets~~). If a is an object, then there exists a set $\{ a \}$ whose only element is a, i.e., for every object y,we have $y \in \{ a \}$ if and only if $y = a$; we refer to $\{ a \}$ as the singleton set whose element is a.
>
> **Axiom 3.5** (Pairwise union). Given any two sets A, B, there exists a set $A \cup B$, called the union of A and B, which consists of all the elements which belong to A or B or both. In other words, for any object x,
>
> $$x \in A \cup B \iff (x \in A \text{ or } x \in B)$$
>
> **Axiom 3.6** (Axiom of specification). Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e., for each $x \in A$, $P(x)$ is either a true statement or a false statement). Then there exists a set, called $\{ x \in A : P(x) istrue \}$ (or simply $\{ x \in A : P(x) \}$ for

short), whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y,

$$y \in \{ x \in A : P(x) \text{ is true} \} \iff (y \in A \text{ and } P(y) \text{ is true})$$

Axiom 3.7 (Replacement). Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which P(x,y) is true. Then there exists a set $\{ y : P(x, y) \text{ is true for some } x \in A \}$, such that for any object z,

$$\begin{gathered} z \in \{ y : P(x, y) \text{ is true for some } x \in A \} \\ \iff P(x, z) \text{ is true for some } x \in A \end{gathered}$$

Axiom 3.9 (Universal specification). (Dangerous!) Suppose for every object x we have a property $P(x)$ pertaining to x (so that for every x, $P(x)$ is either a true statement or a false statement). Then there exists a set $\{ x : P(x) \text{ is true} \}$ such that for every object y,

$$y \in \{ x : P(x) \text{ is true} \} \iff P(y) \text{ is true}$$

Proof. Axiom 3.3 (Empty set): Let $P(x)$ be the property that $x \neq x$.

$$y \in \{x : x \neq x \text{ is true}\} \iff y \neq y \text{ is true.}$$

Axiom 3.4 (Singleton sets ~~and pair sets~~): Let $P(x)$ be the property that $x = a$.

$$y \in \{x : x = a \text{ is true}\} \iff y = a \text{ is true.}$$

Axiom 3.5 (Pairwise union): Let $P(x)$ be the property that $x \in A$ or $x \in B$.

$$y \in \{x : x \in A \text{ or } x \in B \text{ is true}\} \iff y \in A \text{ or } y \in B \text{ is true.}$$

Axiom 3.6 (Axiom of specification). Let $P(x)$ be the property that $x \in A$ and $P(x)$ is true.

$$y \in \{x : x \in A \text{ and } P(x) \text{ is true}\} \iff y \in A \text{ and } P(y) \text{ is true.}$$

Axiom 3.7 (Replacement) Let $P(y)$ be the property that there exists $x \in A$ such that $P(x, y)$ is true.

$$z \in \{y : P(y) \text{ is true}\} = \{x : P(x, y) \text{ is true for some } x \in A\} \iff P(x, z) \text{ is true for some } x.$$

□

7 Exercise 3.2.2

Use the axiom of regularity (and the singleton set axiom) to show that if A is a set, then $A \notin A$. Furthermore, show that if A and B are two sets, then either $A \notin B$ or $B \notin A$ (or both). (One corollary of this exercise is worth noting: given any set A, there exists a mathematical object that is not an element in A, namely A itself. Thus one can always "add one more element" to a set A to create a larger set, namely $A \cup \{ A \}$.)

Axiom 3.10 (Regularity). If A is a non-empty set, then there is at least one element x of A which is either not a set, or is disjoint from A.

Proof. Suppose that A is a non-empty set, and A is an element of A.

Let's consider set $\{ A \}$, which can be constructed through Axiom 3.4:

Obviously, $A \in \{ A \}$

By Axiom 3.10, A is either not a set, or is disjoint from $\{ A \}$, which leads to $A \cap \{ A \} = \emptyset$.

Since $A \in A$ and $A \in \{ A \}$, this implies that $A \cap \{ A \} = A$, a contradiction. While if A is an empty set. By Axiom 3.3, $A \notin A$.

Furthermore, suppose that A, B are two sets and we both have $A \in B$ and $B \in A$. By Axiom 3.4 (and Axiom 3.5), we have $A, B \in \{ A, B \}$. By Axiom 3.10, we have $A \cap \{ A, B \} = \emptyset$ and $B \cap \{ A, B \} = \emptyset$, but this is contradictive with $A \in B, B \in A$ and $A, B \in \{ A, B \}$. $\square$

8 Exercise 3.2.3

Show (assuming the other axioms of set theory) that the universal specification axiom, Axiom 3.9, is equivalent to an axiom postulating the existence of a "universal set" Ω consisting of all objects (i.e., for all objects x,we have $x \in \Omega$). In other words, if Axiom 3.9 is true, then a universal set exists, and conversely, if a universal set exists, then Axiom 3.9 is true. (This helps explain why Axiom 3.9 is called the axiom of universal specification.) Note that if a universal set existed, then we would have $\Omega \in \Omega$ by Axiom 3.1, contradicting Exercise 3.2.2. Thus the axiom of foundation specifically rules out the axiom of universal specification.

Proof. Suppose that Axiom 3.8 it true, then we have $y \in \{x : x \in \Omega\}$. Conversely, if a universal set Ω exists, by Axiom 3.6, we have $y \in \{x \in \Omega : P(x) \text{ is true}\} \implies y \in \{x : P(x) \text{ is true}\}$. Hence we obtain Axiom 3.8. $\square$

9 Thoughts on the set of all sets

Let U as a universal set, i.e. the set of all sets. Does such U exist?

Proof. The proof is clipped from PanSci. We slightly modify the example in Russell' s Paradox ($\{ x \mid x \notin x \}$) to

$$B := \{ x \in U \mid x \notin x \}$$

If $B \in B$, the by the definition of B, there is

$$B \in B \implies B \in V \text{ and } B \notin B \tag{1}$$

If $B \notin B$, since U contains every set, we have $B \in V$, so B complies the definition of B, which means

$$B \in U \text{ and } B \notin B \implies B \in B \tag{2}$$

With (1) and (2), we get

$$B \in B \iff B \in U \text{ and } B \notin B$$

This made the claim above to be true only when it's vacuously true, which is that $B \in V$ is not true. Thus lead to contradiction. $\square$